THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

INVESTIGATION OF A GAUGE THEORY IN GENERAL RELATIVITY

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma

1975

INVESTIGATION OF A GAUGE THEORY IN

GENERAL RELATIVITY

APPROVED BY:

DISSERTATION COMMITTEE

ACKNOWLEDGEMENTS

I would like to express my gratitude to Dr. Jack Cohn not only for suggesting this investigation, but also, by his patience and enthusiasm, for creating an atmosphere which sparked and allowed for the development of whatever creativity I possess. I would also like to thank Dr. R. Kantowski, Dr. D. Shay, Dr. S. Babb, and Dr. W. Huff for their reading of this manuscript.

A special work of thanks is due to my mother for her efforts in preparing this manuscript, to Jaquine Littell for the typing of the manuscript, and to my wife, Cheryl, for her patience and understanding during the construction and preparation of this dissertation.

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CHAPTER I

INTRODUCTION

In a recent paper, henceforth to be referred to as I, Cohn[1] has presented a gauge theory of relativity. In his paper, Cohn re-emphasizes the view put forth by Reichenbach[2], that a physical theory which prescribes a metric via a set of field equations is incomplete unless accompanying these equations there is some prescription as to how lengths and times measured by rods and clocks in inertial reference frames are to be compared at different points in space-time. This prescription is called a congruence definition or simply a gauge. It should be emphasized that this prescription is a definition or convention, since there is no way to compare the lengths and times measured by rods and clocks at different space-time points. However, once a theory containing both field equations and a congruence definition has been set forth, it is possible via the predictions of the theory to check the compatibility of the gauge with the field equations. For a complete discussion of these points, the reader is referred to reference 2.

In the Einstein theory, the prescription is simply that the field equations are to be valid in a gauge where, by definition, the length of a rod as measured from a local inertial rest frame shall remain unchanged when it is transferred from one space-time event to any other. The same statement being true for the time intervals between the ticks of a clock. In what follows, this gauge will be referred to as the customary gauge.

Cohn proposes a different prescription. Namely, that Einstein's field equations are to be valid in some gauge other than the customary one. That is, by definition the length or time measured by a rod or clock in a local inertial rest frame will be different at different space-time points. Henceforth, this gauge will be called the non-customary gauge.

The problem to be dealt with in this work is two-fold. First, to construct an equation determining this non-customary gauge. Second, to compare the predictions in this non-customary gauge theory with the predictions made by the customary gauge theory of ordinary Einstein relativity and to observed phenomena. To this end, we will now briefly outline Cohn's theory. For a more detailed discussion, the reader is referred to I.

1. Cohn's Gauge Theory

We shall assume, along with Cohn, that the length, $\bar{\ell}$, and the period, $\bar{\tau}$, of a rod and clock change via a conformal transformation to ℓ and τ when transported from a local inertial rest frame at an event $\{x_0^\mu\}$ to another local inertial rest frame at $\{x^\mu\}$. That is,

$$\ell = \lambda^{-1}(x^1,\ldots,x^4)\bar{\ell} \tag{I.1.1a}$$

and

$$\tau = \lambda^{-1}(x_1,\ldots x_4)\bar{\tau} \tag{I.1.1b}$$

where

$$\lambda^{-1}(x_0^1,\ldots x_0^4) = 1 \ , \tag{I.1.1c}$$

and $\{x_0^\mu\}$ is arbitrary.

In these equations, λ^{-1} will be taken to be a real, positive, and as yet unspecified function of the coordinates $\{x^\mu\}$.

For the sake of completeness, besides the above transformations on length and time, we shall allow, again by definition, the proper-mass, $\bar{m}$, of a particle to change under this transportation to m according to

$$m = \lambda^{-\beta}\,\bar{m} \tag{I.1.2}$$

β being a constant. Later we shall confine our theory to consider only the case $\beta=0$.

At this point, it should be noted that in the world we live in observations of physical phenomena are, by convention, carried out using the customary congruence definition. So if we are to compare the predictions of our non-customary gauge theory to observed phenomena or to ordinary Einstein theory, we shall need relationships between the basic physical quantities in the two gauges. Realizing this, we further assume that at any single event of our choosing we may assign to the non-customary lengths, masses, and times their values as determined by an observer using the customary gauge. So in (I.1.1), if $\{x_0^\mu\}$ is the selected event, $\bar{\ell}$, $\bar{\tau}$, and $\bar{m}$ are the length of a small rod, the period of a clock, and the mass of a particle respectively as determined by an observer using the customary congruence definition. In what follows, unless otherwise stated, barred quantities will refer to the customary gauge and unbarred quantities the non-customary gauge.

Letting $\bar{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ denote the metric tensors in the two gauges, and setting $\lambda^{-1} = e^{-\sigma}$, from the statements above we have that the relationship between the four-intervals in the customary and non-customary gauges is

$$d\bar{s} = e^{\sigma}\, ds \,, \tag{I.1.3}$$

where

$$d\bar{s}^2 = \bar{g}_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$$

and

$$ds^2 = g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$$

Furthermore, letting space-time coordinates be dimensionless, i.e. gauge invariant, we can now write down several well known properties of conformal transformations.[3]

For the metric tensors and for the Christoffel symbols $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ and $\Gamma^{\alpha}_{\beta\gamma}$ constructed from the customary and non-customary metric tensors respectively, we have:

$$\begin{aligned} \bar{g}_{\mu\nu}\{x^{\alpha}\} &= e^{2\sigma}\, g_{\mu\nu}\{x^{\alpha}\} \\ g^{\mu\nu}\{x^{\alpha}\} &= e^{-2\sigma}\, g^{\mu\nu}\{x^{\alpha}\} \end{aligned} \tag{I.1.4}$$

and

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \delta^{\alpha}_{\beta}\, \sigma,_{\gamma} + \delta^{\alpha}_{\gamma}\, \sigma,_{\beta} - g_{\gamma\beta}\, g^{\alpha\varepsilon}\, \sigma,_{\varepsilon} \tag{I.1.5}$$

where commas donate ordinary differentiation.

Letting the Ricci tensor, $R_{\mu\nu}$, be defined by

$$-R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\gamma} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\beta}\,\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\beta\alpha}\,\Gamma^{\beta}_{\mu\nu} \qquad \text{(I.1.6)}$$

we have for the relationship between the customary and non-customary Einstein tensors:

$$\bar{G}_{\mu\nu} = G_{\mu\nu} - 2\sigma_{,\mu}\,\sigma_{,\nu} - \bar{g}_{\mu\nu}\,\bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\,\sigma_{,\beta} - 2\sigma_{,\mu;\nu}$$
$$+ 2\bar{g}_{\mu\nu}\,\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta}\ . \qquad \text{(I.1.7)}$$

In this last expression, semicolons denote covariant differentiation employing the customary Christoffel symbols.

Following Cohn, we now assume that the Einstein field equations hold in the non-customary gauge. That is,

$$G_{\mu\nu} = K\, T_{\mu\nu} + g_{\mu\nu}\Lambda\ , \qquad \text{(I.1.8)}$$

where K is a constant defined by

$$K = \frac{8\pi G}{c^4}\ , \qquad \text{(I.1.9)}$$

and $T_{\mu\nu}$ is the stress-energy-momentum tensor in a non-customary gauge. Λ in (I.1.8) is a cosmological constant which shall be assumed to be small.

Once again, since our physical observations are carried out using the customary congruence definition, we shall need a relationship between $T_{\mu\nu}$ and the customary stress-energy-momentum tensor $\bar{T}_{\mu\nu}$ in order to make comparisons between our theory, observed phenomena, and Einstein's theory. To this end, we notice that since our space-time coordinates are dimensionless then so is $G_{\mu\nu}$, and hence so are $KT_{\mu\nu}$ and $g_{\mu\nu}\Lambda$. If we now assume that K transforms in accord with its dimensions under the transformations (I.1.1) and (I.1.2) to a quantity $\bar{K}$, we have:

$$\bar{K} = K\, e^{\sigma(1-\beta)}\ . \qquad \text{(I.1.10)}$$

We shall now be interested in constructing a reasonable relationship between $KT_{\mu\nu}$ and $\bar{K}\bar{T}_{\mu\nu}$. In order to accomplish this we consider the customary stress-energy-momentum tensor of a globule of incoherent matter of mass density $\bar{\rho}$ in the customary gauge:

$$\bar{T}_{\mu\nu} = \bar{\rho}\ \bar{g}_{\mu\nu}\ \bar{g}_{\gamma\beta}\ \frac{dx^{\alpha}}{d\bar{s}}\ \frac{dx^{\beta}}{d\bar{s}}\ . \tag{I.1.11}$$

Applying the transformations (I.1.1), (I.1.2), (I.1.3) and (I.1.4) to (I.1.11) and invoking (I.1.10), we have

$$\bar{K}\bar{T}_{\mu\nu} = KT_{\mu\nu} \tag{I.1.12}$$

where

$$T_{\mu\nu} = \rho\ g_{\mu\alpha}\ g_{\mu\beta}\ \frac{dx^{\alpha}}{ds}\ \frac{dx^{\beta}}{ds}$$

and

$$\rho = \bar{\rho}\ e^{\sigma(3-\beta)}\ . \tag{I.1.13}$$

We will assume in all that follows that (I.1.12) holds in general.

Furthermore, we take

$$g_{\mu\nu}\ \Lambda = \bar{g}_{\mu\nu}\ \bar{\Lambda} \tag{I.1.14}$$

where by (I.1.4):

$$\bar{\Lambda} = e^{-2\sigma}\ \Lambda \tag{I.1.15}$$

Using (I.1.12) and (I.1.14) equation (I.1.18) becomes

$$G_{\mu\nu} = \bar{K}\ \bar{T}_{\mu\nu} + \bar{g}_{\mu\nu}\ \bar{\Lambda} \tag{I.1.16}$$

Inserting this last result into (I.1.7) yields

$$\begin{aligned}\bar{G}_{\mu\nu} = \bar{K}\ \bar{T}_{\mu\nu} + \bar{g}_{\mu\nu}\ \bar{\Lambda} - 2\sigma_{,\mu}\ \sigma_{,\nu} - \bar{g}_{\mu\nu}\ \bar{g}^{\alpha\beta}\ \sigma_{,\alpha}\ \sigma_{,\beta} \\ - 2\sigma_{,\mu;\nu} + 2\bar{g}_{\mu\nu}\ \bar{g}_{\alpha\beta}\ \sigma_{,\alpha;\beta}\end{aligned} \tag{I.1.17}$$

This equation points out that σ appears in the non-customary field equations expressed in the customary gauge much like the scalar field of Brans and Dicke appears in their field equation.[4] The relationships between our theory and that of Brans and Dicke will be taken up again in section 2 of this chapter. Before proceeding with this, however, we now note that it is Eq. (I.1.16), or equivalently (I.1.17), together with an equation on σ which is to be compared to observations and to Einstein's theory. In our notation, Einstein's field equations in the customary gauge are

$$\bar{G}_{\mu\nu} = K\,\bar{T}_{\mu\nu} + \bar{g}_{\mu\nu}\,\Lambda \tag{I.1.18}$$

In what follows, the reader is cautioned not to confuse the quantities $\bar{K}$ and $\bar{\Lambda}$ defined by (I.1.10) and (I.1.15) with the constants K and Λ. Also, the reader should note that in the ordinary Einstein theory, $\bar{T}^{\mu\nu}{}_{;\nu} = 0$, whereas in our theory from (I.1.16) or (I.1.17), $\bar{T}^{\mu\nu}{}_{;\nu} \neq 0$. From (I.1.8), we have that in our non-customary gauge theory the statement $\bar{T}^{\mu\nu}{}_{;\nu} = 0$ is to be replaced by the statement

$$T^{\mu\nu}{}_{|\nu} = 0 \tag{I.1.19}$$

where slash means covariant differentiation using the non-customary metric tensor $g_{\mu\nu}$, and the non-customary Christoffel symbols, $\Gamma^{\alpha}_{\beta\gamma}$.

2. Relationship to the Brans-Dicke Theory

We now mention, as can be seen from (I.1.17), that our theory is incomplete without an equation governing σ.

Since in Chapter II we will invoke analogies existing between our theory and the Brans-Dicke theory to develop equations on σ; at this point we will briefly discuss the Brans-Dicke formulation of relativity with a cosmological constant included and point out the similarities and differences existing between this formulation and ours. For a detailed description of the Brans-Dicke theory without a cosmological constant, the reader is referred to their original paper.[4] Also for a more thorough description of the analogies existing between the Brans-Dicke scalar field theory and our non-customary gauge theory without a cosmological constant, the reader is referred to I.

Since the Brans-Dicke theory is formulated via a variational principle the easiest way to proceed then is to develop both theories from variational principles. Since the Brans-Dicke theory is formulated using the customary congruence definition, we shall use barred quantities, where appropriate, in formulas dealing with their theory.

First of all, it is apparent that Cohn's field equations, (I.1.8), can be derived from the following variational principle:

$$\delta \int \left[R + 2KL + 2\Lambda\right] \sqrt{-g}\, d^4x = 0 \qquad \text{(I.2.1a)}$$

where the variations are taken with respect to the non-customary metric tensor $g_{\mu\nu}$, with $\delta\, g_{\mu\nu}$ vanishing on the boundary and $L = L(g_{\mu\nu})$ is so constructed that

$$\frac{\partial(\sqrt{-g}\, L)}{\partial\, g_{\mu\nu}} = \frac{1}{2}\sqrt{-g}\, T^{\mu\nu} \ .$$

If in (I.2.1), instead of carrying out the variations with respect to $g_{\mu\nu}$ we carry them out with respect to the customary metric tensor, $\bar{g}_{\mu\nu}$, holding σ fixed the result is Eq. (I.1.17). To demonstrate this we note that from (I.1.4) and (I.1.7)

$$R = e^{2\sigma}\,\bar{R} + 6e^{2\sigma}\left[\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta} - \bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\,\sigma_{,\beta}\right]$$

and

$$\sqrt{-g} = e^{-4\sigma} \sqrt{-\bar{g}}$$

Inserting these last two statements into (I.2.1), we find that

$$\delta\int\left\{e^{-2\sigma}\,\bar{R} + 6e^{-2\sigma}\left[\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta} - \bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\,\sigma_{,\beta}\right] + 2e^{-4\sigma}\left[KL + \Lambda\right]\right\}$$

$$\times \sqrt{-\bar{g}}\; d^4x = 0 \qquad \text{(I.2.1b)}$$

By carrying out the variations in (I.2.1b) with respect to the $\bar{g}_{\mu\nu}$ holding σ fixed and requiring that $\delta\bar{g}_{\mu\nu}$ vanish on the boundary, the reader may verify that (I.2.1b) yields (I.1.17).*

Furthermore, if the variations in (I.2.1b) are carried out with respect to σ holding the $\bar{g}_{\mu\nu}$ fixed and $\delta\sigma$ is required to vanish on the boundary, it can be easily shown that the resulting equation is the trace of (I.1.17), namely:

$$\bar{R} = -\bar{K}\bar{T} - 4\bar{\Lambda} + 6\bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\,\sigma_{,\beta} - 6\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta} \qquad \text{(I.2.2)}$$

This merely serves to point out that the variations on (I.2.1a) with respect to the $g_{\mu\nu}$ brings out all the information possible. That is, no new equation on σ insues from taking variations on (I.2.1b) with respect to σ.

Next, we briefly consider the Brans-Dicke theory without a cosmological term.(4) It is formulated by first considering the

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*This calculation may be facilitated by making use of the fact that for any scalar ϕ(4)

$$\delta\int\phi\bar{R}\,\sqrt{-\bar{g}}\;d^4x = -\int\left\{\phi\bar{G}^{\mu\nu} - \left[\bar{g}^{\mu\nu}\,\bar{g}^{\alpha\beta}\,\phi_{,\alpha;\beta} - \bar{g}^{\mu\alpha}\,\bar{g}^{\nu\beta}\,\phi_{,\alpha;\beta}\right]\right\}$$

$$\times \sqrt{-\bar{g}}\;\delta\,\bar{g}_{\mu\nu}\;d^4x$$

variational principle from which the ordinary Einstein theory is developed.[5]

$$\delta\int\left\{\bar{R} + \frac{16\pi G}{c^4}\bar{L}\right\}\sqrt{-\bar{g}}\,d^4x = 0 \tag{I.2.3}$$

where by definition $\bar{L} = \bar{L}(\bar{g}_{\mu\nu})$ and $\dfrac{\partial(\bar{L}\sqrt{-\bar{g}})}{\partial\bar{g}_{\mu\nu}} = \dfrac{1}{2}\sqrt{-\bar{g}}\;\bar{T}^{\mu\nu}$.

The idea is then set forth that G, the gravitational constant in (I.2.3) should possibly be replaced by a scalar function ϕ^{-1} and that a more correct description of physical phenomena might be obtained by dividing the Lagrangian in (I.2.3) by ϕ^{-1} and adding on a convenient Lagrangian density for the scalar field, ϕ. The result of this being:[4]

$$\delta\int\left\{\phi\bar{R} + \frac{16\pi}{c^4}\bar{L} - \omega\,\frac{\bar{g}^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}}{\phi}\right\}\sqrt{-\bar{g}}\,d^4x = 0\ . \tag{I.2.4}$$

The variations in (I.2.4) being taken with respect to both ϕ and the $\bar{g}_{\mu\nu}$, ω being a constant.

To obtain a Brans-Dicke formulation of relativity complete with a cosmological constant, we therefore consider the Lagrangian density from which the ordinary Einstein theory with a cosmological constant (I.1.18), can be derived.

$$\delta\int\left\{\bar{R} + \frac{16\pi G}{c^4}\bar{L} + 2\Lambda\right\}\sqrt{-\bar{g}}\,d^4x = 0 \tag{I.2.5}$$

We next identify G with ϕ^{-1}, divide this density by ϕ^{-1} and add on the density for the scalar field, ϕ. The result being:

$$\delta\int\left\{\phi\bar{R} + \frac{16\pi}{c^4}\bar{L} + 2\phi\Lambda - \omega\,\bar{g}^{\alpha\beta}\,\frac{\phi_{,\alpha}\,\phi_{,\beta}}{\phi}\right\}\sqrt{-\bar{g}}\,d^4x = 0 \tag{I.2.6}$$

The results of the variations in (I.2.6) may be expressed in the following two equations:

$$\bar{G}_{\mu\nu} = \frac{8\pi\phi^{-1}}{c^4}\bar{T}_{\mu\nu} + \bar{g}_{\mu\nu}\Lambda + \frac{\omega}{\phi^2}\left[\phi,_{\mu}\phi,_{\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\phi,_{\alpha}\phi,_{\beta}\right] + \frac{1}{\phi}\left[\phi,_{\mu;\nu} - \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\phi,_{\alpha;\beta}\right] \tag{I.2.7a}$$

$$\frac{\bar{g}^{\alpha\beta}\phi,_{\alpha;\beta}}{\phi} - \frac{\bar{g}^{\alpha\beta}\phi,_{\alpha}\phi,_{\beta}}{2\phi^2} + \frac{1}{2\omega}\bar{R} + \frac{\Lambda}{\omega} = 0\ , \tag{I.2.7b}$$

where semicolons again denote covariant derivatives relative to the $\bar{g}_{\mu\nu}$. Combining the trace of the first of these equations with the second equation yields

$$\frac{\bar{g}^{\alpha\beta}\phi,_{\alpha;\beta}}{\phi} = \frac{8\pi\phi^{-1}}{c^4(2\omega+3)}\bar{T} + \left(\frac{2}{2\omega+3}\right)\Lambda\ . \tag{I.2.8}$$

That is, the variations taken on (I.2.6) not only produce a set of field equations akin in form to our non-customary field equation (I.1.17), but also an equation on the scalar field, ϕ.

