A METHOD FOR THE ANALYSIS AND SYNTHESIS OF LINEAR THIRD-ORDER SYSTEMS

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OF LINEAR THIRD-ORDER SYSTEMS

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LIST OF SYMBOLS

a_n . . coefficient of general equation.

B_1 . . coefficient of transformed equation.

B_0 . . coefficient of transformed equation.

j . . square root of minus one.

k . . forward loop gain.

os . . overshoot.

P . . linear transformation complex variable.

s . . Laplace transformation complex variable.

t_r . . rise time.

t_{rt} . . transformed rise time.

t_s . . settling time.

t_{st} . . transformed settling time.

$X(s)$ . . Laplace transform of $x(t)$.

$x(t)$ . . system output variable.

$Y(s)$ . . Laplace transform of $y(t)$.

$y(t)$ . . system input variable.

β . . ratio of real parts of roots.

γ . . phase margin.

ζ . . damping ratio.

θ . . damping angle.

φ . . lag angle.

ω_d . . natural damped frequency.

LIST OF SYMBOLS (Continued)

ω_{dt} . . transformed natural damped frequency.

ω_{n} . . natural undamped frequency.

ω_{nt} . . transformed natural undamped frequency.

CHAPTER I

INTRODUCTION

The dynamic analysis of physical systems is a topic which has received considerable attention in the past twenty years. Due to the demands for higher system performance characteristics, technological areas such as feedback control, mechanical vibrations, electrical networks, and chemical processes have been advanced immensely through the use of procedures developed in this field.

The first step in the dynamic analysis of such systems is the application of the appropriate physical laws, e.g., the laws of conservation of mass, momentum, and energy, which results in mathematical models that are functions of the physical properties of the systems. Such models usually consist of nonlinear integrodifferential equations, difference equations, or sets of these equations which are very difficult, if not impossible, to solve analytically. However, many systems can be adequately described by ordinary linear equations with constant coefficients in which case the work of the analyst is greatly simplified. If the independent variable is time, such equations may be expressed in the general form:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_o x = y(t) \qquad (1\text{-}1)$$

where

a_n, a_{n-1}, ..., a_1, a_o are real numbers

and

$$a_n \neq 0.$$

First and Second-Order Systems

The order of a linear system is determined by the number of independent energy storage elements, thus, the first-order equation

$$a_1 \frac{dx}{dt} + a_o x = y(t) \qquad (1\text{-}2)$$

describes such basic systems as

1. A simple thermometer (one capacitance)
2. R-C and R-L networks
3. A mass in motion against resistance (friction) only

while the second-order equation

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_o x = y(t) \qquad (1\text{-}3)$$

is applicable for

1. A two-capacitance thermometer (example - thermometer in a well)
2. R-L-C network

3. A mass in motion against resistance and inductive elements (springs, etc.).[1]

Since such systems are of considerable importance in engineering, the solutions to these equations are well documented (1, 2, 3, 4, 5).[2]

Of particular interest are the responses to a step change in the forcing function, y(t), (hereafter referred to as the transient responses) and these are shown graphically in many texts with a normalized time base (see Figures 1-1 and 1-2).

Also of interest are the steady-state responses of the systems to a sinusoidal input (referred to as the frequency response).[3] In such cases the output will be a sinusoidal oscillation of the same frequency as the input, but different in magnitude and phase. Information showing the variation of these two variables with frequency is frequently shown graphically with a normalized time base. Diagrams for first-order and second-order systems described by Equations (1-2) and (1-3) are shown in Figures 1-3 and 1-4, respectively.

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[1] "Elements are independent in this sense if they cannot be combined with other elements. An inductance and capacitance in series constitute a second-order system whereas two inductances in series do not." (5).

[2] Numbers in parentheses refer to references in the Selected Bibliography.

[3] The response of a system after transient terms have "died-out."

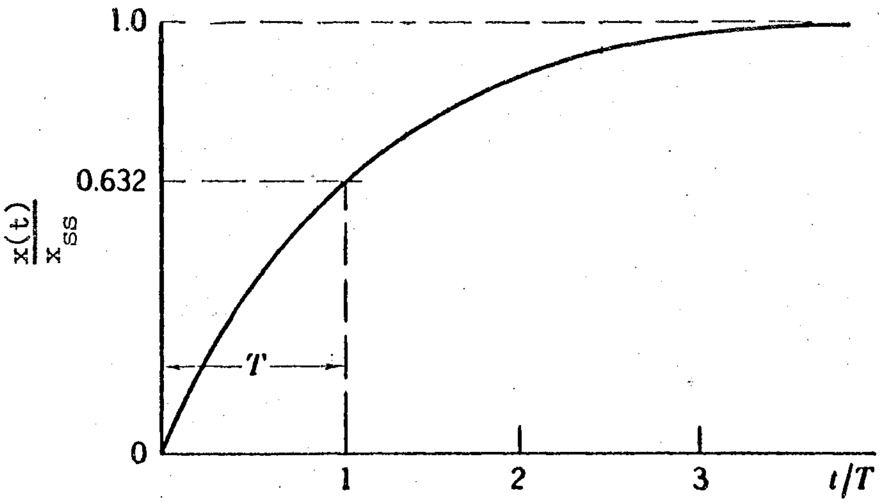

Figure 1-1. Normalized Transient Response of a First-Order System

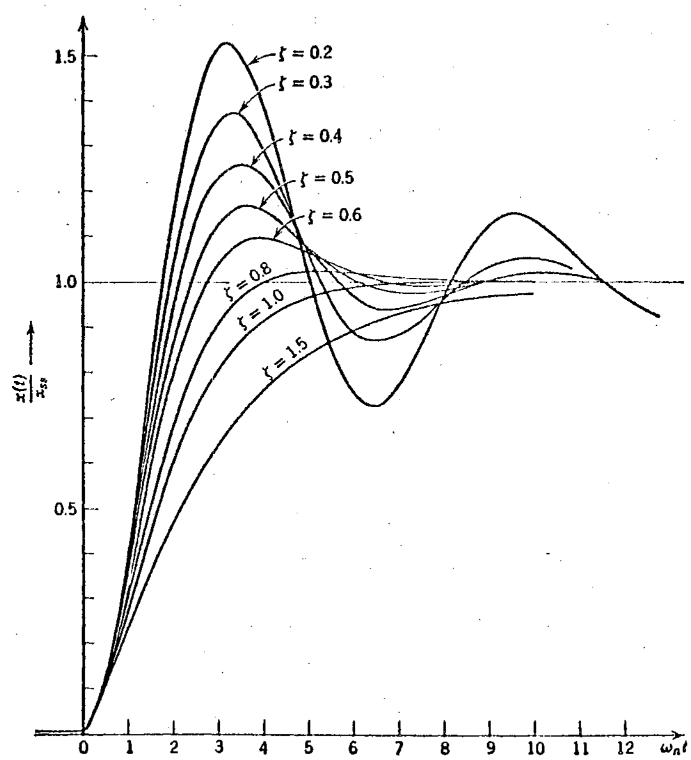

Figure 1-2. Normalized Transient Response Curves of a Second-Order System

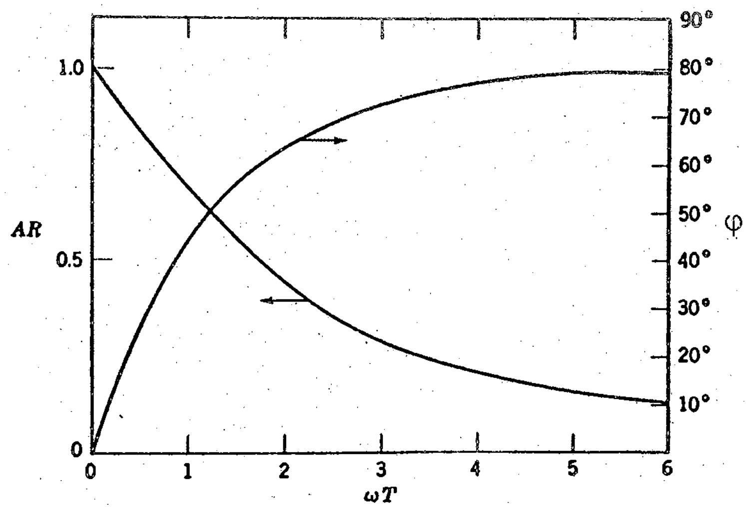

Figure 1-3. Amplitude Ratio and Phase Shift of a First-Order System

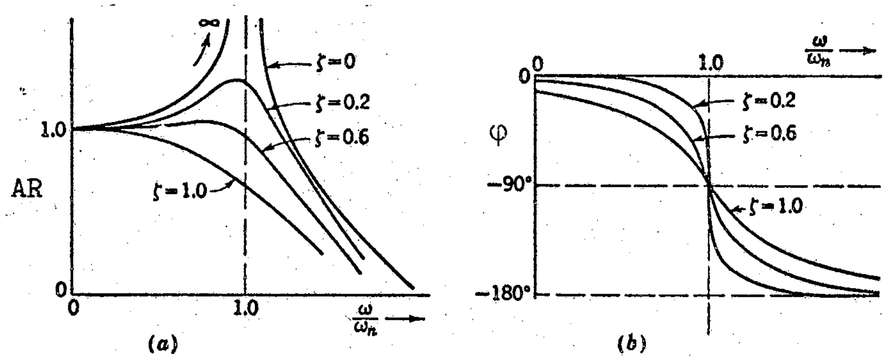

Figure 1-4. Amplitude Ratio and Phase Shift of a Second-Order System

Third-Order Systems

The third-order equation

$$a_3 \frac{d^3x}{dt^3} + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = y(t) \qquad (1\text{-}4)$$

occurs frequently in the field of automatic controls. A typical example is the electro-hydraulic servomechanism with displacement feedback shown in Figure 1-5.

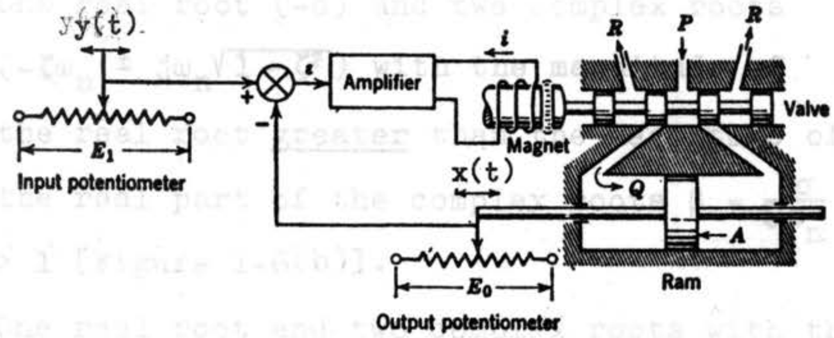

Figure 1-5. Third-Order Electro-Hydraulic Servomechanism

The characteristic equation of such a system is

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \qquad (1\text{-}5)$$

which may be factored to the form

$$a_3 (s + s_1)(s + s_2)(s + s_3) = 0 \qquad (1\text{-}6)$$

where s_1, s_2, s_3 are the characteristic roots (sometimes

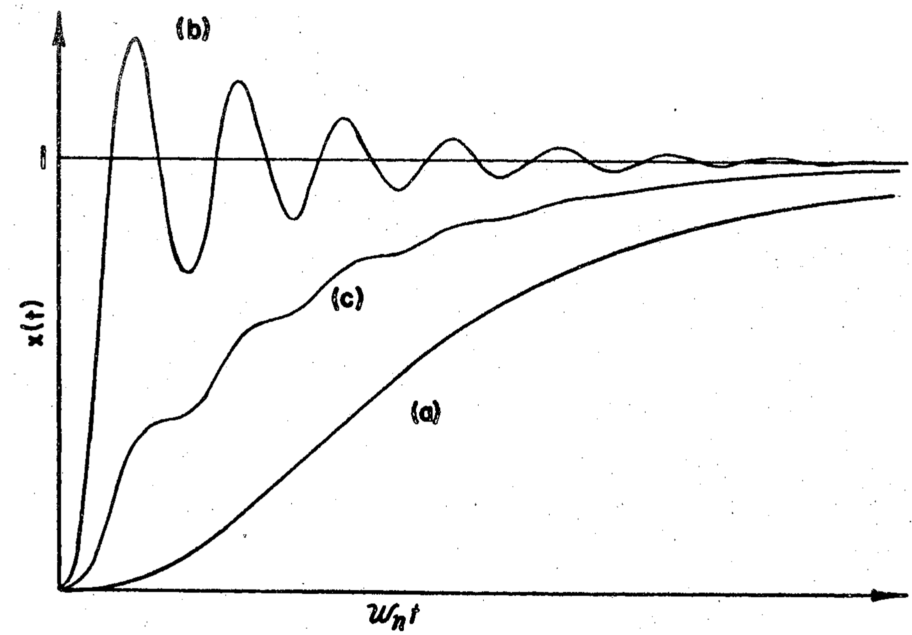

Figure 1-7. Three Basic Types of Third-Order Transient Response

The analyst does not often require knowledge of the complete time history of the transient response, but only certain essential characteristics. Three of these characteristics are defined below and illustrated in Figure 1-9:

Overshoot, os - the difference between the magnitudes of the maximum and final (steady-state) values of the response, expressed as a percentage.

Rise Time, t_r - time from 10% to 90% of final value.

Settling Time, t_s - time for the response to decrease to less than a

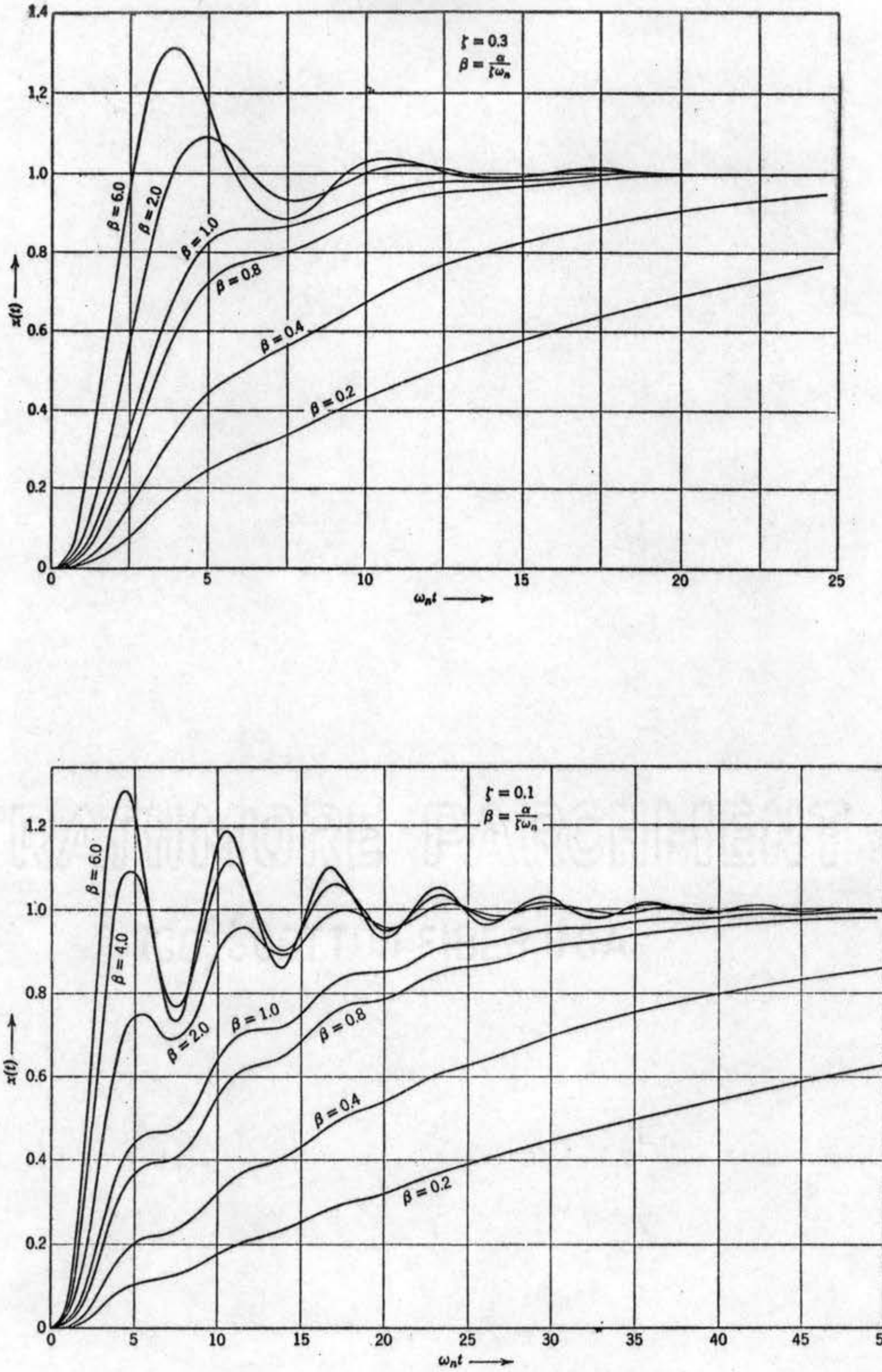

Figure 1-8. Normalized Transient Response Curves for a Third-Order System

specified percentage of the final value (2% and 5% are common values) (7).

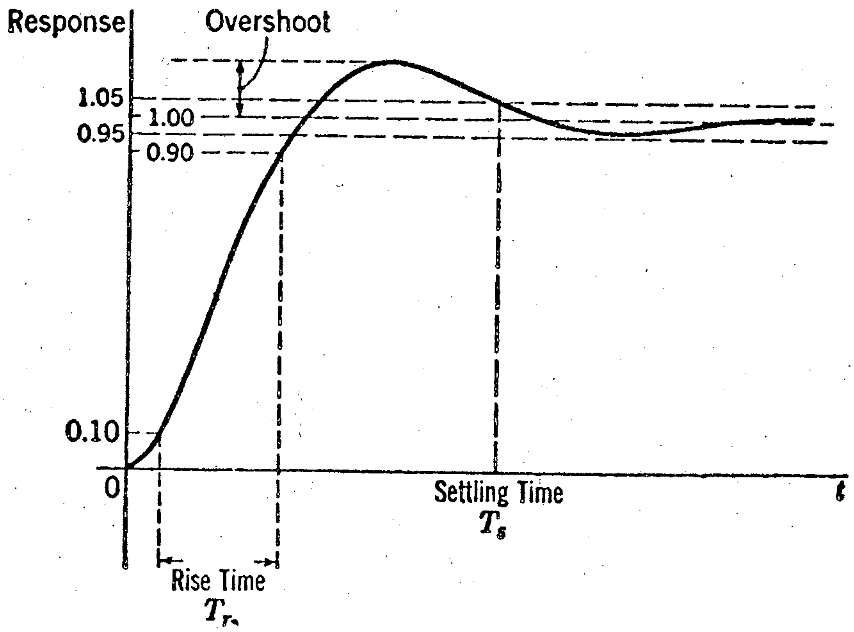

Figure 1-9. Transient Response Specifications

The frequency response of a stable linear system is defined as the steady-state response which will be observed when the forcing function is a sinusoid or disturbance of constant frequency and amplitude (7). Polar plots are frequently used for the analysis of systems in the frequency domain and associated with these diagrams are two important stability criteria which are defined below and illustrated graphically in Figure 1-10:

Gain margin - the ratio of the gain at which the system becomes unstable to the actual system gain.

Phase margin - the amount of negative phase shift which must be introduced to make the system unstable.[6]

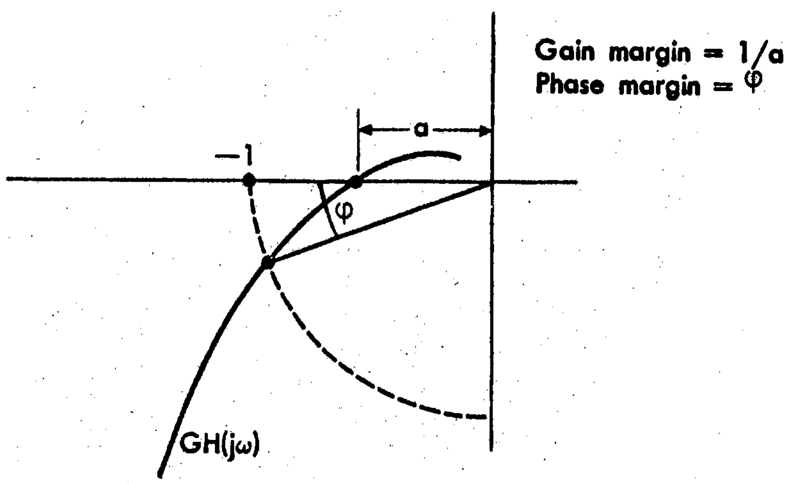

Figure 1-10. Gain and Phase Margin From the Polar Plot

Definition of the Problem

The purpose of this study is to develop a procedure

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[6]Polar plots and frequency domain are discussed in more detail in Chapter III.

which provides the information concerning a linear third-order system that is necessary for analysis and synthesis in either the time or frequency domains without resorting to root determination methods and sets of normalized response graphs.

CHAPTER II

LITERATURE SURVEY

General Methods for the Determination of Roots

Practically all previously published methods for the analysis and synthesis of linear third-order systems have involved some root determination process.[1] For example, the determination of a complete system response [such as given by Equation (3-22)] requires a knowledge of all three roots of the characteristic equation, while frequency response techniques require at least factorization into a single root with a quadratic term. There is a number of standard procedures for determining the roots of an equation; algebraic methods are illustrated in references (8, 9, 10, 11), numerical methods in references (12) and (13), and some basic graphical techniques in references (12, 13, 14, 15, 16).

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[1]There are some published methods concerned with stability criteria which do not require knowledge of the roots, but they give no information about response characteristics.

Third-Order Root Charts

Of particular interest, however, are the methods which have been developed solely for the analysis of third-order systems. One of the first was by H. K. Weiss (17) in 1939 who was analyzing the operation of a "speed-control system" or governor. Weiss developed the cubic charts shown in Figure 2-1(a-c) which illustrate the functional relationship between the system roots and the system parameters ζ (damping ratio) and s (a control constant). He also presented several normalized transient response curves of the type shown in Figure 2-1(d).

A similar paper by Koenig (18) in 1951 dealt with the "Design of Damper-Stabilized Instrument Servomechanisms" and showed the two graphs in Figure 2-2. In regard to the use of these charts for servo design, Koenig stated:

> In the general vicinity of the point r-4, K-0.3, the complex conjugate pair of roots becomes very dominant compared to the real root. Under this condition the system operation approaches the operation of a second order system. Transient curves for a second order system have been published for a step-position input. Therefore, the designer, using Figs. 2 and 3 [Figure 2-2(a) and (b)], can select the set of parameter values corresponding to the amount of overshoot desired.[2]

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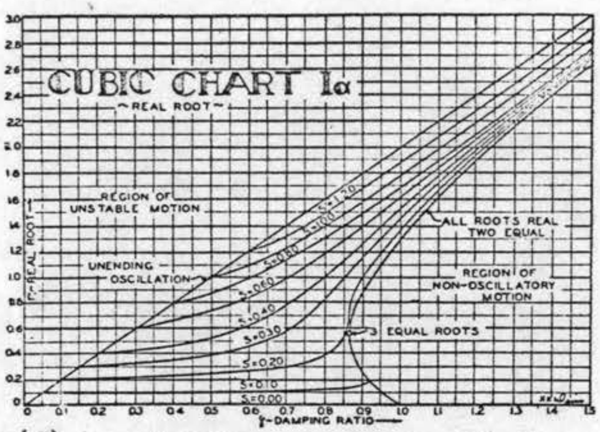
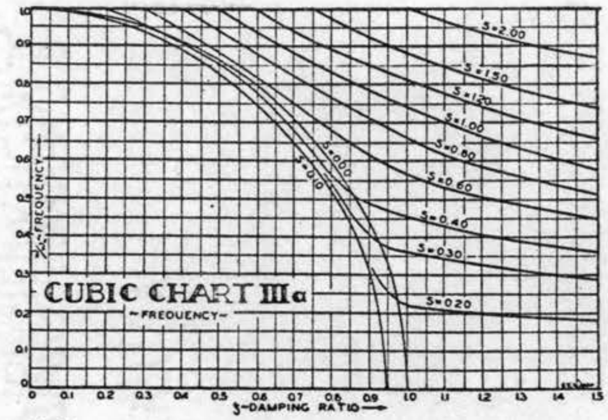
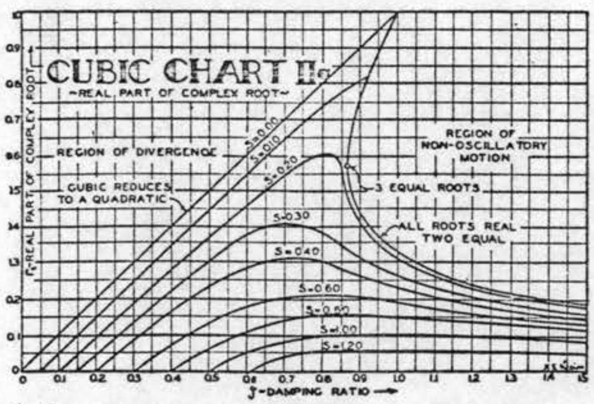
[2]Statements of this type occur often in the literature regarding third-order systems. The method developed by the writer in Chapter III provides the information for design without such restrictions.



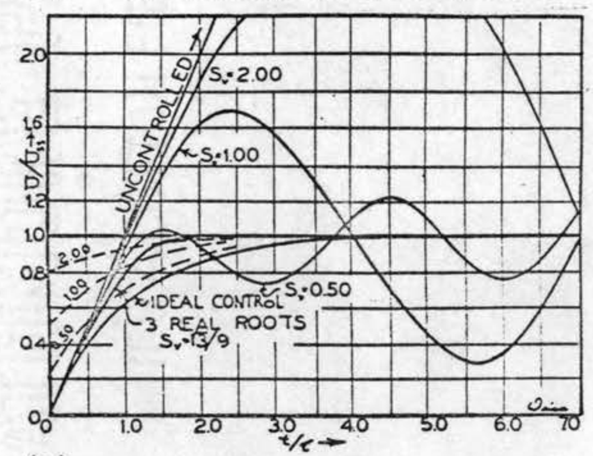
(a) Effect of parameters on real root of cubic.



(b) Effect of parameters on frequency of cubic.

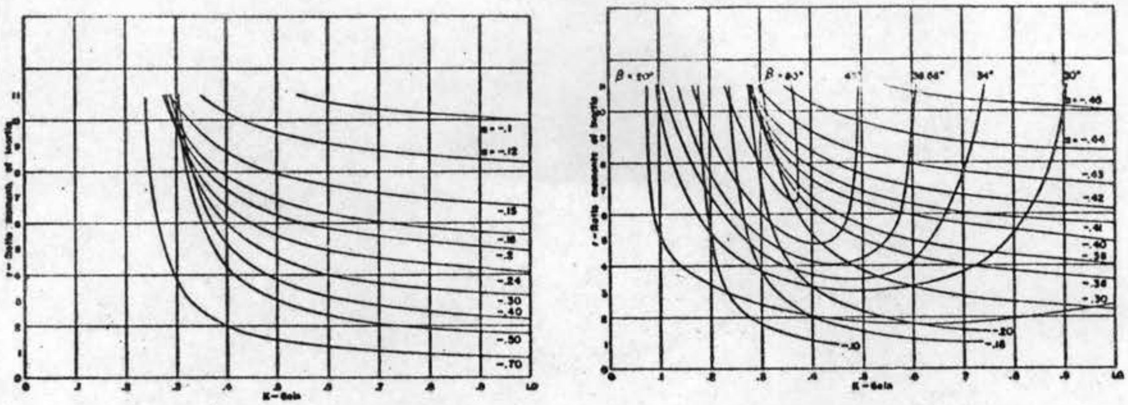


(c) Effect of parameters on real part of complex root of cubic.



(d) Solution of critical cases of error produced by constant load.

Figure 2-1. Third-Order Charts by Weiss



(a) —*Trajectories of the real roots.*

(b) —*Trajectories of the complex conjugate pair of roots.*

K = gain (abscissa)
r = ratio, moments of inertia (ordinate)
γ = real root [trajectories on chart (a) and chart (b)]
β = angle between the imaginary axis and line to complex roots, a measure of the damping [trajectories on chart (b)].

Figure 2-2. Root Trajectory Charts by Koenig

Higgins and Levinthal (19) used linear transformations on a third-order equation to obtain a set of sixty charts relating the coefficients and roots of the equation. They stated that their procedure had two major uses:

> First, in analysis it affords quick ascertainment of the roots of a specified cubic characteristic equation, thus enabling rapid determination of the stability characteristics of the corresponding third-order servo system; conversely, in synthesis it enables selection of the coefficients of a cubic characteristic equation to the end that a third-order servo system can be formulated with a desired degree and kind of stability.

However, the reviewer of their paper considered the number of charts involved to be excessive (and this writer concurs).

The early work of Evans (20) and Liu (21) received widespread distribution when a chapter in "Instrument Engineering" (a three volume work of major importance) was devoted to their graphical methods for solving cubic and quartic equations (22). They dealt with equations with unity coefficients for the highest order term such as Equation (2-1) (obtained from Equation (1-5) by dividing by a_3),

$$s^3 + \gamma_2 s^2 + \gamma_1 s + \gamma_0 = 0 \qquad (2\text{-}1)$$

where

$$\gamma_2 = \frac{a_2}{a_3}, \text{ etc.}$$

The equation was then "expressed as the product of a first-order term and a second-order factor, each with real coefficients" (22); i.e.,

$$(s + \sigma)(s^2 + 2\zeta\omega_n s + \omega_n^2) = 0 \qquad (2\text{-}2)$$

for which the roots are

$$s = -\sigma \qquad (2\text{-}3a)$$

$$s = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2} \qquad (2\text{-}3b)$$

$$s = -\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2} \;. \qquad (2\text{-}3c)$$

The characteristic equations were further simplified

by a substitution of variables which made one of the remaining coefficients also equal to unity; i.e., the first-term coefficient was made equal to unity by factoring $\gamma_1^{3/2}$ out of Equation (2-1) and taking the variable as $\frac{s}{\gamma_1^{1/2}}$.
The transformed equation was

$$\gamma_1^{3/2}\left[\left(\frac{s}{\gamma_1^{1/2}}\right)^3 + \frac{\gamma_2}{\gamma_1^{1/2}}\left(\frac{s}{\gamma_1^{1/2}}\right)^2 + \frac{s}{\gamma_1^{1/2}} + \frac{\gamma_0}{\gamma_1^{3/2}}\right]$$

$$= \gamma_1^{3/2}\left[\frac{s}{\gamma_1^{1/2}} + \frac{\sigma}{\gamma_1^{1/2}}\right]\left[\left(\frac{s}{\gamma_1^{1/2}}\right)^2 + 2\zeta\left(\frac{\omega_n}{\gamma_1^{1/2}}\right)\left(\frac{\lambda}{\gamma_1^{1/2}}\right) + \left(\frac{\omega_n}{\gamma_1^{1/2}}\right)^2\right] \tag{2-4}$$

and the following curves were plotted:

1. Curves of constant damping ratio (ζ) and constant undamped angular natural frequency (ω_n) as functions of the nonunity cubic coefficient ratios, $\frac{\gamma_2}{\gamma_1^{1/2}}$ and $\frac{\gamma_0}{\gamma_1^{3/2}}$ (see Figure 2-3).
2. $\frac{\gamma_2}{\gamma_1^{1/2}}$ and $\frac{\gamma_0}{\gamma_1^{3/2}}$ plotted as functions of ζ and ω_n (see Figure 2-4)
3. Curves of real-part-of cubic root ratios as functions of the previously mentioned ratios (see Figure 2-5).

