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BY

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VIBRATIONS OF RECTANGULAR SANDWICH PLATES

WITH VARIOUS EDGE CONDITIONS

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Siddique M. Dar

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CHAPTER I

RESUME OF DEVELOPMENTS IN THE ANALYSES OF SANDWICH PANELS

In general, a sandwich panel consists of a low density, thick core bonded to two relatively thin, strong face plates. Such a construction can be used to produce stiff, light-weight structural panels that are particularly adaptable to air-craft, ballistic missiles, and to space vehicle construction where least weight requirements are mandatory.

Most of the literature pertaining to sandwich structures treats theoretical and experimental investigations with static loads. Only limited information is available concerning either the theoretical analysis of dynamic loads on sandwich structures, or experimental results pertaining to the natural frequencies and mode shapes of vibrations of sandwich panels. Since vibratory phenomena embody the basic notions and assumptions of the static stress analysis, it seems relevant to give an account of the development of static stress analysis in this resume.

Developments in the Static Stress Analysis

Most of the work in the analyses of sandwich construction had been initiated and continued by the Forest Products Laboratory,* United States Department of Agriculture. Successive developments in the investigations of sandwich structures under the sponsorship of this agency have resulted in some of the basic design criteria for sandwich construction. In the early phase of the investigations, March (1) formulated a differential equation for the deflection on the following assumptions:

1. The facings are so thin that their flexural can be neglected and they can be treated as membranes.

2. The transverse shear stress components in the core are uniformly distributed and all other stress components are negligible.

3. The deflections of the panel are such that a small deflection theory can be employed.

4. The core as well as the facings are isotropic.

Even under these simplifying assumptions, the solutions of the resulting differential equations are very involved for other than simply supported plates.

Ericksen and March (2) extended the energy methods of analysis employed by British investigators like Leggett (3),

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* Maintained at Madison, Wisconsin in cooperation with the University of Wisconsin.

Williams (4), and Hopkins (5) to account for orthotropic cores in the buckling analysis of sandwich panels by assuming the following displacement functions:

$$u^c = -k\ (z-q)\frac{\partial w}{\partial x}$$

$$v^c = -h\ (z-r)\frac{\partial w}{\partial y}$$

$$w^c = w\ (x,y)$$

where u^c, v^c, and w^c are core displacements in the x, y, and z directions, and k, h, and r, free constants. Thus the assumption that plane sections before deformation remain plane after deformation was retained with a slight modification; that is, plane sections rotate about a line $z = q$ for the x-displacements and $z = r$ for the y-displacement. It is obvious that, for a sandwich panel with facings of unequal thickness, the middle plane of the core is not the neutral plane of the panel. Also, the assumption of different proportionality factors k and h is readily understandable by considering that the elastic properties of the core material may differ greatly in both directions. In this connection Chang and Ebcioglu (6) comment:

> ...For pure bending, u^c and v^c are justified. But the authors could not see the need for evaluating some of their constants (such as q and r) by means of minimizing the energy integral. To the authors's knowledge, the value of q and r depends only on the elastic and geometrical arrangement of the sandwich cross section and should not depend on the minimum of energy integral. Fortunately, their choice for w^c was good and their results check with the present theory...

This point will be pursued further in Chapter III where this writer intends to show that q and r can be justifiably construed as unknown parameters whose true values are determined by minimizing the total energy. The Ericksen-March method remained the basic approach at the Forest Products Laboratories where subsequent work has been conducted with the aim of improving the Ericksen-March displacement functions. In this method the total strain energy consists of energy due to shearing strains in the core, and the energies due to membrane and bending strains in the facings. Most of the work, however, has been confined to simply-supported plates (7), (8) which are governed by functions which, including their first and second derivatives, are orthogonal and, additionally, satisfy the boundary conditions approximated by Hoff (9)*.