At this point, the idea suggests itself that there might possibly be a Lagrangian density other than the one used in (I.2.1) which would produce both our non-customary field equations and an equation on σ. The author admits this possibility, but has been unable to construct a density satisfying these conditions.

To continue this discussion, we find that by using (I.1.9) and (I.1.10), and (I.1.15) we can bring (I.1.17) into the form

$$\bar{G}_{\mu\nu} = \frac{8\pi G}{c^4}e^{\sigma(1-\beta)}\bar{\bar{T}}_{\mu\nu} + \bar{g}_{\mu\nu}e^{-2\sigma}\Lambda - 2\sigma,_{\mu}\sigma,_{\nu} - \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\sigma,_{\alpha}\sigma,_{\beta} - 2\sigma,_{\mu;\nu} + 2\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\sigma,_{\alpha;\beta} \tag{I.2.9}$$

We now notice that by setting $\omega = -3/2$ in (I.2.7a) and by setting $\beta = -1$ in (I.2.9) and identifying ϕ^{-1} with $Ge^{2\sigma}$, that in the case $\Lambda=0$, (I.2.7a) and (I.2.9) become identical. We might hope then that since the procedure just outlined makes our field equations the same as those of Brans and Dicke in the case $\omega = -3/2$ and $\Lambda=0$, that we could, by analogy, generate an equation on σ by replacing ϕ in (I.2.7b) by $G^{-1}e^{-2\sigma}$. Unfortunately, this choice of ω makes the source term in (I.2.7b) diverge. Furthermore, in the case $\Lambda \neq 0$ it seems that the presence of the factor $e^{-2\sigma}$ in the term involving Λ in (I.2.9) prohibits (I.2.9) from becoming identical to the Brans-Dicke field equations irregardless of how we select ω and β and of any correspondence we select between ϕ and e^{σ}. However, the fact that in the case $\Lambda = 0$ our field equations can, in one special case, be made identical to those of Brans and Dicke lends credence to the idea of constructing an equation on σ which is analogous to the scalar wave equation of Brans and Dicke. In Chapter II, we shall construct equations on σ using this idea.

3. Equations of Motion

In this section we will demonstrate that Eq. (I.1.19), i.e. $T^{\mu\nu}{}_{|\nu} = 0$, implies the equations of motion for particles in our theory to the same extent that $\bar{T}^{\mu\nu}{}_{;\nu} = 0$ implies the equations of motion in the ordinary Einstein theory. Furthermore, from the results of our study of $T^{\mu\nu}{}_{|\nu} = 0$, we will present what the author considers to be a good argument for only considering the case $\beta = 0$ in Eq. (I.1.2).

To develop the equations of motion for a particle of mass density ρ in the non-customary gauge, we shall make use of the results of a calculation contained in Adler, Bazin and Schiffer's book.[6] In this calculation, it is assumed that for a small globule of incoherent matter with proper-mass density ρ that the following equations are true:

$$T^{\mu\nu}{}_{|\nu} = 0 \tag{I.1.19}$$

and

$$T^{\mu\nu} = \rho V^{\mu} V^{\nu} \tag{I.3.1}$$

where

$$V^{\mu} = \frac{dx^{\mu}}{ds} \tag{I.3.2}$$

By applying a world-tube technique to the globule described by Eqs. (I.1.19), (I.3.1), and (I.3.2), it is then shown that

$$\frac{dmV^{\alpha}}{ds} + m\,\Gamma^{\alpha}_{\mu\nu}\,V^{\mu}\,V^{\nu} = 0 \tag{I.3.3}$$

where

$$m = \int \rho\, dV \tag{I.3.4}$$

and dV is the proper-volume element of the globule.

In order to utilize (I.3.3) we must first determine $\frac{dm}{ds}$. To accomplish this we note that for massive particles

$$V_{\mu} V^{\mu} = 1 \tag{I.3.5}$$

where

$$V_{\mu} = g_{\mu\alpha} V^{\alpha}$$

If we now differentiate (I.3.5) with respect to the curve parameter, s, we find

$$V_{\alpha}\left[\frac{dV^{\alpha}}{ds} + \Gamma^{\alpha}_{\mu\nu}\,V^{\mu}\,V^{\nu}\right] = 0\ . \tag{I.3.6}$$

Finally, contracting (I.3.3) with V_α and combining the result with (I.3.6) yields the desired results:*

$$\frac{dm}{ds} = 0 \qquad (I.3.7)$$

and

$$\frac{dV^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} V^\mu V^\nu = 0 \qquad (I.3.8)$$

At this point, it should be noted that (I.3.7) and (I.3.8) are to hold only where (I.1.19) is true. In our theory, (I.1.19) is true only in the non-customary gauge. We therefore have that in our theory particles move along geodesics, (I.3.8), according to an observer using the non-customary congruence definitions. And, in our theory, the proper-mass of a particle (I.3.4), as determined by observers using the non-customary congruence definition does not change along its trajectory.

To interpret the result (I.3.8) in terms of measurements that would be made by an observer using the customary congruence definition, we note that according to (I.1.3) and (I.3.2) that

$$V^\alpha = \bar{V}^\alpha e^\sigma \qquad (I.3.9)$$

where

$$\bar{V}^\alpha = \frac{dx^\alpha}{d\bar{s}}$$

Now, applying (I.3.9), (I.1.3), and (I.1.5) to (I.3.8) we find

$$\frac{d\bar{V}^\alpha}{d\bar{s}} + \bar{\Gamma}^\alpha_{\mu\nu} \bar{V}^\mu \bar{V}^\nu = \frac{d\sigma}{d\bar{s}} \bar{V}^\alpha - \bar{g}_{\mu\nu} \bar{V}^\mu \bar{V}^\nu \bar{g}^{\alpha\beta} \sigma_{,\beta} \, . \qquad (I.3.10)$$

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*The results, (I.3.7) and (I.3.8), are also derived from a more general standpoint in a paper by Papapetrou.[7]

In this last equation, we shall assume that $\bar{g}_{\mu\nu}\ \bar{V}^{\mu}\ \bar{V}^{\nu}$ is 1 for material particles and 0 for photons.* We see then that particles in our theory do not follow geodesics in the customary gauge. That is, in the customary gauge, the gradients of σ add a force term to the equations of motion which is not present in the ordinary Einstein theory.

We now turn to a discussion of the congruence definition for masses in our theory. According to the discussion presented in (I.1), $\bar{m}$ in (I.1.2) can be identified as the mass of a particle as determined by observers using the customary congruence definitions. So from (I.1.2) we have

$$\bar{m} = e^{\beta\sigma}\ m \tag{I.3.11}$$

From (I.3.7), the proper-mass, m, of a particle as determined by an observer using the non-customary congruence definitions does not change along the particle's trajectory. Consequently, by (I.3.11), an observer using the customary congruence definitions must, for $\beta \neq 0$ and σ non-trivial, see the proper-mass of the particle change as he moves with the particle along its trajectory. That is, for $\beta \neq 0$ and σ non-trivial, our theory only admits particles whose proper-masses as determined by "customary observers" change along their trajectories.

It may indeed be true that in the gauge we make measurements in, the customary gauge, the proper-masses of particles change by

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*At this point, we mention that we are assuming that the speed of light is the same in both gauges. That is, assuming that c transforms in accord with its dimensions we have from (I.1.1) that

$$c = \bar{c}\ .$$

minute amounts along their trajectories. However, since for non-radiating particles this is not observed in nature and since our theory allows us the freedom to select our mass gauge, we will require that β be zero thereby requiring the proper-masses of particles in either gauge to be constant along their trajectories regardless of the congruence definition between lengths and times. It should be noted that the basis of the above argument for β being zero lies in our assumption that in the non-customary gauge $T^{\mu\nu}{}_{|\nu} = 0$. Therefore, if we were to relax this condition, i.e. change our non-customary field equations, then it is conceivable that a theory could be developed in which $\bar{m}$, the customary mass, is constant along the particle trajectory for a non-zero value of β.

So, setting β=0 in (I.1.2), (I.1.10), (I.1.13), and (I.2.9) respectively, we have

$$m = \bar{m} \tag{I.3.12}$$

$$\bar{K} = K\, e^{\sigma} \tag{I.3.13}$$

$$\rho = \bar{\rho}\, e^{3\sigma} \tag{I.3.14}$$

and

$$\begin{aligned} \bar{G}_{\mu\nu} = \frac{8\pi G}{c^4}\, e^{\sigma}\, \bar{T}_{\mu\nu} + \bar{g}_{\mu\nu}\, e^{-2\sigma}\, \Lambda - 2\sigma_{,\mu}\, \sigma_{,\nu} \\ - \bar{g}_{\mu\nu}\, \bar{g}^{\alpha\beta}\, \sigma_{,\alpha}\, \sigma_{,\beta} - 2\sigma_{,\mu;\nu} \\ + 2\bar{g}_{\mu\nu}\, \bar{g}^{\alpha\beta}\, \sigma_{,\alpha;\beta}\, . \end{aligned} \tag{I.3.15}$$

In what follows, we will compare, after coupling with an equation on σ, the predictions made by our non-customary field equations, (I.3.15) or equivantly (I.1.8), and our equations of motion, (I.3.8) or (I.3.10), to the predictions made by the ordinary Einstein theory. Before turning to the construction of an equation on σ, we will now, in order to gain some feeling as to how σ will manifest itself physically, present a qualitative discussion of the "Newtonian limit" of our theory.

4. The Newtonian Limit

In order to discuss qualitatively the roles to be played by $\bar{g}_{\mu\nu}$ and σ in our non-customary gauge theory, we will now discuss the Newtonian limit of our theory in relation to that of the ordinary Einstein theory. To this end, we point out that by inspection of Eqs. (I.1.1), (I.3.10), and (I.3.15) our theory is identical to the ordinary Einstein theory if σ is identically zero. Now, since the ordinary Einstein theory includes the Newtonian gravitational theory as a limiting case and since, especially at large distances from gravitating objects, the ordinary Einstein theory is in close agreement with observation, if our theory is to be sensible σ should become small at large distances from gravitating bodies. Further weight will be lent to the idea of σ becoming small at large distances from a gravitating body in Chapter II where we will see that the source term for σ in the equations presented for investigation is the stress-energy-momentum tensor for matter. Furthermore, from Section 1 of this chapter, σ can, by assumption be set to zero at some arbitrary event in space-time. With these points in mind, we now assert that σ should become small at large distances from a gravitating body and should approach zero as these distances approach infinity. That is, at large distances from a gravitating body, our theory and the ordinary Einstein theory are to be asymptotically the same.

Employing the preceding assertion, we will now demonstrate that the Newtonian gravitational theory is a first approximation of our theory in the event that we identify the Newtonian potential, ψ, with the quantity $c^2\left[\frac{\bar{g}_{44}-1}{2} - \sigma\right]$.* To accomplish this, we must first show that the Newtonian equations of motion are a first approximation of our equations of motion, (I.3.10). Secondly, we must demonstrate that Poisson's equation on ψ is a first approximation to our field equation, (I.3.15). In doing this, we will restrict ourselves to be at a large distance from a quasi-static gravitating object of mass density $\bar{\rho}$ in the customary gauge, and to consider test particles which have velocities small compared to the speed of light.

From the preceding discussion of the relationship between our theory and the ordinary Einstein theory at large distances from gravitating bodies, we can evidently write for the problem at hand

$$\sigma \approx 0 \tag{I.4.1}$$

$$d\bar{s}^2 \approx -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 \tag{I.4.2}$$

$$\bar{g}_{ij} \approx -\delta_{ij} \; ; \; i,j = 1,2,3 \tag{I.4.3}$$

and

$$\bar{g}_{44} \approx 1 \; . \tag{I.4.4}$$

Also, from our restrictions that the gravitating object be quasi-

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*In this development, we will closely follow a discussion on the Newtonian limit for the ordinary Einstein theory as presented by Tolman.(8)

static and the velocities of our test particles be small compared to the speed of light, we shall assume that

$$\frac{\partial\sigma}{\partial x^4} = 0 \tag{I.4.5}$$

$$\frac{\partial\bar{g}_{\mu\nu}}{\partial x^4} = 0 \tag{I.4.6}$$

and

$$\frac{dx^1}{d\bar{s}} \approx \frac{dx^2}{d\bar{s}} \approx \frac{dx^3}{d\bar{s}} \approx 0, \quad \frac{dx^4}{d\bar{s}} \approx 1 . \tag{I.4.7}$$

Employing the constraints (I.4.1) through (I.4.7), we will now show that the Newtonian equations of motion are a first approximation to our equations of motion. From (I.3.9) and (I.3.10), we have that for massive particles our equations of motion in the customary gauge are

$$\frac{d^2x^\alpha}{d\bar{s}^2} + \bar{\Gamma}^\alpha_{\mu\nu}\frac{dx^\mu}{d\bar{s}}\frac{dx^\nu}{d\bar{s}} = \frac{\partial\sigma}{\partial x^\beta}\frac{dx^\beta}{d\bar{s}}\frac{dx^\alpha}{d\bar{s}} - \bar{g}^{\alpha\beta}\frac{\partial\sigma}{\partial x^\beta}$$

In view of the restrictions (I.4.1) through (I.4.7), this last statement can be simplified to

$$\frac{d^2x^i}{d(x^4)^2} = \frac{\partial}{\partial x^i}\left\{\tfrac{1}{2}\,\bar{g}_{44} - \sigma\right\}, \quad i = 1,2,3 . \tag{I.4.8}$$

Setting ψ equal to $c^2[\tfrac{1}{2}\,\bar{g}_{44} - \sigma + \text{constant}]$ in (I.4.8) we arrive at the Newtonian equations of motion for a free particle in a gravitational field.

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\psi}{\partial x^i}$$

Furthermore, from (I.4.1), (I.4.4), and the fact that ψ must be zero at infinity, we have

$$\psi = c^2\left[\frac{\bar{g}_{44}-1}{2} - \sigma\right] . \qquad \text{(I.4.9)}$$

To complete our task of showing that the Newtonian gravitational theory is a first approximation of our theory, we must now show that Poisson's equation on ψ is a first approximation to our field equation, (I.3.15). To accomplish this, we will assume that the components of the stress-energy-momentum tensor in the customary gauge all vanish with the exception of the 4-4 component for which we assume

$$\bar{T}^{44} = \bar{\rho}\ c^2$$

Hence, provided that (I.4.3) and (I.4.4) hold, we have

$$\bar{T}^{44} = \bar{T}^{4}{}_{4} = \bar{T}_{44} = \bar{T} = \bar{\rho}\ c^2 . \qquad \text{(I.4.10)}$$

Now, inserting (I.4.10) into our field equations (I.3.15) and its trace, we find

$$\begin{aligned}\bar{G}_{44} = \bar{R}_{44} - \tfrac{1}{2}\,\bar{g}_{44}\,\bar{R} = \frac{8\pi G}{c^2}\,\bar{\rho}e^{\sigma} + \bar{g}_{44}\,e^{-2\sigma}\,\Lambda \\ - 2\sigma_{,4}\sigma_{,4} - \bar{g}_{44}\,\bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\sigma_{,\beta} \\ - 2\sigma_{,4;4} + 2\bar{g}_{44}\,\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta}\end{aligned} \qquad \text{(I.4.11)}$$

and

$$-\bar{R} = \frac{8\pi G}{c^2}\,\bar{\rho}\ e^{\sigma} + 4e^{-2\sigma}\,\Lambda - 6\bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\sigma_{,\beta} + 6\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta} \qquad \text{(I.4.12)}$$

By applying the restrictions (I.4.1) through (I.4.5) to (I.4.11) and (I.4.12), and by assuming that Λ, the cosmological constant, is small enough to be neglected, we find that

$$\bar{R}_{44} - \tfrac{1}{2}\,\bar{R} = \frac{8\pi G}{c^2}\,\bar{\rho}(1+\sigma) + \sigma_{,i}\sigma_{,i} - 2\sigma_{,4;4} + 2\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta} \qquad \text{(I.4.13)}$$

and

$$-\bar{R} = \frac{8\pi G}{c^2}\,\bar{\rho}(1+\sigma) + 6\sigma_{,i}\sigma_{,i} + 6\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta} \tag{I.4.14}$$

where $i = 1,2,3$.

Combining (I.4.13) and (I.4.14) yields

$$\bar{R}_{44} = \frac{4\pi G}{c^2}\,\rho(1+\sigma) - 2\sigma_{,i}\sigma_{,i} - 2\sigma_{,4;4} - \bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta} \tag{I.4.15}$$

At this point, we will assume that since σ and $\bar{g}_{\mu\nu}$ are essentially constants in the region under consideration that $\sigma_{,\alpha}$ and $\frac{\partial \bar{g}_{\mu\nu}}{\partial x^\alpha}$ are small. With this in mind, we now consider that the terms in (I.4.15) which are quadratic in the derivatives of σ and $\bar{g}_{\mu\nu}$ may be neglected in comparison to the terms that are of first order in these derivatives. From this consideration we have

$$\left.\begin{aligned} \sigma_{,i}\sigma_{,i} &\simeq 0 \\ \sigma_{,4;4} &\simeq 0 \\ \bar{g}^{\alpha\beta}\sigma_{,\alpha;\beta} &\simeq -\frac{\partial^2\sigma}{\partial x^i \partial x^i} = -\nabla^2\sigma \end{aligned}\right\} \tag{I.4.16}$$

and from (I.1.6)

$$\bar{R}_{44} \simeq \tfrac{1}{2}\,\frac{\partial^2\,\bar{g}_{44}}{\partial x^i \partial x^i} = \nabla^2\left(\frac{\bar{g}_{44}}{2}\right) \tag{I.4.17}$$

Utilizing (I.4.16) and (I.4.17) in (I.4.15) yields

$$\nabla^2\left(\frac{\bar{g}_{44}}{2} - \sigma\right) = \frac{4\pi G}{c^2}\,\bar{\rho}(1+\sigma) \simeq \frac{4\pi G}{c^2}\,\bar{\rho}$$

Finally, by using the identification (I.4.9), we arrive at Poisson's equation for the Newtonian gravitational potential

$$\nabla^2\psi = 4\pi G\bar{\rho}$$

The above calculations demonstrate that in our theory it is the quantity $\frac{\bar{g}_{44}}{2} - \sigma$ which is closely associated with the Newtonian potential. In the ordinary Einstein theory, it is the quantity $\frac{\bar{g}_{44}}{2}$ which is associated with this potential.[8] Therefore, at this point we note that whereas in the ordinary Einstein theory, the effects of a gravitational field on a test particle can be transformed away at a point by using a local inertial fram, i.e. a frame in which the quantities $\frac{\partial \bar{g}_{\mu\nu}}{\partial x^{\alpha}}$ vanish at a point, this is not the case in our theory. That is, in our case the vanishing of $\frac{\partial \bar{g}_{\mu\nu}}{\partial x^{\alpha}}$ at a point in no way insures the vanishing of $\frac{\partial \psi}{\partial x^{\alpha}}$. We will discuss this point further in Chapter III where we will show that the principle equivalence between an accelerating frame and a gravitational field does not hold in our theory. We now turn to the construction of equations on σ.

CHAPTER II

EQUATIONS FOR σ

In this chapter, we will concern ourselves with the construction of equations on σ. Before doing this, we point out that there is no guiding physical concept which will allow us to determine the validity of an equation on σ independent of our field equations (I.1.8) or (I.3.15). It is only an equation on σ coupled with our field equations which has physical significance. So the testing of the validity of an equation on σ must be carried out by means of a trial and error technique. Namely, postulate a σ-equation, couple this equation with our field equations (I.1.8) or (I.3.15), and compare the results of this coupling to observation.