Similar charts were developed for the other two cases; namely, unity third- and second-order coefficients, and unity third-order coefficient with unity constant term ($\gamma_0 = 1$). This latter case was also considered by

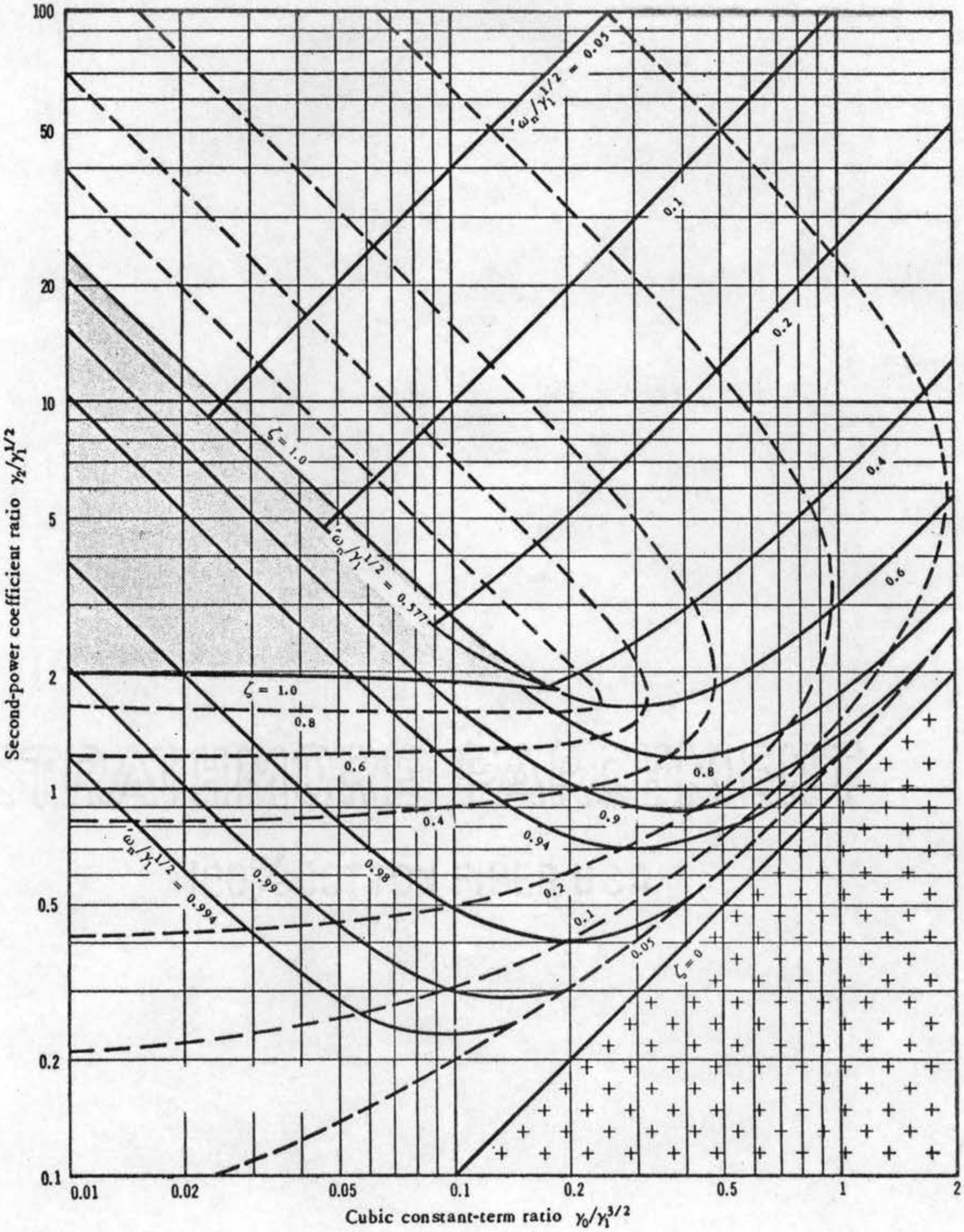


Figure 2-3. Curves of Constant Damping Ratio and Constant Undamped Natural Frequency by Liu

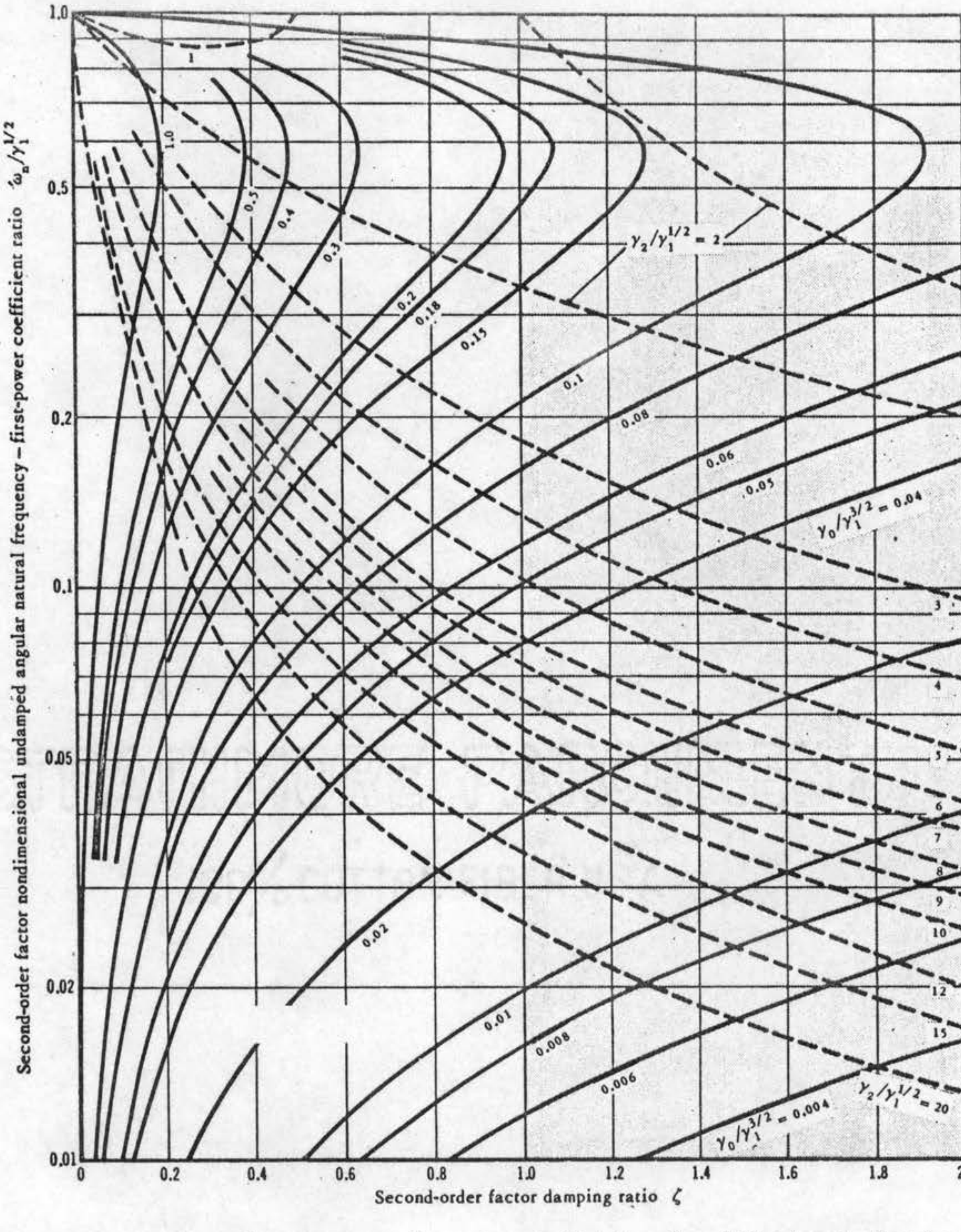


Figure 2-4. Curves of Constant Cubic Coefficient Ratios by Liu

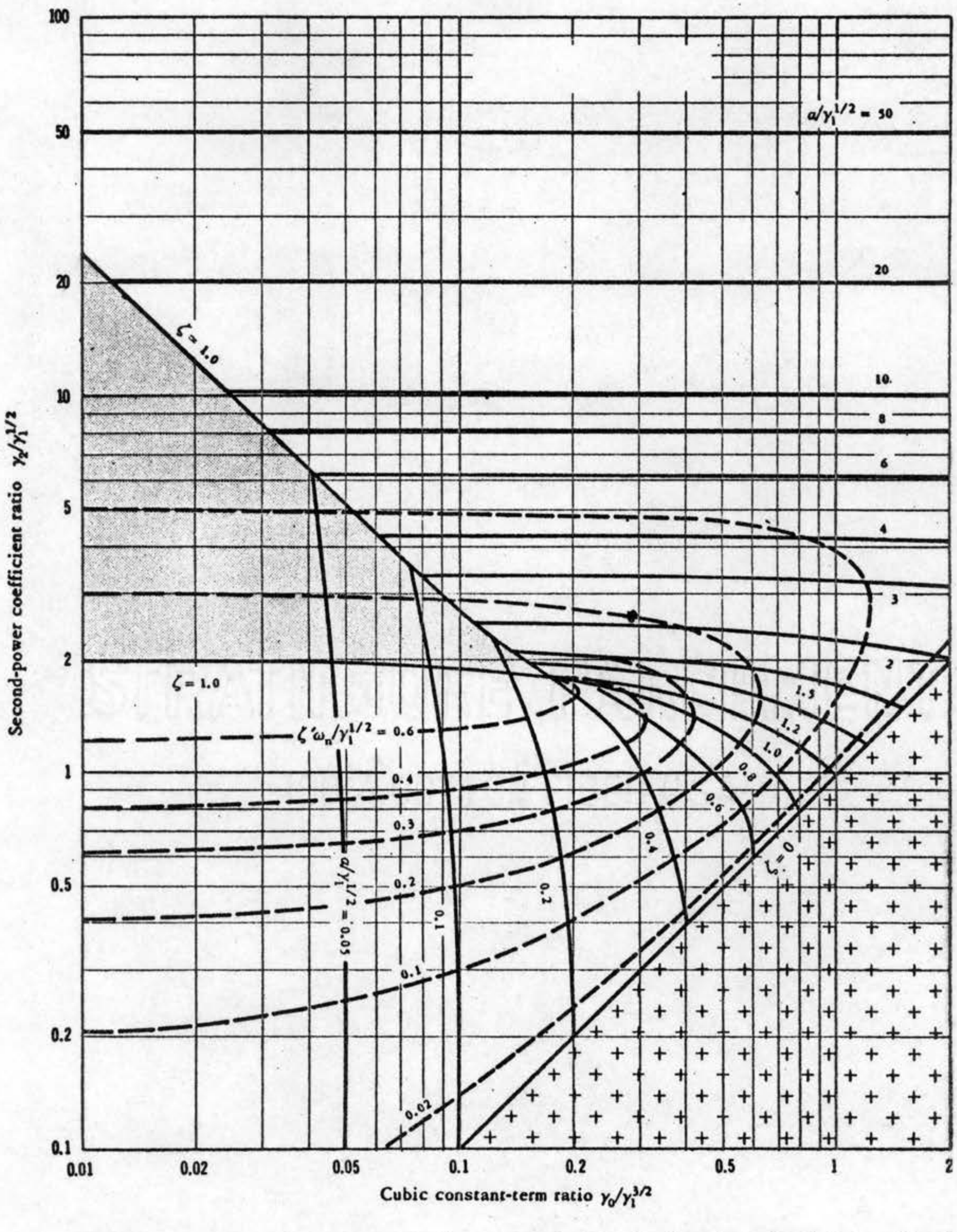


Figure 2-5. Curves of Constant Real-Part-of-Cubic Root Ratios by Liu

Stefaniak (23) with similar results.

In 1959, Mitrovic (24) published a paper on "The Graphical Procedure for the Analysis and Synthesis of Feedback Control System" which was applicable for linear systems of any order. By allowing two coefficients of the system equation to vary, he developed relationships which defined curves along which the system damping ratio, ζ, was constant. As a special case of his method (which is too complex for discussion here) he considered the third-order Equation (1-5):

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \qquad (1\text{-}5)$$

Using a linear transformation, $s = \frac{a_2}{a_3} P$, Equation (1-5) was transformed to a normalized equation with two variables[3]

$$P^3 + P^2 + B_1 P + B_0 = 0 \qquad (2\text{-}5)$$

where

$$B_1 = \frac{a_1 a_3}{a_2^2} \quad \text{and} \quad B_0 = \frac{a_0 a_3^2}{a_2^3}$$

the roots of which were then obtained from his third-order

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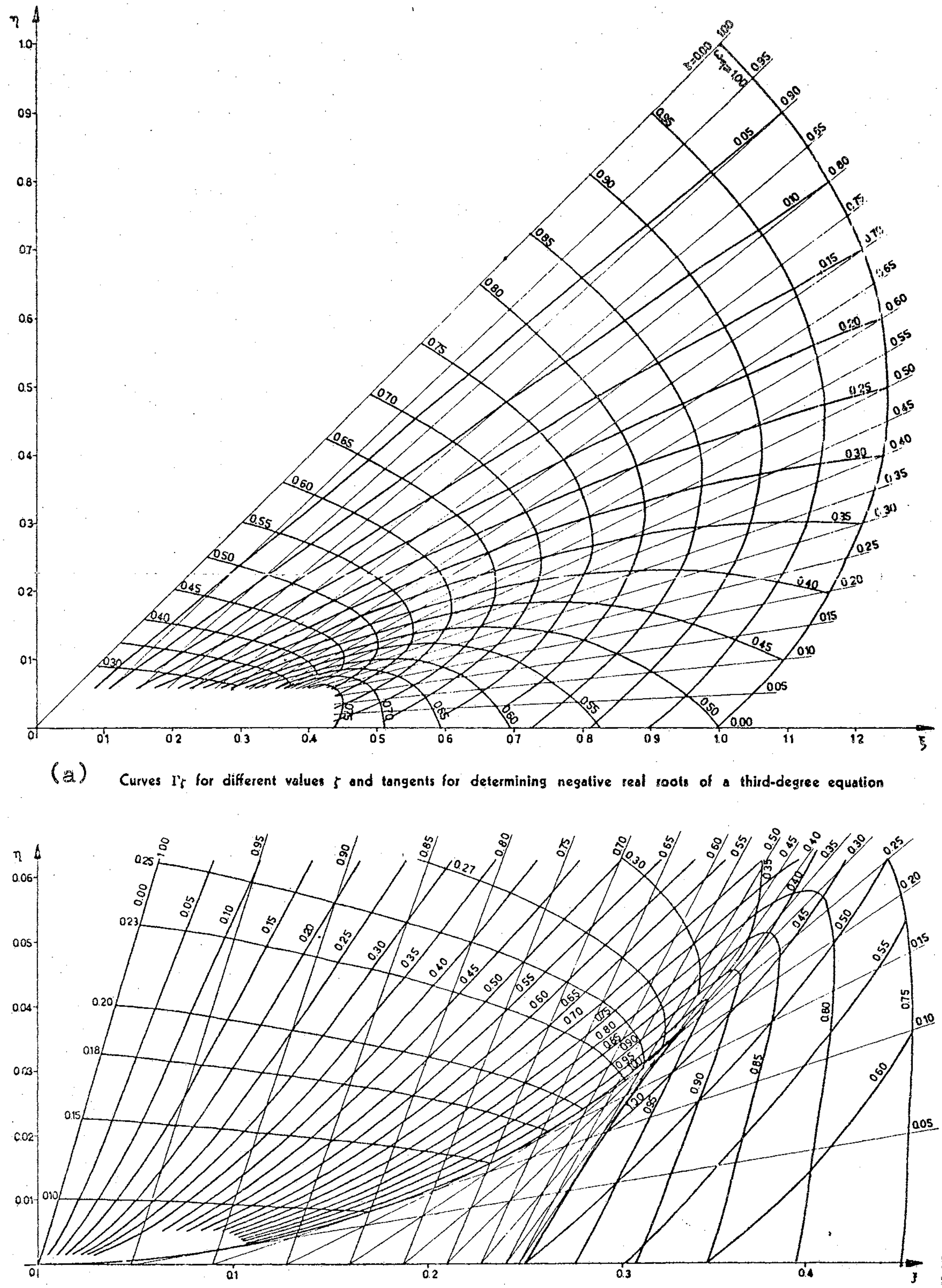
[3]The variable nomenclature here is the same as that used by Thaler and Brown (25) who devoted a complete chapter to Mitrovic's method.



(a) Curves Γ_ζ for different values ζ and tangents for determining negative real roots of a third-degree equation

(b) Enlarged region around the origin of Fig. 5

Figure 2-6. Third-Order Charts by Mitrovic charts (see Figure 2-6).[4] If three real roots were involved, they were readily obtained from the small triangular shaped area in Figure 2-6(b). If two complex conjugate roots were present, they could be determined using Equation (2-5) once ζ and ω_{nt} were known, since

$$P = -\zeta\omega_{nt} \pm j\omega_{nt}\sqrt{1-\zeta^2} \tag{2-6}$$

and the roots of Equation (1-5) were then calculated using the transformation relationship

$$s = -\sigma_t\left(\frac{a_2}{a_3}\right) \tag{2-7a}$$

$$s = -\zeta\omega_{nt}\left(\frac{a_2}{a_3}\right) \pm j\omega_{nt}\sqrt{1-\zeta^2}\left(\frac{a_2}{a_3}\right) \tag{2-7b}$$

A different approach to the problem of root determination of the third-order equation was made by Chu and Yeh (26) using the root-locus method of Evans (27). They concluded that if an equation of the form of (1-5) was stable, it could be written as

$$s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + k = 0 \tag{2-8}$$

and manipulated to obtain

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[4]Similar graphs with a larger scale were developed by the writer and are shown in Appendix A. In addition, the writer developed a set of graphs with constant ω_{dt} and ω_{nt} curves. The use of these graphs circumvents the calculation required by Equation (2-6) and is believed to be unique (also in Appendix A).

$$\frac{k}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = -1 \; . \qquad (2\text{-}9)$$

By equating the phase angle and magnitude on each side of this equation, two relationships were derived which were used to plot the five possible shapes of root locus for cubic characteristic equations. It was found that "the shapes of the root locus for these five cases are all portions of a hyperbola or its degenerated form in addition to a certain portion of the real axis," and that, although Equation (2-8) had three parameters, only the parameters ζ and ω_n determined the exact shape of the root locus (26). The functional relationship was further simplified by the substitution of the quantities

$$k' = \frac{k}{\omega_n^3} \qquad (2\text{-}10a)$$

$$s' = \frac{s}{\omega_n} \qquad (2\text{-}10b)$$

into Equation (2-9), resulting in

$$\frac{k'}{s'(s'^2 + 2\zeta s' + 1)} = -1 \qquad (2\text{-}11)$$

for which the shape of the root locus is determined by ζ alone. Chu and Yeh plotted a family of root loci on the complex s'-plane for various values of ζ (see Figure 2-7) and commented that "all possible roots of a cubic characteristic equation can be extracted from this chart" (26).

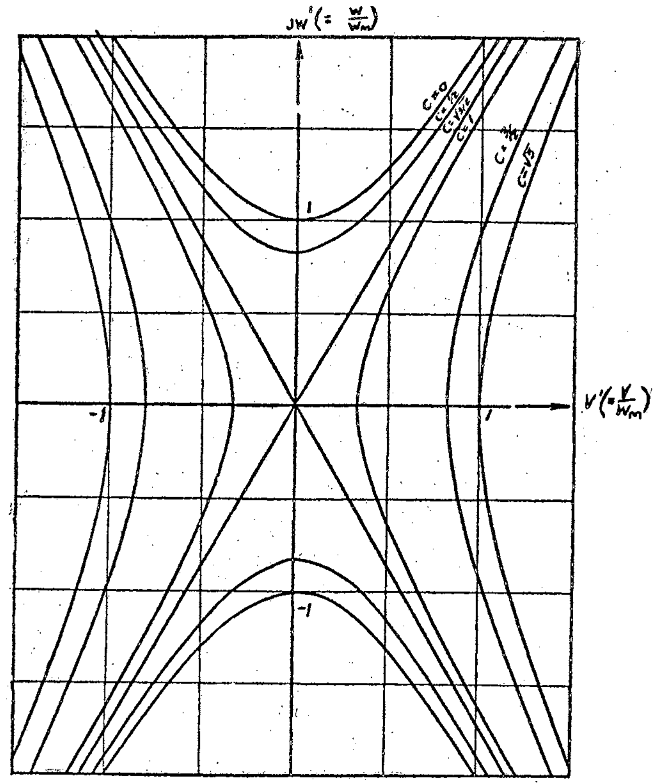

Figure 2-7. Root-Locus Chart by Chu and Yeh

Relationships Between Roots and Transient Response

In addition to the aforementioned references which dealt mainly with the problem of root determination, there have been several papers concerned with the relationships between the roots and the transient response of a system. The problem was considered for general linear systems (no restriction on order) by Mulligan (28) who concluded that,

> ... systems having appreciable separation between poles, particularly along the σ [real] axis relative to the first pair of complex poles, and having no complex poles very close to the σ axis

> and no complex zeros very near to the first pair of complex poles, it is quite likely that the term sin $(\beta_1 t + \lambda_1)$ [the sinusoid in the response equation due to the first pair of complex poles] will provide a very good approximation to g(t) [the time response] at the time corresponding to the first maximum and beyond.

Mulligan presented charts and equations which could be used to calculate the per cent of error involved when poles were neglected, but did not give any fixed value for the magnitude of "appreciable separation" required.[5] He also developed several "constant overshoot-factor charts" (see Figure 2-8) for different dominant-pole angles (a measure of the damping of the dominant poles).

Zemanian (29) later extended the work of Mulligan with a procedure for determining "the rise time from zero to the final value." However, the application of such methods to third-order systems would be of benefit only when one is

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[5]In regard to this problem, D'Azzo and Houpis (2) stated:

> The values of T_p [time to peak overshoot] and M_p [peak overshoot] are therefore sufficiently accurate if the other poles have a negative real part whose magnitude is equal to or greater than $3/T_p$.

And, Truxal (7) presented the design criterion that:

> ... any real poles which are to contribute negligibly to the step-function response should be placed at least six times as far from the jω axis as those poles governing the response. Clearly, it is irrelevant whether such poles far from the jω axis are actually real; complex poles are also negligible as long as the real part is much larger than that of the significant poles.

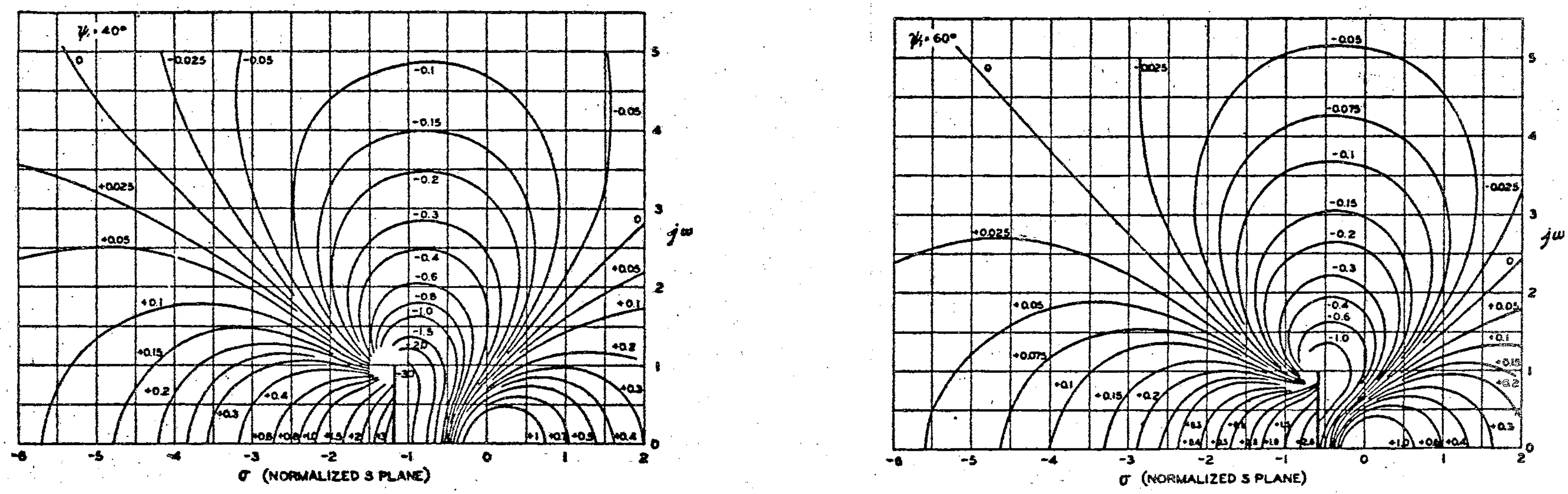

Figure 2-8. Constant Overshoot Charts by Mulligan

dealing with a "category b" (see Figure 1-6) root pattern which satisfies the criteria for "appreciable separation between poles."

Normalized Response Curves

A number of people have made extensive use of normalized transient response curves.[6] One of the first was Bretoi (30) who was concerned with the analysis and synthesis of an automatic flight control system. A third-order equation was developed which described the motion of the aircraft about the yaw axis. The normalized transient responses of the system were shown graphically (see Figure 2-9).

In addition, Elgerd and Stephens (31) developed the responses shown in Figure 2-10 as part of their paper dealing with "nine fundamental pole-zero configurations." According to the authors, the purpose of their charts was to enable the designer "to correlate immediately the connection between pole-zero configurations and real-time response."

Meyfarth (32) made use of the cubic charts of Evans and Liu (20, 21) to produce a total of ninety-six charts (see Figure 2-11) showing normalized step responses, impulse responses, and Bode plots (frequency response) for

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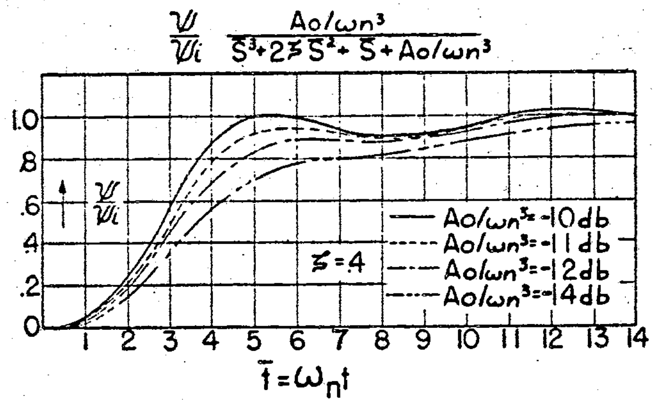
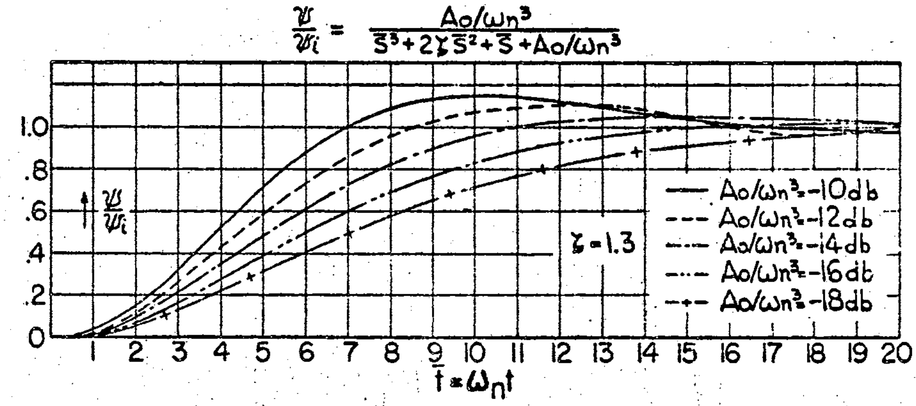
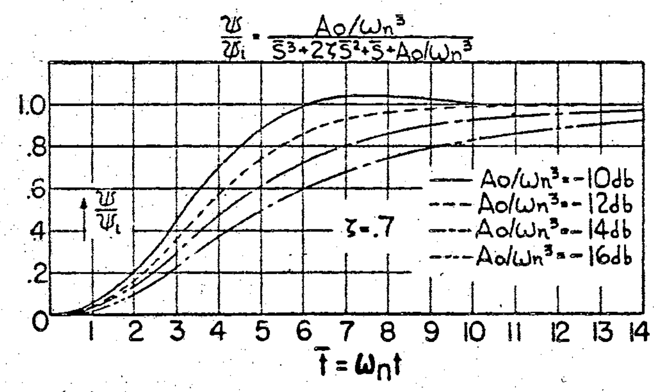
[6]For example, see Figure 1-8 by Clark (4).



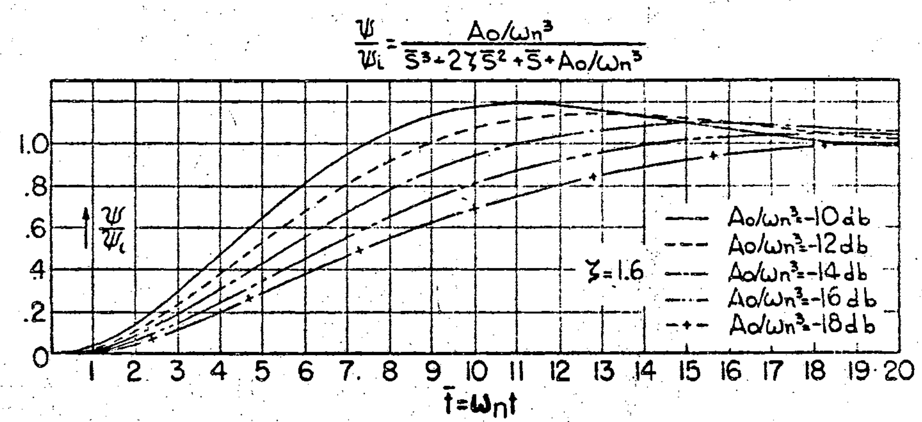
(a) Third-Order Chart—Directional Response to Step Heading Input



(b) Third-Order Chart—Directional Response to Step Heading Input



(c) Third-Order Chart—Directional Response to Step Heading Input



(d) Third-Order Chart—Directional Response to Step Heading Input

Figure 2-9. Normalized Response Curves of Bretoi

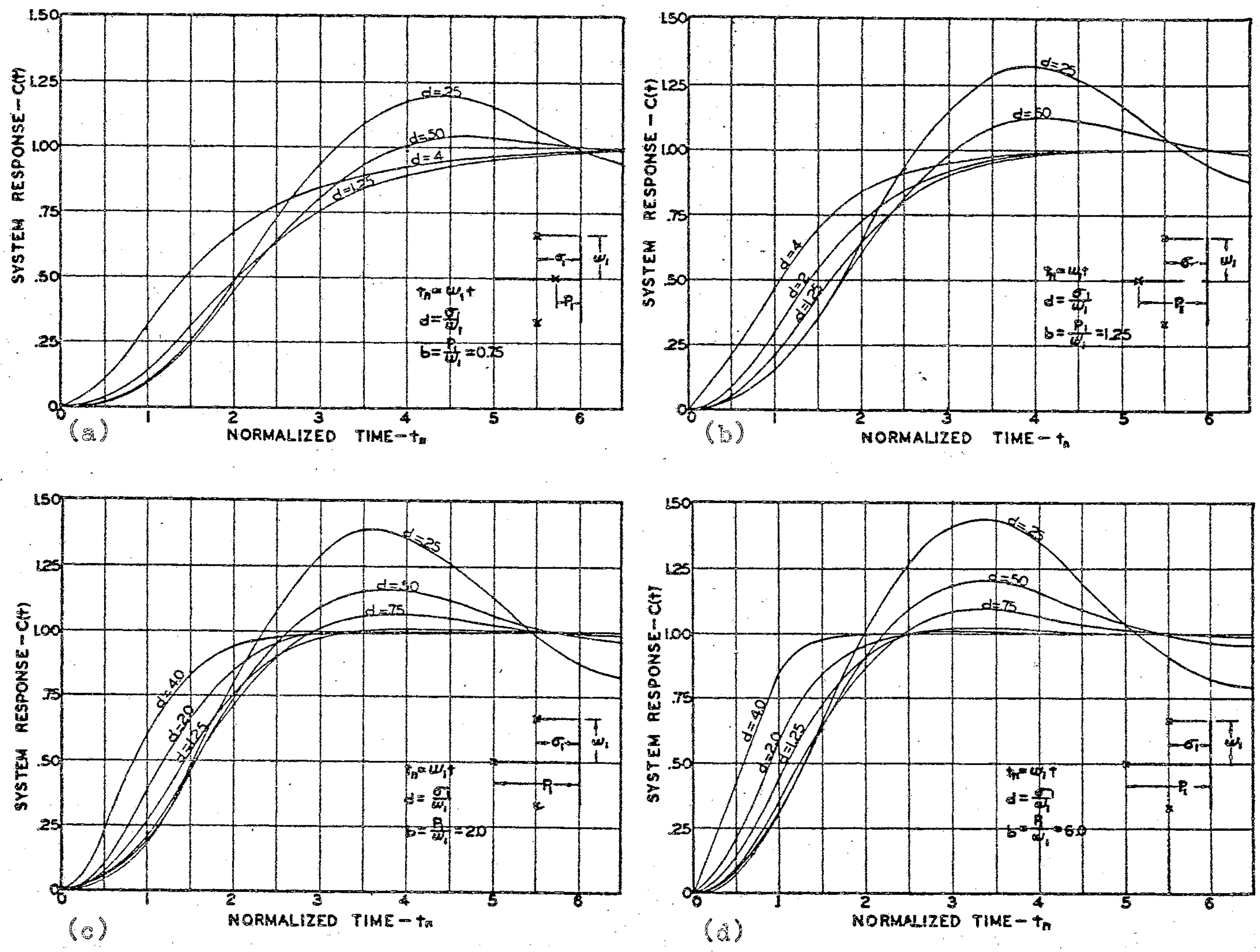
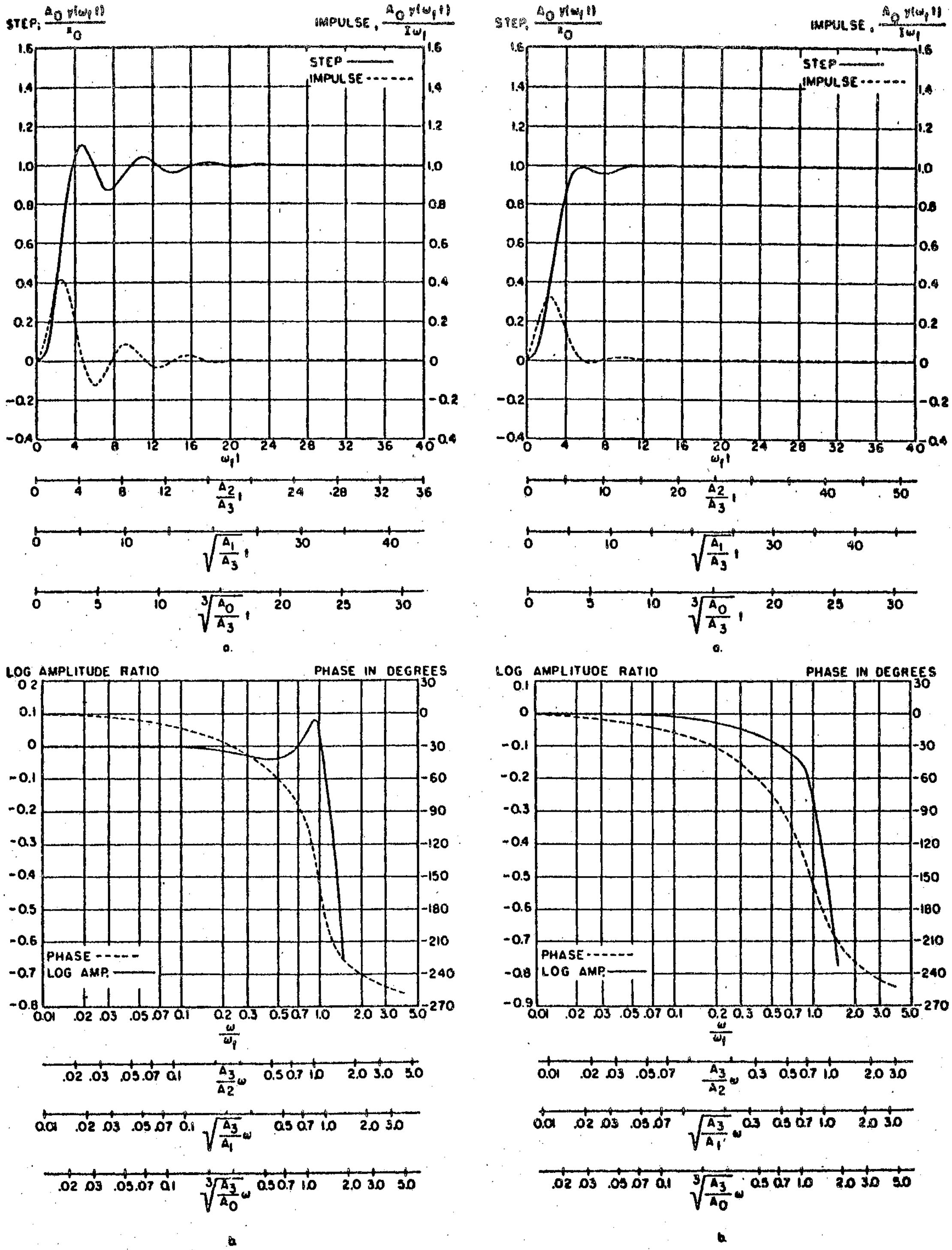


Figure 2-10. Normalized Response Curves of Third-Order System by Elgerd and Stephens



(a) $\tau_f \omega_f = 2.0 \quad \zeta_f = 0.2$

(b) $\tau_f \omega_f = 2.0 \quad \zeta_f = 0.4$

Figure 2-11. Normalized Response Curves by Meyfarth forty-eight different combinations of ζ_f, damping ratio, and $\tau_f\omega_f$, product of time constant of the factor representing the real root and the undamped natural frequency.