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*The exact boundary conditions for a simply-supported plate, obtained by means of variational methods, are:

(1) $\frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} = 0$ at $x=0$, $x=l_x$

(2) $\frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} = 0$ at $y=0$, $y=l_y$

(3) $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$ at $x=0$, $x=l_x$, $y=0$, $y=l_y$

(4) $\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} = 0$ at $x=0$, $x=l_x$

(5) $\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} = 0$ at $y=0$, $y=l_y$

(6) $w=0$ at $x=0$, $x=l_x$, $y=0$, $y=l_y$

where u, v, and w are the displacements in the x, y, and z directions and μ is the Poisson's ratio. Hoff approximated the condition (3) by 3(a) $\equiv$ $u=0$ at $y=0$, $y=l_y$
$v=0$ at $x=0$, $x=l_x$

Thurston (10) applied the Lagrangean Multiplier method to the energy expressions of Ericksen-March (2) to determine deflections and buckling loads of rectangular plates clamped on all four edges. The Lagrangean Multiplier method was first employed by Budiansky and Hu (11) in their buckling analysis of clamped rectangular solid plates. The method is an application of the minimization of functions subject to certain constraints. These constraints arise from the lack of appropriate functions which satisfy the boundary conditions of plates other than simply-supported. A parameter, called Lagrangean Multiplier, is associated with each constraint. Considering, for example, a clamped beam of length 'a' with coordinate origin at one of the extremities, the boundary conditions may be written as:

(a) $w(0) = w(a) = 0$

(b) $\frac{\partial w}{\partial x}(0) = \frac{\partial w}{\partial x}(a) = 0$

where w may be assumed as

$$w = \sum_{m}^{\infty} A_m \sin \frac{m\pi x}{a}$$

It is clear that the boundary condition (a) is satisfied, but not (b). In order to satisfy (b) the equation

$$\sum_{m}^{\infty} \left(\frac{m\pi}{a}\right) A_m = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

must hold, where only even values of m enter. If the total energy of a system is denoted by U, then the Lagrangean method demands that

$$\frac{\partial U}{\partial A_m} - \lambda \sum_{m}^{\infty} \left(\frac{m\pi}{a}\right) A_m = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where λ is the Lagrangean Multiplier. Equations (1) and (2) contain the unknown λ and implicitly in u, the buckling load p_c.

In general, more than one constraining condition exist. In fact, in case of a plate problem involving double summations, there are infinitely many λs. Further, if the analysis considers displacements in x and y directions, more than one infinite set of λ's occur. However, an exact solution for buckling loads or static deflections would lead to an infinite determinant. To obtain approximate results, two alternative procedures have been suggested:

1. An upper bound solution is achieved when all the boundary conditions are satisfied while working with only a finite number of undetermined coefficients. This has the effect of stiffening the plate and raising the buckling load.
2. A lower bound solution is attained when all the coefficients are used, but not all the constraints. This relaxes the restraints on the boundaries and makes the plate more flexible, thus decreasing the buckling load.

Although the Lagrangean Multiplier method has contributed much to the handling of more complex boundary conditions, the amount of work involved is tremendously large. The solutions are not in closed form and the attained accuracy hardly warrants the numerical effort. The results of Thurston (10), for example, are within 3% of the values predicted by Ericksen and March on the basis of methods which do not involve such lengthy computational procedures. In addition, the Lagrangean Multiplier method does not accomodate conditions which are functions of z. Yet it is not uncommon to find the deflections w as functions of z in sandwich construction, for buckling or static deflection investigations. Thus, further simplifying assumptions are necessary for the evaluation of w over the plate thickness at the edges. For simple beam problems, however, the Lagrangean Multiplier method is not too complicated despite the open form solution; Raville (17) applied it to the vibrations of a sandwich beam clamped on both edges, by employing only one constraint and assuming w to be constant over the thickness.