In sections 1 through 3 of this chapter, we will postulate three equations on σ to be tested in later chapters by the procedure just outlined. The first of these equations will be constructed by requiring that the variations on σ holding the $\bar{g}_{\mu\nu}$ fixed in $\int \bar{R}\sqrt{-\bar{g}}\, d^4x$ vanish. That is, the integral of the product of the customary Ricci curvature scalar, $\bar{R}$, with $\sqrt{-\bar{g}}$ over an arbitrary 4-space volume shall be extremal with respect to σ. The other two equations will come from requiring that our theory be analogous to the Brans-Dicke formulation with a cosmological term. The first of these last two equations will be constructed by identifying Ge^{σ} (G being the gravational constant) with ϕ^{-1} in the Brans-Dicke scalar wave equation, (I.2.8). The second will come from requiring that the equation relating ϕ to $\bar{R}$ in the Brans-Dicke formulation, (I.2.7b), hold in our theory if Ge^{σ} is identified with ϕ^{-1}. After making this identification, we will couple the resulting equation to the trace of our field equations, (I.3.15), and arrive at a third equation for σ.

We wish to again stress that there is no physical justification for any of the three procedures just outlined. Nor do the equations produced by these procedures exhaust the possibilities for equations on σ. However, we shall see that the three equations on σ produced by these procedures are of a fairly general nature. Therefore, we feel that these equations will suffice to at least test the feasibility of our non-customary gauge theory. We now proceed with the development of our σ-equations.

1. An Equation on σ From a Variational Technique

For our first example of an equation on σ we postulate that

$$\delta\int\bar{R}\sqrt{-\bar{g}}\; d^4x = 0 \;, \tag{II.1.1}$$

where the variations are to be carried out on σ holding the $\bar{g}_{\mu\nu}$ fixed with the variations of σ, $\delta\sigma$, vanishing on the boundary of the 4-volume and $\bar{R}$ is found by taking the trace of (I.3.15),

$$\bar{R} = -Ke^{\sigma}\bar{T} - 4e^{-2\sigma}\Lambda + 6\bar{g}^{\alpha\beta}\sigma,_{\alpha}\sigma,_{\beta} - 6\bar{g}^{\alpha\beta}\sigma_{,\alpha;\beta} \;, \tag{II.1.2}$$

where from (I.1.9)

$$K = \frac{8\pi G}{c^4}$$

Inserting (II.1.2) into (II.1.1) yields

$$\delta\int\left\{\bar{g}^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta} - \bar{g}^{\alpha\beta}\sigma_{,\alpha;\beta} - \frac{1}{6}Ke^{\sigma}\bar{\bar{T}} - \frac{2}{3}e^{-2\sigma}\Lambda\right\}\sqrt{-\bar{g}}\,d^4x = 0 , \qquad (II.1.3)$$

Carrying out the variations in (II.1.3) term by term, we have

$$\delta\int\bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\,\sigma_{,\beta}\,\sqrt{-\bar{g}}\,d^4x = -2\int\left\{\frac{\partial}{\partial x^\beta}\left[\bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\,\sqrt{-\bar{g}}\right]\right\}\delta\sigma d^4x$$

$$= -2\int\left\{\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta}\right\}\sqrt{-\bar{g}}\,\delta\sigma d^4x , \qquad (II.1.4)$$

$$\delta\int\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta}\,\sqrt{-\bar{g}}\,d^4x = \delta\int\left\{\frac{\partial}{\partial x^\beta}\left|\bar{g}^{\alpha\beta}\,\sigma_{,\alpha}\,\sqrt{-\bar{g}}\right|\right\}d^4x = 0 , \qquad (II.1.5)$$

and

$$\delta\int\left\{\frac{1}{6}Ke^{\sigma}\bar{\bar{T}} + \frac{2}{3}e^{-2\sigma}\Lambda\right\}\sqrt{-\bar{g}}\,d^4x = \int\left\{\frac{1}{6}Ke^{\sigma}\bar{\bar{T}} - \frac{4}{3}e^{-2\sigma}\Lambda\right\}\sqrt{-\bar{g}}\,\delta\sigma d^4x \qquad (II.1.6)$$

Inserting (II.1.4) through (II.1.6) into (II.1.3), we find

$$\int\left\{\bar{g}^{\alpha\beta}\sigma_{,\alpha;\beta} + \frac{1}{12}K\,e^{\sigma}\,\bar{\bar{T}} - \frac{2}{3}e^{-2\sigma}\,\Lambda\right\}\sqrt{-\bar{g}}\;\delta\sigma d^4x = 0 . \qquad (II.1.7)$$

Since this last statement must be true for arbitrary variations in σ we have the first of the three equations on which we will consider.

$$\bar{g}^{\alpha\beta}\,\sigma_{,\alpha;\beta} = -\frac{1}{12}K\,e^{\sigma}\,\bar{\bar{T}} + \frac{2}{3}e^{-2\sigma}\,\Lambda . \qquad (II.1.8)$$

For this equation, we see that the source of the σ field is the trace, $\bar{\bar{T}}$, of the customary stress-energy-momentum tensor. We will see that $\bar{\bar{T}}$ is also the source of σ in our other two equations to be considered. We will now present the last two of our three equations to be investigated, after which we will consider all three in some detail.

2. First Equation on σ From An Analogy to Brans-Dicke Theory

If in the Brans-Dicke field equations, (I.2.7a), ϕ^{-1} is identified with Ge^{σ}, then, as was discussed in (I.2), their field equations become similar in form to our field equation, (I.3.15). Using this as a guide, we now propose another σ-equation to be tested against observation by making the same identification, $\phi^{-1} \rightarrow Ge^{\sigma}$, in the Brans-Dicke scalar wave equation, (I.2.8). Carrying out this identification yields

$$\bar{g}^{\alpha\beta}(e^{-\sigma})_{,\alpha;\beta} = \left(\frac{K}{2\omega+3}\right)\bar{T} + \left(\frac{2}{2\omega+3}\right)e^{-\sigma}\Lambda \; . \qquad \text{(II.2.1)}$$

Eq. (II.2.1) can be rewritten in the form

$$\bar{g}^{\alpha\beta}\sigma_{,\alpha;\beta} - \bar{g}^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta} = -\left(\frac{K}{2\omega+3}\right)e^{\sigma}\bar{T} - \left(\frac{2}{2\omega+3}\right)\Lambda \; . \qquad \text{(II.2.2)}$$

This is the second of the three equations on σ which we will test against observation.

3. Second Equation on σ From An Analogy to Brans-Dicke Theory

In this third and last equation on σ, we shall again require that our theory be analogous to the Brans-Dicke theory by making the same identification on ϕ, $\phi^{-1} \rightarrow Ge^{\sigma}$, used to arrive at (II.2.2). However, we will make this identification in the equation relating ϕ to $\bar{R}$ in the Brans-Dicke formulation rather than in the Brans-Dicke scalar wave equation. Thus, setting $\phi^{-1} = Ge^{\sigma}$ in (I.2.7b) we find

$$\bar{g}^{\alpha\beta}\sigma_{,\alpha;\beta} - \tfrac{1}{2}\,\bar{g}^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta} - \left(\frac{1}{\omega}\right)\Lambda = \left(\frac{1}{2\omega}\right)\bar{R} \; . \qquad \text{(II.3.1)}$$

If we now combine (II.3.1) with our expression (II.1.2) for $\bar{R}$ we arrive at the last of the three equations on σ which we will consider.

$$\bar{g}^{\alpha\beta}\sigma_{,\alpha;\beta} - \left(\frac{\omega+6}{2\omega+6}\right)\bar{g}^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta} = -\left(\frac{K}{2\omega+6}\right)e^{\sigma}\bar{T} + \left(\frac{2}{2\omega+6}\right)\Lambda$$

$$- \left(\frac{4}{2\omega+6}\right)e^{-2\sigma}\Lambda \ . \qquad \text{(II.3.2)}$$

4. Comparison of the σ-Equations

We now wish to discuss and compare our equations on σ, (II.1.8), (II.2.2), and (II.3.2). For the sake of this comparison and for future use, we will re-express these equations in terms of the non-customary Christoffel symbols, $\Gamma^{\alpha}_{\beta\gamma}$, and the non-customary metric tensor, $g_{\mu\nu}$. Using (I.1.4) and (I.1.5), it is easily shown that

$$\bar{g}^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta} = e^{-2\sigma}g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta} \qquad \text{(II.4.1)}$$

and

$$\bar{g}^{\alpha\beta}\sigma_{,\alpha;\beta} = e^{-2\sigma}\left\{g^{\alpha\beta}\sigma_{,\alpha|\beta} + 2g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}\right\} \qquad \text{(II.4.2)}$$

where

$$\sigma_{,\alpha|\beta} \equiv \frac{\partial^2\sigma}{\partial x^{\alpha}\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}\,\sigma_{,\mu} \ . \qquad \text{(II.4.3)}$$

Inserting (II.4.1) and (II.4.2) into (II.1.8), (II.2.2), and (II.3.2), we find

$$\sigma^{,\alpha}{}_{|\alpha} + 2\sigma^{,\alpha}\sigma_{,\alpha} = -\frac{1}{12}Ke^{3\sigma}\bar{T} + \frac{2}{3}\Lambda \ ,$$

$$\sigma^{,\alpha}{}_{|\alpha} + \sigma^{,\alpha}\sigma_{,\alpha} = -\left(\frac{K}{2\omega+3}\right)e^{3\sigma}\bar{T} - \left(\frac{2}{2\omega+3}\right)e^{2\sigma}\Lambda \ , \qquad \text{(II.4.4)}$$

and

$$\sigma^{,\alpha}{}_{|\alpha} + \left(\frac{3\omega+6}{2\omega+6}\right)\sigma^{,\alpha}\sigma_{,\alpha} = -\left(\frac{K}{2\omega+6}\right)e^{3\sigma}\bar{T} + \left(\frac{2}{2\omega+6}\right)e^{2\sigma}\Lambda - \left(\frac{4}{2\omega+6}\right)\Lambda \ , \qquad \text{(II.4.6)}$$

where we have adopted the notation

$$\sigma^{,\alpha} \equiv g^{\alpha\beta}\sigma_{,\beta} \ . \qquad \text{(II.4.7)}$$

Comparing Eqs. (II.4.4) through (II.4.6), we notice that they are all second order, non-linear equations in σ each containing $\bar{\bar{T}}$ as a source term. They differ from each other firstly in the coefficients of the terms involving $\sigma^{,\alpha}\sigma_{,\alpha}$ and $Ke^{3\sigma}\bar{\bar{T}}$ and secondly in the manner in which the cosmological constant, Λ, enters the equations. In the first equation, (II.4.4), the coefficient of is a constant while in the second equation, (II.4.5), Λ is multiplied by a factor involving $e^{2\sigma}$. The third equation, (II.4.6), involves a mixture of the terms involving Λ and $e^{2\sigma}\Lambda$.

In Chapters III and IV, we will solve these equations and investigate problems in relativity under the assumption that the terms involving $\underset{\circ}{\Lambda}$ are negligible in comparison to the terms involving $\bar{\bar{T}}$. In this case, all of our equations can be cast into the form

$$\sigma^{,\alpha}{}_{|\alpha} + P_1\sigma^{,\alpha}\sigma_{,\alpha} = -KP_2e^{3\sigma}\bar{\bar{T}} \tag{II.4.8}$$

where P_1 and P_2 are constants. It is only in Chapter V where we discuss applications to cosmology that the terms involving Λ will make themselves felt. We now turn to the solutions of our σ-equations and our field equations for the case of a static, finite, spherically symmetric distribution of matter.

CHAPTER III

SOLUTIONS TO THE FIELD EQUATIONS AND THE σ-EQUATIONS

In this chapter, we will, by solving our field and gauge equations, lay the foundations which will be necessary in Chapter IV to solve several standard problems in relativity. We will solve these equations in the case of a static, finite, spherically symmetric distribution of matter as determined by an observer using the customary congruence definition. Solutions will be obtained for both the interior and exterior regions of this distribution of matter. For ease of calculation, we will solve for the non-customary metric tensor, $g_{\mu\nu}$, and σ from Eqs. (I.1.8) and (II.4.4) through (II.4.6) respectively. Then, since we wish to compare the predictions made by our theory with observations made by observers employing the customary congruence definition and with the predictions of the ordinary Einstein theory, we will perform the gauge transformation, (I.1.4), to arrive at the customary metric tensor.

1. Assumptions to be Used in the Calculations

In carrying out the procedures just outlined, we will assume that the terms involving Λ, the cosmological constant, in our field equations and in our σ-equations are negligible compared to the terms involving the stress-energy-momentum tensor, $T_{\mu\nu}$, or its trace, T. With this assumption, our field equations, (I.1.8)

$$G_{\mu\nu} = K\, T_{\mu\nu} \tag{III.1.1}$$

For $\Lambda = 0$, our σ-equations, (II.4.4) through (II.4.6), all have the form (II.4.8),

$$\sigma^{,\alpha}{}_{|\alpha} + P_1\, \sigma^{,\alpha}\, \sigma_{,\alpha} = -K\, P_2\, \bar{T}\, e^{3\sigma} \tag{II.4.8}$$

Since from Eqs. (I.1.12) and (I.3.13), we have that $T = e^{3\sigma}\bar{T}$, then (II.4.8) can also be written as

$$\sigma^{,\alpha}{}_{|\alpha} + P_1\, \sigma^{,\alpha}\, \sigma_{,\alpha} = -K\, P_2\, T\ . \tag{III.1.2}$$

In (II.4.8) and (III.1.2), P_1 and P_2 are constants specifying which of the Eqs. (II.4.4) through (II.4.6) we are using. Their values are

$$\left.\begin{aligned} &P_1 = 2,\ P_2 = \frac{1}{12} \\ &P_1 = 1,\ P_2 = \frac{1}{2\omega+3} \\ \text{and}\qquad &P_1 = \left(\frac{3\omega+6}{2\omega+6}\right),\ P_2 = \frac{1}{2\omega+6} \end{aligned}\right\} \tag{III.1.3}$$

for Eqs. (II.4.4) through (II.4.6) respectively. The determination of which set P_1, P_2 is correct must come from comparing the predictions of our theory with observation.

Also, since we are considering a static, finite, spherically symmetric distribution of matter as determined by an observer using the customary congruence definition, we can, by using well established results,* assume that our line element, both inside and outside the distribution of matter, in the customary gauge may be written in the standard form

$$d\bar{s}^2 = -e^{\bar{\lambda}(\bar{r})}\, d\bar{r}^2 - \bar{r}^2 (d\Omega)^2 + e^{\bar{\nu}(\bar{r})}\, (dx^4)^2 \qquad \text{(III.1.4)}$$

$$(d\Omega)^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$$

In keeping with our custom of using bars on quantities associated with the customary gauge, we have denoted the radial coordinate appearing in the standard form line element, (III.1.4), by $\bar{r}$. Shortly, we will see that in order to express the non-customary line element associated with (III.4.1) in standard form, it will be necessary to transform $\bar{r}$ into a new radial coordinate, r.

In (III.1.4), $\bar{\lambda}(\bar{r})$ and $\bar{\nu}(\bar{r})$ are the terms usually solved for in the ordinary Einstein theory, and they are the quantities which allow for comparisons between theory and observations made by observers using the customary congruence definition. Therefore, it will be our task to determine these quantities from our non-customary gauge theory and compare our results to the values for these quantities predicted from the ordinary Einstein theory and to observations.

To carry out the solution for the $g_{\mu\nu}$ from (III.1.1), we shall need to know the form of the line element in the non-customary gauge. To accomplish this, we shall assume that since we are dealing with a static, spherically symmetric distribution

*For a simple discussion of this, the reader is referred to Addler, Bazin, and Schiffer(9) or Tolman(10).

of matter that $\sigma = \sigma(\bar{r})$. Now, applying the transformation $ds = e^{-\sigma} d\bar{s}$ to (III.1.4), we find

$$ds^2 = -e^{\bar{\lambda}(\bar{r})-2\sigma(\bar{r})} d\bar{r}^2 - \bar{r}e^{-2\sigma(\bar{r})} (d\Omega)^2$$
$$+ e^{\bar{\nu}(\bar{r})-2\sigma(\bar{r})} (dx^4)^2 \qquad \text{(III.1.5)}$$

The line element in the non-customary gauge, (III.1.5), can be brought into standard form by applying the coordinate transformation

$$r = \bar{r}\, e^{-\sigma(\bar{r})} \qquad \text{(III.1.6)}$$

The results of this transformation are

$$ds^2 = -e^{\lambda(r)} dr^2 - r^2(d\Omega)^2 + e^{\nu(r)}(dx^4)^2 \qquad \text{(III.1.7a)}$$

where

$$e^{\lambda(r)} = e^{\bar{\lambda}(r)} \left[1 + r\frac{d\sigma}{dr}\right]^2 \qquad \text{(III.1.7b)}$$

and

$$e^{\nu(r)} = e^{\bar{\nu}(r)-2\sigma(r)} . \qquad \text{(III.1.7c)}$$

So, we see that if we can solve our field equations (III.1.1) for $e^{\lambda(r)}$ and $e^{\nu(r)}$, and solve for $\sigma(r)$ from (III.1.2), then, from (III.1.6), (III.1.7b), and (III.1.7c), we will be able to determine $e^{\bar{\lambda}(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$ in our theory and compare our theory to Einstein's and to observation.

To continue, by employing (I.1.6), it can be shown that for the non-customary line element, (III.1.7a), that our field equations, (III.1.1) become[10]

$$G_1^1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = KT_1^1$$

$$G_2^2 = G_3^3 = -\, e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{(\nu')^2}{4} + \frac{\nu'-\lambda'}{2r}\right) = KT_2^2$$

$$= KT_3^3 \qquad \text{(III.1.8)}$$

and

$$G_4^4 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = KT_4^4$$

all other components being zero, and primes denoting differentiation with respect to r. Likewise, for this line element, our σ-equation, (III.1.2), is

$$e^{-\lambda}\left[\sigma'' + \frac{2}{r}\,\sigma' + \left(\frac{\nu'-\lambda'}{2}\right)\sigma' + P_1(\sigma')^2\right] = K\ P_2\ T \qquad \text{(III.1.9)}$$

To solve either (III.1.8) or (III.1.9), we will need to assume a form for the stress-energy-momentum tensor. To this end, we assume that in the customary gauge the stress-energy-momentum tensor, $\bar{T}^{\mu}_{\nu}$, is that of a perfect fluid confined to the spherical region $\bar{r} \leq \bar{a}$.

$$\bar{T}^{\mu}_{\nu} = (\bar{\rho}c^2 + \bar{P})\bar{g}_{\mu\nu}\,\frac{dx^{\alpha}}{d\bar{s}}\,\frac{dx^{\mu}}{d\bar{s}} - \delta^{\mu}_{\nu}\,\bar{P}\ . \qquad \text{(III.1.10)}$$

By applying (I.1.12) and (I.3.13) to (III.1.10), we find that the stress-energy-momentum tensor in the non-customary gauge is

$$T^{\mu}_{\nu} = (\rho c^2 + P)g_{\alpha\nu}\,\frac{dx^{\alpha}}{ds}\,\frac{dx^{\nu}}{ds} - \delta^{\mu}_{\nu}\,P\ , \qquad \text{(III.1.11)}$$

in the region $r \leq a$, where from (III.1.6)

$$a = \bar{a}\ e^{-\sigma(\bar{a})} \qquad \text{(III.1.12)}$$

In Eqs. (III.1.10) and (III.1.11), $\bar{\rho}$, $\bar{P}$ and ρ, P are the proper macroscopic mass densities and pressures in the customary and non-customary gauges respectively. They are related by the equations

$$\rho = \bar{\rho}\ e^{3\sigma}\ , \tag{I.3.14}$$

and

$$P = \bar{P}\ e^{3\sigma}\ . \tag{III.1.13}$$

In (III.1.11), we will take $P = 0$ at $r = a$, and for ease of calculation we assume that ρ is constant in the region $r \leq a$. We will see that this model leads to a radially decreasing customary mass density, $\bar{\rho}$.