Charts With Transient Response Characteristics

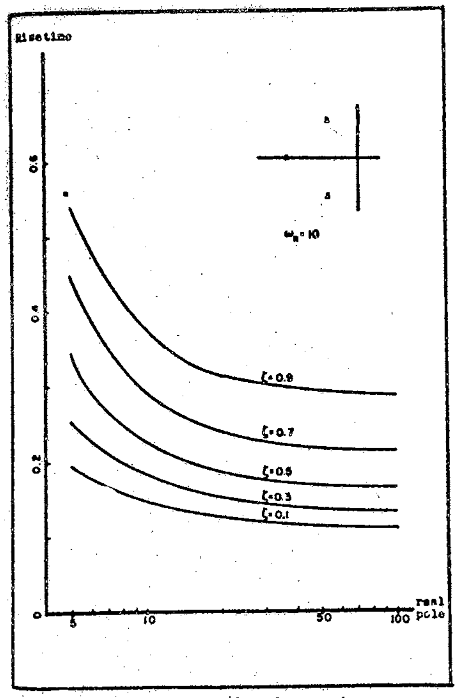
There have been several authors who developed charts showing the functional relationship between the system roots and the transient response characteristics. Burnette and Shumate (33) used a digital computer to calculate the data for the charts shown in Figure 2-12, while Clement (34) used an analog computer for his third-order curves in Figure 2-13. Clark (4) in his textbook included a considerable amount of material on third-order systems among which were the normalized curves previously shown in Figure 1-8 and the overshoot chart in Figure 2-14.

Summary of Current Methods

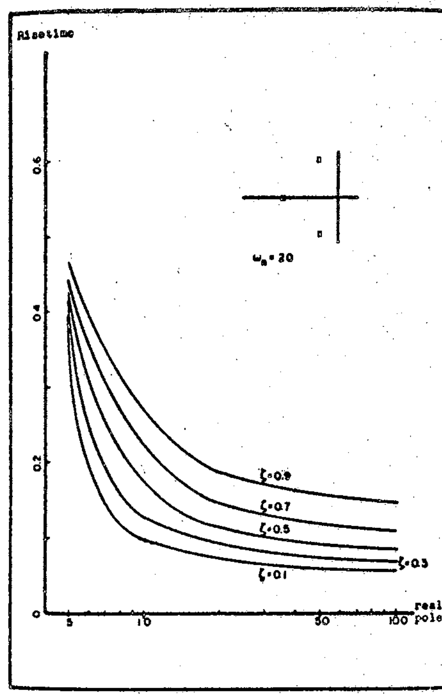
The present methods for the analysis of third-order systems which were cited in the last two sections of this chapter have two basic steps in common:

1. A root-determination process.
2. Association of the root pattern with a normalized response or response characteristic chart.

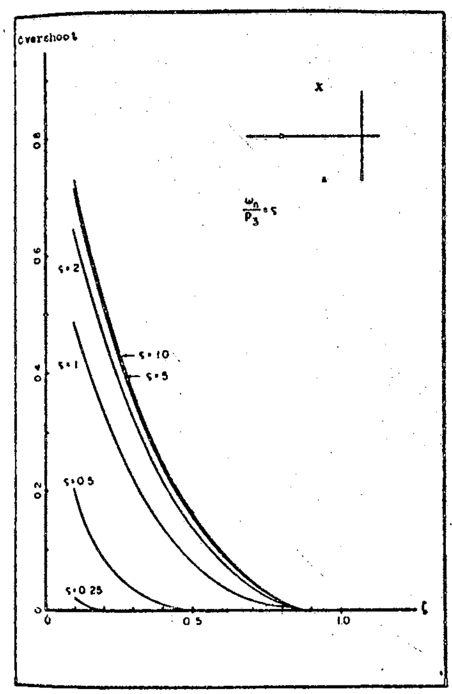
The root determination may or may not take a considerable amount of time, depending on the experience of the analyst,



—Rise time for three-pole case.



—Rise time for three-pole case.



—Overshoot for three-pole case.

t_r = rise time (10% to 90%)

os = per cent overshoot of final value

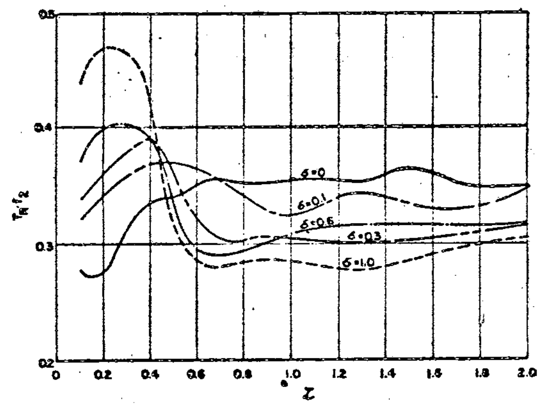
p = magnitude of real pole

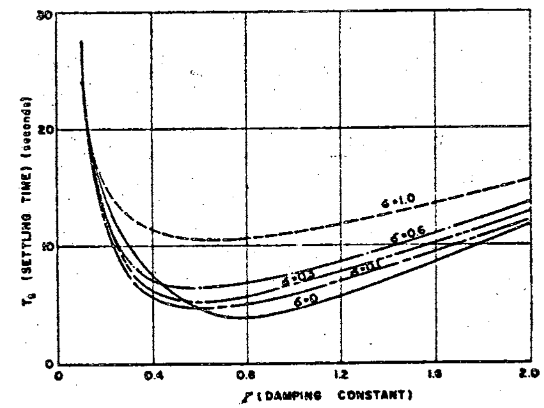
ω_n = natural undamped frequency

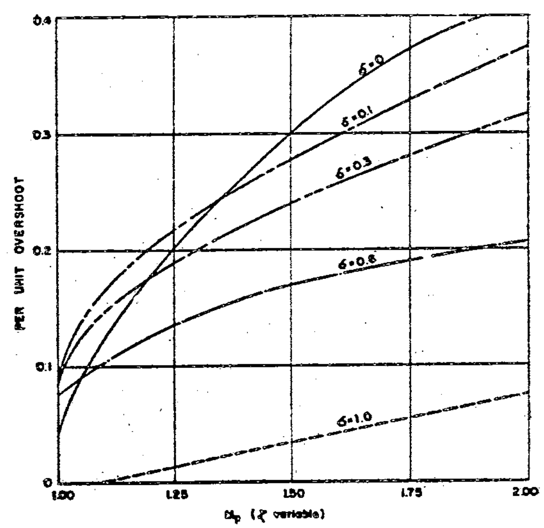
ζ = damping ratio

$\delta = \frac{\omega_n}{p}$

Figure 2-12. Third-Order Curves by Burnette and Shumate







ζ = damping constant

δ = inverse of negative real root

f_2 = upper half-power frequency

M_p = peak value of frequency response

os = per unit overshoot

T_R = rise time (10% - 90%)

T_s = settling time (time to settle within 5% of the final value)

Figure 2-13. Third-Order Curves by Clement

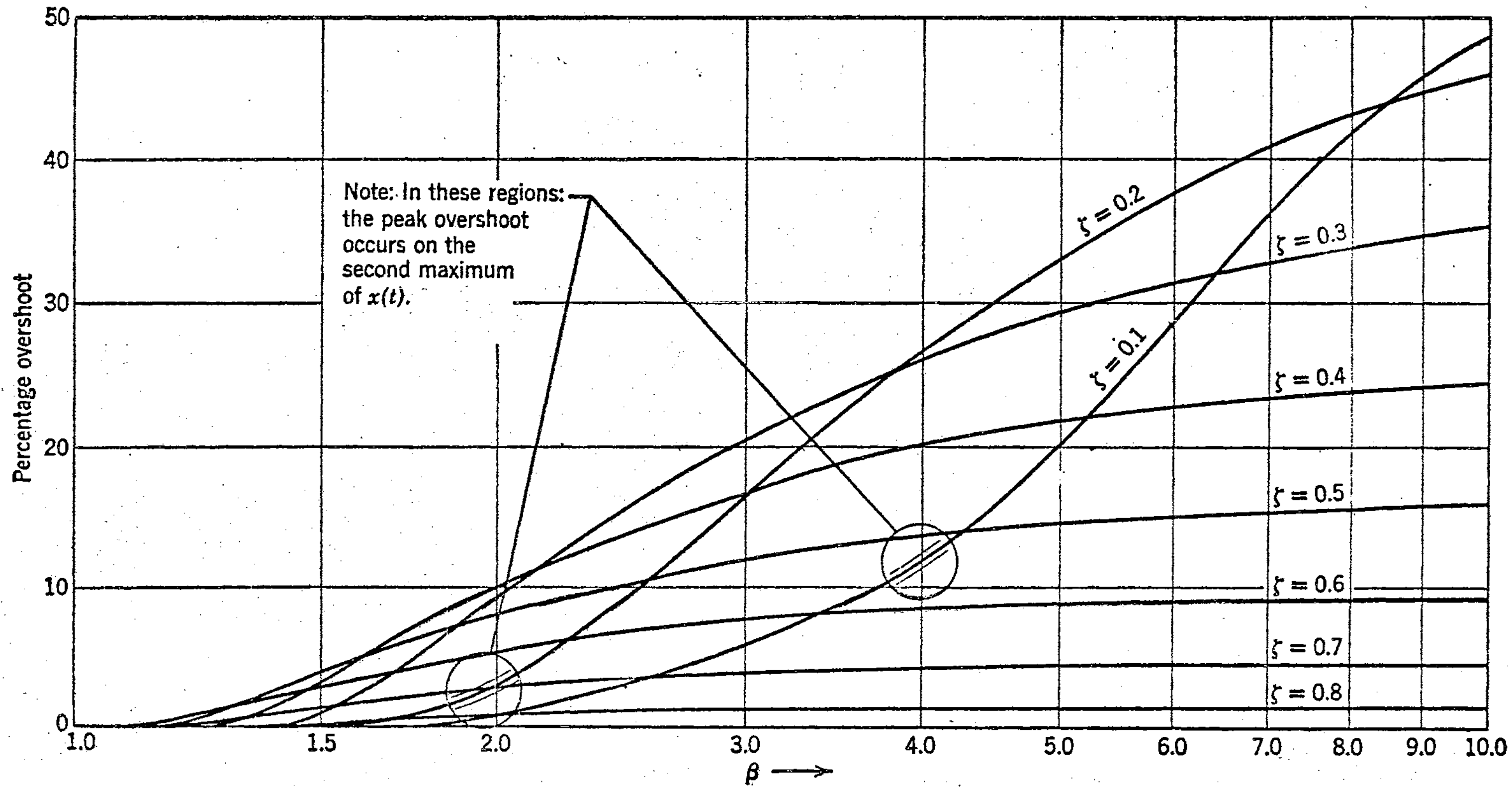


Figure 2-14. Overshoot Chart by Clark

but the major disadvantage of the current methods lies in the fact that, even with a large number of charts, only a few discrete points or curves are of use to the analyst for a particular design problem. For instance, consider the work of Meyfarth (32), who developed ninety-six charts of normalized responses (see Figure 2-11). Of the forty-eight transient response curves, only four showed overshoot between 20 per cent and 40 per cent. Thus, for a designer faced with a transient response limit of 30 per cent, the possible parameter combinations with which he can work is definitely limited. In addition, such characteristics as rise time, settling time, phase margin, etc., have to be scaled off each chart. All of these factors, coupled with the fact that the system roots have to be re-evaluated everytime a parameter is changed, make this method of little practical use in system design.

The work of Burnette and Shumate (33) (see Figure 2-12) is also of limited applicability because:

1. Only three different natural frequencies were used in the development of the rise time charts.
2. Only six different ratios of real root parts were shown on the overshoot chart.

Likewise, Clement (34) included only five negative root trajectories on his response characteristic curves (see Figure 2-13).

The chart by Clark (4) (see Figure 2-14) represents the best approach to date. However, it also requires a knowledge of the roots and the information available is limited to the response characteristic of overshoot, while the values of settling time and rise time must be scaled from the accompanying normalized response curves (see Figure 1-8).

The need for a generalized approach, which will provide in a direct and versatile manner the information required for the analysis and synthesis of linear third-order systems, seems apparent from a review of the literature.

CHAPTER III

DEVELOPMENT OF THE ANALYSIS METHOD

In this chapter a method for the analysis and synthesis of linear third-order systems is presented. The method makes use of a set of charts to determine the following response characteristics of the system as functions of the coefficients of the system characteristic equation:

Transient response characteristics

1. Overshoot
2. Rise time
3. Settling time (output within 5 per cent of final value)

Frequency response characteristics

1. Gain margin
2. Phase margin.

The transient response charts are applicable for systems described by the transfer function[1]

$$G(s) = \frac{k}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \qquad (3\text{-}1)$$

[1]For a definition of transfer function, see reference (5).

where k is a dimensional constant.[2] If the system described by this transfer function is subjected to a unit step input, then

$$Y(s) = \frac{1}{s}$$

and[3]

$$X(s) = \frac{k}{s(a_3 s^3 + a_2 s^2 + a_1 s + a_0)} . \qquad (3\text{-}2)$$

The steady-state solution is[4]

$$x(t)_{ss} = \frac{k}{a_0}. \qquad (3\text{-}3)$$

Thus, if $k = a_0$, $x(t)_{ss} = 1$ and there is no steady-state error $\left(e(t)_{ss} = y(t)_{ss} - x(t)_{ss}\right)$. However, if $k \neq a_0$, there will be a steady-state error equal to

$$e(t)_{ss} = 1 - \frac{k}{a_0}. \qquad (3\text{-}4)$$

The simplest procedure for handling both cases is to consider the ordinate scale of the response plot to be "per cent of $x(t)_{ss}$," then the overshoot, rise time, and settling time quantities derived for the case where $k = a_0$ are

---

[2] k is sometimes referred to as the forward loop gain.

[3] Initial conditions are assumed to be zero throughout this text.

[4] $\lim_{t \to \infty} x(t) = s \lim_{s \to 0} X(s)$ (6).

applicable also for $k \neq a_o$.

The frequency response charts may be used only for systems with the transfer function

$$G(s) = \frac{a_o}{a_3 s^3 + a_2 s^2 + a_1 s + a_o} \qquad (3\text{-}5)$$

Relationships Between Responses

Mitrovic (24) used the linear transformation $S = \frac{a_2}{a_3} P$ to transform

$$a_3 s^3 + a_2 s^2 + a_1 s + a_o = 0 \qquad (1\text{-}5)$$

to the normalized equation

$$P^3 + P^2 + B_1 P + B_o = 0 \qquad (2\text{-}5)$$

where

$$B_1 = \frac{a_1 a_3}{a_2^2} \quad \text{and} \quad B_o = \frac{a_o a_3^2}{a_2^3} .$$

Since the transformation is one of "magnification" only, the root pattern geometry for Equation (1-5) will be the same as that for Equation (2-5), except for a difference in the magnitudes of the roots.[5] Thus, the following relationships between system parameters will hold:

---

[5] a_2 and a_3 are both real numbers; therefore, no rotation or translation is involved.

Parameter	Relationship[6]
Damping ratio ($\zeta = \cos^{-1}\theta$)	$\zeta = \zeta_t$
Undamped natural frequency	$\omega_n = \left(\frac{a_2}{a_3}\right)\omega_{nt}$
Damped natural frequency	$\omega_d = \left(\frac{a_2}{a_3}\right)\omega_{dt}$
Time constant of real root $\left(t_\sigma = \frac{1}{\lvert\sigma\rvert}\right)$	$t_\sigma = \left(\frac{a_3}{a_2}\right) t_{\sigma t}$
Time constant of complex roots $\left(t_{\zeta\omega} = \frac{1}{\lvert\zeta\omega\rvert}\right)$	$t_{\zeta\omega} = \left(\frac{a_3}{a_2}\right) t_{\zeta\omega t}$
Ratio of real parts, $\beta = \frac{\sigma}{\zeta\omega}$	$\beta = \beta_t$

For example, consider a system with the characteristic equation

$$s^3 + 4s^2 + 9s + 10 = 0$$

the roots of which are [see Figure 2-1(a)]

$$s = -2$$

$$s = -1 \pm j2 \ .$$

The transformed equation (using s = 4P) is

$$P^3 + P^2 + \frac{9}{16}P + \frac{5}{32} = 0$$

---

[6]The subscript t denotes a variable of the transformed equation.

with roots [see Figure 3-1(b)]

$$P = -\frac{1}{2}$$

$$P = -\frac{1}{4} \pm j\frac{1}{2}$$

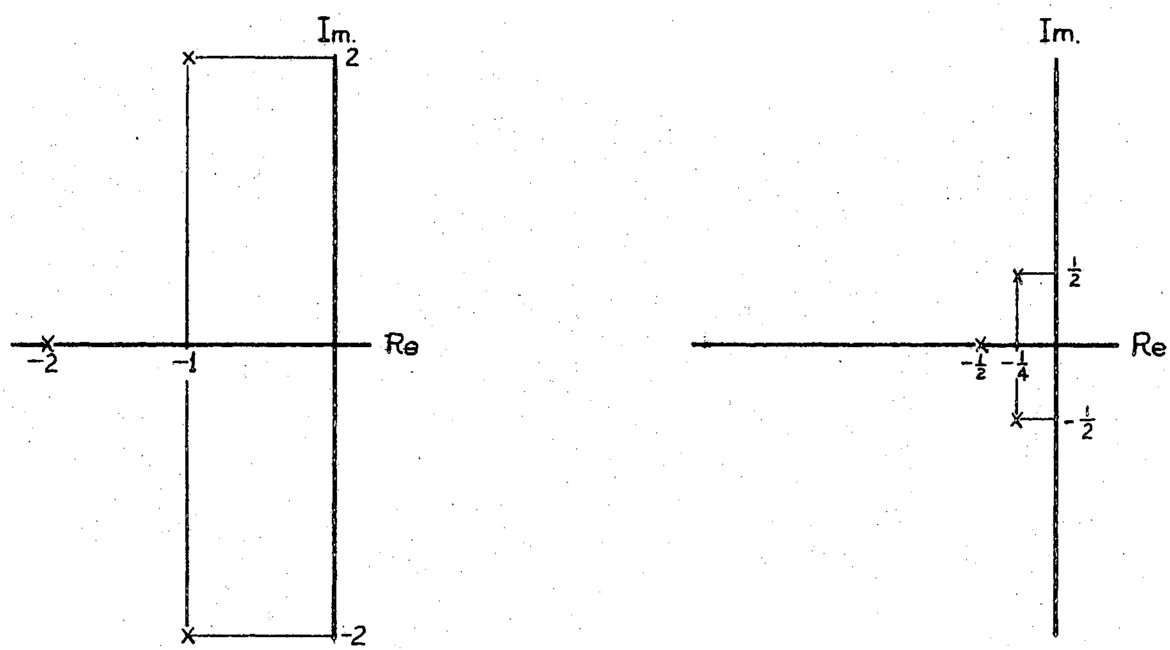


Figure 3-1. Roots of Equations (1-5) and (2-5)

The parameters of the two systems are:

Parameter	(a)	(b)
ζ	.447	.447
ω_n	5	$\frac{5}{4}$
ω_d	2	$\frac{1}{2}$
t_σ	$\frac{1}{2}$	2

Parameter	(a)	(b)
$t_{\zeta\omega}$	1	4
β	2	2

Since ζ and β are the same in both cases, normalized response plots for the two systems will be identical except for the time scales (for example, see Figure 1-8). Hence, in the case of step inputs, the overshoot for the two systems will be the same, while the rise time and settling time are related by $t_r = \left(\frac{a_3}{a_2}\right) t_{rt}$ and $t_s = \left(\frac{a_3}{a_2}\right) t_{st}$, respectively. Because all third-order equations may be normalized by Mitrovic's procedure, the transient response of any third-order system (with no zero in the transfer function) can be ascertained from the transient responses of the system associated with the normalized characteristic equation.

As was mentioned previously, the charts involving gain margin and phase margin are applicable only for systems where $k = a_o$. However, the values of these stability criteria for the system with the normalized characteristic equation are the same as those for the system before normalization, so no transformation is required.[7]

Basis for Coordinate System

The coefficients and the roots ($-P_1$, $-P_2$, $-P_3$) of

---

[7]The reason for this relationship is discussed in more detail later in the chapter.

Equation (2-5) are related as follows:

$$P_1 + P_2 + P_3 = 1 \tag{3-6a}$$

$$P_1 P_2 + P_1 P_3 + P_2 P_3 = B_1 \tag{3-6b}$$

$$P_1 P_2 P_3 = B_o \tag{3-6c}$$

If Equation (3-6a) is solved for P_1, then by substitution

$$B_1 = P_2 P_3 + (P_2 + P_3)(1 - P_2 - P_3) \tag{3-7a}$$

$$B_o = P_2 P_3(1 - P_2 - P_3). \tag{3-7b}$$

Hence, if the system has three real roots ($-\sigma_{t1}$, $-\sigma_{t2}$, $-\sigma_{t3}$)

$$B_1 = \sigma_{t2} \sigma_{t3} + (\sigma_{t2} + \sigma_{t3})(1 - \sigma_{t2} - \sigma_{t3}) \tag{3-8a}$$

$$B_o = \sigma_{t2} \sigma_{t3}(1 - \sigma_{t2} - \sigma_{t3}) \tag{3-8b}$$

If $-\sigma_{t1}$ is a root of Equation (2-5), then

$$-\sigma_{t1}^3 + \sigma_{t1}^2 - B_1\sigma_{t1} + B_o = 0 \tag{3-9}$$

and

$$B_o = \sigma_{t1}(B_1 - \sigma_{t1} + \sigma_{t1}^2) \tag{3-10}$$

which is the equation of a straight line of slope σ_{t1} with an intersection on the B_1 axis at $\sigma_{t1} - \sigma_{t1}^2$.

If Equation (2-5) has only one real root ($\sigma_t = 1 - 2\zeta\omega_{nt}$), the complex roots are $P = -\zeta\omega_{nt} \pm j\omega_{nt}\sqrt{1-\zeta^2}$ and

$$B_1 = \omega^2_{nt} + 2\zeta\omega_{nt} - 4(\zeta\omega_{nt})^2 \qquad (3\text{-}11a)$$

$$B_0 = \omega^2_{nt}(1 - 2\zeta\omega_{nt}) \,. \qquad (3\text{-}11b)$$

If $\zeta = 0$, then

$$B_1 = B_0 \qquad (3\text{-}12)$$

which defines the line separating the stable and unstable regions on the chart in Figure 3-2.

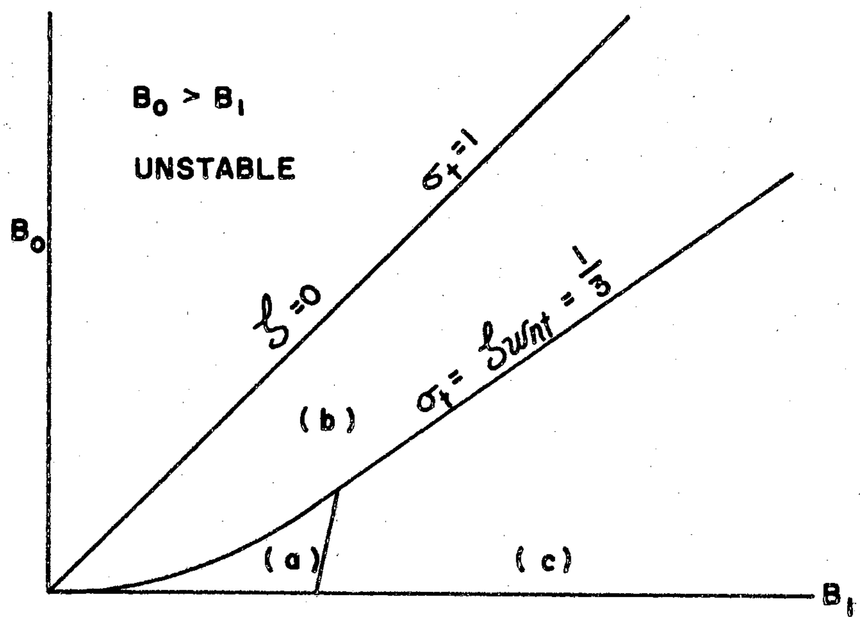


Figure 3-2. Basic Root Pattern Regions of Normalized Third-Order Equations

When $B_1 < B_0$, the complex roots fall in the right one-half of the P-plane and any system disturbance will result in an unstable response. The stable region, $B_1 > B_o$, is divided into three regions corresponding to the type of root pattern involved:

1. Three real roots (enclosed by $\zeta = 1$).
2. Two complex roots with $\beta_t = \frac{\sigma_t}{\zeta\omega_{nt}} > 1$.
3. Two complex roots with $\beta_t < 1$.[8]

The line separating the last two regions is determined using the relationship $\sigma_t = \zeta\omega_{nt} = \frac{1}{3}$ to obtain the equation of the line

$$B_o = \frac{1}{3}\left(B_1 - \frac{1}{9}\right).^{9} \qquad (3\text{-}13)$$

Equations (3-8) and (3-11) are used in the following sections to locate points on the $B_1 - B_o$ charts associated with particular response characteristics.

Transient Response Equations

Consider a system described by the transfer function

$$G(s) = \frac{a_o}{a_3 s^3 + a_2 s^2 + a_1 s + a_o} \qquad (3\text{-}14)$$

---

[8]The stability criteria $B_1 > B_o$ is equivalent to Routh's criteria $a_2 a_1 > a_3 a_o$.

[9]The top part of the $\zeta = 1$ line along with the extension of the line defined in Equation (3-13) separates regions of overshoot and no overshoot for step inputs (see Figure 3-2).

which can be transformed to

$$G(P) = \frac{B_o}{P^3 + P^2 + B_1 P + B_o} . \qquad (3\text{-}15)$$

When complex roots are involved, the latter transfer function may be manipulated to

$$G(P) = \frac{\sigma_t \omega_{nt}^2}{(P + \sigma_t)(P + 2\zeta\omega_{nt}P + \omega_{nt}^2)} \qquad (3\text{-}16)$$

where

$$B_o = \sigma_t \omega_{nt}^2 .$$

If such a system is subjected to a unit step input, then

$$X(P) = \frac{\sigma_t \omega_{nt}^2}{P(P + \sigma_t)(P + 2\zeta\omega_{nt}P + \omega_{nt}^2)} \qquad (3\text{-}17)$$

the inverse transform of which is

$$x(t) = 1 - \frac{\omega_{nt}^2 e^{-\sigma_t t}}{\sigma_t^2 - 2\zeta\omega_{nt}\sigma_t + \omega_{nt}^2}$$

$$+ \frac{\sigma_t e^{-\zeta\omega_{nt} t} \sin(\omega_{nt}\sqrt{1 - \zeta^2}\, t - \varphi)}{\sqrt{(1 - \zeta^2)(\sigma_t^2 - 2\zeta\omega_{nt}\sigma_t + \omega_{nt}^2)}} \qquad (3\text{-}18)$$

where $\varphi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{-\zeta} + \tan^{-1} \frac{\omega_{nt}\sqrt{1 - \zeta^2}}{\zeta\omega_{nt} - \sigma_t}$ (6).

Substituting $\sigma_t = 1 - 2\zeta\omega_{nt}$

$$x(t) = 1 - \frac{\omega_{nt}^{2} e^{-(1 - 2\zeta\omega_{nt})t}}{8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^{2}}$$

$$+ \frac{(1 - 2\zeta\omega_{nt})e^{-\zeta\omega_{nt} t} \sin(\omega_{nt}\sqrt{1 - \zeta^2}\, t - \varphi)}{\sqrt{(1 - \zeta^2)}[8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^{2}]} \qquad (3\text{-}19)$$

and $\varphi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{-\zeta} + \tan^{-1} \frac{\omega_{nt}\sqrt{1 - \zeta^2}}{3\zeta\omega_{nt} - 1}$.

When the system has three real roots, the transfer function is

$$G(P) = \frac{\sigma_{t1}\sigma_{t2}\sigma_{t3}}{(P + \sigma_{t1})(P + \sigma_{t2})(P + \sigma_{t3})} \qquad (3\text{-}20)$$

where

$$B_o = \sigma_{t1}\sigma_{t2}\sigma_{t3} .$$

For a unit step input, the inverse transform of X(P) is

$$x(t) = 1 - \frac{\sigma_{t2}\sigma_{t3} e^{-\sigma_{t1} t}}{(\sigma_{t2} - \sigma_{t1})(\sigma_{t3} - \sigma_{t1})}$$

$$- \frac{\sigma_{t1}\sigma_{t3} e^{-\sigma_{t2} t}}{(\sigma_{t1} - \sigma_{t2})(\sigma_{t3} - \sigma_{t2})}$$

$$- \frac{\sigma_{t1}\sigma_{t2} e^{-\sigma_{t3} t}}{(\sigma_{t1} - \sigma_{t3})(\sigma_{t2} - \sigma_{t3})} \quad (6). \qquad (3\text{-}21)$$

Substituting $\sigma_{t1} = 1 - \sigma_{t2} - \sigma_{t3}$

$$x(t) = 1 - \frac{\sigma_{t2}\sigma_{t3}\, e^{-(1-\sigma_{t2}-\sigma_{t3})t}}{(2\sigma_{t2} + \sigma_{t3} - 1)(2\sigma_{t3} + \sigma_{t2} - 1)}$$

$$- \frac{(1 - \sigma_{t2} - \sigma_{t3})\sigma_{t3}\, e^{-\sigma_{t2} t}}{(1 - 2\sigma_{t2} - \sigma_{t3})(\sigma_{t3} - \sigma_{t2})}$$

$$- \frac{(1 - \sigma_{t2} - \sigma_{t3})\sigma_{t2}\, e^{-\sigma_{t3} t}}{(1 - 2\sigma_{t3} - \sigma_{t2})(\sigma_{t2} - \sigma_{t3})} . \qquad (3\text{-}22)$$

The response Equations (3-19) and (3-22) are used in the following sections concerned with transient response characteristics.

Settling Time Chart

Settling time is defined as the time for the response to decrease to less than a specified percentage of the final value. In this thesis five per cent was selected.

The procedure used by most analysts concerned with settling time is to work with the envelope of the system response rather than with the exact equation, since in most cases only a small error will result and the mathematics is simplified. If this approach is used, then for a specified value of settling time, the following nonlinear equation must be solved when complex roots are involved.[9]

---

[9] F = the envelope of x(t) - 0.95.

$$F(\zeta,\omega_{nt},t_{st}) = .05 - \frac{\omega_{nt}^2 e^{-(1-2\zeta\omega_{nt})t_{st}}}{8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2}$$

$$- \frac{(1 - 2\zeta\omega_{nt})e^{-\zeta\omega_{nt}t_{st}}}{\sqrt{(1-\zeta^2)[8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2]}} = 0 \tag{3-23}$$

while for real roots

$$F(\sigma_{t2},\sigma_{t3},t_{st}) = .05 - \frac{\sigma_{t2}\sigma_{t3} e^{-(1-\sigma_{t2}-\sigma_{t3})t_{st}}}{(2\sigma_{t2}+\sigma_{t3}-1)(2\sigma_{t3}+\sigma_{t2}-1)}$$

$$- \frac{(1-\sigma_{t2}-\sigma_{t3})\sigma_{t3} e^{-\sigma_{t2}t_{st}}}{(1-2\sigma_{t2}-\sigma_{t3})(\sigma_{t3}-\sigma_{t2})}$$

$$- \frac{(1-\sigma_{t2}-\sigma_{t3})\sigma_{t2} e^{-\sigma_{t3}t_{st}}}{(1-2\sigma_{t3}-\sigma_{t2})(\sigma_{t2}-\sigma_{t3})} . \tag{3-24}$$

Since there are two unknowns remaining in these equations after the settling time is specified, one of the other variables must be fixed before a solution is possible.

The writer chose to fix ω_{nt} when the system had complex roots and σ_{t3} when real roots were involved.[10] The Newton-Raphson method was then employed to solve for the

---

[10] ω_{nt} was specified for each calculation because ζ approached a constant value as ω_{nt} was decreased. In this region of the $B_1 - B_0$ chart the third-order system may be effectively approximated by a second-order system.

remaining variable in the equation, afterwhich B_1 and B_0 were determined using the appropriate equations in the previous section.[11] Thus, by varying ω_{nt} (or σ_{t_3}) systematically while keeping the settling time fixed, a set of B_1 - B_0 values were generated which defined a "line of constant settling time." The calculations were performed by an IBM 1410 computer and the results are shown in Figure 3-3.[12]

Overshoot Chart

Overshoot is defined as the difference between the magnitudes of the maximum and final values of the response, expressed as a percentage. Since overshoot may occur only when the system has complex roots, the response Equation (3-22) for real roots is not involved in this section.