Chang and Ebcioglu (6) studied the elastic stability of sandwich panels. Their assumptions do not seem to deviate from those of Thurston (10) and Ericksen-March (2), and their analysis of the temperature differences in the facings could also have been included in the works of previous authors. However, their approach is based on the derivation of three differential equations by means of the principle of virtual displacements. The solution of the differential equations has been achieved by assuming the displacements in the following form:

$$u=C_1 \cos\frac{\pi x}{a} \sin\frac{n\pi y}{b}$$

$$v=C_2 \sin\frac{\pi x}{a} \cos\frac{n\pi y}{b}$$

$$w= {}_3 \sin\frac{\pi x}{a} \sin\frac{n\pi y}{b}$$

where u, v, and w are displacements in x, y, and z directions respectively. These functions do satisfy the boundary conditions of a simply supported plate.* For any other boundary condition, the difficulty of finding suitable displacement functions still continues. Also, when a summation over n is introduced, the functions and their derivatives, in general, do not exhibit orthogonal properties, resulting in open form solutions. Furthermore, the authors admit that whereas rather exact strain distribution are obtained for sandwich panels with weak cores, the results become unfavorable when the stiffness of the core approaches that of the facings. In this connection it is worthwhile to point out that Bijlaard (12) developed equations for bending and torsional moments by assuming the temperature of the upper surface of a rectangular sandwich plate to be higher than that of the lower one. Plates with simply supported edges, and plates with two edges simply supported and the other two free or clamped, have been dealt with. The calculations are very involved as in all investigations which treat the problem by the differential equation approach.

Raville and Kimel (13) investigated the problem of elastic stability of a simply supported beam by both the differential equation and the energy methods. They showed that two methods give identical results under the same set of assumptions. Although neglecting the elasticity of the core

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*See footnote on page (4).

in the plane of the core, Raville and Kimel (13) assumed non-zero strains in the thickness direction of the core. This is perhaps a very desirable improvement, since the core is generally very weak as well as thick in comparison to the facing materials. The assumption that w is independent of z may be justified in solid plates of homogeneous and isotropic material, but may lead to discrepencies in the analysis of sandwich plates, particularly if the core is weak and very thick in comparison to the facings. The authors have termed this state of stress as "antiplane" stress. In other words, the usual two dimensional stress components σ_x, σ_y, and τ_{xy}, are assumed zero, and the non-zero components of stress are σ_z, τ_{xz}, and τ_{yz}. This method of analysis has resulted in not only "Euler type" buckling criteria, but also in "face-wrinkling" mode shapes and their associated critical values. An examination of their results reveals that for relatively short panels with thin facings and very weak cores, face-wrinkling, rather than the usual "Euler type" buckling failure, occurs. However, for sandwich construction of usual proportions and physical properties, the possibility of face-wrinkling failure is very remote. The phenomenon of face-wrinkling has also been investigated by Weikel and Kobayashi (14) in connection with the local buckling of a honeycomb face plate. Their assumptions, besides being numerous, are somewhat more restrictive. For example, their theory applies to honeycomb cells which are square in shape, and in which the cells are oriented such that the load is parallel to the diagonal of the cell. Considering the face plate over each cell as a simply-supported square plate, a deflection surface was assumed for "intercell buckling," i.e., the buckling of the honeycomb face plate within each individual cell. For "wrinkling type" buckling which is the buckling of the honeycomb face plate over a row of several honeycomb cells, another deflection is the superposition of the two deflections from which, by means of energy method, the final results were established. These results show that for a given face and cell size of the core, the predominant mode of buckling failure will be wrinkling for a relatively weak core. With increasing core stiffness, intercell buckling becomes predominant.

Developments in the Dynamic Stress Analysis

As pointed out in the beginning, the literature on the dynamical characteristics of sandwich panels is very limited. Until 1959 almost all the investigations pertained to the bending and buckling phenomena of sandwich construction. In fact, the first publication in connection with the dynamics of sandwich plates did not appear until September, 1959. This publication by Yu (15) dealt with "One Dimensional Theory" of isotropic sandwich panels.