With the above assumptions, we will find that our field equations, (III.1.8), are exactly solvable for the $g_{\mu\nu}$. We will also find that our σ-equation, (III.1.9), is exactly solvable in the exterior region $r > a$. However, in solving our σ-equation in the interior region, $r < a$, we will only find an approximate solution by invoking several more assumptions: Namely, that the $g_{\mu\nu}$ are only slightly perturbed from their Minkowskian values, that σ is only slightly different from zero, and that the pressure term entering T in (III.1.9) is negligible compared to the term involving ρc^2. That is, in the interior region we shall assume that quadratic terms in λ, ν, and σ are negligible compared to first order terms in these quantities and that T may be approximated by ρc^2. So in the interior region, we will solve for σ from the equation

$$\sigma' \ + \frac{2}{r}\,\sigma' = K\ P_2\ \rho\ c^2 \tag{III.1.14}$$

We now turn to the solution of these various equations.

2. Interior and Exterior Solutions

For the applications to be made in Chapter IV, we shall only need the solutions to our field equations in the exterior region. Hence, we shall begin this discussion by solving Eqs. (III.1.8) in the region $r > a$ where $T_{\mu\nu}$ is zero. In the process of solving these equations, we shall find that in order to specify the constants appearing in these solutions we will need to also consider the solution to (III.1.8) in the interior region.

a. Exterior Solution

In the region $r > a$, Eqs. (III.1.8) become

$$\left.\begin{aligned} & e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0 \\ & e^{-\lambda}\left\{\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{(\nu')^2}{4} + \frac{\nu'-\lambda'}{2r}\right\} = 0 \\ \text{and} \quad & \\ & e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \ . \end{aligned}\right\} \qquad \text{(III.2.1)}$$

The solution to this set of equations for the coefficients of the non-customary metric tensor are the well known Schwarzschild exterior solutions,[10]

$$e^{-\lambda} = 1 - \frac{2mG}{rc^2} \qquad \text{(III.2.2)}$$

and

$$e^{\nu} = 1 - \frac{2mG}{rc^2} \qquad \text{(III.2.3)}$$

where m is a constant to be determined by matching the exterior solution to the interior solution at the boundary $r = a$.

b. Interior Solution

Following the procedure presented in Tolman,[10] for a static perfect fluid in the region $r < a$ we have

$$KP = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{III.2.4}$$

$$K\rho c^2 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{III.2.5}$$

and

$$\frac{dP}{dr} = -\frac{(\rho c^2+P)}{2}\,\nu' \tag{III.2.6}$$

For ρ a constant, the solution to this set of equations for the components e^{λ} and e^{ν} of the non-customary metric tensor are

$$e^{-\lambda} = 1 - \frac{Kc^2}{r}\int \rho r^2 dr = 1 - \frac{K\rho c^2}{3} r^2 \tag{III.2.7}$$

$$e^{\frac{1}{2}\nu} = A - B\sqrt{1 - \frac{K\rho c^2}{3} r^2}\ , \tag{III.2.8}$$

and

$$P = \frac{\rho c^2}{3}\left\{\frac{3B\sqrt{1 - \frac{K\rho c^2}{3} r^2} - A}{A - B\sqrt{1 - \frac{K\rho c^2}{3} r^2}}\right\}. \tag{III.2.9}$$

In these equations, A and B are constant to which, in order to make P zero at $r = a$, we assign the values

$$A = \frac{3}{2}\sqrt{1 - \frac{K\rho c^2}{3} a^2}\ , \quad B = \tfrac{1}{2}\ .$$

Matching (III.2.2), to (III.2.7) at the boundary $r = a$ yields, in the case ρ = constant, for m,

$$m = 4\pi \int_0^a \rho r^2 dr = \frac{4}{3}\pi\rho a^3 \quad . \qquad \text{(III.2.10)}$$

Thus, we see from (III.1.7b), (III.1.7c), (III.2.2), and (III.2.3) that in the region $r > a$

$$e^{-\bar{\lambda}(r)} = \left(1 - \frac{2mG}{rc^2}\right)\left(1 + r\frac{d\sigma}{dr}\right)^2 \qquad \text{(III.2.11)}$$

and

$$e^{\bar{\nu}(r)} = \left(1 - \frac{2mG}{rc^2}\right)e^{2\sigma(r)} \qquad \text{(III.2.12)}$$

where m is given by (III.2.10).

These solutions are to be compared with the exterior solutions from the ordinary Einstein theory for $e^{\bar{\lambda}(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$ which are[10]

$$e^{-\bar{\lambda}(\bar{r})} = 1 - \frac{2\bar{m}G}{\bar{r}c^2} \qquad \text{(III.2.13)}$$

and

$$e^{\bar{\nu}(\bar{r})} = 1 - \frac{2\bar{m}G}{\bar{r}c^2} \qquad \text{(III.2.14)}$$

where

$$\bar{m} = 4\pi \int_0^{\bar{a}} \bar{\rho}\, \bar{r}^2\, d\bar{r} \ , \qquad \text{(III.2.15)}$$

$$r = \bar{r}\, e^{-\sigma(\bar{r})} \qquad \text{(III.1.6)}$$

$$a = \bar{a}\, e^{-\sigma(\bar{a})} \qquad \text{(III.1.12)}$$

and

$$\rho = \bar{\rho}\, e^{3\sigma} \qquad \text{(I.3.14)}$$

So, we see that if we are to compare the results of our theory to the results of the ordinary Einstein theory that we must now determine the relationships between r and $\bar{r}$ and m and $\bar{m}$. We will now discuss the solutions of our equations on σ.

3. Solutions for σ

As was the case in the solution of our field equations, we shall, for the application to be made in Chapter IV, only need solutions for σ in the exterior region. However, to specify the constants appearing in this exterior solution and to determine the relationship between m and $\bar{m}$, we shall also need the interior solution for σ.

As was mentioned earlier, Eq. (III.1.9) can be solved exactly for σ in the exterior region. We shall see that this exact solution for σ can be expressed as a power series in $\frac{Gm}{rc^2}$ which shall be assumed small compared to 1. It will then be shown that if we consider the quadratic terms in λ, ν, and σ appearing in (III.1.9) to be negligible compared to the first order terms in these quantities that the resulting solution for σ is the same as the power series solution to first order in $\frac{Gm}{rc^2}$. Furthermore, the fact that the exact solution for σ can be expressed as a power series in $\frac{Gm}{rc^2}$ will mean that the solutions for $e^{\bar{\lambda}(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$ can be expressed as power series in $\frac{G\bar{m}}{\bar{r}c^2}$. In all the practical applications to be made in Chapter IV, we shall only need to know $e^{\bar{\lambda}(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$ to first order in $\frac{G\bar{m}}{\bar{r}c^2}$ and the relationship between m and $\bar{m}$ to first order in $\frac{G\bar{m}}{\bar{a}c^2}$. This will mean that we only need the solution for σ to first order in $\frac{Gm}{rc^2}$. Therefore, for our needs, it will suffice to solve for σ in the interior region from Eq. (III.1.14) and match this "first order" solution at $r = a$ to the exterior power series solution to first order in $\frac{Gm}{rc^2}$ to determine the constant necessary to complete this exterior solution. Also, this first order interior solution will be sufficient to determine the relationship between m and $\bar{m}$.

a. Exterior Solution for σ

In the region $r > a$, Eq. (III.1.9) becomes

$$\sigma'' + \frac{2}{r}\sigma' + \left(\frac{\nu'-\lambda'}{2}\right)\sigma' + P_1(\sigma')^2 = 0 \qquad \text{(III.3.1)}$$

which, assuming $\sigma' \neq 0$, may be rewritten as

$$\frac{d(\ln \sigma')}{dr} + \frac{d(\ln r^2)}{dr} + \tfrac{1}{2}\frac{d(\nu-\lambda)}{dr} + P_1\frac{d\sigma}{dr} = 0 \qquad \text{(III.3.2)}$$

Integrating (III.3.2) twice yields

$$e^{P_1\sigma} = B + A\,P_1 \int r^{-2}\, e^{\left(\frac{\lambda-\nu}{2}\right)}\, dr\ , \qquad \text{(III.3.3)}$$

where A and B are constants of integration. By utilizing the expression (III.2.2) and (III.2.3) for e^{λ} and e^{ν} in the integral appearing in (III.3.3) we find

$$e^{P_1\sigma} = B + \frac{AP_1c^2}{2mG}\ln\left(1 - \frac{2mG}{rc^2}\right)\ . \qquad \text{(III.3.4)}$$

In (I.4), we set forth that at infinite distances from a gravitating body σ should become zero. Invoking this assumption in (III.3.4) implies that $B = 1$. So, inserting this value of B into (III.3.4) yields the exact exterior solution for σ to within the constant factor A.

$$e^{\sigma} = \left\{1 + \frac{AP_1c^2}{2mG}\ln\left(1 - \frac{2mG}{rc^2}\right)\right\}^{\frac{1}{P_1}} . \tag{III.3.5}$$

As previously mentioned, this exact solution can be expressed as a power series in $\frac{Gm}{rc^2}$,

$$e^{\sigma} = 1 + \frac{AP_1c^2}{2mG}\left\{\frac{-2mG}{rc^2} + 0\left(\frac{Gm}{rc^2}\right)^2 + \ldots\right\} \tag{III.3.6}$$

If it is now assumed that the terms of order $\left(\frac{Gm}{rc^2}\right)^2$ and higher on the right-hand side of (III.3.6) are negligible compared to $\frac{Gm}{rc^2}$ itself, and that σ is small enough so that e^{σ} can be approximated by $(1+\sigma)$, we find

$$\sigma \simeq -\frac{A}{r} \tag{III.3.7}$$

We will now demonstrate that (III.3.7) could have been obtained by neglecting the quadratic terms in λ, ν, and σ appearing in (III.3.1) in comparison to the first order terms in these quantities. That is, the solution to the equation

$$\sigma'' + \frac{2}{r}\sigma' = 0 \tag{III.3.8}$$

is

$$\sigma = B' - \frac{A}{r} \tag{III.3.9}$$

where in (III.3.9) B' and A are constants and B' can be set to zero from the assumption that $\sigma \to 0$ as $r \to \infty$.

With the above calculation in mind, we will now solve for σ in the interior region from the first order equation, (III.1.14), and match this solution to the first order exterior solution, (III.3.7), to determine the constant A.

b. Interior Solution for σ

Using the aforementioned approximation, in the region $r < a$ we have,

$$\sigma'' + \frac{2}{r}\sigma' = K\, P_2\, \rho c^2 \qquad \text{(III.14)}$$

where ρ is taken to be a constant.

The solution to (III.1.14) is elementary and is found to be .

$$\sigma = \frac{P_2 mG}{a^3c^2}\, r^2 - \frac{b_1}{r} + b_2 \qquad \text{(III.3.10)}$$

where we have set $m = \frac{4}{3}\pi\rho a^3$, $K = \frac{8\pi G}{c^4}$, and b_1 and b_2 are constants of integration.

To evaluate b_1 in this solution, we shall require that σ be finite at the origin $r = 0$, thereby making $b_1 = 0$. So our interior solution becomes, approximately,

$$\sigma = \frac{P_2 mG}{a^3c^2}\, r^2 + b_2 \qquad \text{(III.3.11)}$$

where b_2 is to be determined by matching this solution to the exterior solution at $r = a$.

c. Matching the Solutions at the Boundary

In matching our interior and exterior solutions at the boundary $r = a$, we must first set $\sigma(a)$ for the exterior solution equal to $\sigma(a)$ for the interior solution. Secondly, since in arriving at both our interior and exterior solutions, we assumed that σ'' existed, then we must have that σ' is continuous at the boundary $r = a$. Invoking these requirements in (III.3.7) and (III.3.11), we find that

$$-\frac{A}{a} = \frac{P_2 mG}{ac^2} + b_2 \qquad \text{(III.3.12)}$$

and

$$\frac{A}{a^2} = \frac{2P_2 mG}{a^2 c^2} \qquad \text{(III.3.13)}$$

Solving (III.3.12) and (III.3.13) for A and b_2 and inserting the results into (III.3.7) and (III.3.11) we find that

$$A = \frac{2P_2 mG}{c^2} \qquad \text{(III.3.14a)}$$

$$b_2 = -\frac{3P_2 mG}{ac^2} \qquad \text{(III.3.14b)}$$

and

$$\sigma = -\frac{2P_2 mG}{rc^2} \quad , \quad r > a \qquad \text{(III.3.15a)}$$

$$\sigma = \frac{P_2 mG}{a^3 c^2} r^2 - \frac{3P_2 mG}{ac^2} \quad , \quad r < a \qquad \text{(III.3.15b)}$$

The reader can easily verify that these solutions for σ and the interior and exterior solutions for e^λ and e^ν, (III.2.7), (III.2.8), (III.2.2), and (III.2.3), imply that the quadratic terms in λ, ν, and σ appearing in (III.1.9) are negligible compared to first order terms in these quantities if it is assumed that $\frac{Gm}{rc^2} << 1$ for $r \geq a$.

It should be noted that our first order solutions, (III.3.15), are not sensitive to the parameter P_1, but only to P_2. As previously mentioned, P_2 must be determined by comparing the predictions made by our theory to physical observation. In order to place ourselves in a position to make these comparisons, we must now determine the quantities $e^{\bar{\lambda}(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$ in the exterior region. To accomplish this, we will have to express σ in terms of $\bar{r}$ and m in terms of $\bar{m}$.

d. Determination of e^{σ} and $r\,\frac{d\sigma}{dr}$ to First Order in $\frac{Gm}{\bar{r}c^2}$ for $\bar{r} > \bar{a}$

To first order in $\frac{Gm}{rc^2}$ in the exterior region, we find by combining (III.3.6) with (III.3.14a) that

$$e^{\sigma} = 1 - \frac{2mGP_2}{rc^2} \qquad \text{(III.3.16)}$$

also, the relationship between r and $\bar{r}$ is

$$r = \bar{r}\, e^{-\sigma(\bar{r})} \qquad \text{(III.1.6)}$$

Inserting (III.1.6) into (III.3.16) and solving for $e^{\sigma(\bar{r})}$, we find, to first order in $\frac{Gm}{\bar{r}c^2}$ in the region $\bar{r} > \bar{a}$, that

$$\left.\begin{aligned} e^{\sigma(\bar{r})} &\simeq 1 - \frac{2mGP_2}{\bar{r}c^2} \\ \text{or} \qquad & \\ \sigma(\bar{r}) &\simeq - \frac{2mGP_2}{\bar{r}c^2} \end{aligned}\right\} \qquad \text{(III.3.17)}$$

From (III.1.7b), we see that in the determination of $e^{\bar{\lambda}(\bar{r})}$

we shall need the quantity $r \frac{d\sigma}{dr}$ expressed in terms of $\bar{r}$. To accomplish this, we note that from (III.3.15a) and (III.1.6) that

$$r \frac{d\sigma}{dr} = \frac{2mGP_2}{\bar{r}c^2} e^{\sigma(\bar{r})} \quad . \tag{III.3.18}$$

Inserting the expression for $e^{\sigma(\bar{r})}$ from (III.3.17) into (III.3.18), we find that to first order in $\frac{Gm}{\bar{r}c^2}$

$$r \frac{d\sigma}{dr} \simeq \frac{2mGP_2}{\bar{r}c^2} \quad . \tag{III.3.19}$$

e. Relationship Between m and $\bar{m}$ and Consequences of This Relation

For m and $\bar{m}$, we have

$$m = \frac{4}{3} \pi \rho a^3 \quad . \tag{III.2.10}$$

and

$$\bar{m} = 4\pi \int_0^{\bar{a}} \bar{\rho} \, \bar{r}^2 \, d\bar{r} \; , \tag{III.2.15}$$

where

$$\rho = \bar{\rho} \, e^{3\sigma} \tag{I.3.14}$$

and ρ is assumed constant.

We now note that m is to be identified with the mass of the gravitating object as determined by observers using the radial coordinate r, whereas $\bar{m}$ is assumed to be the mass as determined by observers using the radial coordinate $\bar{r}$. We, therefore, expect these two quantities to differ.

From (III.1.6) we find that

$$d\bar{r} = \left(1 + r\,\frac{d\sigma}{dr}\right) e^{\sigma}\,dr \qquad \text{(III.3.20)}$$

Now, by applying (I.3.14), (III.1.6), and (III.3.20) to (III.2.15), we find

$$\bar{m} = m + 4\pi \int_0^a \rho r^3\,\frac{d\sigma}{dr}\,dr \;. \qquad \text{(III.3.21)}$$

The integral appearing in this last expression for $\bar{m}$ may be evaluated by noting that from our interior solution for σ, (III.3.15b), that

$$\frac{d\sigma}{dr} = \frac{2P_2 mG}{a^3c^2}\,r \;. \qquad \text{(III.3.22)}$$

Applying this result to (III.3.21) yields

$$\bar{m} = m + \frac{6}{5}\,\frac{P_2 m^2 G}{ac^2} \qquad \text{(III.3.23)}$$

or

$$m = \frac{-1 + \sqrt{1 + \dfrac{24\bar{m}GP_2}{5ac^2}}}{\dfrac{12}{5}\,P_2\,\dfrac{G}{ac^2}} \;. \qquad \text{(III.3.24)}$$

In all our later applications, the term $\frac{G\bar{m}}{ac^2}$ shall be assumed to be small compared to 1 so that (III.3.24) can be approximated by the expression

$$m \simeq \bar{m}\left[1 - \frac{6}{5}\,\frac{P_2\bar{m}G}{ac^2}\right] . \qquad \text{(III.3.25)}$$

For our later applications, we shall need the relationship between m and $\bar{m}$ in terms of $\bar{a}$ rather than a. From (III.16) and (III.3.16), we have that

$$\bar{a} = a\,e^{\sigma(a)} \simeq a - \frac{2mGP_2}{c^2} \qquad \text{(III.3.26)}$$

By inserting (III.3.26) into (III.3.25) and again requiring that $\frac{G\bar{m}}{ac^2}$ be quite small compared to 1, we can, by a series of straightforward approximations, arrive at the result

$$\frac{m}{\bar{m}} \simeq 1 - \frac{6}{5} P_2 \frac{G\bar{m}}{\bar{a}c^2} \qquad \text{(III.3.27)}$$

Inserting the result (III.3.27) into (III.3.17) and (III.3.19), we find to first order in the terms $\frac{G\bar{m}}{\bar{r}c^2}$ and $\frac{G\bar{m}}{\bar{a}c^2}$ that in the exterior region

$$e^{\sigma(\bar{r})} \simeq 1 - \frac{2P_2\bar{m}G}{\bar{r}c^2} \qquad \text{(III.3.28)}$$

and

$$r \frac{d\sigma}{dr} \simeq \frac{2P_2\bar{m}G}{\bar{r}c^2} . \qquad \text{(III.3.29)}$$

Likewise, to first order in $\frac{G\bar{m}}{\bar{a}c^2}$ our interior solution, (III.3.15b), becomes in terms of barred quantities

$$\sigma(\bar{r}) \simeq \frac{P_2\bar{m}G}{\bar{a}^3c^2} \bar{r}^2 - \frac{3P_2\bar{m}G}{\bar{a}c^2} \qquad \text{(III.3.30)}$$

Approximating $\bar{\rho}$ in expression (I.3.14) by $\bar{\rho} \simeq \frac{\rho}{1+3\sigma}$ and applying (III.3.30) yields

$$\bar{\rho} \simeq \rho\left\{1 - \frac{9P_2\bar{m}G}{\bar{a}c^2} + \frac{3P_2\bar{m}G}{\bar{a}^3c^2} \bar{r}^2\right\}^{-1} . \qquad \text{(III.3.31)}$$

Since for the problem being considered, the non-customary mass density, ρ, is constant, this last equation shows the customary mass density, $\bar{\rho}$, is radially decreasing.