At the point of peak overshoot of a system response, the derivative of the response Equation (3-19) is equal to zero. Thus, for a specified value of overshoot, os, the following equations must hold.[13]

---

[11] The Newton-Raphson method is discussed in Appendix B-I.

[12] The Fortran program is reproduced in Appendix B-II.

[13] $F(\zeta, \omega_{nt}, t) = x(t) - 1 - os$
$G(\zeta, \omega_{nt}, t) = \dot{x}(t)$.

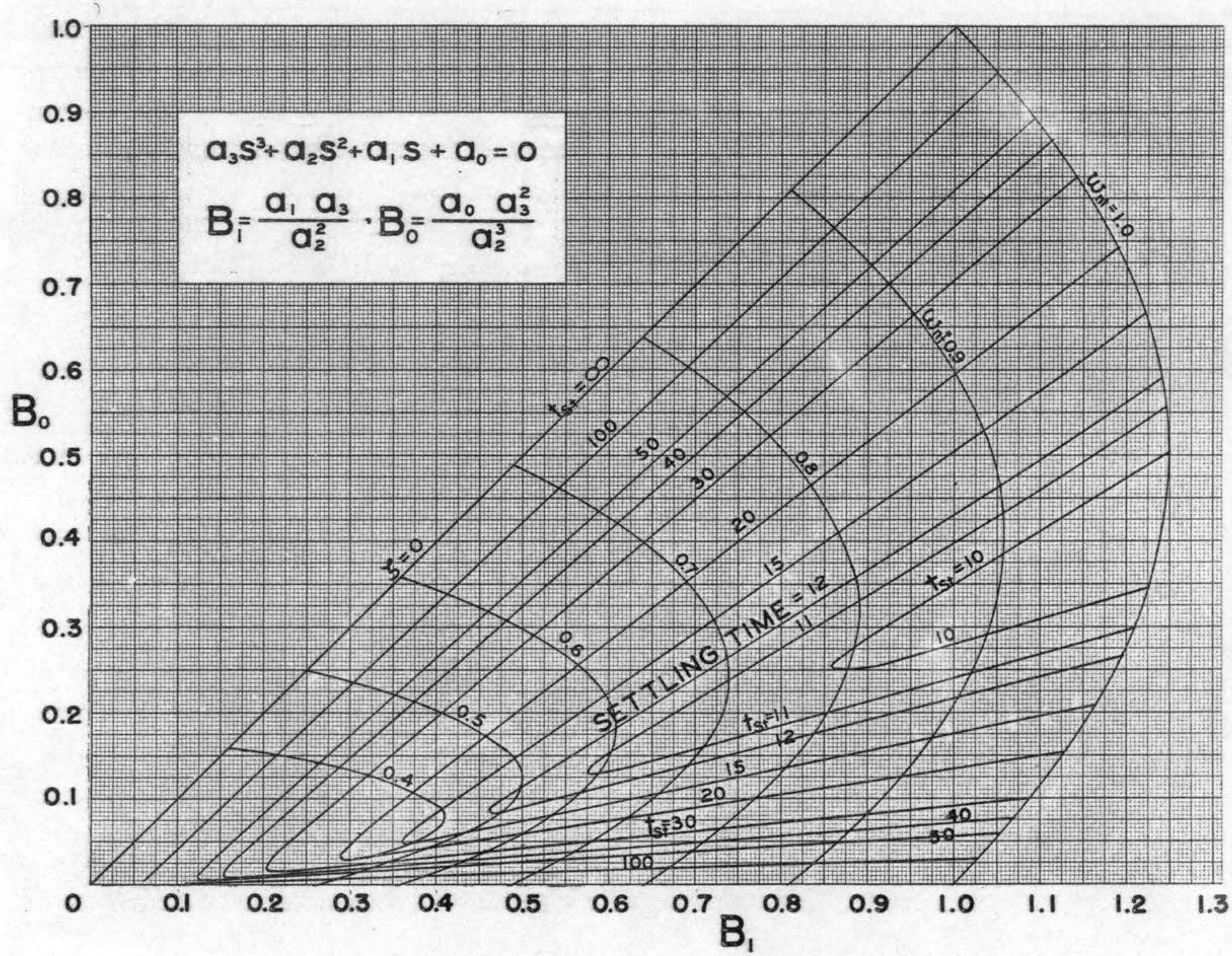


Figure 3-3. Lines of Constant Settling Time

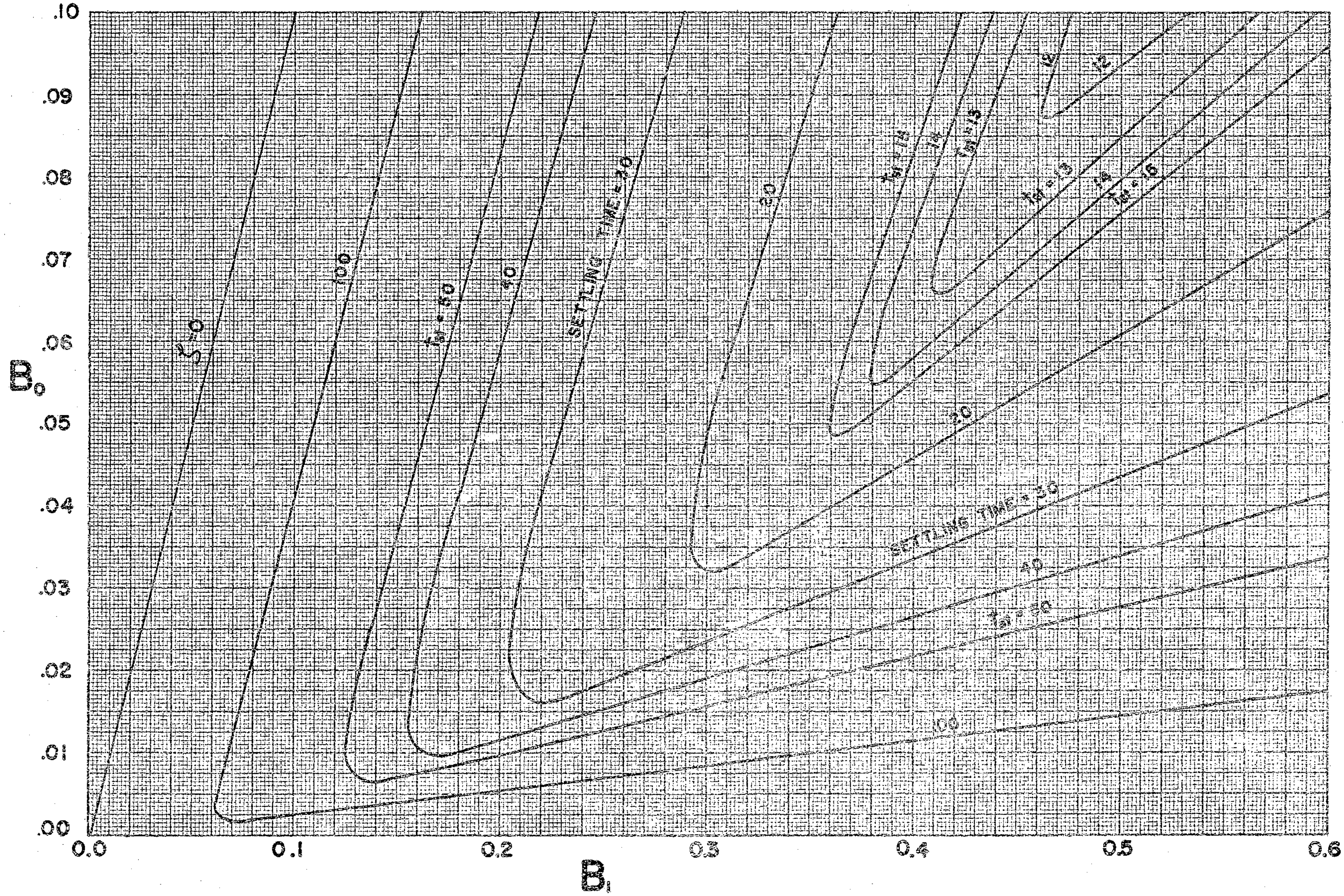


Figure 3-3. (Continued)

$$F(\zeta,\omega_{nt},t) = -os - \frac{\omega_{nt}^2 e^{-(1-2\zeta\omega_{nt})t}}{8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2}$$

$$+ \frac{(1-2\zeta\omega_{nt})e^{-\zeta\omega_{nt}t}\sin(\omega_{nt}\sqrt{1-\zeta^2}\,t - \varphi)}{\sqrt{(1-\zeta^2)}[8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2]} = 0 \qquad (3\text{-}25)$$

$$G(\zeta,\omega_{nt},t) = \frac{(1-2\zeta\omega_{nt})\omega_{nt}^2 e^{-(1-2\zeta\omega_{nt})t}}{8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2}$$

$$-\left[\frac{\omega_{nt}(1-2\zeta\omega_{nt})e^{-\zeta\omega_{nt}t}}{\sqrt{(1-\zeta^2)}[8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2}\right.$$

$$\times [\zeta \sin(\omega_{nt}\sqrt{1-\zeta^2}\,t - \varphi) - \sqrt{1-\zeta^2}$$

$$\left.\cos(\omega_{nt}\sqrt{1-\zeta^2} - \varphi)]\right] = 0. \qquad (3\text{-}26)$$

Holding ω_{nt} as well as overshoot constant, the writer used a generalization of the Newton-Raphson method[14] to solve both equations simultaneously for ζ and t.[15] Using the values of ζ and ω_{nt}, Equations (3-11a) and (3-11b) were then used to calculate B_1 and B_0. By varying ω_{nt} as before, the procedure was repeated for additional $B_1 - B_0$ points

[14]The generalization for the simultaneous solution of two equations and the Fortran program are listed in Appendix C.

[15]Care must be exercised to obtain only points of peak overshoot, since Equations (3-25) and (3-26) may be satisfied by values of ζ, ω_{nt}, and t not associated with a peak overshoot.

and a "line of constant overshoot" was obtained. The results for various values of overshoot are shown in Figure 3-4.

Rise Time Chart

There are several different definitions for the rise time of a system. The definition employed here is the one given in Chapter I; the time from 10 per cent to 90 per cent of the final value. Thus, in order to determine the system parameters for a selected value of rise time, t_{rt}, the following relationships must hold simultaneously:

$$x(t_1) = 0.1 \qquad (3\text{-}27)$$

$$x(t_2) = 0.9 \qquad (3\text{-}28)$$

$$t_2 - t_1 = t_{rt}. \qquad (3\text{-}29)$$

For complex roots, the equations are:

$$F(\zeta,\omega_{nt},t_1) = 0.9 - \frac{\omega_{nt}^2 e^{-(1 - 2\zeta\omega_{nt})t_1}}{8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2}$$

$$+ \frac{(1 - 2\zeta\omega_{nt})e^{-\zeta\omega_{nt}t_1}\sin(\omega_{nt}\sqrt{1-\zeta^2}\,t_1 - \varphi)}{\sqrt{(1-\zeta^2)}[8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2]} = 0$$

$$(3\text{-}30)$$

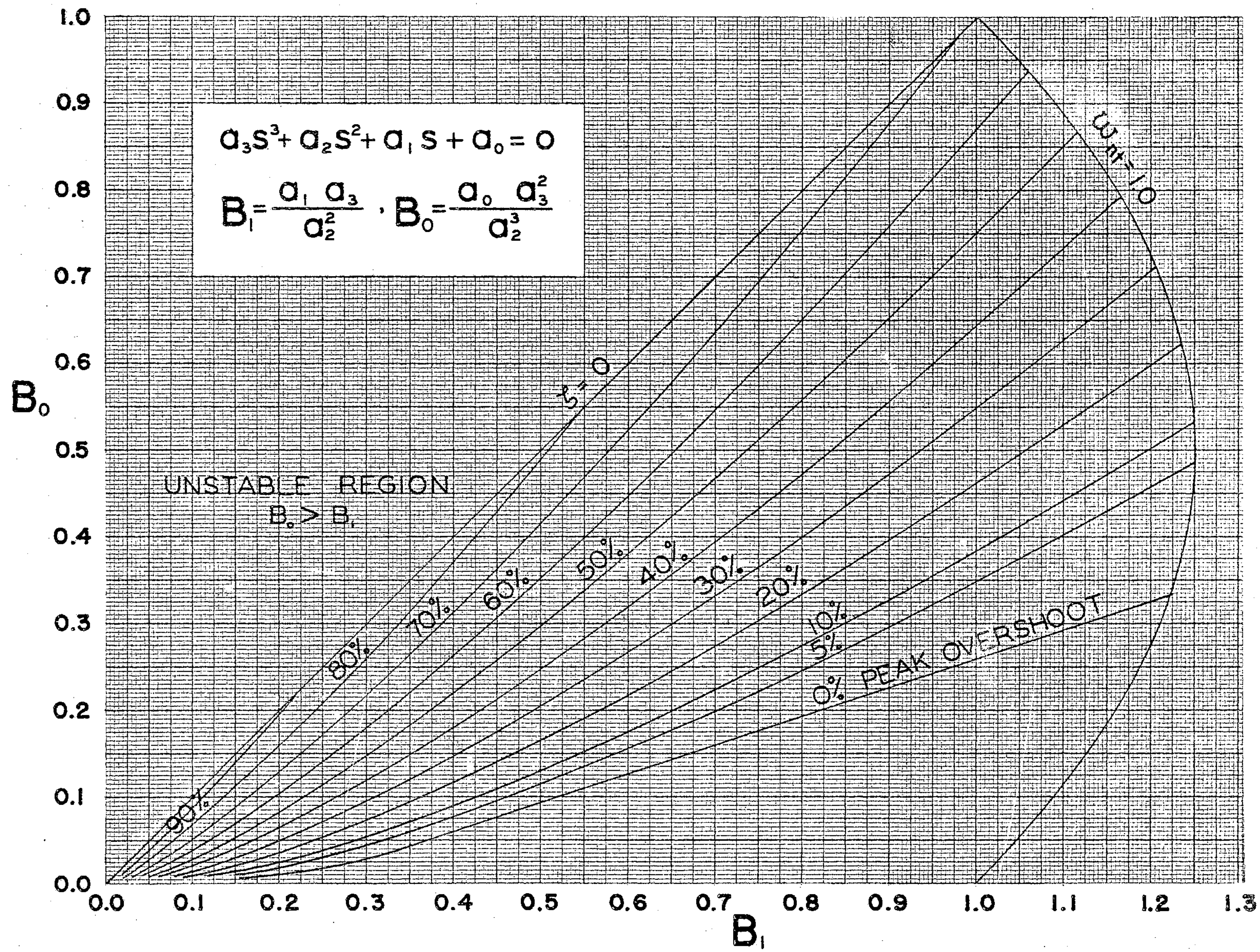


Figure 3-4. Lines of Constant Overshoot

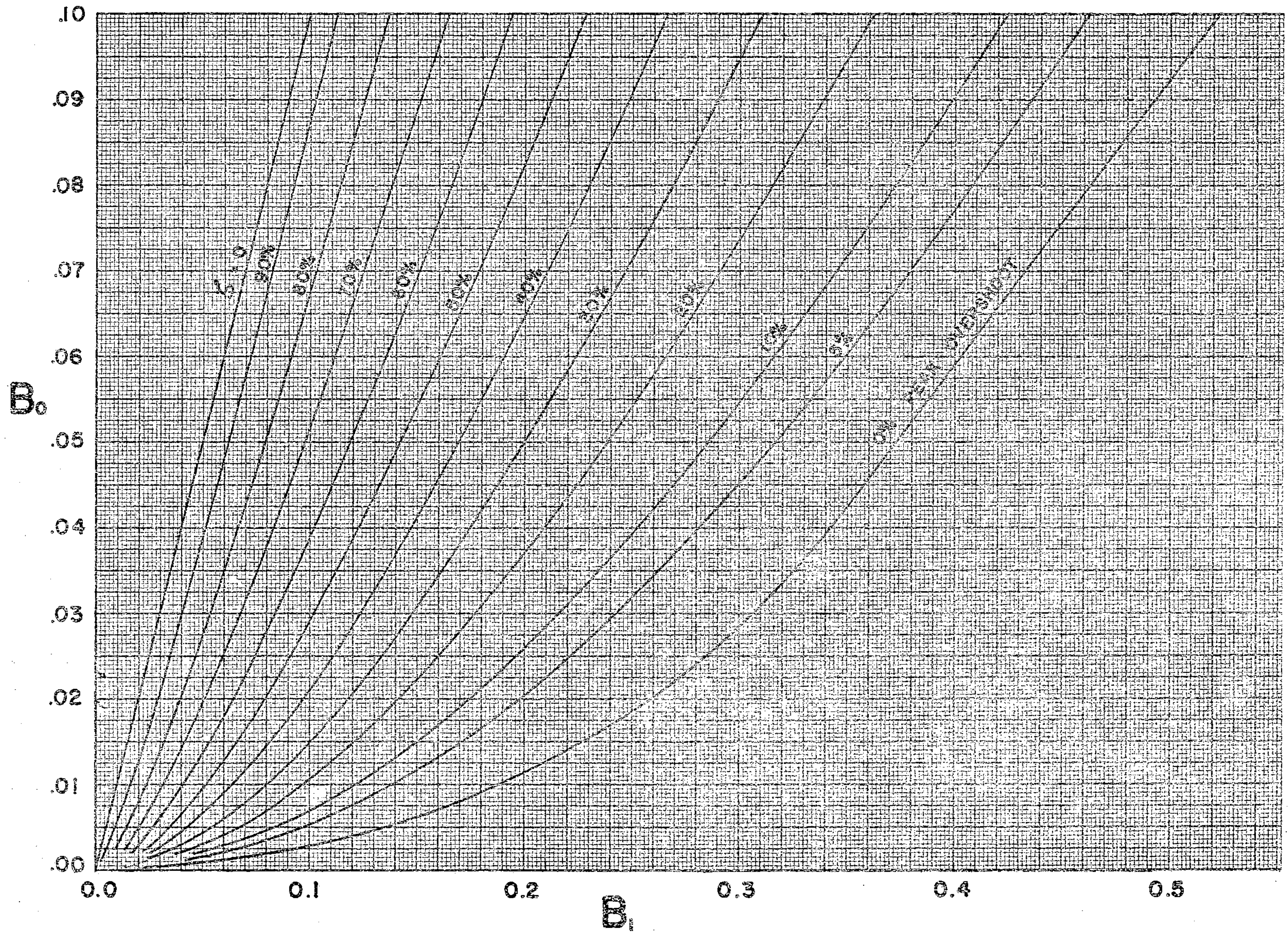


Figure 3-4. (Continued)

$$G(\zeta,\omega_{nt},t_2) = 0.1 - \frac{\omega_{nt}^2 e^{-(1-2\zeta\omega_{nt})t_2}}{8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2}$$

$$+ \frac{(1-2\zeta\omega_{nt})e^{-\zeta\omega_{nt}t_2} \sin(\omega_{nt}\sqrt{1-\zeta^2}\, t_2 - \varphi)}{\sqrt{(1-\zeta^2)}[8(\zeta\omega_{nt})^2 - 6\zeta\omega_{nt} + 1 + \omega_{nt}^2]} = 0$$

(3-31)

$$H(t_{rt},t_1,t_2) = t_2 - t_1 - t_{rt} = 0 \qquad (3\text{-}32)$$

while for real roots

$$F(\sigma_{t2},\sigma_{t3},t_1) = 0.9 - \frac{\sigma_{t2}\,\sigma_{t3}\, e^{-(1-\sigma_{t2}-\sigma_{t3})t_1}}{(2\sigma_{t2}+\sigma_{t3}-1)(2\sigma_{t3}+\sigma_{t2}-1)}$$

$$- \frac{(1-\sigma_{t2}-\sigma_{t3})\sigma_{t3}\, e^{-\sigma_{t2}t_1}}{(1-2\sigma_{t2}-\sigma_{t3})(\sigma_{t3}-\sigma_{t2})}$$

$$- \frac{(1-\sigma_{t2}-\sigma_{t3})\sigma_{t2}\, e^{-\sigma_{t3}t_1}}{(1-2\sigma_{t3}-\sigma_{t2})(\sigma_{t2}-\sigma_{t3})} = 0 \qquad (3\text{-}33)$$

$$G(\sigma_{t2},\sigma_{t3},t_2) = 0.1 - \frac{\sigma_{t2}\,\sigma_{t3}\, e^{-(1-\sigma_{t2}-\sigma_{t3})t_2}}{(2\sigma_{t2}+\sigma_{t3}-1)(2\sigma_{t3}+\sigma_{t2}-1)}$$

$$- \frac{(1-\sigma_{t2}-\sigma_{t3})\sigma_{t3}\, e^{-\sigma_{t2}t_2}}{(1-2\sigma_{t2}-\sigma_{t3})(\sigma_{t3}-\sigma_{t2})}$$

$$- \frac{(1-\sigma_{t2}-\sigma_{t3})\sigma_{t2}\, e^{-\sigma_{t3}t_2}}{(1-2\sigma_{t3}-\sigma_{t2})(\sigma_{t2}-\sigma_{t3})} = 0 \qquad (3\text{-}34)$$

$$H(t_{rt},t_1,t_2) = t_2 - t_1 - t_{rt} = 0. \qquad (3\text{-}35)$$

Numerical procedures similar to those employed in the two previous sections were used for this problem and $B_1 - B_0$ points were determined which defined a family of "lines of constant rise time."[16] They are shown in Figure 3-5.

Chart for Gain Margin and Phase Margin

Consider the system defined by the block diagram configuration in Figure 3-6. The open-loop transfer function is given by the quantity within the block

$$G' = \frac{a_0}{s(a_3 s^2 + a_2 s + a_1)} \qquad (3\text{-}36)$$

while the closed-loop transfer function is the same as that used in previous theory:

$$G(s) = \frac{G'(s)}{1 + G'(s)} \qquad (3\text{-}37a)$$

$$G(s) = \frac{a_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}. \qquad (3\text{-}37b)$$

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[16]The generalization of the Newton-Raphson method for the simultaneous solution of three equations and the associated Fortran program are listed in Appendix D.

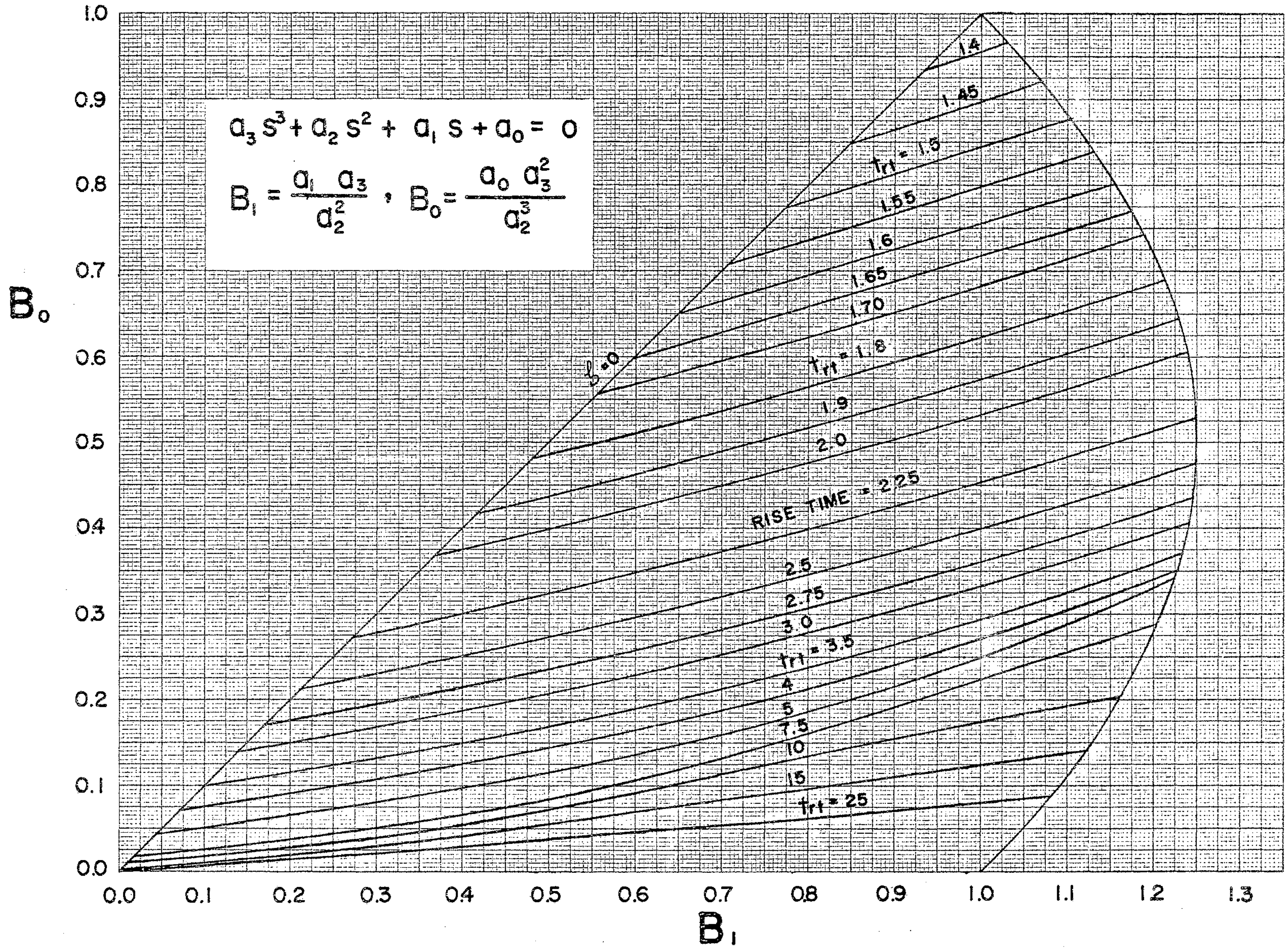

Figure 3-5. Lines of Constant Rise Time

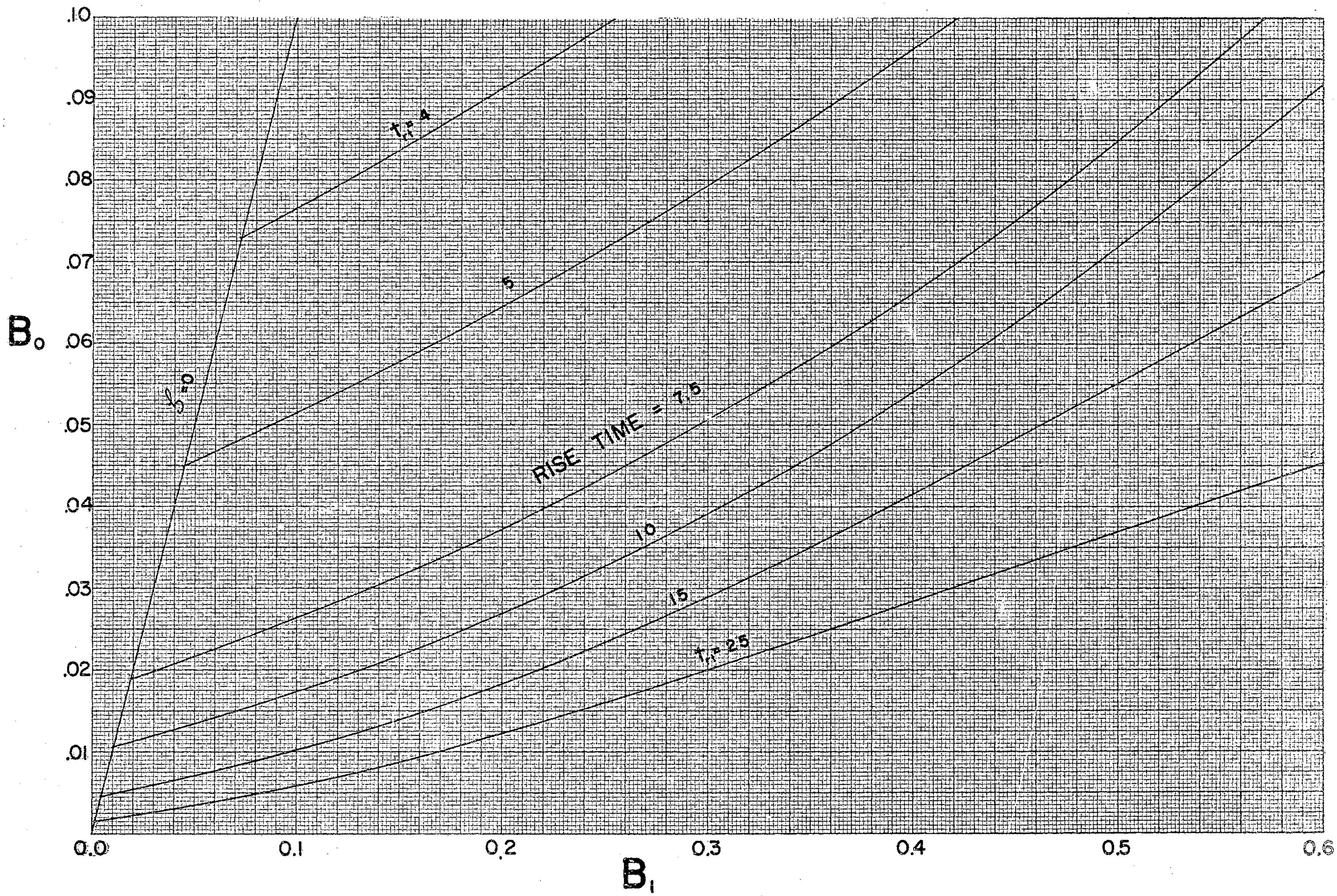

Figure 3-5. (Continued)

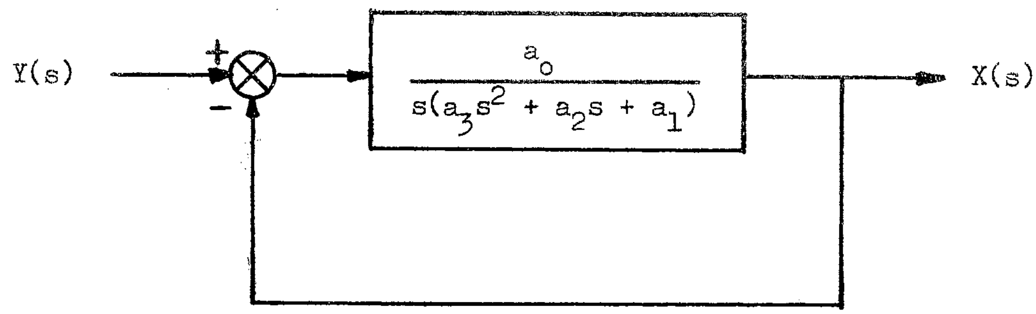

Figure 3-6. Block Diagram of a Third-Order System

Gain margin and phase margin are measures of the stability of such a system which result from a frequency response analysis using Nyquist's stability criterion. This criterion is applied in the following manner:

1. Substitute $j\omega$ for s in the open loop transfer function.
2. Plot the polar curve of $G'(j\omega)$ as ω varies from 0 to $+\infty$.
3. If the system has no poles in the right one-half s plane, the system is stable if the $G'(j\omega)$ contour does not encircle the -1 point (7).

Using this criterion as a basis, the stability criteria as defined by Thaler and Brown (25) are:

> Gain margin is the ratio of the gain at which the system becomes unstable to the actual system gain, assuming that the phase of all vectors remains unchanged ... .
>
> Phase margin is the amount of negative phase shift which must be introduced (without gain increase) to make the curve pass through the -1 + jo point. The vector which must be shifted is obviously one unit long, ..."

The block diagram for the system associated with the normalized equation is shown in Figure 3-7 with the system gain B_o. Using the stability criterion, $B_1 > B_o$, given earlier in the chapter, it is obvious that if the gain of this system, B_o, is a variable while B_1 is constant, the gain margin is given by the ratio $\frac{B_1}{B_o}$. Hence, "lines of constant gain margin" for the system are straight lines from the origin on a $B_1 - B_o$ chart. Several such lines are shown in Figure 3-8.