In his analysis, Yu included the rotary inertia and the transverseshear deformation of the core and of the facings. The analysis is based on the state of plane-strain which yields the following two equations:

$$\int_V \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} - \rho \frac{\partial^2 u}{\partial t^2} \right) \delta u dv = 0$$

$$\int_V \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} - \rho \frac{\partial^2 \omega}{\partial t^2} \right) \delta \omega dv = 0$$

where u and w are displacements in the x and y directions, respectively. In order to express the stresses in terms of strains, the following displacement functions were assumed:

$$u^c = z\psi_1$$

$$u^b = z\psi_2 - \frac{c}{2}(\psi_1 - \psi_2)$$

$$u^a = z\psi_2 + \frac{c}{2}(\psi_1 - \psi_2)$$

$$w^1 = w^a = w^b = w^c = w$$

where u^c, u^a, u^b are the x-displacements in the core, the upper facing and the lower facing, respectively. These assumptions may be summarized as:

1. The displacements in the plane of the panel vary linearly over the thickness of the plate.

2. At each point of the panel, the vertical cross-section of the two facings rotate through the same angle; the rotation of the core is different from that of the facings.

3. The core is incompressible in the thickness direction. On expressing stresses in terms of strains, the resulting differential equations were governed by three variables, ψ_1, ψ_2, and w. Thus the above analysis is applicable to panels having three solid isotropic plates and where the top and the bottom plates have the same thickness and the same physical properties. Yu investigated two cases, one dealing

with the propagation of waves in an infinitely long panel, and the other with the bending phenomenon of a cantilevered isotropic sandwich panel.

Yu's analysis has thrown light on the variation of shearing stresses over the thickness of the sandwich plane by considering bending of the core in the two directions and introducing a correction of coeffieient for a possible variation of shearing stresses along the thickness. This procedure is analagous to that of Mindlin's (16) who determined shear coefficient corrections for homogeneous plates. Yu concludes that the appropriate value for the shear correction is nearly equal to unity for the most common sandwich panels. For present sandwich constructions, a uniform shear distribution across the thickness of the core is assumed, since the core is very weak in resisting the normal stresses in the plane of the plate as compared to the facings.

In a separate paper entitled "Flexural Vibrations of Sandwich Plates," Yu (15a) applied this theory to an infinite plate in plane strain and established a cubic equation for the frequency. The three branches of the frequency curve were explained in the following manner: The first branch is attributed to the assumption of bending and membrane strains in the facings and the transverse shears in the core. The second branch is associated with the rotary inertia of the core and the third branch, which corresponds to the highest frequency, is eliminated when the rotary inertia and transverse shear in the facings are neglected, thereby reducing the cubic to a second degree equation.

In spite of the simplifying assumptions of isotropy, identical facings, and plane strain, the resulting equations were so involved that the author neglected also the rotary inertia and the transverse shear in the facings, and additionally introduced further approximations, for example, that the ratio of facing-to-core thickness is negligible. The simplified analysis was then applied to a simply-supported plate. The established results bear out the importance of the shear effect in the core, particularly in the high frequency range. Yu (15a) concludes:

> Except for the plates with thin faces and sufficiently low frequency ranges, the flexural rigidities of the faces about their own middle planes must be included even though they may be small by themselves. As a consequence, the system of equations of motion has to remain sixth order and cannot be reduced, which at once complicates the vibration problem of finite sandwich plates with boundary conditions other than that of simply-supported...

In order to overcome the difficulty of solving the differential equation, the author, in a later publication entitled "Simplified Vibration Analysis of Sandwich Plates," (15b) modified his theory for sandwich plates with very thin facings. The modified displacement functions were:

$$u^c = z\,\psi_1$$

$$u^a = -h_1\psi_1$$

$$u^b = +h_1\psi_1$$

$$w^i = w$$

Yu (15b) suggested that the rotary inertias and the flexural rigidities may be disregarded for investigations in the low frequency range. Employing these assumptions, he established relatively simple equations of motion. While the analysis was applied to an infinite sandwich plate, no results for any finite plate were given.