We wish to now mention that to first order in $\frac{G\bar{m}}{\bar{a}c^2}$ it can be easily shown that the equation relating P to ρ, (III.2.9), becomes

$$P \approx \frac{\rho c^2}{2} \left(\frac{\bar{m}G}{\bar{a}^3c^2}\right) \left(\bar{a}^2 - \bar{r}^2\right)$$

which, assuming that $\frac{G\bar{m}}{\bar{a}c^2}$ is very small, justifies our assumption that $P << \rho c^2$.

By utilizing the results (III.3.28) and (III.3.29), we will now determine the exterior solutions for $e^{\bar{\lambda}(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$ to first order in $\frac{G\bar{m}}{\bar{r}c^2}$.

4. Exterior Solutions for $e^{\bar{\lambda}(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$

For the relationships between $e^{\bar{\lambda}(\bar{r})}$ and $e^{\lambda(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$ and $e^{\nu(\bar{r})}$, we have

$$e^{\bar{\lambda}(\bar{r})} = \frac{e^{\lambda(\bar{r})}}{\left(1 + r\,\frac{d\sigma}{dr}\right)^2} \tag{III.1.7b}$$

where $r\,\frac{d\sigma}{dr}$ is to be expressed in terms of $\bar{r}$ and

$$e^{\bar{\nu}(\bar{r})} = e^{\nu(\bar{r})-2\sigma(\bar{r})} \tag{III.1.7c}$$

We wish to solve these equations for $e^{\bar{\lambda}(\bar{r})}$ and $e^{\bar{\nu}(\bar{r})}$ to first order in $\frac{G\bar{m}}{\bar{r}c^2}$. To this end, we notice that from (III.1.6), (III.2.2), (III.2.3), (III.3.28), and (III.3.29), that to first order in $\frac{G\bar{m}}{\bar{r}c^2}$ and $\frac{G\bar{m}}{\bar{a}c^2}$

$$e^{\lambda(\bar{r})} = \left[1 - \frac{2mG}{\bar{r}c^2} e^{\sigma(\bar{r})}\right]^{-1} \simeq 1 + \frac{2\bar{m}G}{\bar{r}c^2} \quad , \tag{III.4.1a}$$

$$\left[1 + r\frac{d\sigma}{dr}\right]^{-2} \simeq 1 - \frac{4\bar{m}GP_2}{\bar{r}c^2} \quad , \tag{III.4.1b}$$

$$e^{\nu(\bar{r})} = 1 - \frac{2mG}{\bar{r}c^2} e^{\sigma(\bar{r})} \simeq 1 - \frac{2\bar{m}G}{\bar{r}c^2} \quad , \tag{III.4.1c}$$

and

$$e^{\sigma(\bar{r})} \simeq 1 - \frac{2\bar{m}GP_2}{\bar{r}c^2} \quad . \tag{III.3.28}$$

Applying these approximations to (III.1.7b) and (III.1.7c), we arrive at

$$e^{\bar{\lambda}(\bar{r})} \simeq 1 + \frac{2\bar{m}G}{\bar{r}c^2} - \frac{4\bar{m}GP_2}{\bar{r}c^2} \tag{III.4.2}$$

and

$$e^{\bar{\nu}(\bar{r})} \simeq 1 - \frac{2\bar{m}G}{\bar{r}c^2} - \frac{4\bar{m}GP_2}{\bar{r}c^2} \quad . \tag{III.4.3}$$

So our solutions, to first order in $\frac{G\bar{m}}{\bar{r}c^2}$ and $\frac{G\bar{m}}{\bar{a}c^2}$, (III.4.2) and (III.4.3), differ from the first order solutions obtained from the ordinary Einstein theory by the term $\left(\frac{-4\bar{m}GP_2}{\bar{r}c^2}\right)$.

In Chapter IV, we shall use the results (III.4.2) and (III.4.3) to compare the predictions of our theory to observation and to the predictions of the ordinary Einstein theory. Before proceeding with these calculations, we wish to now discuss the principle of equivalence in our theory.

5. Principle of Equivalence

In the ordinary Einstein theory where test particles move along geodesics, as determined by observers using the customary congruence definition, the following statement from Weinberg[11] is taken to be true:

> At every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation.

In this statement, usually known as the strong principle of equivalence, a locally inertial coordinate system is taken to be one in which the gradients of the metric tensor, and thereby the Christoffel symbols constructed from this metric tensor vanish at the point in question.

It will now be demonstrated that in our theory, from the standpoint of an observer employing the customary congruence definition, that this principle of equivalence does not hold due to the fact that in our theory test particles do not move along geodesics as determined by customary observers.

From the discussion of the equations of motion in our theory, (I.3), we have that according to customary observers, test particles in our theory moving according to the equations

$$\frac{d^2x^\alpha}{d\bar{s}^2} + \bar{\Gamma}^\alpha_{\mu\nu}\frac{dx^\mu}{d\bar{s}}\frac{dx^2}{d\bar{s}} = \frac{\partial\sigma}{\partial x^\beta}\frac{dx^\beta}{d\bar{s}}\frac{dx^\alpha}{d\bar{s}} - \bar{g}^{\alpha\beta}\frac{\partial\sigma}{\partial x^\beta} \qquad \text{(III.5.1)}$$

In a "local inertial frame" as determined by customary observers, i.e. a frame in which the $\bar{\Gamma}^\alpha_{\mu\nu}$ vanish at a point, (III.5.1) becomes

$$\frac{d^2x^\alpha}{d\bar{s}^2} = \frac{\partial\sigma}{\partial x^\beta}\frac{dx^\beta}{d\bar{s}}\frac{dx^\alpha}{d\bar{s}} - \bar{g}^{\alpha\beta}\frac{\partial\sigma}{\partial x^\beta} \qquad \text{(III.5.2)}$$

We now point out that since σ is a scalar, then $\frac{\partial\sigma}{\partial x^\beta}$ is a four-vector. Therefore, the right-hand side of (III.5.1) is also a four-vector, and if it vanished at a point in one frame, it would vanish at this point in every frame. As can be seen from (III.3.28), $\frac{\partial\sigma}{\partial x^\beta}$ or indeed the right-hand side of (III.5.1) does not in general vanish at an arbitrary point in space-time. We, therefore, conclude that in a local inertial frame as determined by customary observers the right-hand side of (III.5.2) does not in general vanish. Hence, in this local inertial frame our equations of motion for a free test particle do not become the same as the Newtonian equations of motion for a free test particle in an unaccelerated Cartesian coordinate system in the absence of a gravitational field. Therefore, the principle of equivalence, as stated above, does not hold in our theory.

We see, however, that if we had considered a frame in which

$$\bar{\Gamma}^\alpha_{\mu\nu}\frac{dx^\mu}{d\bar{s}}\frac{dx^\nu}{d\bar{s}} = \frac{\partial\sigma}{\partial x^\beta}\frac{dx^\beta}{d\bar{s}}\frac{dx^\alpha}{d\bar{s}} - \bar{g}^{\alpha\beta}\frac{\partial\sigma}{\partial x^\beta}$$

at a point, then in this frame our equations of motion for a free test particle as determined by a customary observer would have been the same as the Newtonian equations of motion in an unaccelerated Cartesian coordinate system in the absence of a gravitational field, $\frac{d^2x^\alpha}{d\bar{s}^2} = 0$. From (I.4.8), we see that in the Newtonian limit of our theory the above frame would be one in which $\frac{\partial \bar{g}_{44}}{\partial x^i} = 2 \frac{\partial \sigma}{\partial x^i}$. Since $\frac{\partial \sigma}{\partial x^i}$ can not in general be set to zero at a point in space-time, we conclude that this frame is not the local inertial frame referred to in the above statement of the principle of equivalence.

In Chapter IV, we will discuss the consequences of the fact that the principle of equivalence does not hold in our theory when we apply our theory to the problem of the "Gravitational shift in spectral lines" to which we now turn.

CHAPTER IV

APPLICATIONS TO RELATIVITY

In the framework of our non-customary gauge theory, we will now, by applying the calculations made in Chapter III, solve the following standard problems in relativity:

1. The gravitational shift in spectral lines.
2. The precession of the perihelia of planets.
3. The defection of light by the sun.
4. The time delay of radar echos passing the sun.

We will then compare our solutions to these problems to those obtained from the ordinary Einstein theory and to observation.*

With the exception of the first and last of these problems we will find that there is no "sensible" difference between the solutions obtained from our theory and those obtained from the ordinary Einstein theory.

1. The Gravitational Shift in Spectral Lines

We shall now investigate, from the standpoint of an observer

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*In obtaining our solution to (1.), we will follow closely the method set forth by Adler, Bazin, and Schiffer.[12] For our solutions to (2.), (3.), and (4.), we will follow the procedures presented by Weinberg.[13]

using the customary congruence definition, the shift in spectral lines of light emitted from a stationary source at a point $(\bar{r}_s, t_s)$ in space-time and received by a stationary observer at another point $(\bar{r}_0, t_0)$ in space-time. We will assume that both the source and the observer are in the exterior gravitational field of a static spherically symmetric distribution of matter having mass $\bar{m}$ as determined by customary observers and being confined to the region $\bar{r} \leq \bar{a}$. Furthermore, we confine ourselves to considering the case in which the light beams emitted from the source travel radially outward toward the observer.

For photons moving along the radial direction, $\bar{r}$, our Schwarzschild line-element in the customary gauge, (III.1.4), is

$$d\bar{s}^2 = 0 = -e^{\bar{\lambda}(\bar{r})}\, d\bar{r}^2 + e^{\bar{\nu}(\bar{r})}\, (dx^4)^2 \qquad \text{(IV.1.1)}$$

where to first order in $\frac{G\bar{m}}{\bar{r}c^2}$

$$e^{\bar{\lambda}(\bar{r})} = 1 + 2(1 - 2P_2)\frac{G\bar{m}}{\bar{r}c^2} = 1 - 2(1 - 2P_2)\frac{\psi}{c^2} \qquad \text{(IV.1.2)}$$

and

$$e^{\bar{\nu}(\bar{r})} = 1 - 2(1 + 2P_2)\frac{G\bar{m}}{\bar{r}c^2} = 1 + 2(1 + 2P_2)\frac{\psi}{c^2} \qquad \text{(IV.1.3)}$$

(See Eqs. (III.4.2) and (III.4.3).) In (IV.1.2) and (IV.1.3), we have denoted the Newtonian gravitational potential, $-\frac{G\bar{m}}{\bar{r}}$, by ψ.

In what follows, we will denote the coordinate time intervals between the successive wavefronts emitted by the source and those received by the observer by δt_s and δt_0 respectively. The proper-time intervals and proper-frequencies for these "beats" as determined by customary inertial observers will be denoted by $\delta\bar{\tau}_s$ and $\delta\bar{\tau}_0$ and $\bar{f}_s$ and $\bar{f}_0$ respectively.

In the ordinary Einstein theory, the shift in the spectral lines may be determined by invoking the field equations, (I.1.18). The results being[12]

$$\frac{\bar{f}_0}{\bar{f}_s} = \sqrt{\frac{1 + 2\,\frac{\psi_s}{c^2}}{1 + 2\,\frac{\psi_0}{c^2}}} \tag{IV.1.4}$$

our expanding (IV.1.2) to first order in $\frac{G\bar{m}}{\bar{r}_s c^2}$ and $\frac{G\bar{m}}{\bar{r}_0 c^2}$

$$\frac{\Delta\bar{f}}{\bar{f}_s} \simeq \frac{\psi_s - \psi_0}{c^2} \tag{IV.1.5}$$

where

$$\Delta\bar{f} \equiv \bar{f}_0 - \bar{f}_s\ . \tag{IV.1.6}$$

We note that these results could have also been obtained by invoking the principle of equivalence.[12] Since the principle of equivalence does not hold in our theory, we therefore expect that our solution to this problem will yield results which differ from (IV.1.4) or (IV.1.5).

According to (IV.1.1), we have that for a light beam moving radially outward from the source to the observer

$$\int_{t_s}^{t_0} dt = \frac{1}{c}\int_{\bar{r}_s}^{\bar{r}_0} e^{\frac{\bar{\lambda}-\bar{\nu}}{2}}\, d\bar{r}\ . \tag{IV.1.7}$$

Since both the source and the observer are assumed stationary, it is readily deduced from (IV.1.7) that the coordinate time intervals between wavefronts emitted by the source and received by the observer are the same.

$$\delta t_0 = \delta t_s \tag{IV.1.8}$$

From (IV.1.1), the relationships between the proper-time intervals between these beats and the coordinate time intervals is

$$\delta\bar{\tau}_s = \frac{1}{c}\, e^{\bar{\nu}_s/2}\, \delta t_s \tag{IV.1.9}$$

and

$$\delta\bar{\tau}_0 = \frac{1}{c}\, e^{\bar{\nu}_0/2}\, \delta t_0 \tag{IV.1.10}$$

Combining (IV.1.8) with (IV.1.9) and (IV.1.10) and applying (IV.1.3), we find that to first order in $\frac{G\bar{m}}{\bar{r}_s c^2}$ and $\frac{G\bar{m}}{\bar{r}_0 c^2}$ that

$$\frac{\delta\bar{\tau}_s}{\delta\bar{\tau}_0} = \frac{\bar{f}_0}{\bar{f}_s} = \frac{e^{\bar{\nu}_s/2}}{e^{\bar{\nu}_0/2}} \simeq \sqrt{\frac{1 + 2(1 + 2P_2)\frac{\psi_s}{c^2}}{1 + 2(1 + 2P_2)\frac{\psi_0}{c^2}}} \tag{IV.1.11}$$

Expanding (IV.1.11) to first order in $\frac{G\bar{m}}{\bar{r}_s c^2}$ and $\frac{G\bar{m}}{\bar{r}_0 c^2}$, we arrive at

$$\frac{\Delta\bar{f}}{\bar{f}_s} \simeq (1 + 2P_2)\, \frac{\left(\psi_s - \psi_0\right)}{c^2} \tag{IV.1.12}$$

We see by comparing our result for the fractional shift in frequency, (IV.1.12), to that obtained from the ordinary Einstein theory (or the principle of equivalence), (IV.1.5), that the ratio of the difference of the two results to the result from the ordinary Einstein theory is $2P_2$.

In order to obtain an approximate value for P_2, we note that in the Mossbauer experiment performed by Pound and Rebka[14], their result for the frequency shift, $\frac{\Delta \bar{f}}{\bar{f}_s}$, was to within about one percent of the value predicted by the ordinary Einstein theory. Accepting this result, we conclude then that $|P_2| \leq .005$. This result for P_2 seemingly rules out our equation on σ, (III.1.8), derived from the variational principle (III.1.1), for in this equation $P_2 = \frac{1}{12} =$ .083. It should be noted, however, that in other admittedly less accurate experiments, notably the determination of the red shift of light from 40 Eridani B,[15] that the observed red shifts do not rule out the value $P_2 = \frac{1}{12}$. Furthermore, in Chapter V where we apply our theory to cosmology, we will show that our equation (II.1.8) for σ yields results which are compatible with the observed mass density of the universe. So, for the sake of giving a complete view of our theory, we shall not discard Eq. (II.1.8) at this time.

We now continue with the applications of our theory to problems in relativity by considering the precession of the perihelia of planets.

2. The Precession of the Perihelia of Planets

For the ordinary Einstein theory, it is shown in the discussion presented by Weinberg,[13] that by employing a line element in the standard form

$$d\bar{s}^2 = -e^{\bar{\lambda}(\bar{r})}\, d\bar{r}^2 - \bar{r}^2 d\Omega^2 + e^{\bar{\nu}(\bar{r})}\, (dx^4)^2 \qquad \text{(III.1.4)}$$

where

$$e^{-\bar{\lambda}(\bar{r})} = e^{\bar{\nu}(\bar{r})} \qquad \text{(III.2.13)}$$

$$= 1 - \frac{2\bar{m}G}{\bar{r}c^2} , \qquad \text{(III.2.14)}$$

that the geodesic equations of motion imply that the precision of the perihelion, Δ_0, of a planet in orbit about a uniform spherically symmetric distribution of matter of mass $\bar{m}$ confined to the region $\bar{r} \leq \bar{a}$ is, to first order in $\frac{G\bar{m}}{\bar{r}c^2}$, given by

$$\Delta_0 = \frac{3\pi G\bar{m}}{c^2} \left[\frac{1}{\bar{r}_+} + \frac{1}{\bar{r}_-}\right] \qquad \text{(IV.2.1)}$$

In this expression $\bar{r}_+$ and $\bar{r}_-$ are taken to be the maximum and minimum coordinate distances of the planet from the center of the mass $\bar{m}$, where

$$\bar{m} = \int_0^{\bar{a}} \bar{\rho}\, \bar{r}^2 \, d\bar{r} \; .$$

The error in taking $\bar{r}_+$ and $\bar{r}_-$ to be measured distances being of the order $\frac{G\bar{m}}{\bar{r}_\pm c^2}$.

We point out, that if we had applied the geodesic equations of motion to a line element in the isotropic form

$$d\bar{s}^2 = -e^{\bar{\mu}(r')}\left[(dr')^2 + (r')^2 d\Omega^2\right] + e^{\bar{\nu}(r')}(dx^4)^2 \qquad \text{(IV.2.2)}$$

then the resulting expression for the precession of the perihelion would have been

$$\Delta_0 = \frac{3\pi G\bar{m}}{c^2} \left[\frac{1}{r'_+} + \frac{1}{r'_-}\right] \qquad \text{(IV.2.3)}$$

The relationship between $\bar{r}$ and r' in (III.1.4) and (IV.2.2) is[16]

$$\bar{r} = \left(1 + \frac{G\bar{m}}{2r'c^2}\right)^2 r' \; ;^* \tag{IV.2.4}$$

implying that

$$e^{\bar{\mu}(r')} = \left(1 + \frac{G\bar{m}}{2r'c^2}\right)^4 \tag{IV.2.5a}$$

and

$$e^{\bar{\nu}(r')} = \left[\frac{1 - \frac{G\bar{m}}{2r'c^2}}{1 + \frac{G\bar{m}}{2r'c^2}}\right]^2 . \tag{IV.2.5b}$$

In the discussion of the precession of perihelia from the standpoint of our non-customary gauge theory, we shall find it convenient to employ a line element in the non-customary gauge which is in isotropic form. To this end, we note that by applying the gauge transformation $ds^2 = e^{-2\sigma}d\bar{s}^2$ to the line element (IV.2.2) our non-customary line element can be written in the isotropic form

$$ds^2 = -e^{\mu(r')}\left[(dr')^2 + (r')^2 d\Omega^2\right] + e^{\nu(r')}(dx^4)^2 \tag{IV.2.6}$$

where

$$e^{\mu(r')} = e^{\bar{\mu}(r') - 2\sigma(r')} \tag{IV.2.7a}$$

and

$$e^{\nu(r')} = e^{\bar{\nu}(r') - 2\sigma(r')} . \tag{IV.2.7b}$$

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*The result (IV.2.3) is implied by (IV.2.4) since the error introduced into (IV.2.1) by substituting r' for $\bar{r}$ is of second order in $\frac{G\bar{m}}{\bar{r}c^2}$.

The reader is cautioned not to confuse the values (IV.2.5) for $e^{\bar{\mu}(r')}$ and $e^{\bar{\nu}(r')}$ obtained from the ordinary Einstein theory with the quantities $e^{\bar{\mu}(r')}$ and $e^{\bar{\nu}(r')}$ appearing in (IV.2.7) which could be determined within the framework of our non-customary gauge theory.