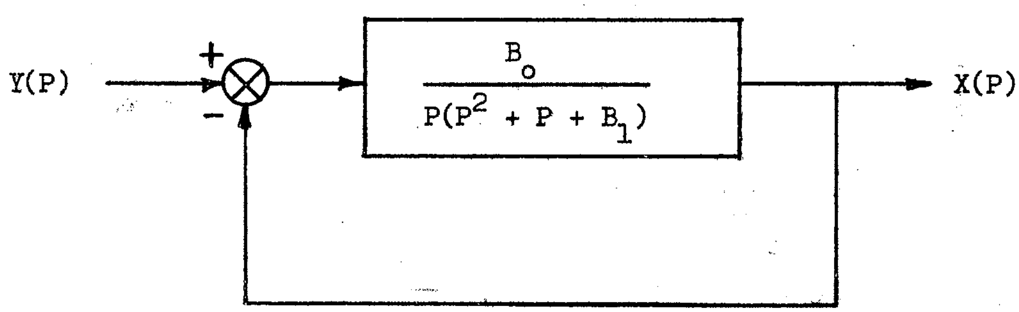

Figure 3-7. Block Diagram of a Normalized Third-Order System

Substitution of $j\omega$ into the open loop transfer function of the normalized system yields

$$G(j\omega) = \frac{B_o}{-\omega^2 + j(B_1\omega - \omega^3)}, \qquad (3\text{-}38)$$

the magnitude of which is

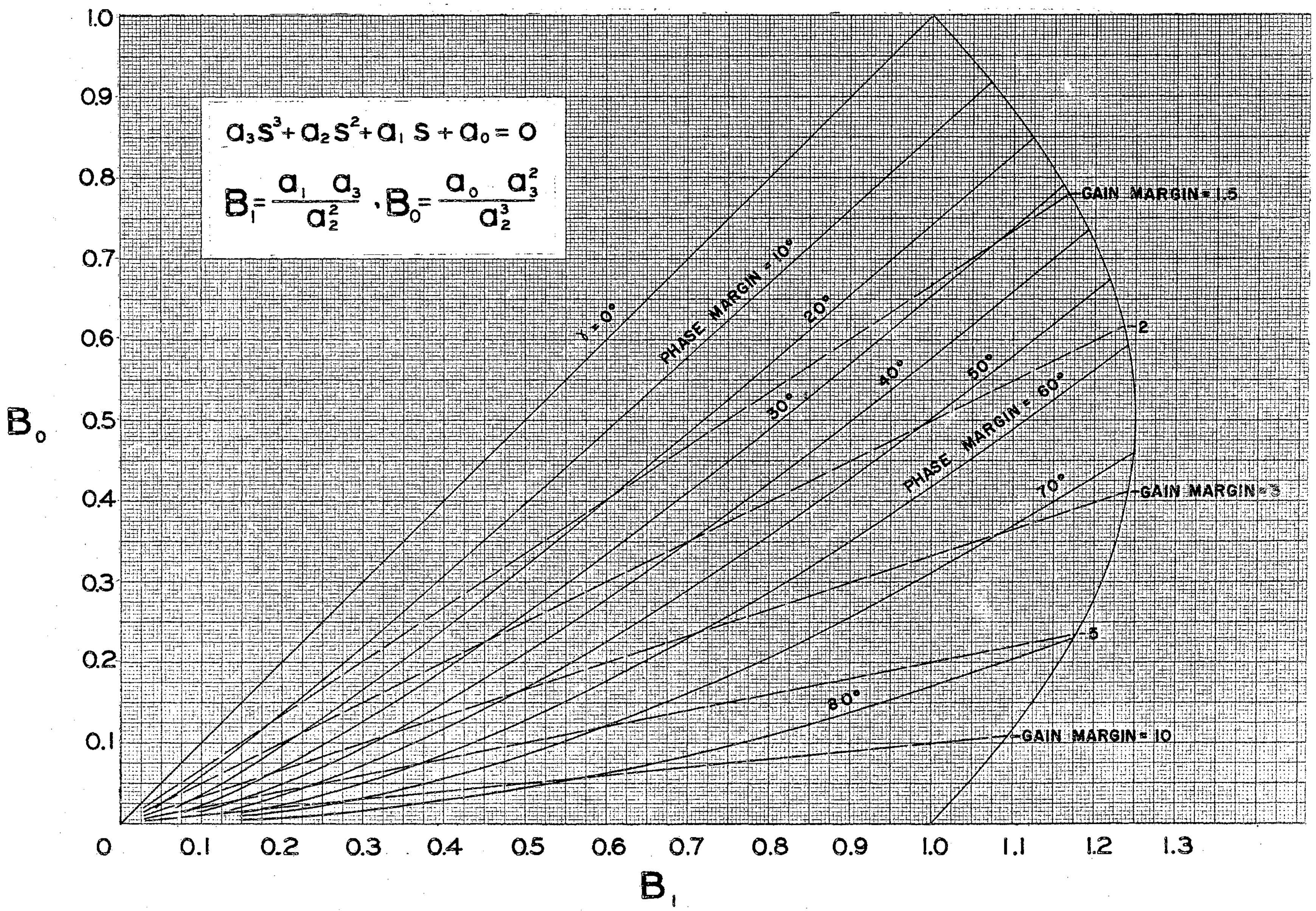

Figure 3-8. Lines of Constant Gain Margin and Constant Phase Margin

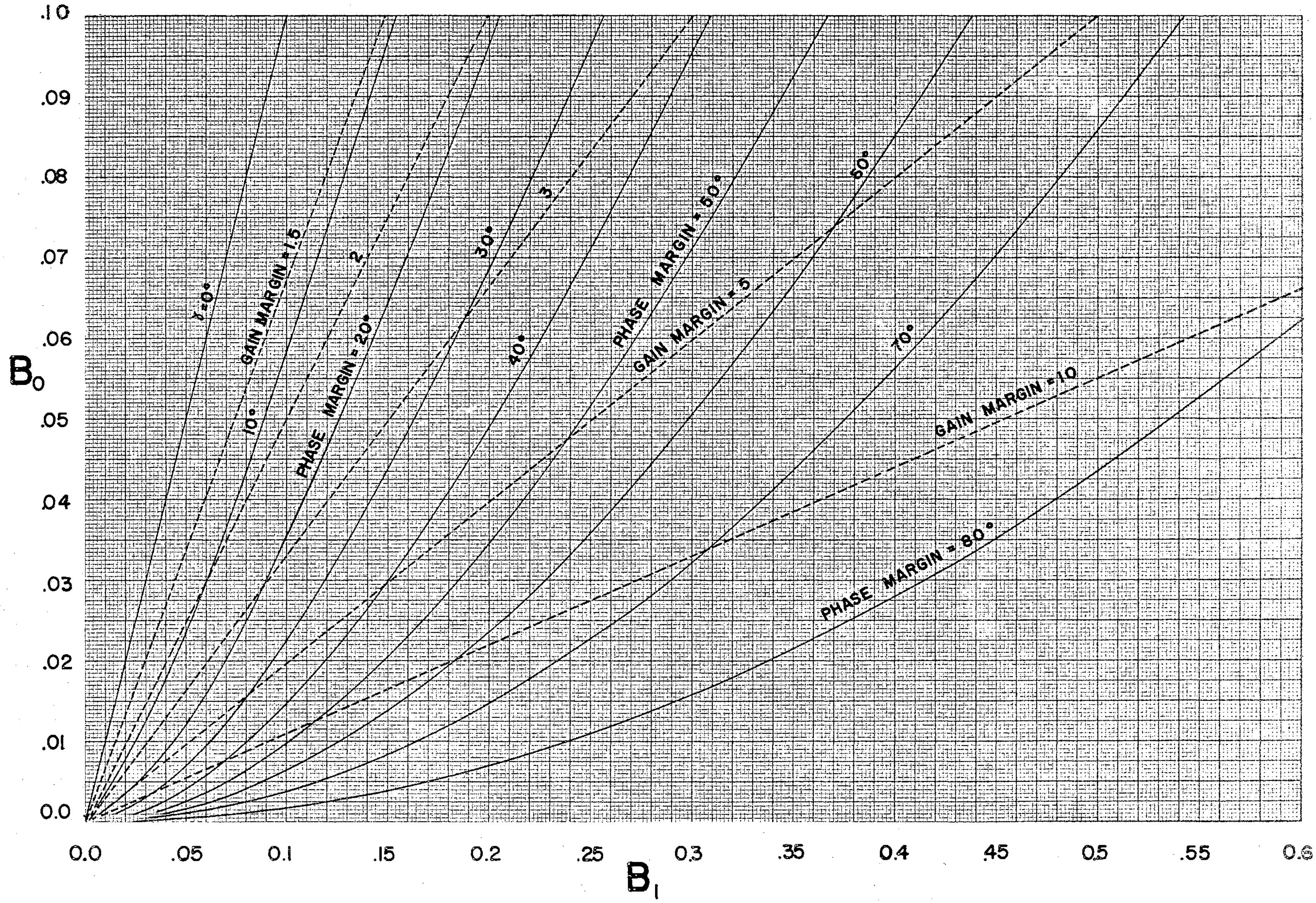

Figure 3-8. (Continued)

$$|G(j\omega)| = \frac{B_o}{\sqrt{\omega^4 + B_1^2\omega^2 - 2B_1\omega^4 + \omega_6}} \qquad (3\text{-}39)$$

and the phase angle

$$\angle G(j\omega) = -\tan^{-1}\frac{B_1 - \omega^2}{-\omega} \;. \qquad (3\text{-}40)$$

Since

$$\angle G(j\omega) = -\pi + \tan^{-1}\frac{B_1 - \omega^2}{\omega} \;, \qquad (3\text{-}41)$$

the phase margin, γ , is given by

$$\gamma = \tan^{-1}\frac{B_1 - \omega^2}{\omega} \qquad (3\text{-}42)$$

when the magnitude relationship for a unit vector is satisfied

$$B_o = \sqrt{\omega^4 + B_1^2\omega^2 - 2B_1\omega^4 + \omega^6} \qquad (3\text{-}43)$$

Thus, in order to solve the system parameters associated with a particular value of phase margin, B_1 was specified and Equation (3-42) solved for ω. Substituting the values of B_1 and ω into Equation (3-43) yielded B_o and as a result a point on the $B_1 - B_o$ chart for a particular value of phase margin was defined. By varying B_1 while holding γ constant, subsequent computations yielded data for a "line of constant phase margin."[17] A family of such lines is shown in Figure 3-8.

---

[17]The Fortran program is listed in Appendix E.

Since the transformation relating the two systems is one of the magnitude only ($s = \frac{a_2}{a_3} P$), then a polar plot in the s-plane will be geometrically similar to the one in the P-plane and, as a result, the gain margins and phase margins of the two systems are identical. Therefore, Figure 3-8 may be used to determine directly the gain margin and phase margins of systems described by the open loop transfer function in Equation (3-36).

Procedure for the Analysis of Systems With Known Coefficients

If the coefficients of the system transfer function

$$G(s) = \frac{k}{a_3 s^3 + a_2 s^2 + a_1 s + a_o} \qquad (3\text{-}1)$$

are known, then the method for the determination of the transient and frequency response characteristics consists of the following steps:

1. Calculate $B_1 = \frac{a_1 a_3}{a_2^2}$ and $B_o = \frac{a_o a_3^2}{a_2^3}$
2. Read t_{st} and t_{rt} from Figures 3-3 and 3-5, respectively.
3. Calculate $t_s = \left(\frac{a_3}{a_2}\right) t_{st}$, $t_r = \left(\frac{a_3}{a_2}\right) t_{rt}$
4. Read overshoot from Figure 3-4.

   } (steps 3–4) transient response characteristics.

5. Calculate $\frac{B_1}{B_o}$ (the gain margin).
6. Read γ (phase margin) from Figure 3-8.

   } (steps 5–6) frequency response stability criteria.

Procedure for the Synthesis of Systems Restricted by Design Specifications

If one or more parameters of the system are considered to be variable, then the following procedure is suggested for the synthesis of a system which will satisfy design specifications:

1. Determine B_1 and B_0 as functions of the unknown parameter(s).
2. Using the appropriate charts, select a point such that the design specifications are satisfied. Note the $B_1 - B_0$ coordinates of this selected point.
3. Solve the equation(s) developed in step 1 for the value(s) of the parameter(s).

CHAPTER IV

EXPERIMENTAL VERIFICATION OF THEORY

The theoretical analysis presented in Chapter IV is applicable to third-order systems in general. Thus, mathematically at least, the choice of a system to be used for the experimental phase of this investigation was restricted only to the order involved and not to any particular type of physical system. However, in order to verify the theoretical results over a wide range of $B_1 - B_0$ values, practical limitations on the choice of a system existed. The selection of a mechanical or hydraulic control system, while being an intriquing possibility, was not considered feasible because of the great amount of difficulty and expense in changing system parameters such as piston area, valve gain, and damping. In addition, the measuring and recording of the input and output variables of such systems would present problems.

The system which was selected was a third-order electrical network which, when designed with variable elements, provides the simplest and most accurate means of parameter variation along with an easily recorded output variable, voltage. The network was established using an electronic differential analyzer, commonly referred to as an "analog computer." In this case, however, the use of the word "analog" is not applicable, since the third-order electrical network is one of the class of systems for which the theory of this dissertation is applicable. Therefore, the response records which were obtained from the analog computer (see Figures 4-2 through 4-9) are the experimental data of this investigation.

Experimental System and Procedure

The third-order electrical network which was set-up on the analog computer (EAI Model TR-10) is shown in block diagram form in Figure 4-1. The transfer function of the system is

$$G(P) = \frac{B_o}{P^3 + P^2 + B_1 P + B_o} \qquad (4\text{-}1)$$

where B_1 and B_o are determined by the wiper position of two separate potentiometers.

The experimental procedure consisted of the following steps:

1. Several $B_1 - B_o$ points associated with the particular response characteristics to be investigated were selected from the digital computer data.
2. The $B_1 - B_o$ potentiometers were set using the coordinate values of one point.

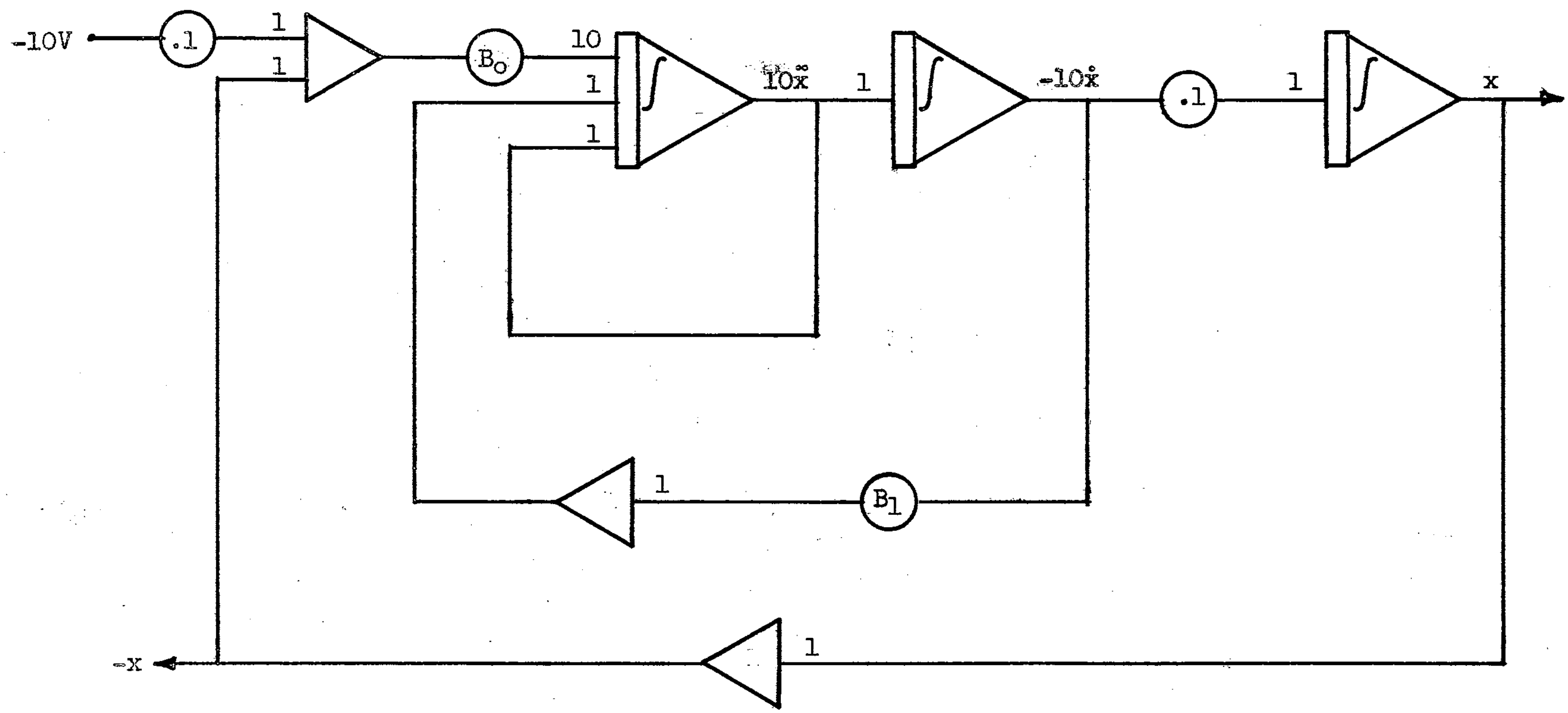

Figure 4-1. Circuit Diagram for Third-Order Electrical System

3. The system was subjected to a step input in voltage.
4. The output voltage of the system was recorded with a T-Y plotter (EAI Model 1110).
5. Steps 2-4 were repeated for the other $B_1 - B_0$ points selected in Step 1.

A large number of tests were run in each phase of the investigation of response characteristics and in every instance the experimental response checked exactly (within the accuracy of the analog computer and recorder).[1] Some of the results are shown in the following sections.

Constant Settling Time Responses

A "line of constant settling time" falls in the area of no overshoot on the $B_1 - B_0$ chart (see Figure 3-3) as well as the area where overshoot occurs. This is accounted for by the fact that in the first case the real root is dominant while in the second case the complex roots are dominant and the settling time is largely determined by the real part of the complex roots. Thus, the $B_1 - B_0$ points which were selected for testing are representative of both root patterns as well as a wide range of system natural frequencies.

---

[1]It was not possible to test directly for phase margin, however several textbook problems were checked using Figure 3-8 and the results agreed with the text answer in each case.

The results of the tests for settling times of 20 and 40 are shown in Figures 4-2 and 4-3, respectively.[2] Examination of these traces reveals that in all cases the system response settled to within five per cent of the final value in the amount of time predicted. The reason that the traces do not cross the ± 5% lines exactly at the predicted time is due to the fact that the equation of the response envelope was used in the analysis and not the exact response equation. However, the amount of error involved is small and is considered to be acceptable for system design.

Constant Overshoot Responses

The second phase of the testing was concerned with verifying the accuracy of the "lines of constant overshoot." Each curve in Figure 3-4 represents an infinite number of system parameter combinations, all having the same peak overshoot value in response to a step input.

Several points were selected from the data derived by the digital computer for 5 per cent and 40 per cent peak overshoot conditions and tested according to the previously outlined procedure. The results of the tests are shown in Figures 4-4 and 4-5 and, as mentioned previously, checked with the values predicted by the theory of Chapter III. Note the wide range in natural frequency.

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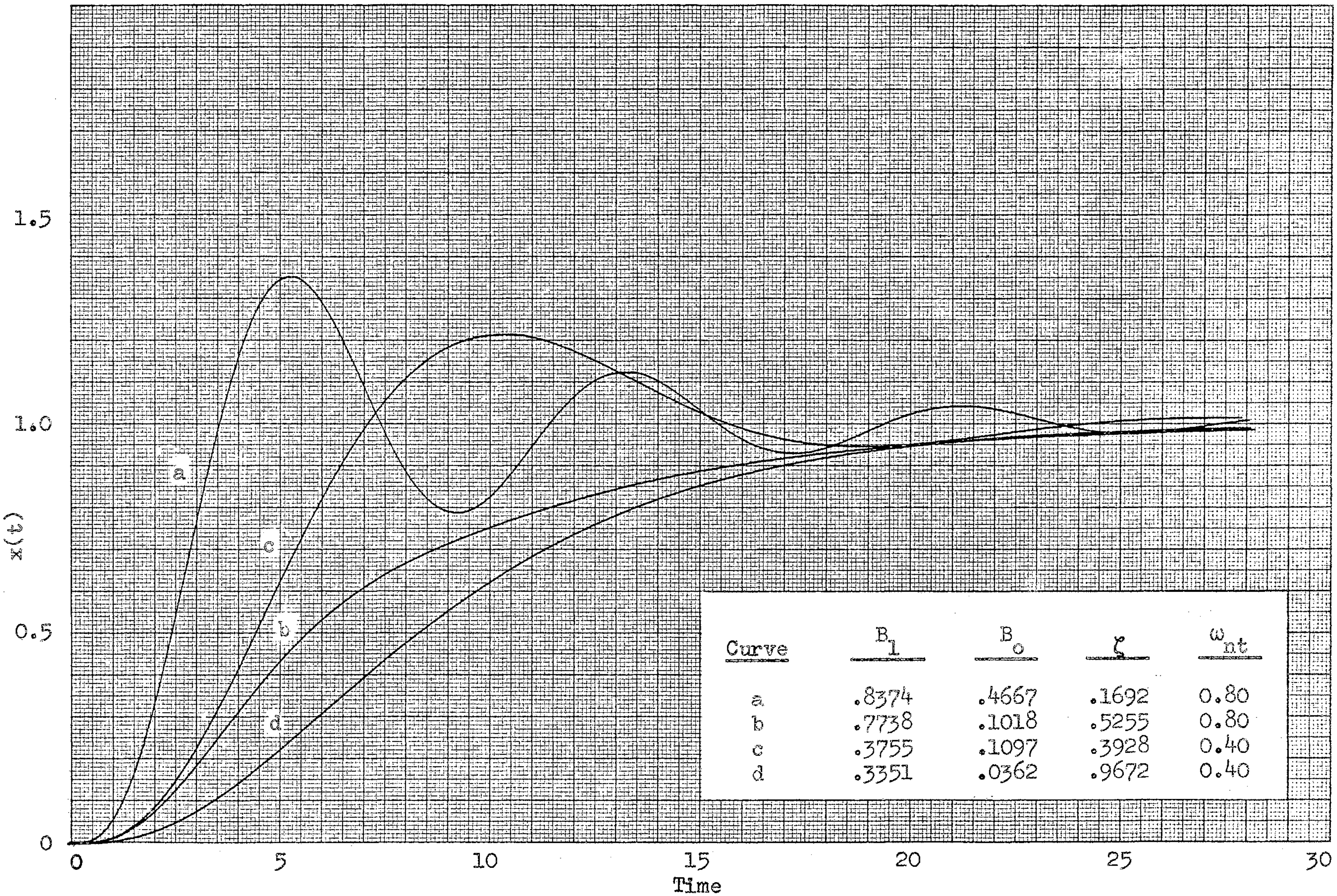
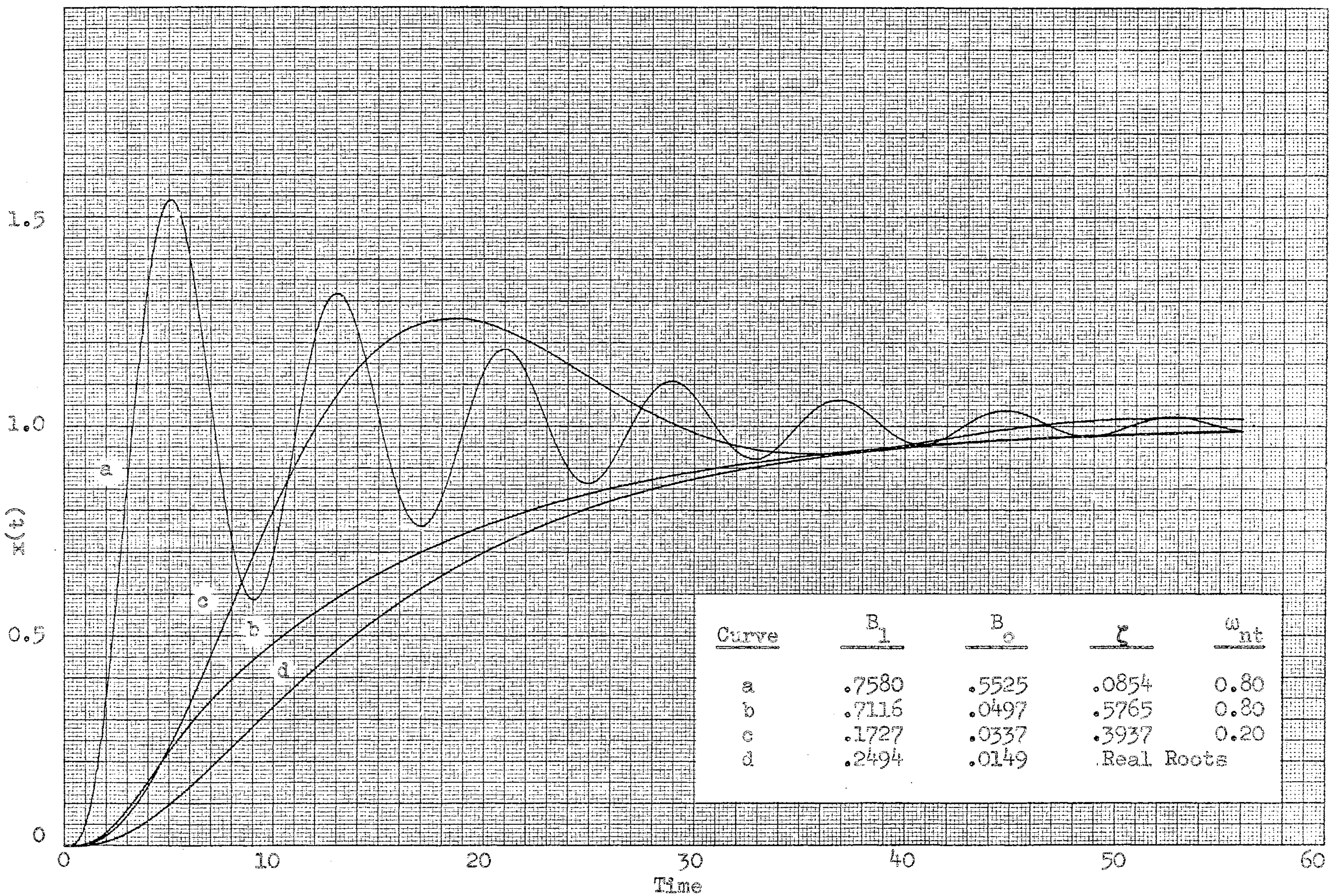
[2]The system which was analyzed had been normalized; therefore, the response time is dimensionless.



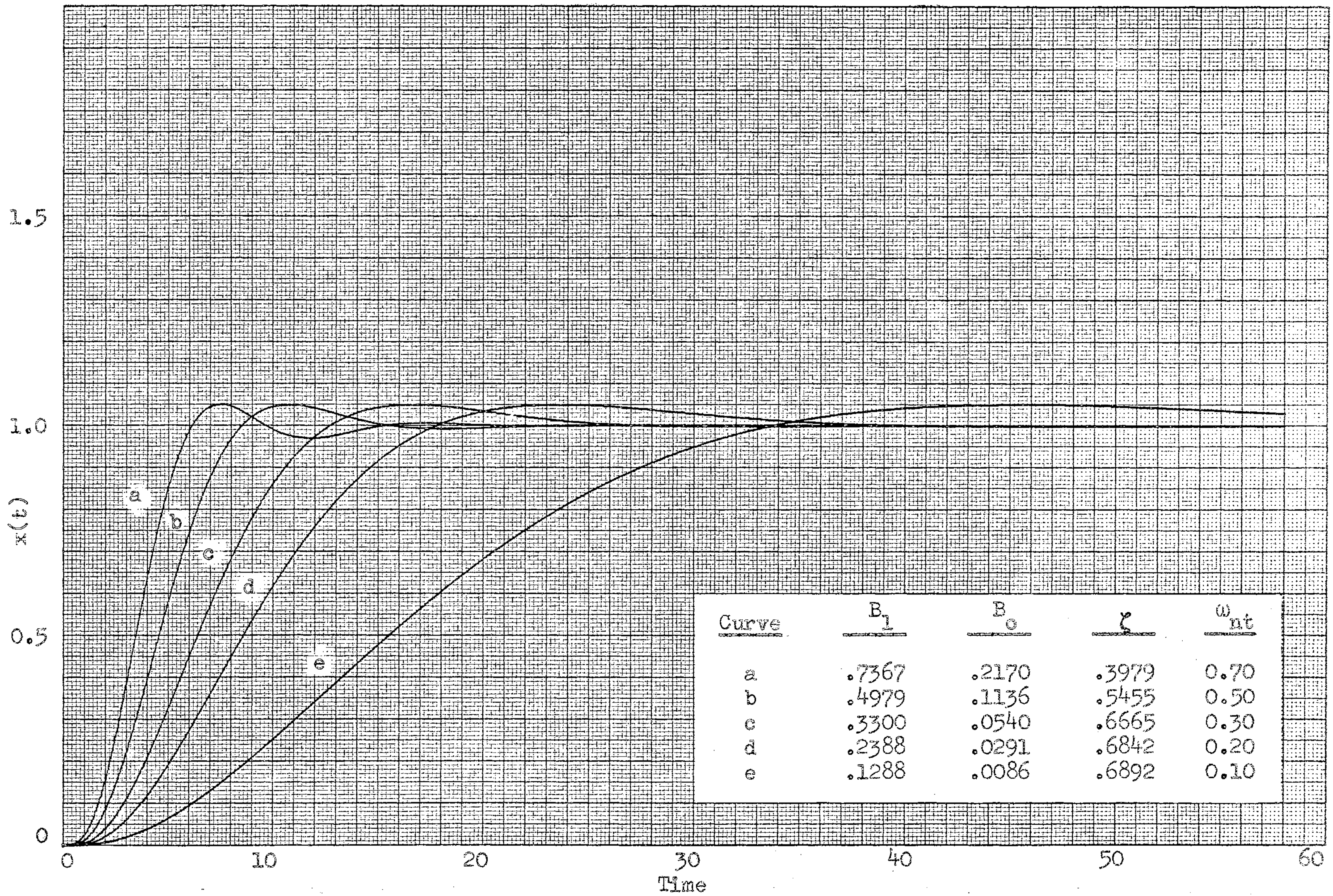
Curve	B_1	B_0	ζ	ω_{nt}
a	.8374	.4667	.1692	0.80
b	.7738	.1018	.5255	0.80
c	.3755	.1097	.3928	0.40
d	.3351	.0362	.9672	0.40

Figure 4-2. Transient Responses With $t_{st} = 20$



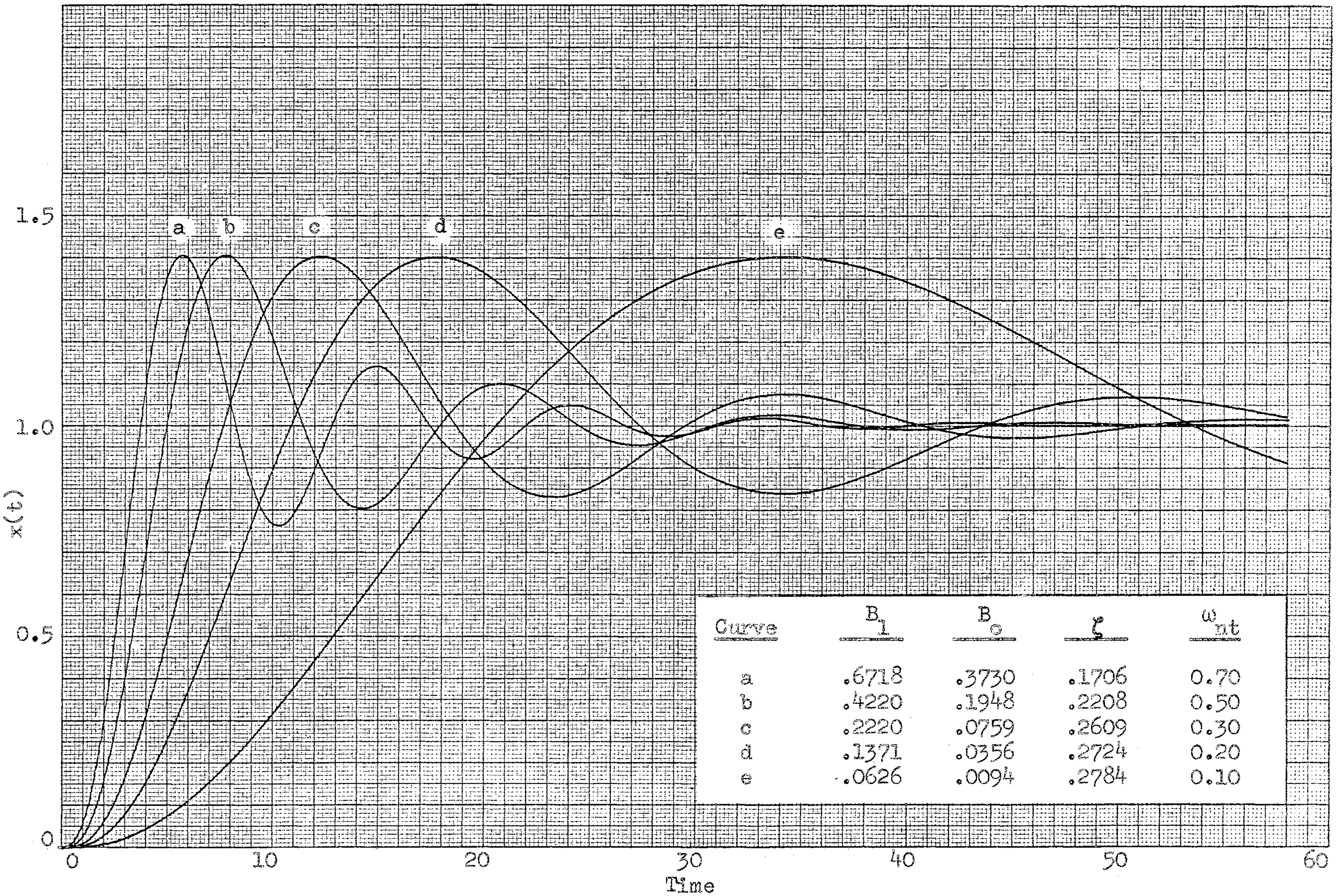
Curve	B_1	B_o	ζ	ω_{nt}
a	.7580	.5525	.0854	0.80
b	.7116	.0497	.5765	0.80
c	.1727	.0337	.3937	0.20
d	.2494	.0149	Real Roots	

Figure 4-3. Transient Responses With $t_{st} = 40$



Curve	B_1	B_0	ζ	$\omega_n t$
a	.7367	.2170	.3979	0.70
b	.4979	.1136	.5455	0.50
c	.3300	.0540	.6665	0.30
d	.2388	.0291	.6842	0.20
e	.1288	.0086	.6892	0.10

Figure 4-4. Transient Responses With 5% Overshoot



Curve	B_1	B_o	ζ	ω_{nt}
a	.6718	.3730	.1706	0.70
b	.4220	.1948	.2208	0.50
c	.2220	.0759	.2609	0.30
d	.1371	.0356	.2724	0.20
e	.0626	.0094	.2784	0.10

Figure 4-5. Transient Responses With 40% Overshoot

Constant Rise Time Responses

In Chapter III the three conditions which are necessary for a system to have a specified rise time were defined. They are:

$$x(t_1) = 0.1 \qquad (3\text{-}27)$$

$$x(t_2) = 0.9 \qquad (3\text{-}28)$$

$$t_{rt} = t_2 - t_1 . \qquad (3\text{-}29)$$

Thus, the $B_1 - B_0$ points which fall on a particular line in Figure 3-5 represent systems that, although they have different system parameter combinations, will respond to step inputs in such a way that the elapsed time from 10 per cent to 90 per cent of the final value will be the same. In other words, the transient responses may differ greatly as far as overshoot (or no overshoot) and settling time are concerned, but in all cases $t_2 - t_1$ is a constant value.