Chang and Fang (18) deal with the periodic response of a loaded sandwich panel. Their approach, though similar to that of Yu (15b), is not as general. Among other commonly used assumptions, they neglect the bending rigidity of the facing, but consider the rotary inertia. As in reference (15a), the resulting differential equations are not easy to solve for plates with boundary conditions other than the simply-supported. The authors (18) demonstrated the justification for neglecting the rotary inertia for long wave lengths.

The frequency response functions have also been investigated by Bieniek and Freudenthal (19). Their analysis is confined to a simply-supported case and takes into account material damping of the two identical facings and the core by using complex moduli of elasticity.

$$\bar{E}_{xx} = E_{xx}\,(1 + i\eta_{xx})$$

$$\bar{G}_{xz} = G_{xz}\,(1 + i\mu_{xz})$$

where η_{xx}, and μ_{xz}, are the damping coefficients. The remaining assumptions were the same as in (15) and (18). The three parameters were determined by using the displacement functions for a simply-supported plate.*

Recently Chu (20) and Freudenthal (20) assumed non-linear displacement functions with an aim of analyzing the large deflection problem of sandwich panels. According to this theory, the strains were expresses as:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2;$$

where u and w are displacements in the x and y directions, respectively, and ϵ_{xx} is the strain in the x direction; the strains in the facings are then

$$\epsilon^a_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \varphi}{\partial x}$$

The discussion was limited to very low frequencies of a simply-supported beam. No variation of w over the thickness was considered. Chu (20) concluded that in non-linear vibration analysis of honeycomb sandwich construction, the influence of transverse shear deformation can, as a rule be neglected.

Raville (17), utilizing his earlier assumptions, (see 13) and disregarding the elasticity of the core in the thickness direction, investigated the natural frequencies of a clamped-clamped beam by an energy approach in which the Lagrangean

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*See footnote on page (4).

Multiplier method was applied to satisfy the boundary conditions. As stated earlier, the solution is in the form of infinite series and loses its simplicity when applied to two demensional problems. However, the energy approach does not present the difficulties that are inherent in Yu's (15) approach even though the assumptions of Raville do not differ from those of Yu with the exception of the inclusion of rotary inertia. Excellent correlation between theory and experiment was established with most of the theoretical values agreeing within 5% of the experimental values.

Conclusions

From the developments in the foregoing resume one concludes that most investigators pursued either the differential equation approach, or the energy approach.

The differential equation method, though potentially more accurate than the energy method, often presents difficulties in finding exact solutions of the equations, so that generally, additional simplifying assumptions become necessary. The existing literature on the subject does not contain investigations of finite plates with complex boundary conditions, especially in regard to the vibrations. The assumption of a state of stress in the core which is similar to that of the facings simplifies the analysis, but in the opinion of this writer, confines it to a particular type of core material whose elastic behaviour must not differ much from these of the face plates.

Generally, however, this is not the case in sandwich construction. If the plane strain assumption is neglected, the differential equation approach becomes an almost impossible task, on the other hand, the energy apprcach has been applied with relative ease. Additional assumptions for further refinement can be incorporated into the theory with little difficulty. The displacement functions generally imply summations of infinite series. However, these can be approximated by modern computing techiques. Yet, excepting simple beam structures, the energy methods have not been fully utilized for vibrations of sandwich plate structures. In the opinion of this writer, the energy approach affords a much better tool for solving plate problems with complex edge conditions.

It is with this hope that the writer has undertaken an investigation of vibrations of rectangular sandwich plates for establishing the normal mode frequencies. The investigation is carried out in such a manner that various edge conditions can be accomodated and that no restrictions are placed on the strains in the core, particularly in the thickness direction.

CHAPTER II

EXPOSITION OF METHOD OF ANALYSIS

INTRODUCTION

This dissertation aims to present an analysis of the normal mode frequencies of rectangular plates with various edge conditions. Since the fundamental frequency is usually the frequency of main interest to the structural engineer, the emphasis of this investigation will be laid on the low-frequency ranges. As indicated in Chapter I, most of the work in this field has been confined to simply-supported cases. These analyses become impractical when applied to finite plates with complex edge conditions. However, from a practical viewpoint, the situations under which a plate may be treated as simply-supported are very limited. In general, one is confronted with cases where other than simply-supported edge conditions correspond more closely to actuality. A vibratory analysis of plates with clamped, free, or simply-supported edges, or arbitrary combinations, will prove more significant in meeting the growing demands for an economical utilization of sandwich structures, particularly where strength-weight ratios are important as in air-craft or ballistic missiles.