The solution to our field equations, $G^{\mu}_{\nu} = 0$, for $e^{\mu(r')}$ and $e^{\nu(r')}$ must be of the same form as (IV.2.5), since the solutions (IV.2.5) came from solving $\bar{G}^{\mu}_{\nu} = 0$. Therefore,

$$e^{\mu(r')} = \left(1 + \frac{Gm}{2r'c^2}\right)^4 \qquad \text{(IV.2.8a)}$$

and

$$e^{\nu(r')} = \left(\frac{1 - \frac{Gm}{2r'c^2}}{1 + \frac{Gm}{2r'c^2}}\right)^2 \qquad \text{(IV.2.8b)}$$

where m in (IV.2.8) is related to $\bar{m}$ in (IV.2.5) by

$$\frac{m}{\bar{m}} \simeq 1 - \frac{6}{5} P_2 \frac{\bar{m}G}{\bar{a}c^2} \qquad \text{(III.3.27)}$$

or from (IV.2.4), to first order in $\frac{G\bar{m}}{a'c^2}$

$$\frac{m}{\bar{m}} \simeq 1 - \frac{6}{5} P_2 \frac{\bar{m}G}{a'c^2} \qquad \text{(IV.2.9)}$$

Now since in our theory particles follow geodesics in the non-customary gauge we must have then that to first order in $\frac{Gm}{r'c^2}$ that the precession of the perihelion of a planet is Δ, where

$$\Delta = \frac{3\pi Gm}{c^2}\left[\frac{1}{r'_+} + \frac{1}{r'_-}\right] \qquad \text{(IV.2.10)}$$

or applying (IV.2.9)

$$\Delta \simeq \frac{3\pi G\bar{m}}{c^2}\left[1 - \frac{6}{5} P_2 \frac{\bar{m}G}{a'c^2}\right]\left[\frac{1}{r'_+} + \frac{1}{r'_-}\right] \tag{IV.2.11}$$

It can be seen from (IV.2.11) and (IV.2.3) that there is a slight difference between our value of the precession, Δ, and the ordinary Einstein value, Δ_0.

$$\Delta - \Delta_0 = -\frac{6}{5} P_2 \frac{\bar{m}G}{a'c^2}\left\{\frac{3\pi\bar{m}G}{c^2}\left[\frac{1}{r'_+} + \frac{1}{r'_-}\right]\right\}$$

Therefore, we see that the difference between our theory and the ordinary Einstein theroy is of the order $\left(\frac{G\bar{m}}{c^2}\right)^2 \frac{1}{a'r'_\pm}$, which is larger than the $\left(\frac{G\bar{m}}{r'_\pm c^2}\right)^2$ error introduced into Δ_0, (IV.2.3), by identifying $r'_\pm$ with measured distances. However, this difference, (IV.2.12), is too small to be detected at the present.

From (IV.2.11), to first order in $\frac{G\bar{m}}{a'c^2}$ and $\frac{G\bar{m}}{r'_\pm c^2}$ we have

$$\Delta \simeq \frac{3\pi G\bar{m}}{c^2}\left[\frac{1}{r'_+} + \frac{1}{r'_-}\right] = \Delta_0 \tag{IV.2.12}$$

So we see that, since the error introduced by using isotropic coordinates in place of standard coordinates is negligible in this calculation, our result for the precession of the perihelia of planets is, to first order in the terms $\frac{G\bar{m}}{r'_\pm c^2}$ and $\frac{G\bar{m}}{a'c^2}$ or $\frac{G\bar{m}}{\bar{r}_\pm c^2}$ and $\frac{G\bar{m}}{\bar{a}c^2}$, the same as the result from the ordinary Einstein theory. We shall see that this same statement is true for the deflection of light by the sun, to which we now turn.

3. The Deflection of Light Rays by the Sun

From the ordinary Einstein theory, it is shown by Weinberg[13] that the deflection of a light beam passing by the sun is, to first order in $\frac{G\bar{m}}{\bar{r}_0c^2}$, $(\Delta\phi)_0$ where

$$(\Delta\phi) = \frac{4\bar{m}G}{\bar{r}_0c^2} \quad . \tag{IV.3.1}$$

In this expression, $\bar{m}$ is taken to be the customary mass of the sun and $\bar{r}_0$ is the standard form Schwarzschild coordinate appearing in (III.1.4) and represents the point of closest approach of the light beam to the center of the sun. Since the difference between this Schwarzschild coordinate and the measured distance of closest approach is of the order $\frac{G\bar{m}}{\bar{r}_0c^2}$, then $\bar{r}_0$ in (IV.3.1) can be identified with the measured distance of closest approach.

As in the discussion of the precession of the perihelia of planets, since the difference between the standard form radial coordinate, $\bar{r}$, and the isotropic form radial coordinate, r', is negligible in this calculation, we can, in place of (IV.3.1), write for the solution of this problem in isotropic coordinates

$$(\Delta\phi)_0 = \frac{4\bar{m}G}{r_0'c^2} \tag{IV.3.2}$$

By applying the discussion presented in (IV.2) for the solution in our theory to the problem of the precession of the perihelia of planets, we see that the result predicted by our theory for the deflection of light passing the sun expressed in isotropic coordinates is, to first order in $\frac{Gm}{r_0'c^2}$, $\Delta\phi$, where

$$\Delta\phi = \frac{4mG}{r_0'c^2} \simeq \frac{4\bar{m}G}{r_0'c^2}\left[1 - \frac{6}{5}P_2\frac{\bar{m}G}{a'c^2}\right] \tag{IV.3.3}$$

In this expression, a' is to be identified with the radius of the sun in the customary gauge. From this result, (IV.3.3), we see that the difference between our theory's predicted value for the deflection angle and that predicted by the ordinary Einstein theory, (IV.3.2), is of the order $\left(\frac{G\bar{m}}{c^2}\right)^2 \frac{1}{a'r_0'}$. As noted in the discussion of the precession of the perihelia of planets, this difference is less than the $\left(\frac{G\bar{m}}{r_0'c^2}\right)^2$ error introduced into by identifying r_0' with the measured distance of closest approach, but is still too small to be detected.

To first order in $\frac{G\bar{m}}{a'c^2}$ and $\frac{G\bar{m}}{r_0'c^2}$, (IV.3.3) becomes

$$\Delta\phi = \frac{4\bar{m}G}{r_0'c^2} = (\Delta\phi)_0 \tag{IV.3.4}$$

So, once again we see that since the difference between standard and isotropic coordinates is negligible in this calculation, that our result for the deflection of light passing the sun is the same to first order in $\frac{G\bar{m}}{a'c^2}$ and $\frac{G\bar{m}}{r_0'c^2}$ or $\frac{G\bar{m}}{\bar{a}c^2}$ and $\frac{G\bar{m}}{\bar{r}_0c^2}$ as the result from the ordinary Einstein theory.*

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*The author wishes to mention that he has carried out the calculations for the precession of perihelia and for light deflection by using the standard form coordinate, $\bar{r}$, and that the results are the same as those mentioned above. It should be noted, however, that to carry out the calculation for the precession of perihelia from this standpoint that expressions to second order in $\frac{G\bar{m}}{\bar{r}c^2}$ for the components of the metric tensor must be used(13) which necessitates calculating $e^{\sigma(\bar{r})}$ to second order in $\frac{G\bar{m}}{\bar{r}c^2}$. This can be carried out in a straight-

We shall now calculate, from the standpoint of our non-customary gauge theory, the time delay of radar echos passing the sun and compare our results to the result from the ordinary Einstein theory. Since, as pointed out by Weinberg,[13] the difference between standard and isotropic coordinates is not negligible in this calculation, we will, in order to compare our results to those from the ordinary Einstein theory presented by Weinberg,[13] carry out our calculations in terms of the standard form coordinate, $\bar{r}$.

4. The Time Delay of Radar Echos Passing the Sun

In this section, we will be interested in determining from the standpoint of our non-customary gauge theory the time required for radar signals to travel to the inner planets and be refelcted back to the earth. We will compare our result to the result obtained from the ordinary Einstein theory. To this end, we note that from the discussion presented by Weinberg,[13] wherein it is shown that since particles in the ordinary Einstein theory follow geodesics (in the customary gauge) it follows that the time required for light to travel from $\bar{r}_0$ to $\bar{r}$ or from $\bar{r}$ to $\bar{r}_0$ is

$$t_0(\bar{r}\ \bar{r}_0) = \frac{1}{c}\int_{\bar{r}_0}^{\bar{r}} \left\{\frac{e^{\bar{\lambda}-\bar{\nu}}}{1 - \left(\frac{\bar{r}_0}{\bar{r}}\right)^2 e^{\bar{\nu}-\bar{\nu}_0}}\right\}^{\frac{1}{2}} d\bar{r} \qquad \text{(IV.4.1)}$$

where

$$e^{-\bar{\lambda}(\bar{r})} = e^{\bar{\nu}(\bar{r})} = 1 - \frac{2\bar{m}G}{\bar{r}c^2} \qquad \text{(III.2.13),(III.2.14)}$$

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forward but tedious manner, so we chose to present the results by the method just employed.

and $\bar{r}$ is the Schwarzschild coordinate appearing in the customary standard form line element, (III.1.4). In this expression, (IV.4.1), $\bar{r}_0$ will be identified with the distance of closest approach of the radar signal to the center of the sun according to an observer using the customary standard form line element, (III.1.4).

By expanding the integrand of (IV.4.1) to first order in $\frac{G\bar{m}}{\bar{r}c^2}$ and $\frac{G\bar{m}}{\bar{r}_0 c^2}$, it can be shown(13) that

$$t_0(\bar{r},\bar{r}_0) \approx \frac{1}{c}\left\{\sqrt{\bar{r}^2-\bar{r}_0^2} + \frac{2\bar{m}G}{c^2}\ln\left(\frac{\bar{r}+\sqrt{\bar{r}^2-\bar{r}_0^2}}{\bar{r}_0}\right) + \frac{\bar{m}G}{c^2}\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}+\bar{r}_0}\right)^{\frac{1}{2}}\right\} \qquad \text{(IV.4.2)}$$

The leading term in this expression, $\frac{1}{c}\sqrt{\bar{r}^2-\bar{r}_0^2}$, is the time we would expect for the trip if space-time were flat. Therefore, the "excess" time required for a trip from a general $\bar{r}_1$ to a general $\bar{r}_2$ and back again due to the curvature of space-time is

$$\Delta t_0 = \frac{2\bar{m}G}{c^3}\left\{2\ln\left(\frac{\bar{r}_1+\sqrt{\bar{r}_1{}^2-\bar{r}_0{}^2}}{\bar{r}_0}\right) + 2\ln\left(\frac{\bar{r}_2+\sqrt{\bar{r}_2{}^2-\bar{r}_0{}^2}}{\bar{r}_0}\right)\right.$$

$$\left. + \left(\frac{\bar{r}_1-\bar{r}_0}{\bar{r}_1+\bar{r}_0}\right)^{\frac{1}{2}} + \left(\frac{\bar{r}_2-\bar{r}_0}{\bar{r}_2+\bar{r}_0}\right)^{\frac{1}{2}}\right\} \qquad \text{(IV.4.3)}$$

We now wish to calculate this excess time delay from our non-customary gauge theory. To accomplish this, we note that for our non-customary line element in the standard form

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2d\Omega^2 + e^{\nu(r)}(dx^4)^2 \qquad \text{(III.1.7)}$$

we have

$$e^{-\lambda(r)} = e^{\nu(r)} = 1 - \frac{2mG}{rc^2} . \qquad \text{(III.2.2),(III.2.3)}$$

From this statement, we see that since particles in our theory follow geodesics in the non-customary gauge that our result for the time required for light to travel from r to r_0 or r_0 to r must be of the same form as (IV.4.1). So we have

$$t(r,r_0) = \frac{1}{c}\int_{r_0}^{r} \left\{\frac{e^{\lambda-\nu}}{1-\left(\frac{r_0}{r}\right)^2 e^{\nu-\nu_0}}\right\}^{\frac{1}{2}} dr \ . \qquad \text{(IV.4.4)}$$

where r_0 is to be identified with the distance of closest approach to the center of the sun as determined by non-customary observers (observers using the line element (III.1.7a).)

In order to compare our result, (IV.4.4), to the result from the ordinary Einstein theory, (IV.4.3), we must express the integrand of (IV.4.4) in terms of $\bar{r}$ to first order in $\frac{G\bar{m}}{\bar{r}c^2}$ and $\frac{G\bar{m}}{\bar{r}_0c^2}$. This can be accomplished by noting that

$$r = \bar{r}\, e^{-\sigma(\bar{r})} \ , \qquad \text{(III.1.6)}$$

$$\frac{m}{\bar{m}} \simeq 1 - \frac{6}{5} P_2 \frac{\bar{m}G}{\bar{a}c^2} \qquad \text{(III.3.27)}$$

and

$$e^{\sigma(\bar{r})} \simeq 1 - 2P_2 \frac{\bar{m}G}{\bar{r}c^2} \qquad \text{(III.3.28)}$$

Using these expressions, we find to first order in $\frac{G\bar{m}}{\bar{r}c^2}$, $\frac{G\bar{m}}{\bar{r}_0c^2}$, and $\frac{G\bar{m}}{\bar{a}c^2}$ that

$$\left.\begin{aligned} \frac{dr}{d\bar{r}} &\simeq 1 \\ e^{\lambda(r)} &\simeq 1 + \frac{2\bar{m}G}{\bar{r}c^2} \end{aligned}\right\} \qquad \text{(IV.4.5)}$$

$$\left.\begin{aligned} e^{\nu(r)} &\approx 1 - \frac{2\bar{m}G}{\bar{r}c^2} \\ \left(\frac{r_0}{r}\right)^2 &\approx \left\{1 + 4P_2\frac{\bar{m}G}{c^2}\left(\frac{1}{\bar{r}_0} - \frac{1}{\bar{r}}\right)\right\}\left(\frac{\bar{r}_0}{\bar{r}}\right)^2 \end{aligned}\right|$$

and

$$1 - \left(\frac{r_0}{r}\right)^2 e^{\nu-\nu_0} \approx \left(1 - \frac{\bar{r}_0^{\,2}}{\bar{r}^2}\right)\left\{1 - 2(1+2P_2)\frac{\bar{m}G\bar{r}_0}{c^2\bar{r}(\bar{r}+\bar{r}_0)}\right\} \tag{IV.4.5}$$

Applying (IV.4.5) to the integrand of (IV.4.4), we find to first order in $\frac{G\bar{m}}{\bar{r}c^2}$ and $\frac{G\bar{m}}{\bar{r}_0c^2}$ that

$$t(\bar{r},\bar{r}_0) \approx \frac{1}{c}\int_{\bar{r}_0}^{\bar{r}} \left(1 - \frac{\bar{r}_0^{\,2}}{\bar{r}^2}\right)^{-\frac{1}{2}}\left\{1 + \frac{2\bar{m}G}{\bar{r}c^2} + (1+2P_2)\frac{\bar{m}G\bar{r}_0}{c^2\bar{r}(\bar{r}+\bar{r}_0)}\right\}d\bar{r}, \tag{IV.4.6}$$

where in the limits on this we integral have neglected the term $\frac{G\bar{m}}{c^2}$ in comparison to $\bar{r}$ and $\bar{r}_0$.

Carrying out the integration in (IV.4.6) yields

$$t(\bar{r},\bar{r}_0) \approx \frac{1}{c}\left\{\sqrt{\bar{r}^2-\bar{r}_0^{\,2}} + \frac{2\bar{m}G}{c^2}\ln\left(\frac{\bar{r}+\sqrt{\bar{r}^2-\bar{r}_0^{\,2}}}{\bar{r}_0}\right) + (1+2P_2)\frac{G\bar{m}}{c^2}\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}+\bar{r}_0}\right)^{\frac{1}{2}}\right\}. \tag{IV.4.7}$$

We see then that in our theory the excess time delay due to the curvature of space-time for a trip from $\bar{r}_1$ to $\bar{r}_2$ and back is

$$\Delta t = \frac{2\bar{m}G}{c^3}\left\{2\ln\left(\frac{\bar{r}_1+\sqrt{\bar{r}_1^{\,2}-\bar{r}_0^{\,2}}}{\bar{r}_0}\right) + 2\ln\left(\frac{\bar{r}_2+\sqrt{\bar{r}_2^{\,2}-\bar{r}_0^{\,2}}}{\bar{r}_0}\right) + (1+2P_2)\left[\left(\frac{\bar{r}_1-\bar{r}_0}{\bar{r}_1+\bar{r}_0}\right)^{\frac{1}{2}} + \left(\frac{\bar{r}_2-\bar{r}_0}{\bar{r}_2+\bar{r}_0}\right)^{\frac{1}{2}}\right]\right\}. \tag{IV.4.8}$$

Therefore, the difference between our result for the excess time delay, (IV.4.8), and the result from the ordinary Einstein theory, (IV.4.3), is

$$\Delta t - \Delta t_0 = \frac{4P_2\bar{m}G}{c^3}\left\{ \left(\frac{\bar{r}_1 - \bar{r}_0}{\bar{r}_1 + \bar{r}_0}\right)^{\frac{1}{2}} + \left(\frac{\bar{r}_2 - \bar{r}_0}{\bar{r}_2 + \bar{r}_0}\right)^{\frac{1}{2}} \right\} \tag{IV.4.9}$$

The maximum excess delay for a signal traveling from Earth to Mercury and back occurs when Mercury is at superior conjunction so that $\bar{r}_0$ can be taken as about equal to the sun's radius, $\bar{r}_0 \simeq \bar{a}$. For this Earth-Mercury trip, we have from the ordinary Einstein theory

$$(\Delta t_0)_{max} \simeq 240 \ \mu\text{sec} \ . \tag{IV.4.10}$$

and for the difference between the results of our theory and the ordinary Einstein theory

$$(\Delta t)_{max} - (\Delta t_0)_{max} \simeq 20 \ P_2 \ \mu\text{sec} \ . \tag{IV.4.11}$$

If the value $P_2 = 1/12$ associated with the σ-equation, (II.1.8), is used in (VI.4.11), then the difference between our result and the ordinary Einstein result would be approximately 1.7 μsec. If the constraint on P_2 stemming from the Pound-Rebka Mossbauer experiment is used, $P_2 \leq .005$, then this difference would be less than .1 μsec.

The experiments concerning radar echo delay carried out to date do not seem to be sufficiently accurate to enable us to assign a value to P_2 from these experiments with much confidence. Some of the experiments[17] incorporate the ordinary theory of general relativity itself in the determination of delay times, and, therefore, the results of these experiments perhaps can not be used to distinguish our theory from Einstein's.

To conclude this chapter, we simply state that in the problems we have considered only the gravitational shift in spectral lines and the "time delay of radar echos passing the sun" have predicted values in our theory which are "sensibly" different than those predicted from the ordinary Einstein theory. We shall now apply our theory to problems in cosmology.

CHAPTER V

AN APPLICATION TO COSMOLOGY

In this chapter, we devote our attention to the application of our non-customary gauge theory to the description of the universe in the large. From the outset, we assume that in the large the universe, from the standpoint of observers employing the customary congruence definition, is both homogeneous and isotropic. From this assumption, and from our field equations, (I.1.8), we demonstrate in section 1 that the line elements in both the customary and non-customary gauges can be written in the Robertson-Walker form.[18]

In section 2, by making use of the Robertson-Walker line element in the non-customary gauge and our field equations, (I.1.8), we present the equations which govern our universe for the case where the stress-energy-momentum tensor in the customary gauge is that of a perfect fluid.

Section 3 will be devoted to a discussion of the line element in the customary gauge. We demonstrate that since both our customary line element and the line element employed in the ordinary Einstein theory can be written in the Robertson-Walker form, then any calculations based solely on the form of these line elements must yield the same results, in form, in both theories.

In section 4, we return to the equations governing the universe presented in section 2, and discuss them for the case of a matter-dominated universe.