The values of rise time chosen for the tests were 2, 5, and 15 and the results are shown in Figures 4-6, 4-7, and 4-8, respectively. The values for t_1 and t_2 determined by the numerical program are shown on the charts with the appropriate curve notation. The experimental values are in agreement with the predicted values.

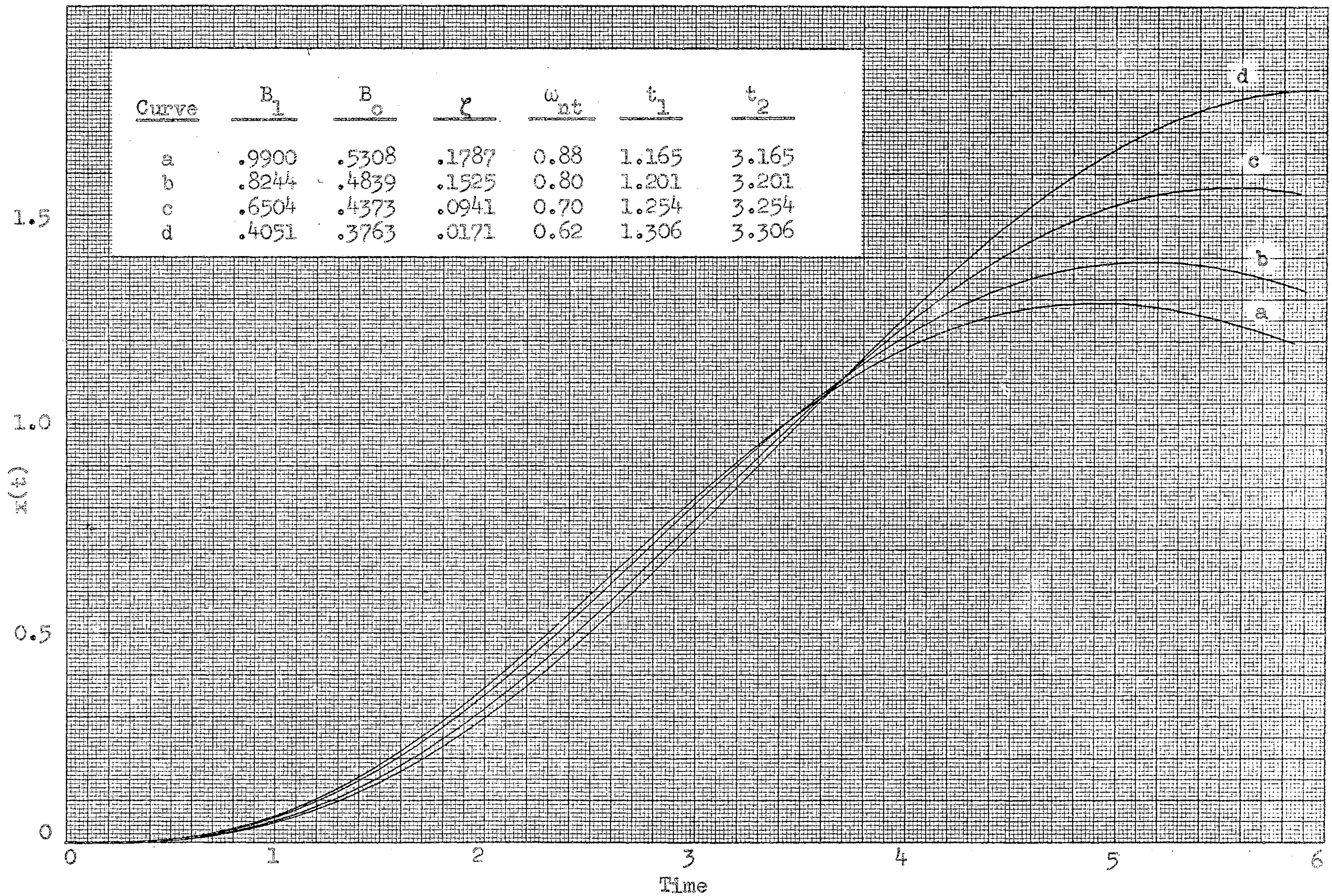


Figure 4-6. Transient Responses With $t_{rt} = 2$

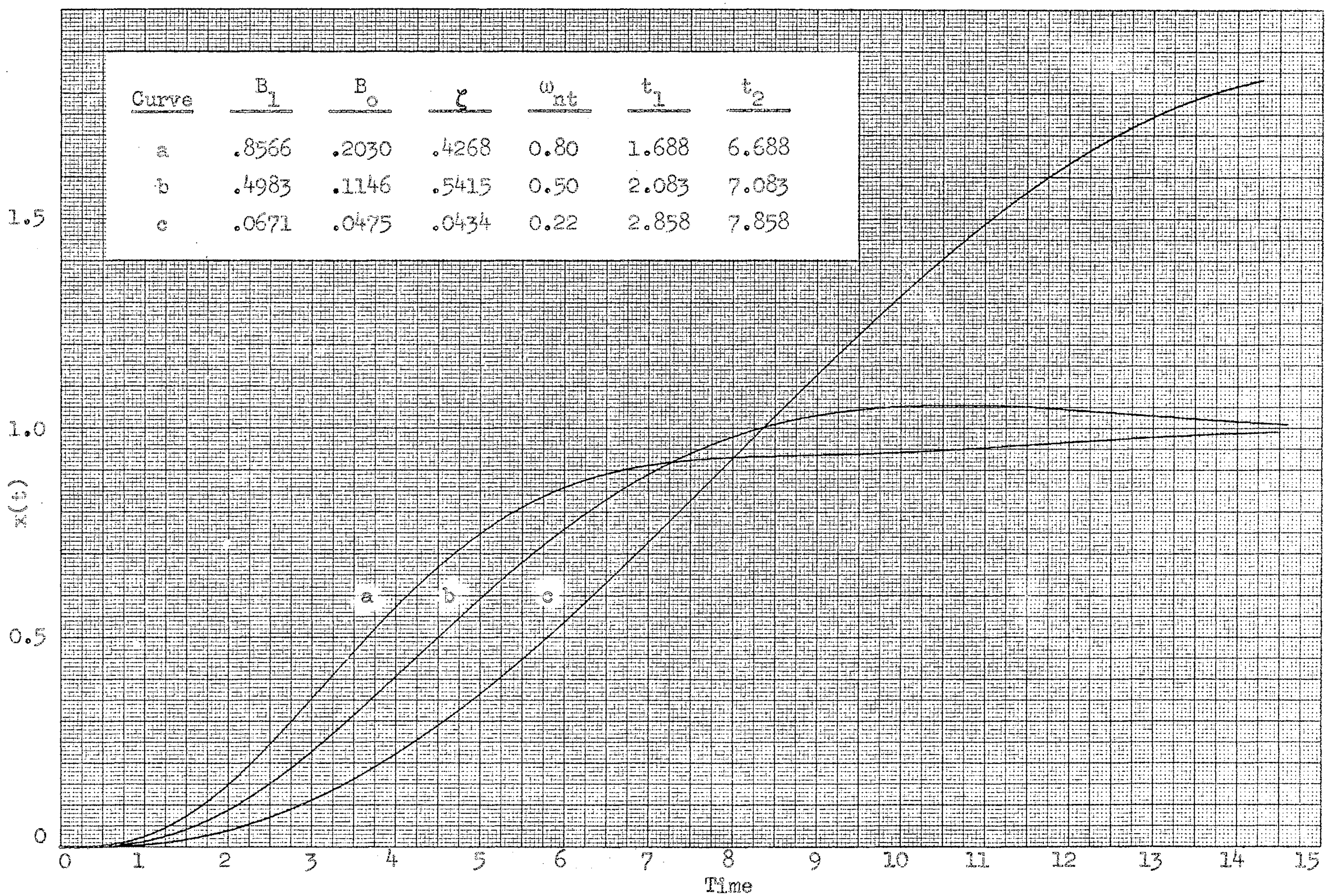
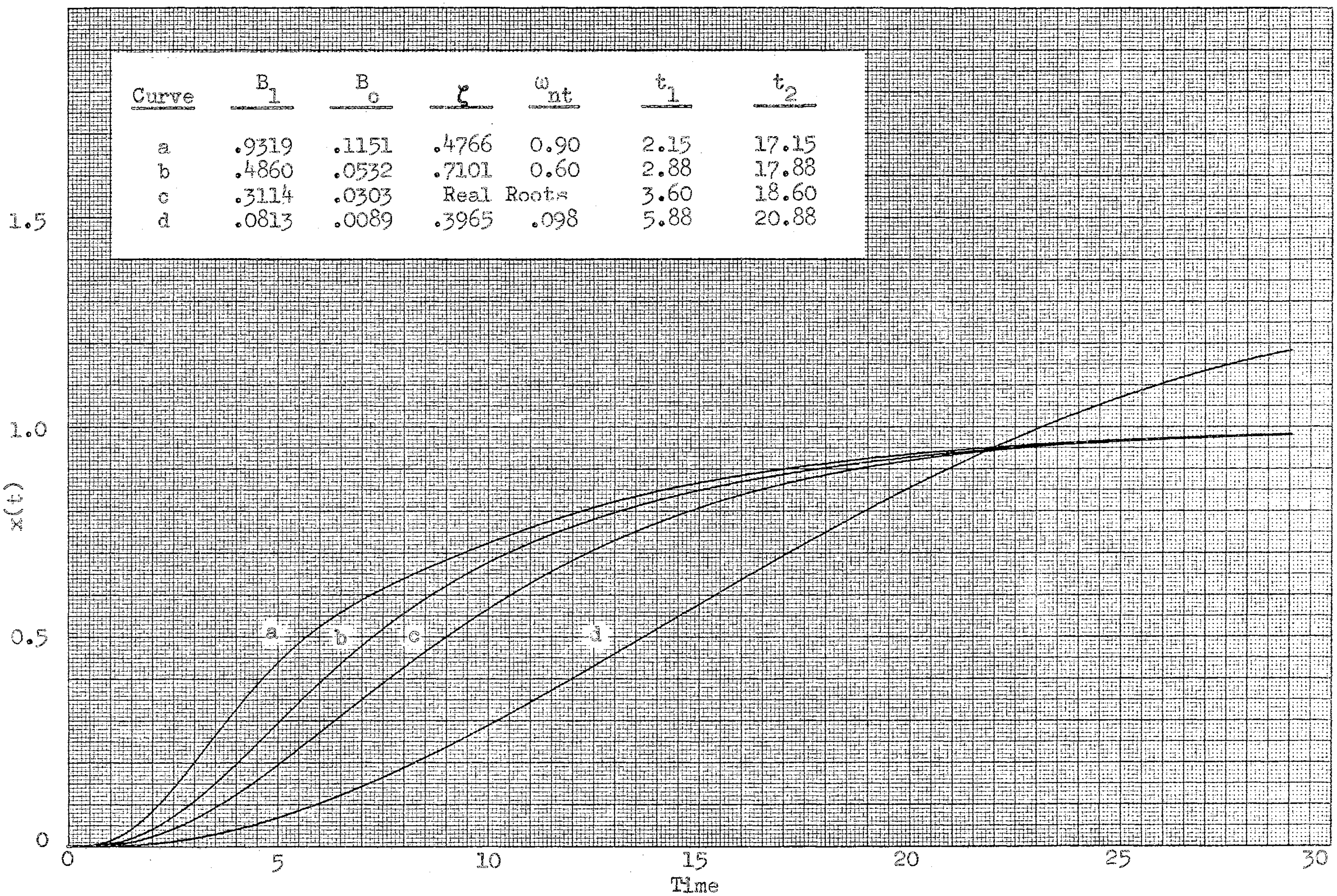
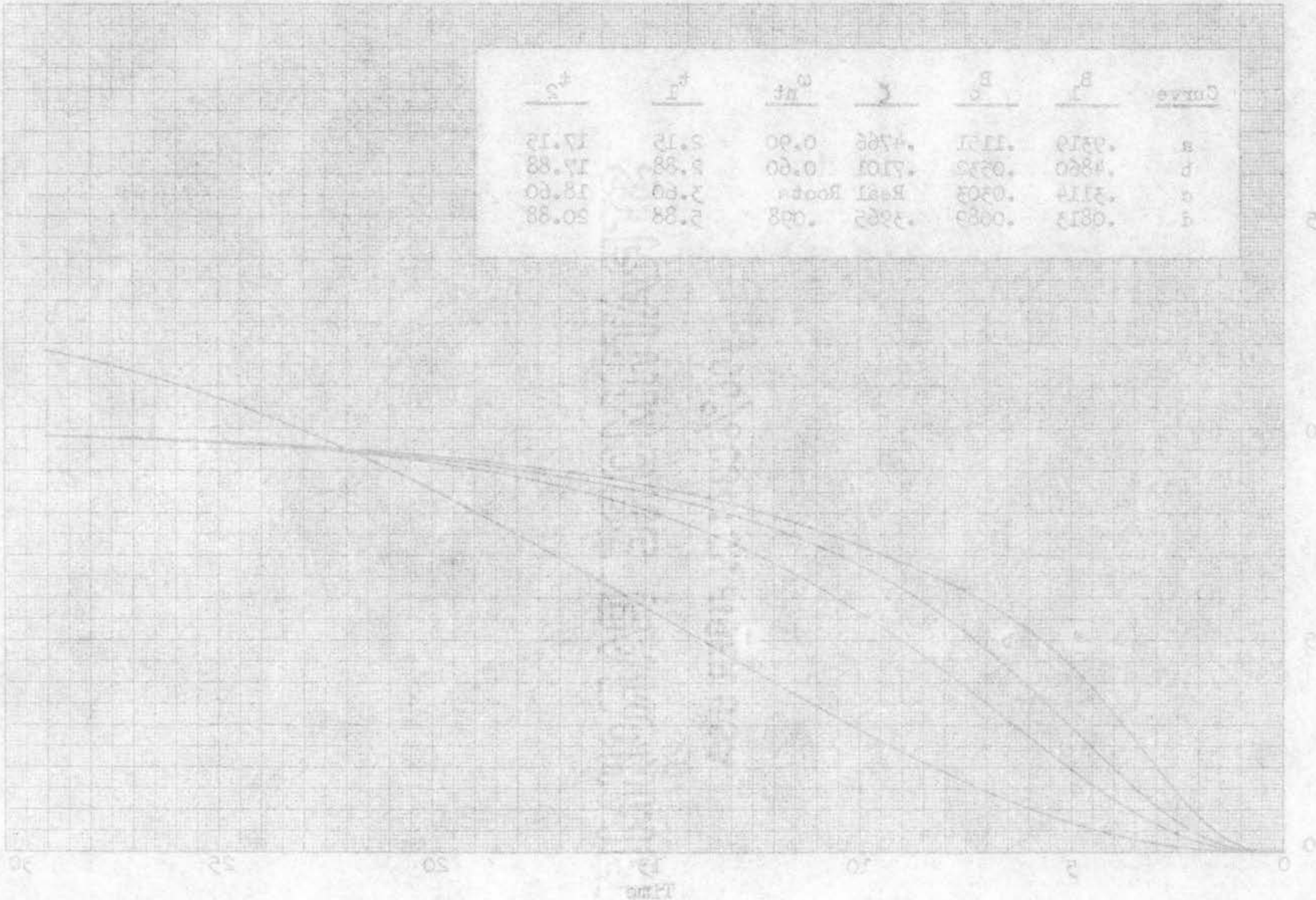


Figure 4-7. Transient Responses With $t_{rt} = 5$



Curve	B_1	B_o	ζ	ω_{nt}	t_1	t_2
a	.9319	.1151	.4766	0.90	2.15	17.15
b	.4860	.0532	.7101	0.60	2.88	17.88
c	.3114	.0303	Real Roots		3.60	18.60
d	.0813	.0089	.3965	.098	5.88	20.88

Figure 4-8. Transient Responses With t_{rt} = 15



Curve	B_1	B_c	ζ	ω_{nt}	t_1	t_2
a	.9519	.1151	.4766	0.90	2.15	17.15
b	.4860	.0352	.7101	0.60	2.88	17.88
c	.3114	.0303	Real Roots		3.60	18.60
d	.0813	.0089	.3965	.098	5.88	20.88

Figure 4-8. Transient Responses With t_{rt} = 15

Effect of Gain Variation

If the gain of a third-order system, a_o, is increased while the other parameters of the system are held constant, then $B_1\left(\frac{a_1 a_3}{a_2^2}\right)$ will remain constant while $B_O\left(\frac{a_O a_3^2}{a_2^3}\right)$ increases. The trajectory on a B_1 - B_O chart associated with a gain variation is, thus, a line parallel to the B_O axis. Therefore, in order to investigate the effect of gain variation, B_1 must be held constant while B_O is varied.

For this series of tests, the B_1 potentiometer was set at 0.50 and the setting of the B_o potentiometer varied in steps from 0.05 to 0.55. The responses for these tests are shown in Figure 4-9. At the value of gain, B_O = 0.05, the system responded with no overshoot (a characteristic in agreement with Figure 3-5). However, as the gain was increased, peak overshoots of increasing magnitudes occurred until, with B_o = 0.50, a steady-state sinusoidal response occurred. When the B_o potentiometer was set at 0.55, the system response to a step input was of an unstable character (the response envelope was growing exponentially). Thus, the line $B_1 = B_o$ separates the regions of stability and unstability as predicted theoretically. In addition, the gain margin of the system associated with the initial point (B_1 = 0.50, B_o = 0.05) was $\frac{0.50}{0.05}$ or 10 which also agrees with the results of the previous chapter.

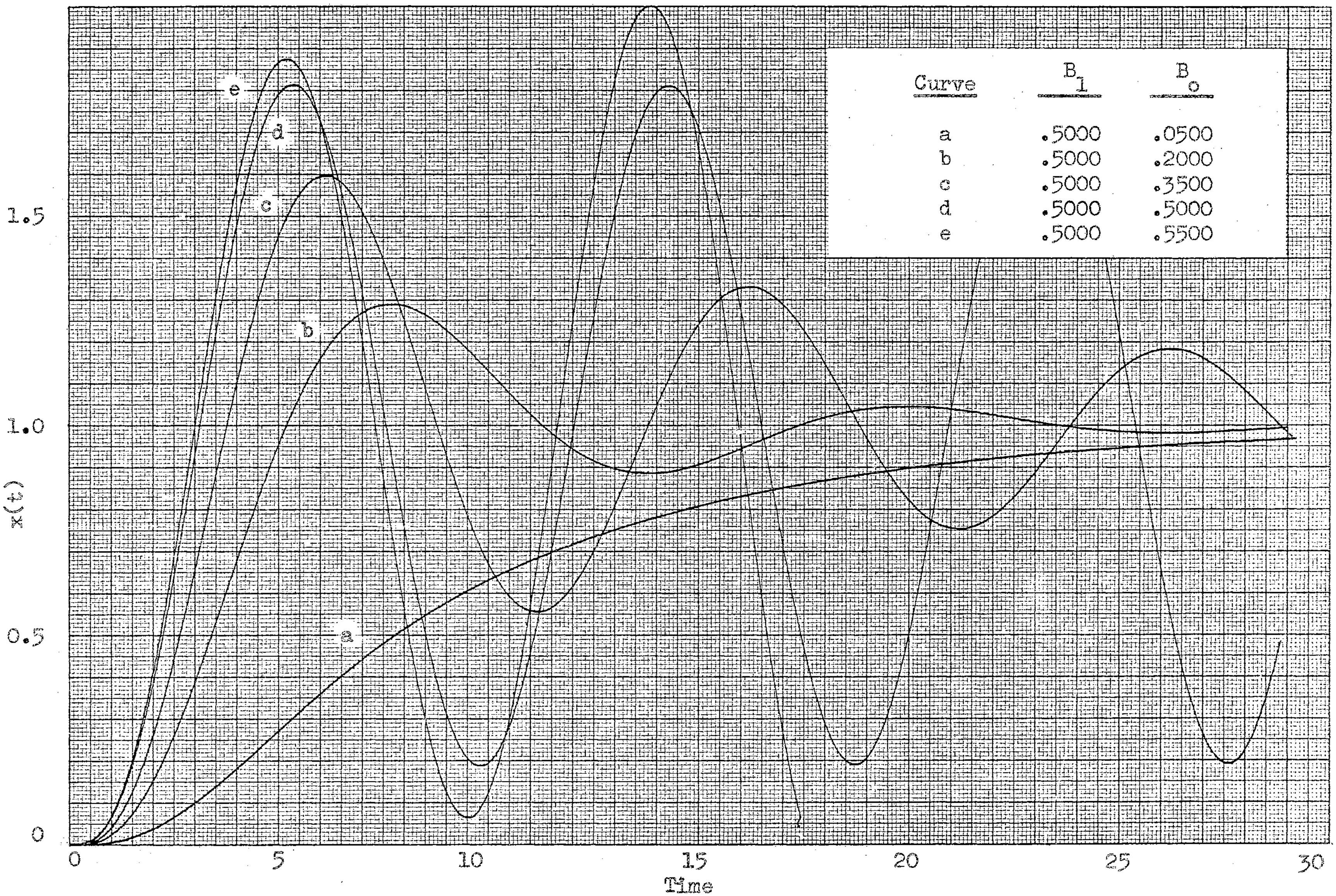


Figure 4-9. Responses Illustrating the Effect of Gain Variation

Summary of Tests Results

A wide range of system parameter values were used in the tests reported in this chapter to determine the accuracy of the response characteristic curves in Chapter III. Since the experimental results in all tests agreed with the predicted values (within the accuracy of the computer and recorder), it may be concluded that the theoretical analysis was correct and, therefore, the charts may be used with confidence for the analysis and synthesis of linear third-order systems.

CHAPTER V

APPLICATIONS OF THE ANALYSIS AND SYNTHESIS PROCEDURES

The procedures outlined in the last two sections of Chapter III specify the steps which are to be followed in applying the response characteristic charts for the analysis and synthesis of a linear third-order system. Several applications are made in this chapter of the utility of these procedures and for convenience the steps involved are repeated here.

Procedure for the Analysis of Systems With Known Coefficients

If the coefficients of the system transfer function

$$G(s) = \frac{k}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \qquad (3\text{-}1)$$

are known, then the method for the determination of the transient and frequency response characteristics consists of the following steps:

1. Calculate $B_1 = \frac{a_1 a_3}{a_2^2}$ and $B_0 = \frac{a_0 a_3^2}{a_2^3}$.
2. Read t_{st} and t_{rt} from Figures 3-3 and 3-5, respectively.

3. Calculate $t_s = \left(\frac{a_1}{a_2}\right) t_{st}$ } transient response characteristics.

   $t_r = \left(\frac{a_1}{a_2}\right) t_{rt}$

4. Read overshoot from Figure 3-4.

5. Calculate $\frac{B_1}{B_0}$ (the gain margin). } frequency response stability criteria.

6. Read Υ (phase margin) from Figure 3-8.

Procedure for the Synthesis of Systems
Restricted by Design Specifications

If one or more parameters of the system are considered to be variable, then the following procedure is suggested for the synthesis of a system which will satisfy design specifications:

1. Determine B_1 and B_0 as functions of the unknown parameter(s).
2. Using the appropriate charts, select a point such that the design specifications are satisfied. Note the $B_1 - B_0$ coordinates of this selected point.
3. Solve the equation(s) developed in step 1 for the value(s) of the parameter(s).

System With Fixed Parameters

Assume that the parameters of a particular servo system have been specified and that the closed loop transfer function is

$$G(s) = \frac{8}{s^3 + 4s^2 + 4s + 8} \,.$$

Problem:

What are the transient and frequency response characteristics of the system?

Solution:

Using the first procedure outline above:

1. $B_1 = \frac{4}{(4)^2} = \frac{1}{4}$ $\qquad B_0 = \frac{8}{(4)^3} = \frac{1}{8}$.
2. $t_{st} = 49$ from Figure 3-3.
   $t_{rt} = 3.45$ from Figure 3-5.
3. $t_s = \left(\frac{1}{4}\right) 49 = 12\frac{1}{4}$ units of time
   $t_r = \left(\frac{1}{4}\right) 3.44 = 0.86$ units of time.
4. $os = 54\%$ from Figure 3-4.
5. Gain margin $= \frac{1}{4} \Big/ \frac{1}{8} = 2$.
6. Phase margin, $\gamma = 21°$ from Figure 3-8.

The system response using $B_1 = \frac{1}{4}$ and $B_0 = \frac{1}{8}$ is shown in Figure 5-1.

Supplementary Information: If the analyst needs to know ζ, ω_n and the roots of the system, the charts in Appendix A may be used.

1. $\zeta = .163$ and $\omega_{nt} = .377$ from Appendix A-I.

   $\omega_n = \left(\frac{a_2}{a_3}\right) \omega_{nt} = 4(.377) = 1.508$.
2. $\sigma_t = .876$, $\zeta\omega_{nt} = .062$, $\omega_{dt} = .374$ from Appendix A-II.

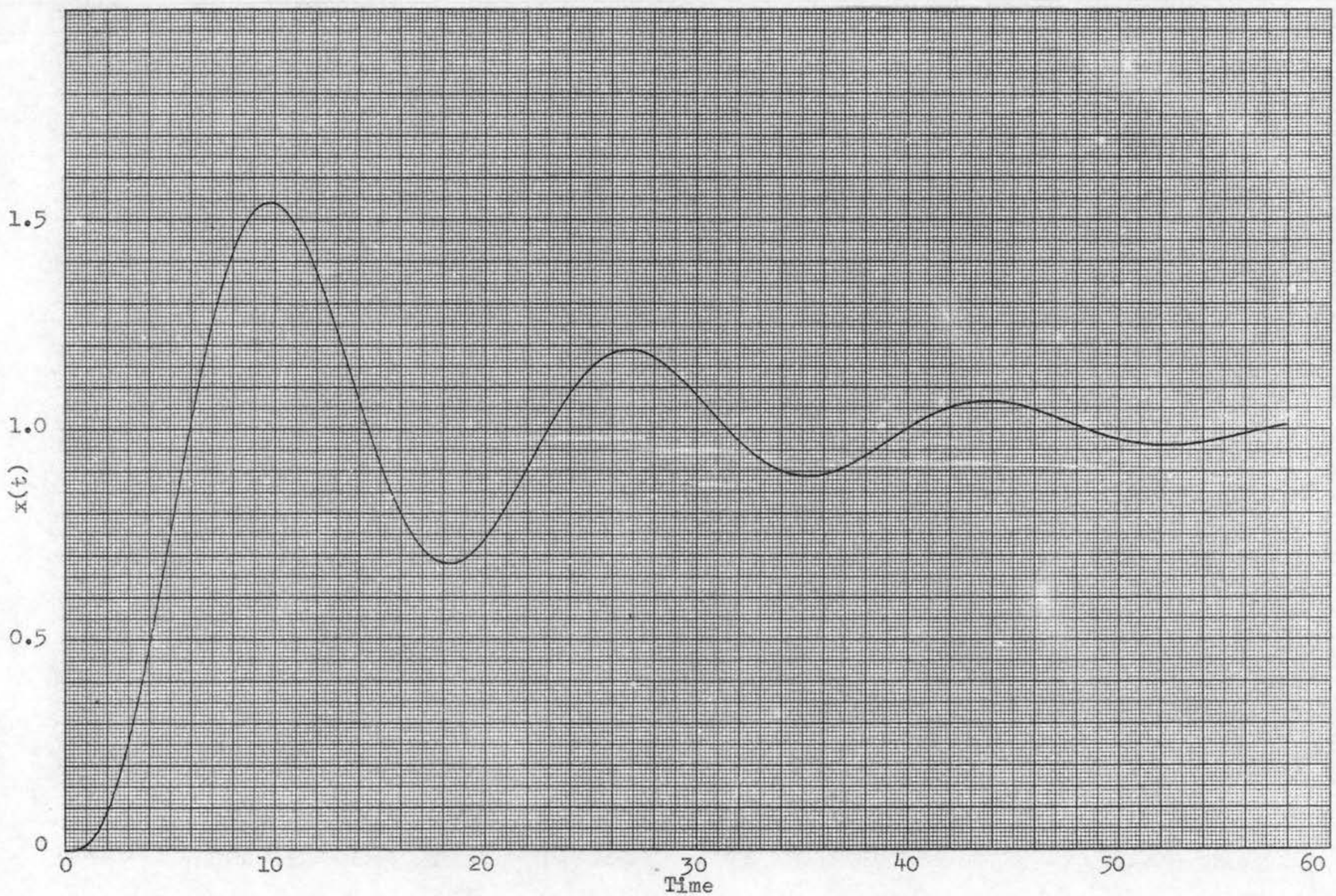


Figure 5-1. System Response With $B_1 = \frac{1}{4}$, $B_o = \frac{1}{8}$

The roots are $s = \left(\frac{a_2}{a_3}\right)\left(-\sigma_t\right) = 4(-.876)$

$= \underline{-3.504}$

$$s = \left(\frac{a_2}{a_3}\right)\left(-\zeta\omega_{nt} \pm j\omega_{dt}\right)$$

$$= 4(-.062 \pm j.374)$$

$$= \underline{-.248 \pm j1.496}.$$

3. $\beta = \frac{3.504}{.248} = 14.1.$

This system is badly underdamped as evidenced by the overshoot of 54% and the large ratio of settling time to rise time. In addition, the ratio of real root parts, β, is quite high indicating that the complex roots are dominant. If the analyst had decided to neglect the real root by assuming a second order model, then the predicted overshoot would have been 59.5%, an error of approximately 10%.

System With a Variable Parameter

The block diagram for the electro-hydraulic servomechanism illustrated in Chapter I is shown in Figure 5-2.

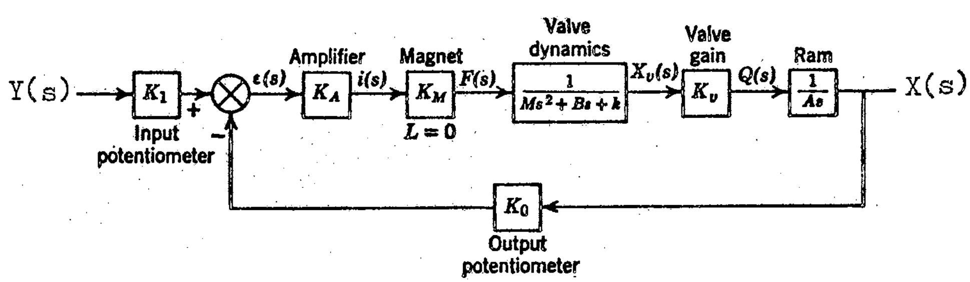


Figure 5-2. Block Diagram for Electro-Hydraulic Servomechanism

Third-order systems, such as that shown in Figure 5-1, with simple gain elements in the feedback loop are in the class of systems for which this dissertation is applicable. However, for convenience the system can be changed to one with unity feedback where $K_R = K_A K_M K_O K_V / A$ (see Figure 5-3).

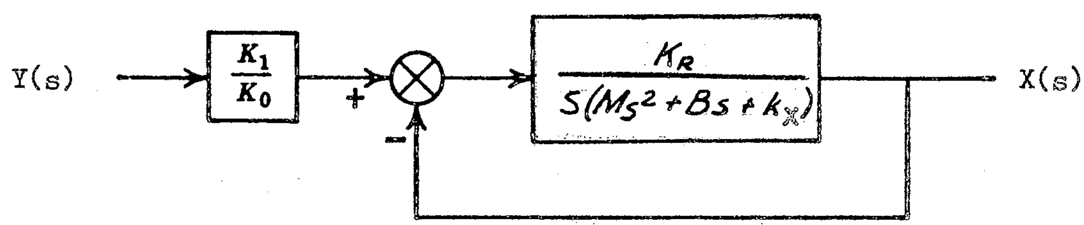


Figure 5-3. Block Diagram in Figure 5-2 Redrawn as a Unity Feedback System

Problem:

If $K_1 = 1$ volt/cm

$K_O = 0.2$ volt/cm

$K_M = 27{,}400$ dynes/ma

$M = 50$ gm.

$k_x = 17.25 \times 10^6$ dynes/cm

$B = 30{,}000$ dynes/cm/sec

$K_V = 2760$ cm³/sec/cm

$A = 5$ cm²,

what is the highest value of amplifier gain, K_A, which can be used without causing overshoot in response to a step input of one volt at the summing junction?[1]

Solution:

$a_3 = 50$

$a_2 = 30{,}000$

$a_1 = 17.25 \times 10^6$

$a_0 = K_A(27{,}400)(2760)(0.2)/5 = 3.02\ K_A \times 10^6.$

Using the procedure set forth at the start of this chapter:

1. $$B_1 = \frac{a_1 a_3}{a_2^2} = \frac{(17.25 \times 10^6)(50)}{(30{,}000)^2} = .959$$

$$B_0 = \frac{a_0 a_3^2}{a_2^3} = \frac{(3.02\ K_A \times 10^6)(50)^2}{(30{,}000)^3} = 2.80\ K_A \times 10^{-4}.$$

2. From Figure 3-4 with $B_1 = 0.959$, $B_0 = 0.245$ for no overshoot.[2]

3. $$K_A = \frac{0.245}{2.8 \times 10^{-4}} = \underline{\underline{874\ \text{ma/V}}}.$$

The analog computer response with $B_1 = 0.959$, $B_0 = 0.245$ is shown in Figure 5-4.

---

[1]The figures and problem in this section are from "Introduction to Automatic Control Systems" by Clark (4).