It is also desirable that a unified approach to the vibrations of sandwich plates be developed in such a way that the analysis is directly applicable to all possible edge conditions.

In the opinion of this writer, the existing theories can be further improved by disregarding the simplifying assumption of core rigidity in the thickness direction, resulting in more exact strain distribution in the core and better accuracy for vibrational characteristics of the plates.

Assumptions

The proposed analysis of sandwich plates will be based on the following assumptions. Some of these are recognized as standard assumptions for common types of sandwich construction. More rigorous assumptions regarding the core and the facings are included here, thus rendering this analysis more general in comparison to the vibratory investigations outlined in Chapter I.

Facings:

(a). The facings of the sandwich plate are of two different materials and thicknesses.

(b). The facings are thin, orthotropic elastic plates, and are subject to membrane as well as bending strains.

Core:

(c). The core is an orthotropic elastic continuum. The two axes of orthotropy are characterized by "transverse" and "ribbon" directions as shown in Fig. 1.

(d). The bending stiffness of the core about the transverse axis is negligible.

(e). The displacements in the core vary linearly across the thickness of the core.

(f). The core carries no shearing stress in the plane of plate.

(g). The core is subject to transverse shearing stresses in (Z-T) and (Z-R) planes which are uniformly distributed over the thickness of the core.

This assumption of constant shearing stresses over the thickness of the core is justified in the light of Yu's investigation.

Bond:

(h). All bonds are strong enough to assure continuity of stresses.

Yu and Raville analyzed vibrations of sandwich panels by assuming isotropic facings of the same material. Chang (18) conducted his analysis by considering facings of two different materials, but with the same Poisson's ratio. He, in addition, neglected the bending rigidity of the facings. The assumption of zero normal stresses in the core reduced the orthotropy of the core to two shear moduli in the work of Chang and to one shear modulus in Raville's one dimensional case.

To this author's knowledge, no literature exists in which non-zero strains in the thickness direction of the core are considered in connection with the vibratory analysis. The consideration of finite elasticity of the core in its thickness direction will not only include in the analysis symmetrical modes of vibration of the plate about its neutral plane, but will also establish more accurate results, particularly for weak cores.

Materials commonly used for sandwich construction exhibit no isotropic behaviour. Not only is the core anistropic, but the facing materials in general also possess anistropic characteristics. In order to facilitate the analysis of stress and strain, some further simplifying assumptions become necessary with regard to the anistropy. For common types of sandwich materials, the assumption of orthotropy is frequently employed. The necessary stress-strain relations based on orthotropy are described in Appendix (B).

Method of Analysis

Regarding the method of analysis, this writer feels that the energy approach is best suited for establishing a general and a refined vibratory analysis of sandwich plates. No proofs of energy principles and variational methods are included here; detailed accounts of energy methods are given in any standard text like those of Timoshenko (21), Bleich (22), Langhaar (23), and Wang (24). Suffice it to say that the energy method for the solutions of the problems of vibrations is founded on an extremum principle of mechanics utilizing an energy criterion which characterized the conditions of equilibrium in an elastic system. This energy criterion, generally called the "Minimum of the Potential Energy," states (24):

> Of all displacements satisfying given boundary conditions, those which satisfy the equilibrium conditions make the potential energy V assume a stationary value. For stable equilibrium V is minimum.

As a consequence of this minimal principle, the governing differential equations are obtained by means of the calculus of variation. (See 15a, 18, 19). However, instead of solving the differential equations together with the boundary conditions, an often difficult mathematical task, one may interpret the problem as that which seeks functions that minimize and satisfy the potential energy of the system. Several approaches have been suggested for finding the solutions of boundary-value problems of which the methods of Rayleigh and Ritz are of prime importance.