Section 5 will be devoted to solving the equations presented in section 4 for the matter-dominated universe. In this section, we shall discuss a static universe as viewed from the standpoint of the non-customary gauge. In order to compare the results of these calculations to observations carried out by customary observers, we shall also need to consider an equation for σ. We now mention that it will not be our purpose to give an exhaustive discussion of all alternatives available in our theory which stem from the three equations postulated on σ in Chapter II. Rather, we shall make it our purpose to demonstrate as simply as possible the workings of our theory in its application to cosmology. Therefore, we will confine ourselves to the discussion of the results which stem from selecting one particular equation for σ. For this equation, we choose what is in the author's opinion the simplist equation of the three in form--(II.4.4). We shall see that the application of this equation to the static "non-customary universe" yields results which are compatible with observations carried out by customary observers.

1. Line Elements

Assuming the universe to be homogeneous and isotropic from the standpoint of observers using the customary congruence definition, it can be shown[18] that the line element in the customary gauge can be written in the form

$$d\bar{s}^2 = -e^{\bar{\mu}(\bar{r},\bar{t})}\, d\bar{\gamma}^2 + (d\bar{x}^4)^2 \tag{V.1.1}$$

where

$$d\bar{\gamma}^2 = d\bar{r}^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{V.1.2}$$

and

$$\bar{\mu}(\bar{r},\bar{t}) = f(\bar{r}) + g(\bar{t}) \tag{V.1.3}$$

The development of this form of the line element, (V.1.1), is independent of the field equations. However, $f(\bar{r})$ and $g(\bar{t})$ are to be determined by the field equations, (I.1.8).*

Now, as can be seen from our field equations expressed in the form (I.3.15), σ appears as a field in the customary gauge. Therefore, in order for the universe to appear both homogeneous and isotropic from the viewpoint of customary observers, σ can be at most a function of time.[19]

$$\sigma = \sigma(\bar{t}) \tag{V.1.4}$$

So, applying the gauge transformation $ds^2 = e^{-2\sigma}\, d\bar{s}^2$ to (V.1.1), we conclude that our line element in the non-customary gauge is of the form

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*Our reason for using bars on the coordinates $\bar{r}$ and $\bar{t}$ will become evident shortly.

$$ds^2 = -e^{\mu(\bar{r},t)} d\bar{\gamma}^2 + (dx^4)^2 \tag{V.1.5}$$

where

$$dx^4 = cdt = ce^{-\sigma(\bar{t})} d\bar{t} \tag{V.1.6}$$

and

$$\mu(\bar{r},t) = f(\bar{r}) + g(t) - 2\sigma(t) \ . \tag{V.1.7}$$

In order to determine $f(\bar{r})$, we once again invoke our assumption of homogeneity and isotropy in the customary gauge, this time in reference to the distribution of energy-momentum in the universe, and conclude that [18]

$$\bar{T}_1^1 = \bar{T}_2^2 = \bar{T}_3^3 \ . \tag{V.1.8}$$

Since from (I.1.12) and (I.3.13) we have $T_\nu^\mu = e^{3\sigma} \bar{T}_\nu^\mu$, it follows from (V.1.8) that for the distribution of energy-momentum in the non-customary gauge

$$T_1^1 = T_2^2 = T_3^3 \ . \tag{V.1.9}$$

Now, from (I.1.8), our field equations in the non-customary gauge can be expressed as

$$G_\nu^\mu = K\ T_\nu^\mu + \delta_\nu^\mu\ \Lambda \ . \tag{V.1.10}$$

where we have now included the cosmological constant Λ.

By following the procedure presented by Adler, Bazin, and Schiffer,[18] we deduce from the coupling of our line element, (V.1.5), with (V.1.10) and from the constraints (V.1.9) that

$$e^{f(\bar{r})} = \left(1 + \frac{k\bar{r}^2}{4\bar{r}_0{}^2}\right)^{-1} \tag{V.1.11}$$

where $\bar{r}_0$ is a constant and $k = +1$, 0, or -1.

Inserting (V.1.11) into (V.1.5), and applying the coordinate transformation $\bar{r}' = \bar{r}/\bar{r}_0$ we have, after suppressing the primes,

$$ds^2 = \frac{-R^2(t)}{\left(1 + \frac{k\bar{r}^2}{4}\right)} d\bar{\gamma}^2 + (dx^4)^2 \qquad \text{(V.1.12)}$$

where

$$R^2(t) = \bar{r}_0{}^2 \, e^{g(t)-2\sigma(t)} \quad . \qquad \text{(V.1.13)}$$

Again following Adler, Bazin, and Shiffer(18), we note that R(t) in (V.1.13) is to be interpreted as the radius in the non-customary gauge of a three-dimensional hypersphere imbedded in a four-dimensional Euclidean space.

For later purposes, we shall need to have our non-customary line element, (V.1.12), expressed in a different form. Setting $r = \left(1 + \frac{k\bar{r}^2}{4}\right)^{-1}$ in (V.1.12) yields

$$ds^2 = \frac{-R^2(t)}{1 - kr^2} dr^2 - R^2(t)\, d\Omega^2 + (dx^4)^2 \qquad \text{(V.1.14)}$$

where

$$d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

In section 3, we shall need the <u>customary</u> line element associated with (V.1.14). So, applying the inverse of the gauge transformation used to arrive at (V.1.5), to (V.1.14), yields for the line element in the customary gauge

$$d\bar{s}^2 = \frac{-\bar{R}^2(\bar{t})}{1 - kr^2} dr^2 - \bar{R}^2(\bar{t}) d\Omega^2 + (d\bar{x}^4)^2 \qquad \text{(V.1.15)}$$

where

$$\bar{R}(\bar{t}) = R(\bar{t}) e^{\sigma(\bar{t})} \qquad \text{(V.1.16)}$$

and

$$d\bar{x}^4 = e^{\sigma(t)}\, dx^4 \tag{V.1.17}$$

It should be noticed that we are using the same radial coordinate, r, in both gauges, while the time coordinates, x^4 and $\bar{x}^4$, appearing in the line elements (V.1.14) and (V.1.15) respectively are related by (V.1.17).

So, we have found that by assuming the universe to be homogeneous and isotropic from the standpoint of customary observers, that both our customary and non-customary line elements are of the Robertson-Walker form.

We mention now that, from (V.1.16) and the identification made above for $R(t)$, $\bar{R}(t)$ is to be identified as the radius in the customary gauge of a three-dimensional hypersphere imbedded in a four-dimensional Euclidean space. Therefore, $\bar{R}(\bar{t})$ is the quantity which determines the time development of the universe from the standpoint of observers using the customary congruence definition and thus will be of primary importance to us in comparing our theory to observation and to the ordinary Einstein theory. For ease of calculation, we shall in the succeeding sections for the most part carry out our computations in the non-customary gauge, i.e. we will determine $R(t)$ and then $\bar{R}(\bar{t})$. Accordingly, from (V.1.16) and (V.1.17), we see that in order to obtain $\bar{R}(\bar{t})$ we will need to determine $\sigma(t)$. In section 5, we will carry out a solution for $\sigma(t)$ in the case of a matter-dominated universe.

Before leaving this section and proceeding with an investigation of our field equations, we point out, again by following the discussion by Adler, Bazin and Schiffer,[18] that k in (V.1.13), (V.1.14), and (V.1.15) determines the sign for the curvature of the universe in either the customary or non-customary gauge: k = +1, 0, or -1 corresponding to positive, zero, or negative curvature respectively.

2. Field Equations for a Perfect Fluid

In this section, we discuss consequences resulting from selecting the stress-energy-momentum tensor in the customary gauge to be that of a perfect fluid. We will see that in our theory in the non-customary gauge that the equations governing the time development of the universe are of the same form as those governing this development in the ordinary Einstein theory.

For a perfect fluid, the stress-energy-momentum tensor in the customary gauge is

$$\bar{T}^{\mu}_{\nu} = (\bar{\rho}c^2 + \bar{P})\bar{g}_{\alpha\nu} \frac{dx^{\alpha}}{d\bar{s}} \frac{dx^{\mu}}{d\bar{s}} - \delta^{\mu}_{\nu} \bar{P} \qquad \text{(III.1.10)}$$

where $\bar{\rho}$ and $\bar{P}$ are the customary proper-mass density and proper-pressure of the fluid. As previously mentioned, we wish to carry out our calculations in the non-customary gauge. To this end, we note from our discussion in (III.1) that the stress-energy-momentum tensor in the non-customary gauge, T^{μ}_{ν}, associated with (III.1.10) is

$$T^{\mu}_{\nu} = (\rho c^2 + P)g_{\alpha\nu} \frac{dx^{\alpha}}{ds} \frac{dx^{\mu}}{ds} - \delta^{\mu}_{\nu} P \qquad \text{(III.1.11)}$$

where

$$\rho = \bar{\rho}\, e^{3\sigma} \qquad \text{(I.3.14)}$$

and

$$P = \bar{P}\, e^{3\sigma} \tag{III.3.13}$$

In these expressions, ρ, P, $\bar{\rho}$, and $\bar{P}$ are, from our assumptions of homogeneity and isotropy, only functions of time, t or $\bar{t}$.

If we assume that in the coordinate system we are using, particles making up our perfect fluid have constant spacial coordinates, then for these particles, we have

$$\frac{dx^i}{ds} = \frac{dx^i}{ds} = 0 \ ; \quad i = 1, 2, 3$$

and

$$\frac{dx^4}{ds} = 1 \ , \quad \frac{dx^4}{d\bar{s}} = e^{-\sigma} \ .$$

These statements are, as may be verified by the reader, consistent for the line element being employed, (V.1.14), with both our geodesic equations in the non-customary gauge, (I.3.8), and our equation of motion in the customary gauge, (I.3.10).

So in our "comoving coordinate system", we have

$$\left.\begin{aligned} T^1_1 = T^2_2 = T^3_3 &= -P \\ T^4_4 &= \rho c^2 \ . \end{aligned}\right\} \tag{V.2.1}$$

Thus, for the problem being considered, our field equations in the non-customary gauge, (V.1.10), become

$$\left.\begin{aligned} G^1_1 = G^2_2 = G^3_3 &= -KP + \Lambda \\ G^4_4 &= K\rho c^2 + \Lambda \end{aligned}\right\} \tag{V.2.2}$$

These field equations coupled with our non-customary line element, (V.1.14) are of the same form as those used in the ordinary Einstein theory to arrive at expressions for $\bar{\rho}(\bar{t})$ and $\bar{P}(\bar{t})$ in terms of $\bar{R}(\bar{t})$. Therefore, our expressions for $\rho(t)$ and $P(t)$ in terms of $R(t)$ must be of the same form as those obtained in the ordinary Einstein theory for the customary mass density and pressure in terms of the customary radius of the universe. So, from the line element (V.1.14) and the field equations (V.2.2), we have

$$-KP = \frac{k}{R^2} + \frac{1}{c^2}\left[\frac{2\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2\right] - \Lambda \tag{V.2.3}$$

$$K\rho c^2 = \frac{3k}{R^2} + \frac{3}{c^2}\left(\frac{\dot{R}}{R}\right)^2 - \Lambda\ , \tag{V.2.4}$$

and, by combining (V.2.3) and (V.2.4)

$$\frac{d(\rho c^2 R^3)}{dt} = -P\,\frac{dR^3}{dt}\ , \tag{V.2.5}$$

where dots denote differentiation with respect to the "non-customary time", t.

In section 4, we will discuss these equations for the case of a matter-dominated universe. Before doing this, we wish to now discuss some consequences stemming from the form of our line element in the customary gauge, (V.1.15).

3. Consequences Stemming from the Form of the Customary Line Element

Since our line element in the customary gauge, (V.1.15), is of the same form as that employed in the ordinary Einstein theory, then any calculation whose basis lies solely in the form of this line element (i.e. does not depend on the field equations) must yield the same results, in form, in our theory as in the ordinary Einstein theory. Therefore, rather than carrying out lengthy calculations, we refer the reader to Weinberg[20] wherein the following relationships are shown to be true:

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\bar{R}(\bar{t}_0)}{\bar{R}(\bar{t}_1)} - 1 \ , \tag{V.3.1}$$

$$\bar{R}(\bar{t}) = \bar{R}(\bar{t}_0)\ \left[1 + \bar{H}_0(\bar{t} - \bar{t}_0) - \tfrac{1}{2}\,\bar{q}_0 H_0{}^2(\bar{t} - \bar{t}_0)^2 + \ldots\right] \tag{V.3.2}$$

$$d_L = r_1\, \frac{\bar{R}^2(\bar{t}_0)}{\bar{R}(\bar{t}_1)} = \frac{c}{\bar{H}_0}\ \left[z + \tfrac{1}{2}(1 - \bar{q}_0)z^2 + \ldots\right] \tag{V.3.3}$$

In these results, z is the red-shift parameter defined as the fractional increase in the wavelength, λ, of light emitted from a stationary source at $(r_1, \bar{t}_1)$ traveling along the $-r$ direction to an observer located at $(0, \bar{t}_0)$. The luminosity distance, d_L, is expressed in terms of the Hubble constant, $\bar{H}_0$, and the deceleration parameter, $\bar{q}$, which are defined by

$$\bar{H}_0 = \frac{\bar{R}_0{}'}{\bar{R}_0} \tag{V.3.4}$$

and

$$\bar{q}_0 = -\frac{\bar{R}_0'' \bar{R}_0}{(\bar{R}_0')^2} \tag{V.3.5}$$

where $\bar{R}_0 = \bar{R}(\bar{t}_0)$ etc., and primes denote differentiation with respect to $\bar{t}$; $\bar{R}' = d\bar{R}/d\bar{t}$.

We point out that it is (V.3.1) and (V.3.3) (or equations based on (V.3.3)) from which determinations of $\bar{H}_0$ and $\bar{q}_0$ are made from observational data. Therefore, the dependence of $\bar{H}_0$ and $\bar{q}_0$ on z and d_L in our theory is the same as in the ordinary Einstein theory. However, we shall find that $\bar{H}_0$ and $\bar{q}_0$ do not enter our field equations in the same manner as they do in the ordinary Einstein theory. In particular, whereas $\bar{q}_0$ determines whether the universe has positive, zero, or negative curvature for a matter-dominated universe with the cosmological constant set to zero in the ordinary Einstein theory, we shall find that in our theory it is a quantity q_0 defined by

$$q_0 = - \frac{\ddot{R}_0 R_0}{\dot{R}_0^{\,2}}$$

which determines this curvature.

We will now discuss Eqs. (V.2.3), (V.2.4), and (V.2.5) for the case of a matter-dominated universe.

4. The Matter-Dominated Universe

In this section, we present a quantitative discussion of our field equations in the case $P \ll \rho c^2$. For this situation, we have from (V.2.5) that in the non-customary gauge

$$\rho c^2 R^3 = \text{constant} = \frac{3A}{K} \tag{V.4.1}$$

or, making use of (I.3.14) and (V.1.16), for the constant A, we find

$$A = \frac{1}{3} K \rho_0 c^2 R_0^3 = \frac{1}{3} K \bar{\rho}_0 c^2 \bar{R}_0^3 \tag{V.4.2}$$

where the naught subscripts indicate that the quantities are to be evaluated at the present time.

Utilizing (V.4.1) in (V.2.4) yields

$$\frac{1}{c^2} \dot{R}^2 = \frac{A}{R} - k + \frac{1}{3} \Lambda R^2 \tag{V.4.3}$$

We will discuss this last equation further at a later point.

To continue our discussion of the matter-dominated universe, we note that by neglecting P in (V.2.3)

$$\frac{k}{R_0{}^2} = \frac{1}{c^2}\left[2q_0 - 1\right]H_0{}^2 + \Lambda \ . \tag{V.4.4}$$

where H_0 and q_0 are defined by

$$H_0 = \frac{\dot{R}_0}{R_0} \tag{V.4.5}$$

and

$$q_0 = -\frac{\ddot{R}_0 R_0}{\dot{R}_0{}^2} \tag{V.4.6}$$

The relationships between H_0 and q_0 and the Hubble constant, $\bar{H}_0$, and the deceleration parameter, $\bar{q}_0$, may be found by noting that from (V.1.16) and (V.1.17)

$$\dot{R} = \bar{R}' - \bar{R}\,\sigma' \tag{V.4.7}$$

and

$$\ddot{R} = e^{\sigma}\left[\bar{R}'' - \bar{R}'\sigma' - \bar{R}\sigma''\right] \ . \tag{V.4.8}$$

Inserting (V.1.16), (V.4.7), and (V.4.8) into (V.4.5) and (V.4.6) and making use of the definitions of $\bar{H}_0$ and $\bar{q}_0$, (V.3.4) and (V.3.5), we find

$$H_0 = (\bar{H}_0 - \sigma_0')e^{\sigma_0} \tag{V.4.9}$$

and

$$(\bar{H}_0 - \sigma_0')^2 q_0 = \bar{H}_0{}^2\bar{q}_0 + \bar{H}_0\sigma_0' + \sigma_0'' \tag{V.4.10}$$

To continue, by inserting (V.4.4) into (V.2.4), we find that at the present time

$$K\rho_0 c^2 = 6q_0 \frac{H_0^2}{c^2} + 2\Lambda \qquad \text{(V.4.11)}$$

By using this last expression in (V.4.4), we find for the curvature constant, k,

$$k = \frac{R_0^3}{3} Kc^2 \left[\rho_0 - \frac{3H_0^2}{Kc^4}\right] + \frac{1}{3} R_0^2 \Lambda \qquad \text{(V.4.12)}$$

If we consider the case $\Lambda = 0$, and define a non-customary critical mass density, ρ_c, to be

$$\rho_c = \frac{3H_0^2}{Kc^4} \qquad \text{(V.4.13)}$$

we see from (V.4.12) that the universe has positive, zero, or negative curvature depending upon whether

$$\left.\begin{array}{l} \rho_0 > \rho_c \\ \rho_0 = \rho_c \\ \text{or} \quad \rho_0 < \rho_c \end{array}\right\} \qquad \text{(V.4.14)}$$

respectively.

Using (V.4.13) in (V.4.11), we see for the case $\Lambda = 0$

$$\frac{\rho_0}{\rho_c} = 2q_0 \qquad \text{(V.4.15)}$$

implying that for

$$\left.\begin{array}{l} q_0 > \tfrac{1}{2} \\ q_0 = \tfrac{1}{2} \\ q_0 < \tfrac{1}{2} \end{array}\right\} \qquad \text{(V.4.16)}$$

the universe has positive, zero, or negative curvature respectively.

The results obtained so far are the same form as those obtained in the ordinary Einstein theory.[21] As we have shown, in the case $\Lambda = 0$ for our theory it is the "non-customary deceleration parameter", which determines the sign of the curvature of space-time in contrast to the ordinary Einstein theory in which the "customary deceleration parameter", $\bar{q}_0$, determines this sign.

We now return to (V.4.3) and investigate one possible solution to this equation.

5. A Static Non-Customary Universe

a. Formulation of the Problem

From (V.4.3),

$$\frac{1}{c^2}\dot{R}^2 = \frac{A}{R} - k + \frac{1}{3}\Lambda R^2 \ , \qquad \text{(V.4.3)}$$

we see that one possible solution in our theory is

$$R = \text{constant} \ . \qquad \text{(V.5.1)}$$

That is, our theory allows for a static non-customary universe.

To investigate this solution, we now insert (V.5.1) into (V.2.3) and (V.2.4) for the negligible pressure case and find

$$\Lambda = \frac{k}{R^2} = \tfrac{1}{2}K\rho c^2 \qquad \text{(V.5.2)}$$

and, by requiring that $\rho > 0$*

$$k = +1 \ . \qquad \text{(V.5.3)}$$

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*It should be remembered that since e^{σ} is taken to always be positive, that the requirement $\rho > 0$ implies from (I.3.14) that $\rho > 0$.