[2]If the equation coefficients are such that B_1 is larger than the range of Figure 3-5, then for a no overshoot response the following criterion may be used:

$$B_1 > 3B_0 + \frac{2}{9}.$$

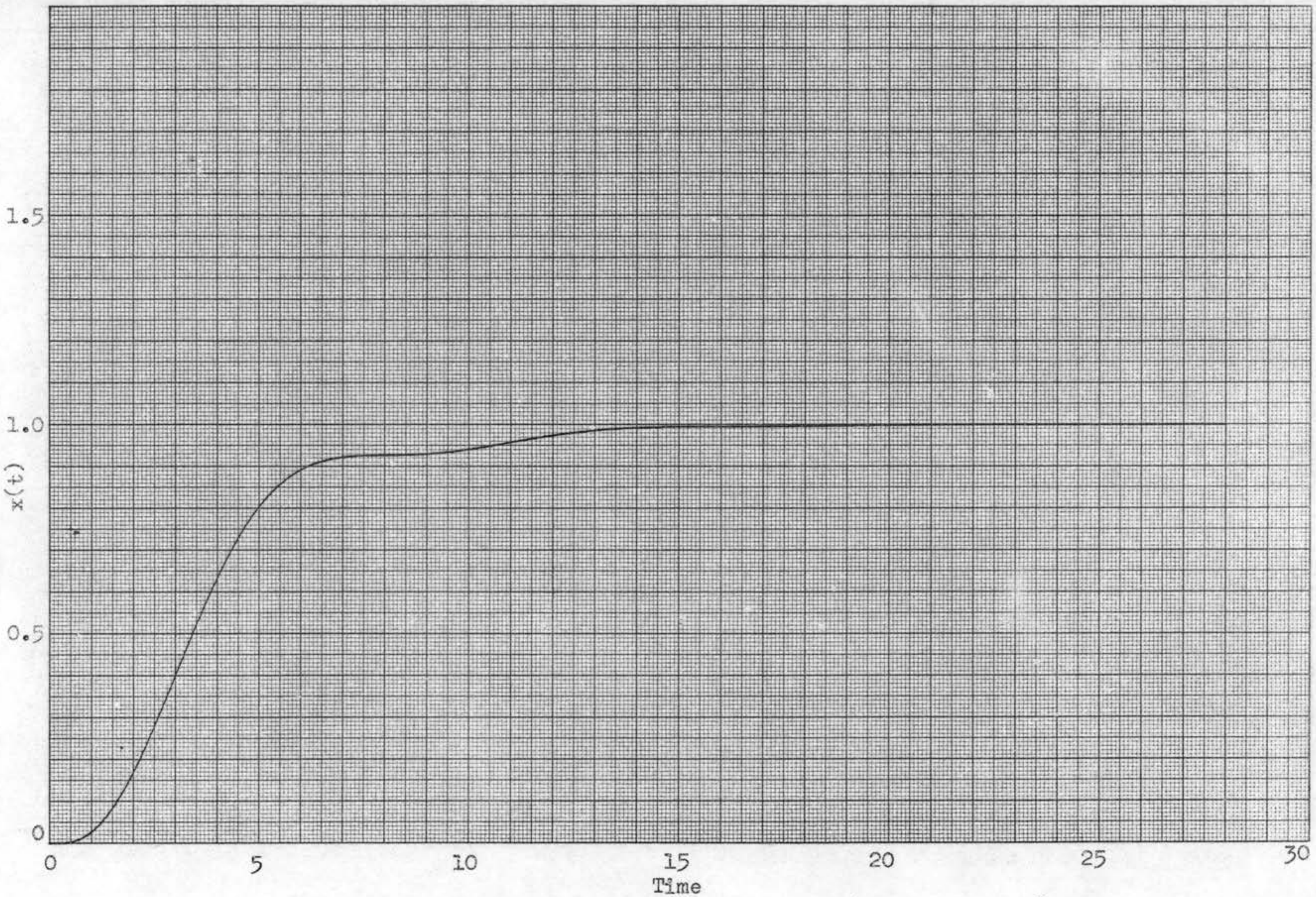


Figure 5-4. System Response With $B_1 = 0.959$, $B_0 = 0.245$

Supplementary Information:

1. Settling time

From Figure 3-3, $t_{st} = 10.3$

$$t_s = \left(\frac{50}{30,000}\right)\left(10.3\right) = .0171 \text{ seconds.}$$

2. Rise time

From Figure 3-5, $t_{rt} = 4.5$

$$t_r = \left(\frac{50}{30,000}\right)\left(4.5\right) = .0075 \text{ seconds.}$$

It is obvious from the settling time and rise time values that the system responds very rapidly to system disturbances. It is, however, a very stable system with a gain margin of $\frac{B_1}{B_o} = 3.92$. This means that the amplifier gain could be increased to (3.92)(874) or 3430 ma/V before an unstable condition will occur.

System With Two-Variable Parameters

Problem:

Assume that the designer wishes to reduce the size of the valve spring in the servo system of the previous section, but the design specifications restrict the settling time to a maximum of .02 seconds for a step input of one volt. With K_A variable, what is the smallest spring constant, k_x, which can be utilized to meet the dual design specifications of no overshoot with $t_s \leq .02$ seconds?

Solution:

1. $B_1 = \dfrac{k_x(50)}{(30,000)^2} = 6.22 k_x \times 10^{-8}$

$$B_0 = \frac{(3.02 K \times 10^6)(50)^2}{(30,000)^3} = 2.80\ K_A \times 10^{-4}$$

2. $t_{st} = \left(\dfrac{30,000}{50}\right)\left(.02\right) = 12.$

Using Figures 3-3 and 3-4 simultaneously, the minimum value of B_1 which satisfies the design criteria occurs at the point

$$B_1 = 0.475 \qquad B_0 = 0.084.$$

3. $k_x = \dfrac{B_1 \times 10^8}{6.22} = \dfrac{0.475 \times 10^8}{6.22} = \underline{\underline{7.63 \times 10^6 \text{ dynes/cm}}}.$

$$K_A = \frac{B_0 \times 10^4}{2.80} = \frac{0.084 \times 10^4}{2.80} = \underline{\underline{300 \text{ ma/V}}}$$

The analog response using $B_1 = 0.475$, $B_0 = 0.084$ is shown in Figure 5-5.

To the writer's knowledge, there are no other methods of solution, outside of trial and error procedures, for a problem of this type.

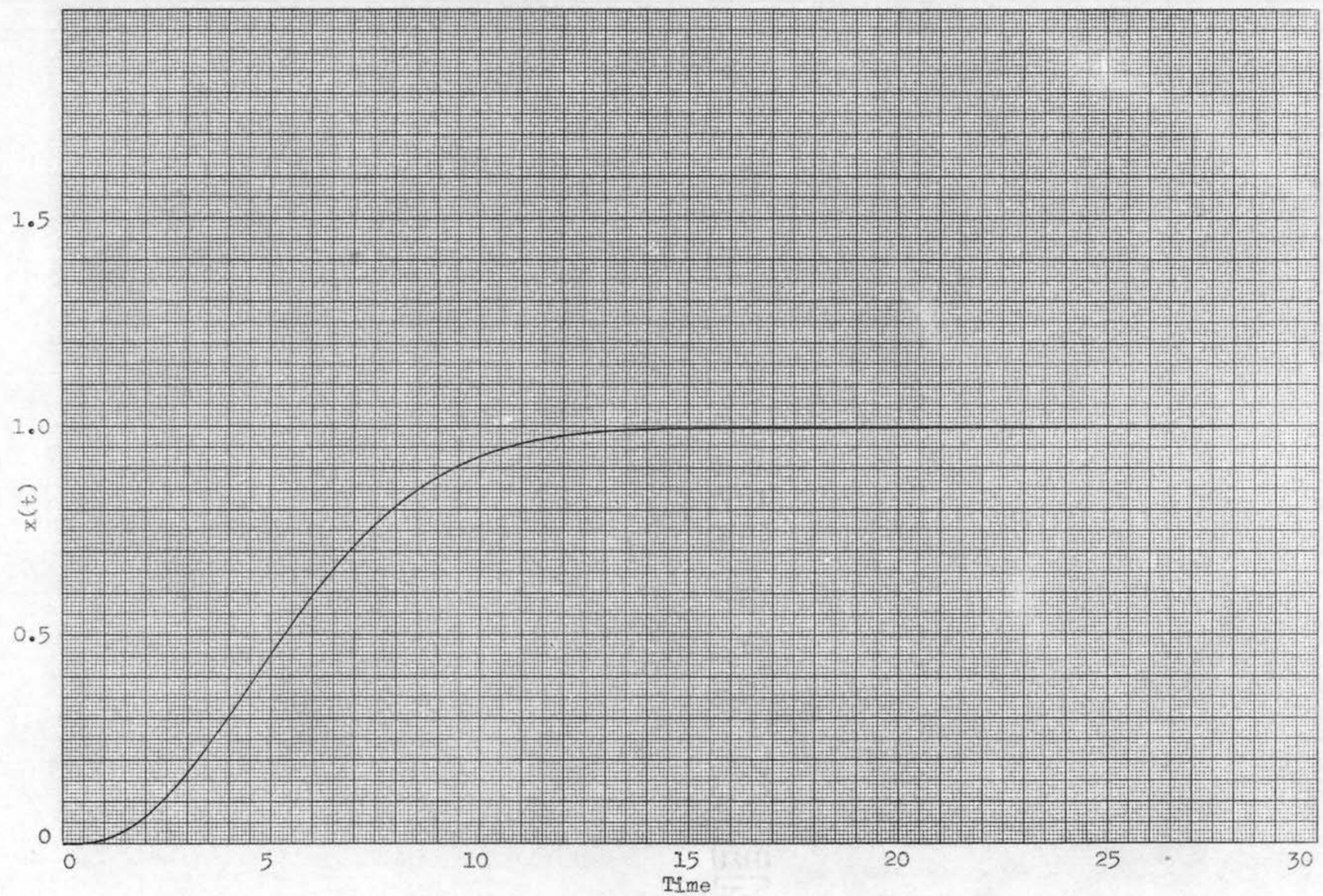


Figure 5-5. System Response With $B_1 = 0.475$, $B_0 = 0.084$

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The analysis and synthesis of linear third-order systems is a topic which is of considerable interest in the area of feedback controls, particularly hydraulic control systems. The synthesis methods which are currently available are limited in their usefulness because of the difficulty of relating system parameters to response characteristics such as overshoot, settling time, gain margin, etc. The major disadvantages of these methods are:

1. A root determination process is required in all cases.
2. Only a small number of system parameter combinations are depicted by the normalized response curves which are used.

This dissertation is concerned with the development of a method for the analysis and synthesis of linear third-order systems which makes use of the root-pattern and response relationships between the generalized third-order equation and the normalized third-order equation with unity second-order and third-order coefficients. The results of the theoretical analysis are illustrated graphically and incorporated into analysis and synthesis procedures which offer the following advantages over current methods:

1. Direct functional relationships between system parameters and response characteristics are available. Thus, the root determination process is obviated completely.
2. The functional relationships are continuous rather than discrete. Therefore, an infinite number of system parameter combinations which satisfy design criteria are available for the synthesis procedure.

There appear to be several promising areas for the extension of the work initiated in this investigation. In particular, the following recommendations are made:

1. The system response equations be analyzed with the objective of defining exactly the system variable combinations (and, thus, areas on a $B_1 - B_0$ chart) which may be adequately described by first-order and second-order equations. The procedure for making the appropriate reduction in order would be an integral part of this work along with an error analysis of the approximation involved.
2. An investigation be made of the response characteristic relationships in order to develop a procedure to be used especially for evaluation

of third-order system parameters from experimental response data.

3. Investigate the possibility of extending the theory of this dissertation into the analysis of higher-order systems with limited ranges of parameter values.

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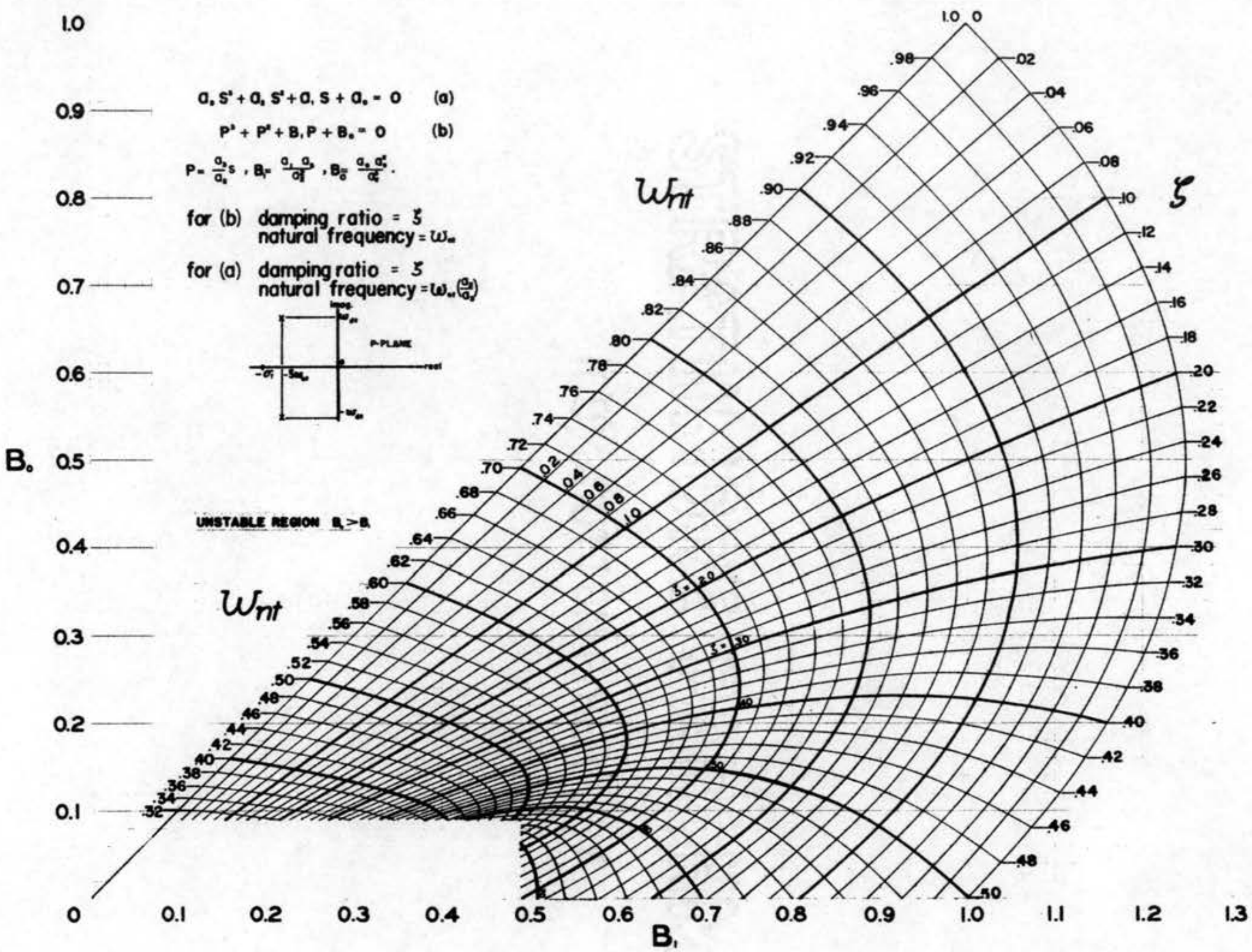
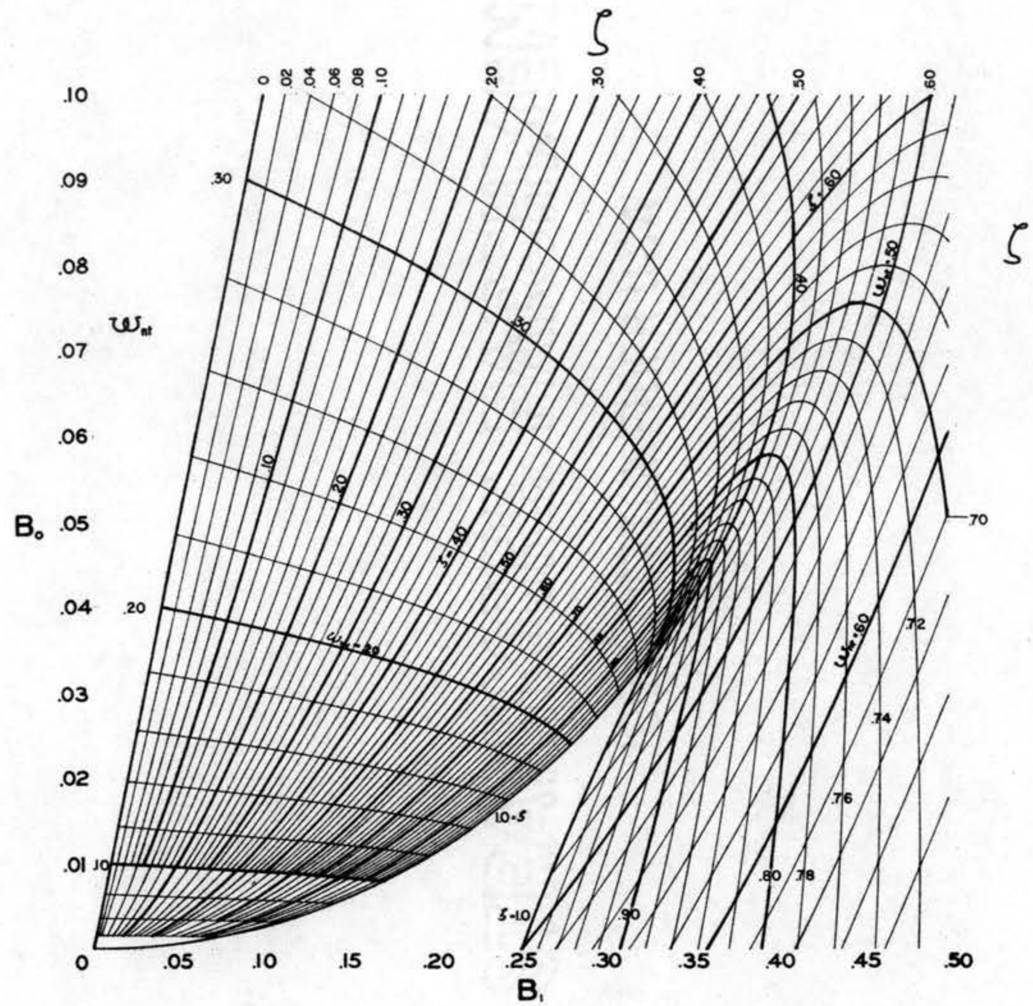
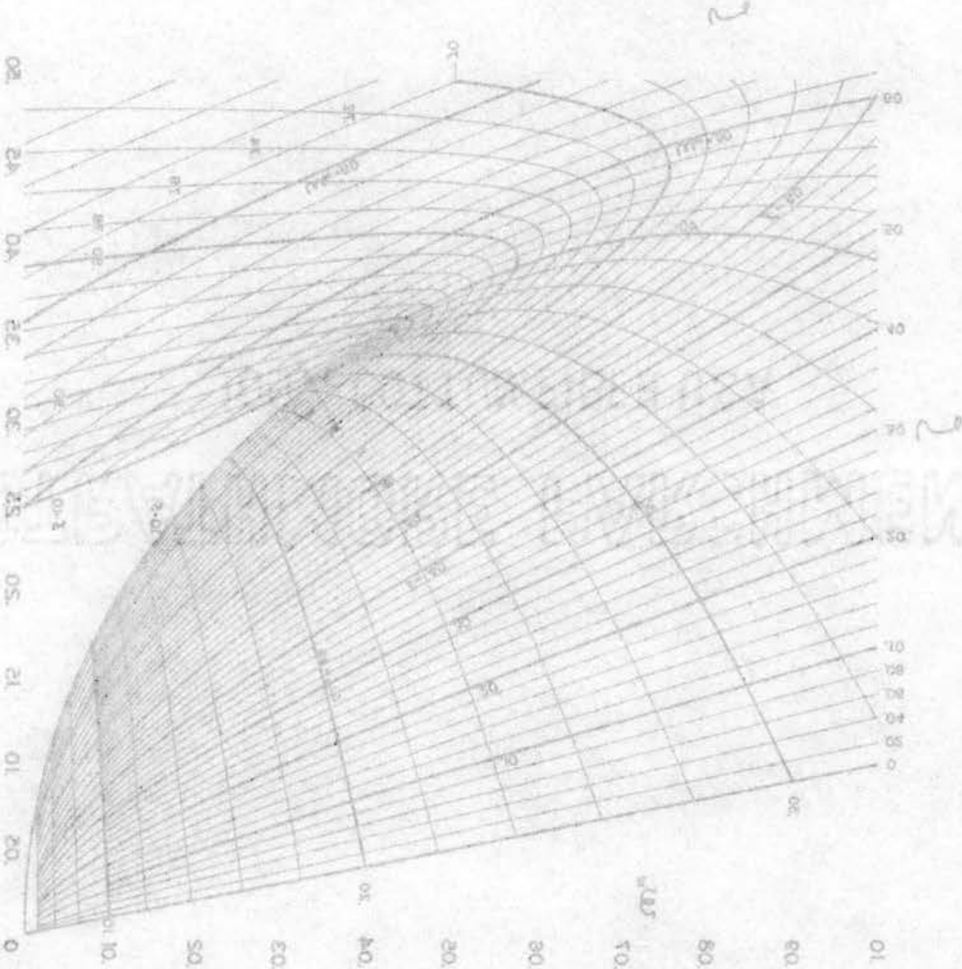
APPENDIX A-I

MITROVIC'S CHART



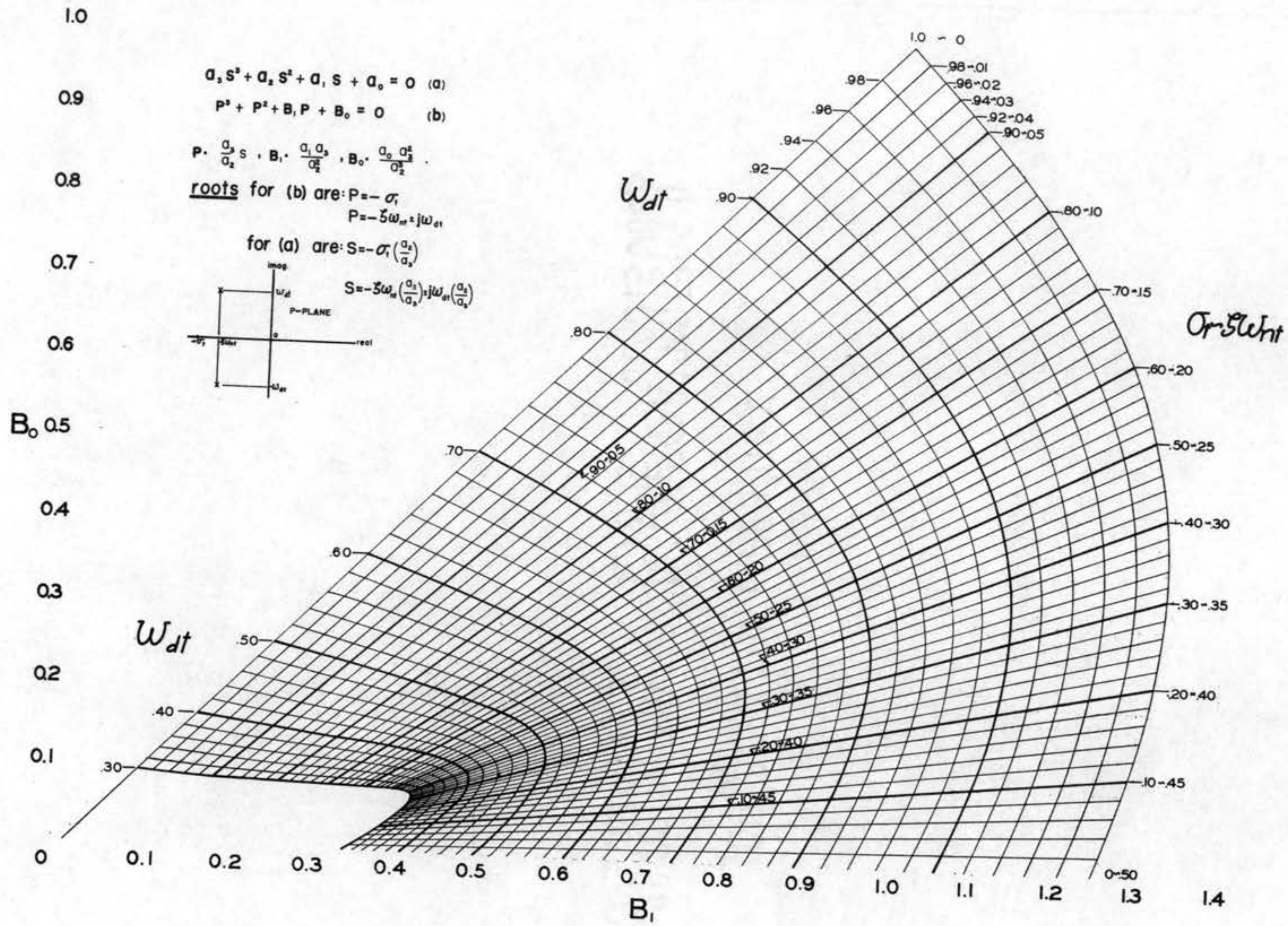
a3 S3 + a2 S2 + a1 S + a0 = 0 (a)
P3 + P2 + B1 P + B0 = 0 (b)
for (b) damping ratio = ζ
natural frequency = ωn
for (a) damping ratio = ζ
P-PLANE
real
UNSTABLE REGION
ωnt
ζ
B0
B1

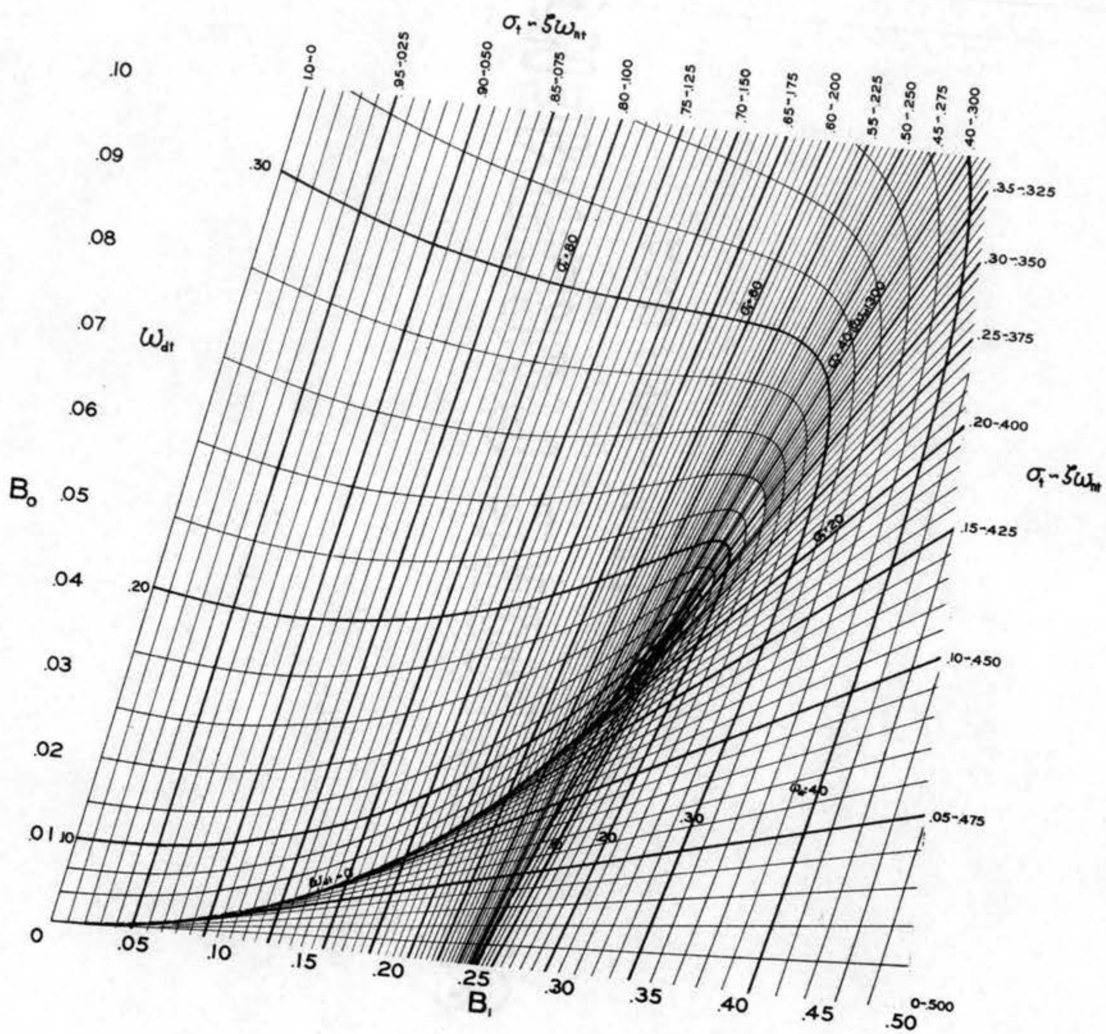

ζ
0
.02
.04
.06
.08
.10
.20
.30
.40
.50
.60
ζ
ω_{nt}
B_o
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
.05
.10
.15
.20
.25
.30
.35
.40
.45
.50
B_1
.70
.72
.74
.76
.78
.80
.90
ζ=1.0
1.0-ζ
ζ=.40
ζ=.60
ω_{nt}=.20
ω_{nt}=.50
ω_{nt}=.60



APPENDIX A-II

ROOT CHART


a₃S³ + a₂S² + a₁S + a₀ = 0 (a)
P³ + P² + B₁P + B₀ = 0 (b)
P = (a₃/a₂) S , B₁ = a₁a₃/a₂² , B₀ = a₀a₃²/a₂³ .
roots for (b) are: P = −σ₁
P = −ζω_nt ± jω_dt
for (a) are: S = −σ₁(a₂/a₃)
S = −ζω_nt(a₂/a₃) ± jω_dt(a₂/a₃)
imag.
P-PLANE
real
ω_dt
σ₁-ζω_nt
B₀
B₁
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4



$\sigma_t - \zeta\omega_{nt}$
1.0-0
.95-.025
.90-.050
.85-.075
.80-.100
.75-.125
.70-.150
.65-.175
.60-.200
.55-.225
.50-.250
.45-.275
.40-.300
.35-.325
.30-.350
.25-.375
.20-.400
.15-.425
.10-.450
.05-.475
0-.500
ω_{dt}
B_0
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
B_1
.05
.10
.15
.20
.25
.30
.35
.40
.45
.50
.30
.20
.10

APPENDIX B-I

NEWTON-RAPHSON METHOD FOR ONE EQUATION

NEWTON-RAPHSON METHOD FOR ONE EQUATION

Given:

$$F(\zeta, \omega_{nt}, t_{st}) = 0$$

with ω_{nt} and t_{st} constant.

Development of Method:

1. Expand $F(\zeta, \omega_{nt}, t_{st})$ in a Taylor series about $\zeta = \zeta_k$ and truncate after two terms.

   $$F(\zeta, \omega_{nt}, t_{st}) = F(\zeta_k, \omega_{nt}, t_{st}) + (\zeta - \zeta_k)\, F'(\zeta_k, \omega_{nt}, t_{st})$$

2. Solve for ζ.

   $$\zeta = \zeta_k - \frac{F(\zeta_k, \omega_{nt}, t_{st})}{F'(\zeta_k, \omega_{nt}, t_{st})} .$$

Procedure:

1. Let ζ_k be an initial approximation to the solution of the given equation.
2. As the next approximation, take

   $$\zeta_{k+1} = \zeta_k - \frac{F(\zeta_k, \omega_{nt}, t_{st})}{F'(\zeta_k, \omega_{nt}, t_{st})}$$

3. Continue iterations until the process

converges to a solution for ζ.

Reference: McCracken and Dorn (13), pp. 133, 156.

APPENDIX B-II

FORTRAN PROGRAM FOR LINES OF CONSTANT SETTLING TIME

```
      MON$$      JOB  252740040    K K GOWDY   SETTLING TIME
      MON$$      ASGN MGO,A2
      MON$$      ASGN MJB,A3
      MON$$      MODE GO,TEST
      MON$$      EXEQ FORTRAN
C      PROGRAM TO COMPUTE LINES OF CONSTANT SETTLING TIME
C      FOR A LINEAR THIRD ORDER SYSTEM
C      Z=DAMPING RATIO    W=NATURAL FREQUENCY
500    FORMAT(4F10.5)
501    FORMAT(///11X,4HZETA,6X,4HFREQ,7X,4HTIME,6X,4HBETA,6X,2HB1,8X,
      12HB0,6X,1HK)
502    FORMAT(F16.5,F9.4,F11.4,5X,15HINPUT CARD DATA)
600    FORMAT(F16.5,F9.4,F11.4,3F10.4,I5)
700    FORMAT(//2X,2HTS,8X,1HA,9X,1HB,9X,1HC,9X,3HSUM,7X,2HB1,8X,2HB0,6X,
      11HK)
701    FORMAT(F6.1,6F10.4,I5)
       Y=.00005
       ROFF=.00005
1      READ(1,500)Z,W,T,D1
2      M=1
       N=1
       I=1
       WRITE(3,501)
       WRITE(3,502)Z,W,T
5      K=1
       IF(Z.GE.1.0)GO TO 150
       L=1
10     W2=W*W
       W3=W2*W
       ZW=Z*W
       IF(ZW.LT.0.)ZW=0.
       A=1.-(Z*Z)
       AR=SQRT(A)
       U=1.-(2.*ZW)
       IF(U.LT.0.)U=0.
       V=U-ZW
       IF(L.GE.3)GO TO 100
       B=(8.*ZW*ZW)-(6.*ZW)+1.+W2
       BR=SQRT(B)
       ARBR=AR*BR
       ZW83=8.*ZW-3.
       E1=EXP(-ZW*T)
       E2=EXP(-U*T)
       FP1=W2*E2/B
       FP2=U*E1/ARBR
       F=.05-FP1-FP2
       IF(L.EQ.2)GO TO 95
       FRZ1=-2.*W*(B*T+ZW83)*FP1/B
       FRZ2=-FP2*(-W*(2./U+T)+(W*ZW83/B)-(Z/A))
       FRZ=FRZ1+FRZ2
       DELTAZ=-F/FRZ
       Z=Z+DELTAZ
       IF(ABS(DELTAZ).LE.Y)L=3
       IF(Z.LT.0.0)Z=0.
       IF(Z.GT.1.0)Z=1.
       K=K+1
       IF(K.LT.100)GO TO 10
80     Z=ZHOLD
       W=W+D1
       K=1
       L=2
       GO TO 10
```

```
85    Z=(1.-(3./T))/2.
      W=1.
      M=1
      N=2
      GO TO 5
95    FRW1=-2.*W*E2*(B*(ZW*T+1.)-(8.*ZW*ZW-(3.*ZW)+W2))/(B*B)
      FRW2=E1*(Z*(U*T+2.)+(U*(8.*ZW*Z-(3.*Z)+W)/B))/ARBR
      FRW=FRW1+FRW2
      DELTAW=-F/FRW
      W=W+DELTAW
      IF(ABS(DELTAW).LE.Y)L=4
      IF(W.LT.0.0)W=0.
      IF(W.GT.1.0)W=1.
      K=K+1
      IF(K.LT.100)GO TO 10
      GO TO (85,1),N
100   BETA=(1./ZW)-2.
      B1=(2.*ZW)-(3.*W2)+(4.*W2*A)+ROFF
      B0=W2*U+ROFF
      IF(L.EQ.4)GO TO 145
      ZHOLD=Z
      WRITE(3,600)Z,W,T,BETA,B1,B0,K
      Z=Z+((Z-Z2)*(Z-Z2)/(Z2-Z1))
      Z1=Z2
      Z2=ZHOLD
      IF(M.LE.2)Z=ZHOLD
      M=M+1
      W=W-D1
      IF(Z.GE.1.)GO TO (80,150,150),N
      IF(W.GE.0.01)GO TO 5
145   WRITE(3,600)Z,W,T,BETA,B1,B0,K
      K=1
      L=2
      Z=Z+D1
      IF(Z.LT.1.0)GO TO 10
150   Z=1.
      B=Z*W
      BHOLD=B
      K=1
      ET=EXP(-T)
160   BT=B*T
      B1=1.-B
      B2=B1-B
      B3=B2-B
      B4=B3-B
      B32=B3*B3
      B33=B32*B3
      EBT=EXP(-BT)
      IF(B2.LT.0.0)B2=0.
      EB2T=EXP(-B2*T)
      BP1=B*B*EB2T/B32
      BP2=(B2*BT*B3+B2*B4)*EBT/B32
      F=.05-BP1-BP2
      FRB=-2.*B*EB2T*(1.+BT-3.*B*BT)/B33+T*BP2-EBT*((T-4.*BT+6.*B*BT)*B3
     1-2.*B)/B33
      DELTAB=-F/FRB
      B=B+DELTAB
      IF(B.GT.1.5*BHOLD)B=0.75*B
165   IF(ABS(DELTAB).LT.Y)GO TO 170
      IF(B.LT.0.0)B=0.
      K=K+1
      IF(K.GT.100)GO TO (85,180),N
```

```
      GO TO 160
170   ZW=B
      W=B
      BETA=(1./ZW)-2.
      W2=W*W
      U=1.-(2.*ZW)
      B1=(2.*ZW)-(3.*W2)+ROFF
      B0=W2*U+ROFF
      WRITE(3,600)Z,W,T,BETA,B1,B0,K
      GO TO (85,180),N
180   WRITE(3,700)
      C=B
      B=B-.01
      A=1.-C-B
184   K=1
185   C=1.-A-B
      IF(C.LT.0.)C=0.
      IF(C.GT.1.)C=1.
      IF(B.LT.A)GO TO 1
      AB=A*B
      AC=A*C
      BC=B*C
      AMB=A-B
      AMC=A-C
      BMC=B-C
      EA=EXP(-A*T)
      EB=EXP(-B*T)
      EC=EXP(-C*T)
      DA=AMB*AMC
      DB=AMB*BMC
      DC=AMC*BMC
      F=.05-B*C*EA/DA+A*C*EB/DB-A*B*EC/DC
      FRA3=-B*EC*(DC*(A*T+1.)-A*(4.*A+5.*B-3.))/(DC*DC)
      FRA2=-EB*AMC*(DB+AC)/(DB*DB)
      FRA1=B*EA*(DA*(C*T+1.)+C*(4.*A-B-1.))/(DA*DA)
      FRA=FRA1+FRA2+FRA3
      DELTAA=-F/FRA
      A=A+DELTAA
      IF(A.LT.0.)A=0.
      IF(A.GT.1.)A=1.
      K=K+1
      IF(K.GT.100)GO TO (195,1),I
      IF(ABS(DELTAA).GT.Y)GO TO 185
      AHOLD=A
      BHOLD=B
      CHOLD=C
      B1=A*B+C*(A+B)+ROFF
      B0=A*B*C+ROFF
      SUM=A+B+C+ROFF
      WRITE(3,701)T,A,B,C,SUM,B1,B0,K
      B=B-D1
      GO TO 184
195   I=2
      D1=.001
      A=AHOLD
      B=BHOLD-D1
      C=CHOLD
      GO TO 184
      END
     MON$$     EXEQ LINKLOAD
               PHASEENTIREPROG
               CALL MAINPGM
     MON$$     EXEQ ENTIREPROG,MJB
```

APPENDIX C-I

GENERALIZATION OF NEWTON-RAPHSON METHOD FOR TWO EQUATIONS

GENERALIZATION OF NEWTON-RAPHSON METHOD FOR TWO EQUATIONS

Given:

$$F(\zeta, \omega_{nt}, t) = 0$$

$$G(\zeta, \omega_{nt}, t) = 0$$

with ω_{nt} constant.

Development of Method:

1. Expand both $F(\zeta,\omega_{nt},t)$ and $G(\zeta,\omega_{nt},t)$ in Taylor series about $\zeta = \zeta_k$, $t = t_k$ and truncate after two terms. For simplification let

$$F = F(\zeta,\omega_{nt},t) \qquad F_k = F(\zeta_k,\omega_{nt},t_k)$$

$$G = G(\zeta,\omega_{nt},t) \qquad G_k = G(\zeta_k,\omega_{nt},t_k),$$

then

$$F = F_k + (\zeta - \zeta_k)\frac{\partial F_k}{\partial \zeta} + (t - t_k)\frac{\partial F_k}{\partial t} + \ldots = 0$$

$$G = G_k + (\zeta - \zeta_k)\frac{\partial G_k}{\partial \zeta} + (t - t_k)\frac{\partial G_k}{\partial t} + \ldots = 0$$

2. Rearrange with F_k and G_k on the right

$$(\zeta - \zeta_k)\frac{\partial F_k}{\partial \zeta} + (t - t_k)\frac{\partial F_k}{\partial t} = -F_k$$

$$(\zeta - \zeta_k) \frac{\partial G_k}{\partial \zeta} + (t - t_k) \frac{\partial G_k}{\partial t} = -G_k$$

3. Use Cramer's rule to solve for ζ and t.

$$\zeta = \zeta_k - \left[F_k \frac{\partial G_k}{\partial t} - G_k \frac{\partial F_k}{\partial t}\right] \Big/ J$$

$$t = t_k + \left[F_k \frac{\partial G_k}{\partial \zeta} - G_k \frac{\partial F_k}{\partial \zeta}\right] \Big/ J$$

where

$$J = \frac{\partial F_k}{\partial \zeta} \frac{\partial G_k}{\partial t} - \frac{\partial F_k}{\partial t} \frac{\partial G_k}{\partial \zeta} .$$

Procedure:

1. Let ζ_k and t_k be initial approximations for the solution of the given equations.
2. Use the equations above to solve for the next approximations ζ_{k+1} and t_{k+1}.
3. Continue the iterations until two successive approximations are found to be sufficiently close to each other.

Reference: McCracken and Dorn (13), pp. 144-145, 156-157.

APPENDIX C-II

FORTRAN PROGRAM FOR LINES OF CONSTANT OVERSHOOT

```
     MON$$      JOB  252740040     K K GOWDY            OVERSHOOT
     MON$$      ASGN MGO,A2
     MON$$      ASGN MJB,A3
     MON$$      MODE GO,TEST
     MON$$      EXEQ FORTRAN
C      PROGRAM TO COMPUTE LINES OF CONSTANT PEAK OVERSHOOT
C      FOR A LINEAR THIRD ORDER SYSTEM
C      Z=DAMPING RATIO    W=NATURAL FREQUENCY
       REAL J,NPEAK
500    FORMAT(5F10.5)
501    FORMAT(///6X,2HOS,5X,4HZETA,6X,4HFREQ,6X,4HTIME,7X,5HSIGMA,5X,
     14HREAL,7X,4HIMAG,8X,4HBETA,5X,2HB1,8X,2HB0,6X,1HK,4X,5HNPEAK)
502    FORMAT(F8.2,2F10.5,F11.4,5X,15HINPUT CARD DATA)
601    FORMAT(F8.2,2F10.5,F11.4,3F10.5,F11.4,2F10.4,I5,I7)
699    FORMAT(//5X,1HF,9X,3HFRZ,7X,3HFRT,7X,1HG,9X,3HGRZ,7X,3HGRT,7X,1HJ,
     19X,1HZ,9X,6HDELTAZ,4X,1HT,9X,6HDELTAT)
700    FORMAT(11F10.5)
701    FORMAT(/5X,2HAR,8X,1HB,9X,2HBR,8X,1HU,9X,1HV,9X,2HE1,8X,2HE2,8X,1H
     1P,9X,1HR,9X,1HS,9X,3HFP1,7X,3HFP2)
702    FORMAT(12F10.5)
1      READ(1,500)OS,Z,W,T,D1
       Y=.00005
       ROFF=.00005
       PI=3.141593
       WRITE(3,501)
       WRITE(3,502)OS,Z,W,T
       L=1
       M=1
5      K=1
       W2=W*W
       W3=W2*W
10     ZW=Z*W
       IF(K.GT.100)GO TO (150,1),L
20     A=1.-(Z*Z)
       AR=SQRT(A)
       WD=W*AR
       B=(8.*ZW*ZW)-(6.*ZW)+1.+W2
       IF(B.LT.0.)B=0.
       BR=SQRT(B)
       ARBR=AR*BR
       ZW83=8.*ZW-3.
       U=1.-(2.*ZW)
       V=1.-(3.*ZW)
       E1=EXP(-ZW*T)
       E2=EXP(-U*T)
       E3=EXP(-V*T)
       P=PI-ATAN(AR/Z)+ATAN(WD/V)
       IF(V.LT.0.)P=2.*PI-ATAN(AR/Z)-ATAN(WD/ABS(V))
       R=WD*T-P
40     S=SIN(R)
       C=COS(R)
       RRZ=((ZW-3.*W2)/B-ZW*T-1.)/AR
       FP1=W2*E2/B
       FP2=U*E1/ARBR
       F=-OS-FP1+FP2*S
       FRZ1=-2.*W3*E2*(T*B-W*ZW83)/(B*B)
       FRZ2=E1*(U*C*RRZ-W*S*(T*U+2.))/ARBR
       FRZ3=-U*E1*S*(W*ZW83*A-Z*B)/(A*B*ARBR)
       FRZ=FRZ1+FRZ2+FRZ3
       FRT=U*FP1-W*FP2*(Z*S-AR*C)
       G=FRT
       GRZ1=2.*W3*E2*(B*(U*T-1.)-U*ZW83)/B
```

```
      GRZ2=-W*E1*(S*(U*(1.-RRZ*AR)-ZW*(U*T+2.)))/ARBR
      GRZ3=U*E1*(Z*S-AR*C)*(W*ZW83/(ARBR*B)-Z/(ARBR*A))
      GRZ=GRZ1+GRZ2+GRZ3
      GRT=-U*U*W2*E2/B-W2*U*E1*(S*(1.-2.*Z*Z)+C*2.*Z*AR)/ARBR
      J=FRZ*GRT-FRT*GRZ
      IF(L.EQ.1)GO TO 70
      IF(ABS(J).LT.1.5)J=SIGN(1.5,J)
70    DELTAZ=(G*FRT-F*GRT)/J
      DELTAT=(F*GRZ-G*FRZ)/J
      Z=Z+DELTAZ
      IF(Z.GT.1.)Z=1.
      IF(Z.LT.0.)Z=0.
      T=T+DELTAT
      IF(T.LT.0.)T=0.
      K=K+1
      IF(ABS(DELTAZ).GE.Y)GO TO 10
      IF(ABS(DELTAT).GT.Y)GO TO 10
95    R3=R+(2.*PI)
      T3=(R3+P)/WD
      S3=SIN(R3)
      E13=EXP(-ZW*T3)
      E23=EXP(-U*T3)
      F2=1.-FP1+FP2*S
      F2=1.-W2*E23/B+U*E13*S3/ARBR
      IF(F2.GE.F3)GO TO 100
      T=T3
      GO TO 5
100   R1=R-(2.*PI)
      T1=(R1+P)/WD
      IF(T1.LE.0.0)GO TO 110
      S1=SIN(R1)
      E11=EXP(-ZW*T1)
      E21=EXP(-U*T1)
      F1=1.-W2*E21/B+U*E11*S1/ARBR
      IF(F1.LT.F2)GO TO 110
      T=T1
      GO TO 5
      BETA=(1./ZW)-2.
110   ZW=Z*W
      U=1.-2.*ZW
      B1=2.*ZW+W2-4.*ZW*ZW+ROFF
      B0=W2*U+ROFF
      NPEAK=(R+(2.*PI))/(2.*PI)
      NP=NPEAK
      WRITE(3,601)OS,Z,W,T,U,ZW,WD,BETA,B1,B0,K,NP
      ZHOLD=Z
      Z=Z+((Z-Z2)*(Z-Z2)/(Z2-Z1))
      Z1=Z2
      Z2=ZHOLD
      IF(T.LT.THOLD)M=1
      THOLD=T
      T=T+((T-T2)*(T-T2)/(T2-T1))
      T1=T2
      T2=THOLD
      IF(M.GT.2)GO TO 125
120   Z=ZHOLD
      T=THOLD
125   M=M+1
      W=W-D1
      IF(W.GE.0.02)GO TO 5
      GO TO 1
150   K=1
```

```
 L=2
 M=1
 Z=ZHOLD
 T=THOLD
 W=W+D1
 GO TO 5
 END
MON$$      EXEQ LINKLOAD
           PHASEENTIREPROG
           CALL MAINPGM
MON$$      EXEQ ENTIREPROG,MJB
```

APPENDIX D-I

GENERALIZATION OF NEWTON-RAPHSON METHOD FOR THREE EQUATIONS

GENERALIZATION OF NEWTON-RAPHSON METHOD FOR THREE EQUATIONS

Given:

$$F(\zeta, \omega_{nt}, t_1) = 0$$

$$G(\zeta, \omega_{nt}, t_2) = 0$$

$$H(t_{rt}, t_1, t_2) = 0$$

with ω_{nt} and t_{rt} constant.

Development of Method:

1. Expand $F(\zeta, \omega_{nt}, t_1)$, $G(\zeta, \omega_{nt}, t_2)$ and $H(t_{rt}, t_1, t_2)$ in Taylor series about $\zeta - \zeta_k$, $t_1 = t_{1k}$, $t_2 = t_{2k}$ and truncate after two terms. For simplification let

$$F = F(\zeta, \omega_{nt}, t_1) \qquad F_k = F(\zeta_k, \omega_{nt}, t_{1k})$$

$$G = G(\zeta, \omega_{nt}, t_2) \qquad G_k = G(\zeta_k, \omega_{nt}, t_{2k})$$

$$H = H(t_{rt}, t_1, t_2) \qquad H_k = H(t_{rt}, t_{1k}, t_{2k})$$

then

$$F = F_k + (\zeta - \zeta_k)\frac{\partial F_k}{\partial \zeta} + (t_1 - t_{1k})\frac{\partial F_k}{\partial t_1} + \ldots = 0$$

$$G = G_k + (\zeta - \zeta_k)\frac{\partial F_k}{\partial \zeta} + (t_2 - t_{2k})\frac{\partial G_k}{\partial t_2} + \ldots = 0$$

$$H = H_k + (t_1 - t_{1k})\frac{\partial H_k}{\partial t_1} + (t_2 - t_{2k})\frac{\partial H_k}{\partial t_2} + \ldots = 0$$

2. Rearrange with F_k, G_k, H_k on the right.

$$(\zeta - \zeta_k)\frac{\partial F_k}{\partial \zeta} + (t_1 - t_{1k})\frac{\partial F_k}{\partial t_1} + 0 = -F_k$$

$$(\zeta - \zeta_k)\frac{\partial G_k}{\partial \zeta} + 0 + (t_2 - t_{2k})\frac{\partial G_k}{\partial t_2} = -G_k$$

$$0 + (t_1 - t_{1k})\frac{\partial H_k}{\partial t_1} + (t_2 - t_{2k})\frac{\partial H_k}{\partial t_2} = -H_k$$

3. Use Cramer's rule to solve for ζ, t_1, and t_2.

$$\zeta = \zeta_k - \left[F_k \frac{\partial G_k}{\partial t_2} - G_k \frac{\partial F_k}{\partial t_1} + H \frac{\partial F_k}{\partial t_1}\frac{\partial G_k}{\partial t_2}\right] \Big/ J$$

$$t_1 = t_{1k} + \left[F_k \frac{\partial G_k}{\partial \zeta} - G_k \frac{\partial F_k}{\partial \zeta} + H \frac{\partial F_k}{\partial \zeta}\frac{\partial G_k}{\partial t_2}\right] \Big/ J$$

$$t_2 = t_{2k} + \left[F_k \frac{\partial G_k}{\partial \zeta} - G_k \frac{\partial F_k}{\partial \zeta} + H \frac{\partial F_k}{\partial t_1}\frac{\partial G_k}{\partial \zeta}\right] \Big/ J$$

where $\frac{\partial H_k}{\partial t_1} = -1$ $\qquad$ $\frac{\partial H_k}{\partial t_2} = 1$

$$J = \frac{\partial F_k}{\partial \zeta}\frac{\partial G_k}{\partial t_2} - \frac{\partial F_k}{\partial t_1}\frac{\partial G_k}{\partial \zeta}.$$

Procedure:

1. Let ζ_k, t_{1k}, t_{2k} be initial approximations for the

solution of the given equations.

2. Use the equations above to solve for the next approximations ζ_{k+1}, $t_{1,k+1}$, $t_{2,k+1}$.

3. Continue the iterations until two successive approximations are found to be sufficiently close to each other.

Reference: McCracken and Dorn (13), pp. 136-157.

APPENDIX D-II

FORTRAN PROGRAM FOR LINES OF CONSTANT RISE TIME

```
      MON$$      JOB  252740040     K K GOWDY     RISE TIME
      MON$$      ASGN MGO,A2
      MON$$      ASGN MJB,A3
      MON$$      MODE GO,TEST
      MON$$      EXEQ FORTRAN,,,,,,,MAINPGM
C      PROGRAM TO COMPUTE LINES OF CONSTANT RISE TIME
C      FOR A LINEAR THIRD ORDER SYSTEM
C      Z=DAMPING RATIO    W=NATURAL FREQUENCY
       REAL J
499    FORMAT(6F10.5)
500    FORMAT(///10X,5HZETA=,F5.2,5X,5HFREQ=,F5.2,5X,3HTR=,F5.2,5X,3HT2=,
      1F5.2,5X,3HT1=,F5.2,5X,3HD1=,F5.2,10X,15HINPUT CARD DATA)
501    FORMAT(///25X,4HZETA,7X,4HFREQ,6X,2HTR,8X,2HT2,8X,2HT1,8X,4HBETA,
      16X,2HB1,8X,2HB0,6X,1HK)
600    FORMAT(20X,F10.5,7F10.4,I5)
601    FORMAT(//20X,F10.5,4F10.4,10X,17HSOLUTION ESTIMATE)
800    FORMAT(//16X,1HA,9X,1HB,9X,1HC,9X,2HTR,8X,2HT2,8X,2HT1,8X,3HSUM,7X
      1,2HB1,8X,2HB0,6X,1HK)
801    FORMAT(10X,9F10.4,I5)
       PI=3.141593
       Y=.00005
       ROFF=.00005


1      READ(1,499)Z,W,TR,T2,T1,D1
       WRITE(3,500)Z,W,TR,T2,T1,D1
       L=1
       M=1
       N=1
       IF(Z.EQ.1.)GO TO 150
       IF(Z.EQ.0.)GO TO 200
       WRITE(3,501)
5      K=1
       W2=W*W
       W3=W2*W
10     ZW=Z*W
       IF(K.GT.100)GO TO 1
15     A=1.-Z*Z
       AR=SQRT(A)
       B=(8.*ZW*ZW)-(6.*ZW)+1.+W2
       BR=SQRT(B)
       ARBR=AR*BR
       V=U-ZW
       U=1.-(2.*ZW)
       WD=W*AR
       ZW83=8.*ZW-3.
       P=PI-ATAN(AR/Z)+ATAN(WD/V)
       IF(V.LT.0.)P=2.*PI-ATAN(AR/Z)-ATAN(WD/ABS(V))
       R1=WD*T1-P
       R2=WD*T2-P
       S1=SIN(R1)
       S2=SIN(R2)
       C1=COS(R1)
       C2=COS(R2)
       E11=EXP(-ZW*T1)
       E12=EXP(-ZW*T2)
       E21=EXP(-U*T1)
       E22=EXP(-U*T2)
       FP1=W2*E21/B
       GP1=W2*E22/B
       FP2=U*E11/ARBR
```

```
      GP2=U*E12/ARBR
      F=.9-FP1+(FP2*S1)
      G=.1-GP1+(GP2*S2)
      H=T2-T1-TR
      PRZ1=((ZW-3.*W2)/B-ZW*T1-1.)/AR
      PRZ2=((ZW-3.*W2)/B-ZW*T2-1.)/AR
      FRZ1=-2.*W3*E21*(T1*B-(W*ZW83))/(B*B)
      GRZ1=-2.*W3*E22*(T2*B-(W*ZW83))/(B*B)
      FRZ2=E11*((U*C1*PRZ1)-(W*S1*(T1*U+2.)))/ARBR
      GRZ2=E12*((U*C2*PRZ2)-(W*S2*(T2*U+2.)))/ARBR
      FRZ3=-U*E11*S1*(W*ZW83*A-(Z*B))/(A*B*ARBR)
      GRZ3=-U*E12*S2*(W*ZW83*A-(Z*B))/(A*B*ARBR)
      FRZ=FRZ1+FRZ2+FRZ3
      GRZ=GRZ1+GRZ2+GRZ3
      FRT1=U*FP1-W*(Z*S1-AR*C1)*FP2
      GRT2=U*GP1-W*(Z*S2-AR*C2)*GP2
      J=(FRZ*GRT2)-(FRT1*GRZ)
      IF(ABS(J).LT.0.05)J=SIGN(0.05,J)
      DELTAZ=((-F*GRT2)+(G*FRT1)-(H*FRT1*GRT2))/J
      DELTT1=((F*GRZ)-(G*FRZ)+(H*FRZ*GRT2))/J
      Z=Z+DELTAZ
      IF(Z.GT.1.)Z=1.
      IF(Z.GT.0.)GO TO 90
      Z=0.
      GO TO (90,200),N
90    T1=T1+DELTT1
      IF(T1.LT.0.)T1=0.
      IF(T1.GT.P/WD)T1=P/WD
      T2=T1+TR
      K=K+1
      IF(ABS(DELTAZ).GT.Y)GO TO 10
      IF(ABS(DELTT1).GT.Y)GO TO 10
100   ZW=Z*W
      BETA=1./ZW-2.
      B1=2.*ZW+W2-4.*ZW*ZW+ROFF
      B0=W2*(1.-2.*ZW)+ROFF
      WRITE(3,600)Z,W,TR,T2,T1,BETA,B1,B0,K
      ZHOLD=Z
      T1HOLD=T1
      T2HOLD=T2
      IF(M.LE.1)GO TO 120
      DZ=Z-Z1
      Z=Z+DZ
      IF(Z.LE.0.)GO TO 200
      IF(Z.GT.1.)GO TO 150
      T1=T1+(T1-T11)
      T2=T2+(T2-T21)
120   Z1=ZHOLD
      T11=T1HOLD
      T21=T2HOLD
      M=M+1
      N=2
      W=W-D1
      IF(W.GE.0.01)GO TO 5
150   B=W
151   Z=1.
152   K=1
153   B1=1.-B
      B2=B1-B
      B3=B2-B
      B4=B3-B
      B32=B3*B3
```

```
      B33=B32*B3
      BT1=B*T1
      BT2=B*T2
      ET1=EXP(-T1)
      ET2=EXP(-T2)
      EBT1=EXP(-BT1)
      EBT2=EXP(-BT2)
      IF(B2.LT.0.)B2=0.
      EB2T1=EXP(-B2*T1)
      EB2T2=EXP(-B2*T2)
      BP11=B*B*EB2T1/B32
      BP12=B*B*EB2T2/B32
      BP21=(B2*BT1*B3+B2*B4)*EBT1/B32
      BP22=(B2*BT2*B3+B2*B4)*EBT2/B32
      F=.9-BP11-BP21
      G=.1-BP12-BP22
      H=T2-T1-TR
      FRB=-2.*B*EB2T1*(1.+BT1-3.*B*BT1)/B33+T1*BP21-EBT1*((T1-4.*BT1+6.*
     1B*BT1)*B3-2.*B)/B33
      GRB=-2.*B*EB2T2*(1.+BT2-3.*B*BT2)/B33+T2*BP22-EBT2*((T2-4.*BT2+6.*
     1B*BT2)*B3-2.*B)/B33
      FRT1=B2*BP11+B*BP21-EBT1*B2*B/B3
      GRT2=B2*BP12+B*BP22-EBT2*B2*B/B3
      J=FRB*GRT2-FRT1*GRB
      IF(ABS(J).LT.0.05)J=SIGN(0.05,J)
      DELTAB=(-F*GRT2+G*FRT1-H*FRT1*GRT2)/J
      DELTT1=(F*GRB-G*FRB+H*FRB*GRT2)/J
      B=B+DELTAB
      IF(B.LT.0.)B=0.
      IF(B.GT.1.)B=1.
      T1=T1+DELTT1
      IF(T1.LT.0.)T1=0.
      IF(T1.GT.TR)T1=TR
      T2=T1+TR
      K=K+1
      IF(K.GT.100) GO TO 1
      IF(ABS(DELTAB).GT.Y)GO TO 153
      IF(ABS(DELTT1).GT.Y)GO TO 153
      ZW=B
      W=B
      BETA=1./ZW-2.
      B1=2.*ZW-3.*W*W+ROFF
      B0=W*W*(1.-2.*ZW)+ROFF
      WRITE(3,600)Z,W,TR,T2,T1,BETA,B1,B0,K
      GO TO (180,178),L
178   W=B-D1
      M=1
      GO TO 5
180   WRITE(3,800)
      C=B
      B=B-D1
      A=1.-C-B
184   K=1
185   C=1.-A-B
      IF(C.LT.0.)C=0.
      IF(C.GT.1.)C=1.
      AB=A*B
      AC=A*C
      BC=B*C
      AMB=A-B
      AMC=A-C
      BMC=B-C
```

```
      DA=AMB*AMC
      DB=AMB*BMC
      DC=AMC*BMC
      EAT1=EXP(-A*T1)
      EAT2=EXP(-A*T2)
      EBT1=EXP(-B*T1)
      EBT2=EXP(-B*T2)
      ECT1=EXP(-C*T1)
      ECT2=EXP(-C*T2)
      F=.9-B*C*EAT1/DA+A*C*EBT1/DB-A*B*ECT1/DC
      G=.1-B*C*EAT2/DA+A*C*EBT2/DB-A*B*ECT2/DC
      H=T2-T1-TR
      FRA1=B*EAT1*(DA*(C*T1+1.)+C*(4.*A-B-1.))/(DA*DA)
      GRA1=B*EAT2*(DA*(C*T2+1.)+C*(4.*A-B-1.))/(DA*DA)
      FRA2=-EBT1*AMC*(DB+AC)/(DB*DB)
      GRA2=-EBT2*AMC*(DB+AC)/(DB*DB)
      FRA3=-B*ECT1*(DC*(A*T1+1.)-A*(4.*A+5.*B-3.))/(DC*DC)
      GRA3=-B*ECT2*(DC*(A*T2+1.)-A*(4.*A+5.*B-3.))/(DC*DC)
      FRA=FRA1+FRA2+FRA3
      GRA=GRA1+GRA2+GRA3
      FRT1=A*B*C*(EAT1/DA-EBT1/DB+ECT1/DC)
      GRT2=A*B*C*(EAT2/DA-EBT2/DB+ECT2/DC)
      J=FRA*GRT2-FRT1*GRA
      IF(ABS(J).LT.0.05)J=SIGN(0.05,J)
      DELTAA=(-F*GRT2+G*FRT1-H*FRT1*GRT2)/J
      DELTT1=(F*GRA-G*FRA+H*FRZ*GRT2)/J
      A=A+DELTAA
      IF(A.LT.0.)A=0.
      IF(A.GT.1.)A=1.
      T1=T1+DELTT1
      IF(T1.LT.0.)T1=0.
      IF(T1.GT.TR)T1=TR
      T2=T1+TR
      K=K+1
      IF(K.GT.100)GO TO 198
      IF(ABS(DELTAA).GT.Y)GO TO 185
      IF(ABS(DELTT1).GT.Y)GO TO 185
      C=1.-A-B
      B1=A*B+C*(A+B)+ROFF
      B0=A*B*C+ROFF
      SUM=A+B+C+ROFF
      WRITE(3,801)A,B,C,TR,T2,T1,SUM,B1,B0,K
      B=B-D1
      IF(B.GT.A)GO TO 184
198   L=2
      WRITE(3,501)
      GO TO 151
200   Z=0.
      K=1
202   W2=W*W
      ET1=EXP(-T1)
      ET2=EXP(-T2)
      S1=SIN(W*T1)
      S2=SIN(W*T2)
      C1=COS(W*T1)
      C2=COS(W*T2)
      D=1.+W2
      F=.9-(W*S1+C1+W2*ET1)/D
      G=.1-(W*S2+C2+W2*ET2)/D
      H=T2-T1-TR
      FRW=(S1*(-1.+T1+W2+W2*T1)+W*C1*(2.-T1-W2*T1)-2.*W*ET1*D)/(D*D)
      GRW=(S2*(-1.+T2+W2+W2*T2)+W*C2*(2.-T2-W2*T2)-2.*W*ET2*D)/(D*D)
```

```
 FRT1=W*(S1-W*(C1-ET1))/D
 GRT2=W*(S2-W*(C2-ET2))/D
 J=FRW*GRT2-FRT1*GRW
 DELTAW=(-F*GRT2+G*FRT1-H*FRT1*GRT2)/J
 DELTT1=(F*GRW-G*FRW+H*FRW*GRT2)/J
 W=W+DELTAW
 IF(W.LT.0.)W=0.
 IF(W.GT.1.)W=1.
 T1=T1+DELTT1
 IF(T1.LT.0.)T1=0.
 IF(T1.GT.TR)T1=TR
 T2=T1+TR
 K=K+1
 IF(K.GT.100)GO TO 1
 IF(ABS(DELTAW).GT.Y)GO TO 202
 IF(ABS(DELTT1).GT.Y)GO TO 202
 B1=W*W+ROFF
 B0=B1
 BETA=0.
 WRITE(3,600)Z,W,TR,T2,T1,BETA,B1,B0,K
 GO TO 1
 END
MON$$      EXEQ LINKLOAD
           PHASEENTIREPROG
           CALL MAINPGM
MON$$      EXEQ ENTIREPROG,MJB
```

APPENDIX E

FORTRAN PROGRAM FOR LINES OF CONSTANT PHASE MARGIN

```
     MON$$      JOB  252740040     K K GOWDY  PHASE MARGIN
     MON$$      ASGN MGO,A2
     MON$$      ASGN MJB,A3
     MON$$      MODE GO,TEST
     MON$$      EXEQ FORTRAN,,,,,,,MAINPGM
C     PROGRAM TO COMPUTE LINES OF CONSTANT PHASE MARGIN
C     FOR A LINEAR THIRD ORDER SYSTEM
C     PM=PHASE MARGIN     WC=CROSSOVER FREQUENCY
100   FORMAT(///29X,6HMARGIN,4X,2HWC,9X,2HB1,8X,2HB0)
101   FORMAT(29X,F5.0,F10.5,2F10.4)
      ROFF=.00005
      PM=10.
      PI=3.141593
3     GAM=PM*PI/180.
      WRITE(3,100)
      B1=1.25
      TANG=SIN(GAM)/COS(GAM)
5     WC=(-TANG+SQRT(TANG*TANG+4.*B1))/2.
      WC2=WC*WC
      B0=WC*SQRT(B1*B1+(1.-2.*B1)*WC2+WC2*WC2)+ROFF
      IF(B0.LT.0.)GO TO 60
      IF(B0.GT.B1)GO TO 60
      WRITE(3,101)PM,WC,B1,B0
      B1=B1-.05
      IF(B1.GT.0.)GO TO 5
      PM=PM+10.
      IF(PM.LT.90.)GO TO 3
60    STOP
      END
     MON$$      EXEQ LINKLOAD
                PHASEENTIREPROG
                CALL MAINPGM
     MON$$      EXEQ ENTIREPROG,MJB
```

VITA

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Candidate for the Degree of

Doctor of Philosophy

Thesis: A METHOD FOR THE ANALYSIS AND SYNTHESIS OF LINEAR THIRD-ORDER SYSTEMS

Major Field: Engineering

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