We note that this solution corresponds to the static zero-pressure "Einstein universe" encountered in the ordinary Einstein theory.[22] However, in our theory, we have from (V.1.16) and (I.3.14) that

$$\bar{R}(\bar{t}) = R\, e^{\sigma(\bar{t})}$$

and

$$\bar{\rho}(\bar{t}) = \rho e^{-3\sigma(\bar{t})}$$

This demonstrates that in our theory a static non-customary universe will in general be viewed as non-static by customary observers.

As can be seen from (V.1.16), (V.1.17) and (I.3.14), in order to discuss the solutions (V.5.1) and (V.5.2) from the viewpoint of customary observers, we must be able to determine σ. To satisfy our purpose stated at the beginning of this chapter of demonstrating the workings of our theory in a simple manner, we now select, for the investigation of the problem at hand, Eq. (II.4.4),

$$\sigma^{,\alpha}{}_{,\alpha} + 2\sigma^{,\alpha}\sigma_{,\alpha} = -\frac{1}{12}KT + \frac{2}{3}\Lambda\ , \tag{II.4.4}$$

as the governing equation for σ.

For the line element, (V.1.14), Eq. (II.4.4) becomes, in the negligible pressure case,

$$\frac{1}{c^2}\left\{\ddot{\sigma} + 3\frac{\dot{R}}{R}\dot{\sigma} + 2\dot{\sigma}^2\right\} = -\frac{1}{12}K\rho c^2 + \frac{2}{3}\Lambda\ . \tag{V.5.4}$$

For the R = constant case, we have, from (V.5.2) and (V.5.3), that (V.5.4) is

$$\ddot{\sigma} + 2\dot{\sigma}^2 = \tfrac{1}{2}\frac{c^2}{R^2} \tag{V.5.5}$$

or, from (V.1.16) and (V.1.17), in terms of derivatives with respect to the customary time, $\bar{t}$,

$$\sigma'' + 3(\sigma')^2 = \tfrac{1}{2}\,\frac{c^2}{R^2 e^{2\sigma}} = \tfrac{1}{2}\,\frac{c^2}{\bar{R}^2} \;. \tag{V.5.6}$$

We will discuss the solution to (V.5.5) a little later. At this point, we wish to demonstrate that (V.5.6) combined with (V.5.1) and (V.5.3) will allow us to relate that present "customary mass-density to the universe", $\bar{\rho}_0$, to $\bar{H}_0$, and $\bar{q}_0$.

b. The Relationship between $\bar{\rho}_0$, $\bar{H}_0$, and $\bar{q}_0$

Inserting our static solution, R = constant, into (V.4.9) and (V.4.10) yields the relations between $\bar{H}_0$, $\bar{q}_0$, σ_0', and σ_0'':

$$\left.\begin{aligned} \bar{H}_0 &= \sigma_0' \\ \bar{q}_0 &= -\left[1 + \frac{\sigma_0''}{\bar{H}_0^2}\right] \end{aligned}\right\} \tag{V.5.7}$$

By evaluating (V.5.6) at the present time, and using the relationships (V.5.7), we can now relate $\bar{R}_0$ to $\bar{q}_0$ and $\bar{H}_0$.

$$\frac{1}{\bar{R}_0^2} = 2(2 - \bar{q}_0)\,\frac{\bar{H}_0^2}{c^2} \tag{V.5.8}$$

Making use of (V.5.2) and (V.5.3), and remembering that $R = \bar{R}e^{-\sigma}$ and $\rho = \bar{\rho}e^{3\sigma}$, we find by inserting (V.5.8) into (V.5.1) at the present time

$$K\bar{\rho}_0 c^2 = 4(2 - \bar{q}_0)\,\frac{\bar{H}_0^2}{c^2}\,e^{-\sigma_0} \tag{V.5.9}$$

At this point, we invoke our assumption presented in (I.1) that at an event of our choosing we may set our gauge factor, e^{σ}, to one. We now select this event to be at the present instant so that

$$e^{\sigma_0} = 1 \; . \tag{V.5.10}$$

Using (V.5.10) in (V.5.9), then, yields for the relationship between $\bar{\rho}_0$, $\bar{H}_0$ and $\bar{q}_0$, the expression

$$K\bar{\rho}_0 c^2 = 4(2 - \bar{q}_0)\,\frac{\bar{H}_0^{\,2}}{c^2} \tag{V.5.11}$$

which should be compared to the relationship between these quantities obtained from the ordinary Einstein theory for the negligible pressure case,[21]

$$K\bar{\rho}_0 c^2 = 6\bar{q}_0\,\frac{\bar{H}_0^{\,2}}{c^2} \; .$$

From this result, (V.5.11), we see that since $\bar{\rho}_0$ is assumed positive that

$$\bar{q}_0 < 2 \tag{V.5.12}$$

Before examining (V.5.11) further, we will now present the solution for σ from (V.5.5) and discuss several implications of this solution.

c. <u>Solution to the σ-equation</u>

By making use of (V.1.16), (V.5.1), (V.5.8), and (V.5.10), we see that (V.5.5) may be expressed in the form

$$\frac{d^2 e^{2\sigma}}{dt^2} = \alpha^2 e^{2\sigma} \tag{V.5.13}$$

where

$$\alpha \equiv \left[2(2 - \bar{q}_0)\bar{H}_0^{\,2}\right]^{\frac{1}{2}} \tag{V.5.14}$$

Therefore, letting a and b be constants of integration, we have

$$e^{2\sigma} = \frac{\bar{R}^2(t)}{\bar{R}_0{}^2} = a\ e^{\alpha t} + b\ e^{-\alpha t} \tag{V.5.15}$$

where in relating e^{σ} to $\bar{R}(t)$, we have again made use of (V.1.16), (V.5.1), and (V.5.10). By requiring $\bar{R}(0) \simeq 0$ in this last expression, we conclude that

$$e^{2\sigma} = \frac{\bar{R}^2(t)}{\bar{R}_0{}^2} = d\ \sinh(\alpha t) \tag{V.5.16}$$

where d is a constant.

From (V.5.16), we see that as a function of the non-customary time, t, $\bar{R}$ increases without bound. But, from our assumption that e^{σ} is positive, we see from (V.1.17) that t is an increasing function of $\bar{t}$. Therefore, as function of the "customary time", $\bar{t}$, $\bar{R}$ must also increase without bound.

To determine whether the rate of change of the expansion rate of our static non-customary universe is positive or negative according to observers using the customary congruence, we now make use of (V.1.17) and (V.5.10) and differentiate (V.5.16) with respect to $\bar{t}$ and find

$$\frac{\bar{R}'}{\bar{R}_0} = \tfrac{1}{2}\,\alpha\,\sqrt{1 + d^2\left(\frac{\bar{R}_0}{\bar{R}}\right)^4} \tag{V.5.17}$$

From this we see that since $\bar{R}$ is an increasing function of $\bar{t}$, then we must have that $\bar{R}'$ is decreasing with $\bar{t}$,

$$\bar{R}'' < 0\ . \tag{V.5.18}$$

So, according to customary observers, the rate of change of the expansion rate of the universe is negative. By following the discussion presented by Weinberg,[21] we conclude, since $\bar{R}'' < 0$ for $0 < \bar{t} < \bar{t}_0$, that the customary age of the universe, $\bar{t}_0$, must be less than the Hubble time, $\bar{H}_0^{-1} = \bar{R}_0/\bar{R}_0'$.

$$\bar{t}_0 < (\bar{H}_0)^{-1} \tag{V.5.19}$$

Furthermore, we see from (I.3.14) and (V.5.16) that

$$\bar{\rho} = \rho\left(\frac{\bar{R}_0}{\bar{R}}\right)^3 \tag{V.5.20}$$

which demonstrates that, since $\bar{R}$ is an increasing function of $\bar{t}$ and ρ is a constant, $\bar{\rho}$ must decrease in time. The present rate of change of $\bar{\rho}$ with respect to $\bar{t}$ may be determined by differentiating (I.3.14) with respect to $\bar{t}$, evaluating the result at $\bar{t}_0$, and applying the first of Eqs. (V.5.7). The result is

$$\bar{\rho}_0' = -3\bar{H}_0\bar{\rho}_0 \tag{V.5.21}$$

To obtain an order of magnitude estimate of $\bar{\rho}_0'$, we will now take the "galactic" mass density[21]

$$\bar{\rho}_G = 3.1\times10^{-31}\left(\frac{g}{cm^3}\right)\left(\frac{\bar{H}_0}{75\ km/sec/mpc}\right)^2 \tag{V.5.22}$$

as a rough estimate of $\bar{\rho}_0$ and take $\bar{H}_0$ to be approximately 75 km/sec/mpc $\approx 7.5\times10^{-11}$ years^{-1}. With these assumptions we find

$$\bar{\rho}_0' \approx -7\times10^{-41}\ \frac{g}{cm^3\text{-}year}\ ,$$

which demonstrates that for our theory, the customary mass density of the universe is decreasing at an extremely slow rate.

Returning to (V.5.17), and evaluating this expression at the present time, we find from (V.3.4) and (V.5.14), that d can be expressed in terms of $\bar{q}_0$ by

$$d^2 = \frac{\bar{q}_0}{2-\bar{q}_0} . \qquad \text{(V.5.23)}$$

This leads us to conclude that

$$0 < \bar{q}_0 < 2 \qquad \text{(V.5.24)}$$

which is in agreement with (V.5.12). With these limits on $\bar{q}_0$ in mind, we now turn to a discussion of the correlation between our theory and observation by investigating (V.5.11).

d. Limits Placed on $\bar{\rho}_0$ from our Theory

To see the effects of the limits placed on $\bar{q}_0$, (V.5.24), we now rewrite (V.5.11) in the form

$$\bar{\rho}_0 = \frac{4}{3}(2 - \bar{q}_0)\frac{3\bar{H}_0{}^2}{Kc^4} \approx 1.46\times10^{-29}(2 - \bar{q}_0)\left(\frac{\bar{H}_0}{75 \text{ km/sec/mpc}}\right)^2 . \qquad \text{(V.5.25)}$$

By inserting the conditions (V.5.24) into (V.5.25), we find that according to our theory the present mass density of the universe as determined by customary observers is

$$0 < \bar{\rho}_0 < 2.9\times10^{-29}\left(\frac{g}{cm^3}\right)\left(\frac{\bar{H}_0}{75 \text{ km/sec/mpc}}\right)^2 . \qquad \text{(V.5.26)}$$

This upper limit placed on $\bar{\rho}_0$ from our theory is compatible with the presently accepted galactic mass density,

$$\bar{\rho}_G = 3.1\times10^{-31}\left(\frac{g}{cm^3}\right)\left(\frac{\bar{H}_0}{75 \text{ km/sec/mpc}}\right)^2 .$$

Accordingly, if we take this value of the galactic mass density as a rough estimate of the present mass density of the universe, $\bar{\rho}_0$, then we find from (V.5.25) that the predicted value of $\bar{q}_0$ from our theory is

$$\bar{q}_0 \approx 1.98 \ . \tag{V.5.27}$$

There is at present a high degree of inaccuracy in the values of $\bar{q}_0$ obtained "observationally" from (V.3.3) (or equations based on (V.3.3)). However, it seems that most of these values for $\bar{q}_0$ fall within our limits (V.5.24).* We feel, therefore, that our value for $\bar{q}_0$, based upon equating the galactic mass density to the present mass density of the universe, $\bar{q}_0 = 1.98$, if not correct is at least reasonable.

So, we have demonstrated thus far that our non-customary gauge theory with (II.4.4) selected as the defining equation for σ leads to, in the case of a static non-customary universe, what seems to be reasonable constraints on the present mass density of the universe and the deceleration parameter. To further test the validity of the static non-customary universe with (II.4.4) as the governing equation for σ, we will conclude this section by demonstrating that our solution for σ, (V.5.16), taken together with (V.1.17) predicts a reasonable lower limit for customary age of the universe, $\bar{t}_0$.

e. <u>The Age of the Universe</u>

From (V.1.17) and (V.5.16), we have

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*See Peach[(23)] or Sandage.[(24)]

$$\bar{t}_0 = d^{\frac{1}{2}} \int_0^{t_0} \left[\sinh(\alpha t)\right]^{\frac{1}{2}} dt$$

where we have assumed $\bar{t}$ to zero at $t = 0$. The integral appearing in this expression involves elliptic integrals of both the first and second kind[25] and is therefore rather complicated. So, rather than trying to evaluate the integral directly, we feel that for our purposes it will be sufficient to determine a lower limit for $\bar{t}_0$ by making use of the Cauchy-Schwarz inequality for integrals,[26]

$$\left| \int_a^b f(x)g(x)dx \right|^2 \leq \int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx \qquad \text{(V.5.29)}$$

where the equality sign holds only in the case of $f(x)/g(x)$ = constant. In carrying out the calculation of this lower limit for $\bar{t}_0$, we will take $\bar{\rho}_0 = \bar{\rho}_G$ thereby making $\bar{q}_0 = 1.98$.

By setting $g(x) = (\sinh\alpha t)^{\frac{1}{4}}$ and $f(x) = \cosh\alpha t$ in (V.5.29), we find

$$\bar{t}_0 > d^{\frac{1}{2}} \frac{\left\{ \int_0^{t_0} \left[\cosh(\alpha t)\right] \left[\sinh(\alpha t)\right]^{\frac{1}{4}} dt \right\}^2}{\int_0^{t_0} \left[\cosh(\alpha t)\right]^2 dt} . \qquad \text{(V.5.30}$$

Carrying out the integrations in (V.5.30) yields

$$\bar{t}_0 > \frac{32}{25} \frac{d^{\frac{1}{2}}}{\alpha} \frac{\left[\sinh(\alpha t_0)\right]^{\frac{5}{2}}}{\left\{\alpha t_0 + \left[\sinh(\alpha t_0)\right]\left[\cosh(\alpha t_0)\right]\right\}} . \qquad \text{(V.5.31)}$$

To obtain a lower limit for $\bar{t}_0$ in terms of $\bar{H}_0$ and $\bar{q}_0$, we notice from (V.5.16), (V.5.17), and (V.3.4) that

$$\left.\begin{aligned} \sinh(\alpha t_0) &= \frac{1}{d} \\ \alpha &= 2\bar{H}_0(1 + d^2)^{-\frac{1}{2}} \end{aligned}\right\} \qquad (V.5.32)$$

Inserting these relations into (V.5.31), we find

$$\bar{t}_0 > \frac{16}{25}\,\bar{H}_0^{-1}\left\{\frac{(1 + d^2)^{\frac{1}{2}}}{d^2\sinh^{-1}\left(\frac{1}{d}\right) + (1 + d^2)^{\frac{1}{2}}}\right\} \qquad (V.5.33)$$

where from (V.5.23), $d^2 = \dfrac{\bar{q}_0}{2 - \bar{q}_0}$.

If we now assume that $\bar{\rho}_0 = \bar{\rho}_G$ which in our theory implies that $\bar{q}_0 = 1.98$, we find

$$\left.\begin{aligned} d^2 &\simeq 99 \\ &\text{and} \\ \sinh^{-1}\left(\frac{1}{d}\right) &\simeq .1 \end{aligned}\right\} \qquad (V.5.34)$$

Using these values in (V.5.33) yields

$$\bar{t}_0 > \frac{8}{25}\,\bar{H}_0^{-1} \qquad (V.5.35)$$

So, from (V.5.19) and (V.5.35) we conclude

$$\frac{8}{25}\,\bar{H}_0^{-1} < \bar{t}_0 < \bar{H}_0^{-1} \qquad (V.5.36)$$

The best estimates of $\bar{H}_0^{-1}$ to date[20] place it in the limits

$$20\times10^9 \text{ years} \geq \bar{H}_0^{-1} \geq 7.5\times10^9 \text{ years} \qquad (V.5.37)$$

Using (V.5.37) in (V.5.36), we find that the largest value for our lower limit on $\bar{t}_0$ is (6.6×10^9 years).

Weinberg summarizes the results of radioactive dating experiments for the age of the galaxy and concludes that it is safe to assume that the age of the universe is at least (7×10^9 years).

So, at the present, all we can say about the lower limit placed on the age of the universe from our theory is that the least value of the age of the universe obtained from radioactive dating, (7×10^9 years), falls within the limits placed on the age of the universe by our theory.

We have demonstrated that in one instance, the static non-customary universe, our field equations coupled with an equation for σ yield results which are compatible with present observations. In particular, it has been shown that by taking $\bar{\rho}_G$ as a rough estimate of the present mass density of the universe our theory predicts a value of $\bar{q}_0$ which is in accord with the values for this parameter obtained by observational calculations. This is exactly the opposite of what is encountered in the ordinary Einstein theory wherein the "observed" values of $\bar{q}_0$ predict a present mass density of the universe on the order of 100 times greater than $\bar{\rho}_G$.[21]

At this point, we mention that besides the static case we have considered, there are several other solutions to (V.4.3) which could be investigated by our theory. In fact, the static case we have considered could be re-examined in the light of σ-equations other than (II.4.4). However, we will now conclude this discussion for we feel that the presentation just delivered is a fair demonstration of the application of our theory to cosmology and that we have therefore accomplished the purpose stated at the beginning of this chapter.

CHAPTER VI

SUMMARY

Einstein's field equations have been assumed valid in a gauge in which the congruence definition between lengths and times is different from the customary one. With this assumption, it was shown that insofar as the field equations imply geodesic motion in the customary gauge for particles in the ordinary Einstein theory, they imply geodesic motion in the non-customary gauge for particles in our theory. From this it was shown that in our theory according to customary observers particles do not follow geodesics and hence the principle of equivalence between an accelerating frame and a gravitational field does not hold in our theory. That is, due to the fact that σ appears as a field in the customary gauge, particles in our theory "know" where they are in a gravitational field.

In order to test our theory, several equations were postulated for the gauge field. These equations were then coupled with the Einstein field equations in the non-customary gauge and applied to four problems in relativity.

For two of these problems, the precession of the perihelia of planets and the deflection of light by the sun, it was demonstrated that there was no sensible difference between the predictions of our theory and those of the ordinary Einstein theory.

In the case of the time delay of radar echos passing the sun, it was shown that there is, in principle, a detectable difference between our theory and Einstein's. However, it seems that to ascertain whether our theory or Einstein's is more correct for this problem, we will have to wait until more accurate experimental evidence has been obtained.

Also, for the case of the gravitational shift in spectral lines, our theory's result is, in principle, detectably different than the result obtained from the ordinary Einstein theory (or the principle of equivalence). The degree of accuracy of one experiment concerning this phenomenon (Pound-Rebka Mossbauer experiment) seemingly rules out one of our equations for σ (II.1.8).

After considering these problems in general relativity, we then applied our theory to cosmology. It was shown that for one model (a static non-customary matter-dominated universe which is red-shifted in the customary gauge) with a particular σ equation, (II.1.8), our theory yielded results which were compatible with observation. In particular, we found, by using the galactic mass density as a rough estimate of the present mass density of the universe, that our theory predicted a value for the decelerration parameter which is in accord with "observational" values for this quantity. The exact opposite of this being true for the ordinary Einstein theory. So, for this one example, our theory seems to yield results more in accord with observation than Einstein's. Unfortunately, the equation used for σ in arriving at this result is the same equation which is seemingly ruled out by the Pound-Rebka Mossbauer experiment. To correct this incompatibility, possibly different equations could be used for σ, and/or the field equation could be altered in such a manner as to allow the principle of equivalence to hold in our theory.

In conclusion, we feel that the investigations carried out herein give a fair demonstration of the possibilities entailed in a non-customary gauge theory of relativity and cosmology. It seems feasible that with some modifications the formulation presented could be used to develop a theory of relativity and of cosmology which are compatible both with each other and with observation.

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