Rayleigh's Method: A mechanical system with infinitely many degrees of freedom may be reduced to a system with finite degrees of freedom by means of assumptions regarding the nature of deformation. This idea was first employed by Lord Rayleigh (27) in his studies of vibrations.

According to Rayleigh's method, when a conservative system vibrates freely, the total mechanical energy is constant. Assuming the system vibrates in a normal mode, the particles will execute simple harmonic motions and if the mean values of the kinetic and potential energies, i.e., averages over a long period of time, are (1/2)T and (1/2)V, the conservation of the total energy leads to $T_{max} = V_{max}$ which in essence is Rayleigh's frequency criterion.

It was Rayleigh's idea to assume a configuration close to the actual configuration of the vibrating system for determining the kinetic energy, T_{max}, and the potential energy V_{max}, of the system. The choice of a definite shape for a deflection curve in this method is equivalent to introducing additional constraints which reduce the system to one having a single degree of freedom. Such additional constraints can only increase the rigidity of the system and lead to a frequency of vibration in excess of the exact value.

Rayleigh-Ritz Method: In order to achieve better accuracy for frequencies as well as closer estimates for mode shapes Ritz, in 1909, refined and generalized Rayleigh's method which has since then been called the Rayleigh-Ritz method. Basically, Ritz suggested that assumed deflection curves be expressed by the sum of several functions in the form

$$w = C_1\phi_1 + C_2\phi_2 + \cdots\cdots, \qquad (2.1)$$

in which the ϕ – terms represent an arbitrarily chosen set of functions of x and y, satisfying the same boundary conditions as the deflection w and where the coefficients C_i are undetermined parameters. Substituting w in the expression for total energy of the system and performing the required mathematical operations leads to the relation

$$V_{max} - T_{max} = F_1\ (C_1, C_2 \ldots\ \ n) - F_2\left[\Omega^2(C_1, C_2, \ldots C_n)\right]$$

in which F_1 and F_2 are quadratic forms of the parameters C_i. If w is to be regarded as a solution of the extremum problem it must satisfy the condition that the function $V_{max} - T_{max}$ assume a stationary value. Consequently, the parameters C_i must be selected to make the expression $F_1 - \Omega^2 F_2$ stationary. The problem therefore becomes an ordinary extremum problem in which C_1, C_2, ...C_n are variables to be obtained from the n conditions, $\frac{\partial}{\partial C_i}(F_1 - \Omega^2 F_2) = 0$; $i = 1, 2, \cdots n$. This operation results in a system of n homogeneous equations which are linear in the n parameters, C_i. For such a set of equations, a nontrivial solution is only possible when the determinant of the coefficients vanishes. This establishes the required frequency criterion. Thus the importance of the energy criterion for the solution of vibration problems becomes evident in the light of the Rayleigh-Ritz method which leads to a direct solution of the extremum problem.

When based upon an appropriate set of co-ordinate functions, the Ritz method furnishes a sequence of parameters C_i which diminish in many cases so rapidly that only a few terms of the series (2-1) suffice to determine the frequency with the required degree of accuracy.

Timoshenko (25), p. 371, while investigating the vibrations of a string, found that with only one parameter C_1, the result for the fundamental frequency was 0.66% higher than the exact frequency. By taking two such parameters, the error was reduced to less than 0.001%. When three terms of the expression (2-1) were considered, the error for the third mode of vibration was found to be less than $\frac{1}{2}$%.

It is seen that by using the Ritz method, not only the fundamental frequency but also the frequencies of higher modes of vibrations can be obtained with good accuracy by taking a sufficient number of terms of the expression (2-1). The major advantage of the Ritz method lies in the fact that it provides approximate solutions of the extremum problems in those cases where an exact solution of the characteristic-value problem becomes too difficult or is not practical. The method can also be applied with much advantage to the frequency calculation in less difficult problems, since it requires less effort than the solution of a complex transcendental equation.

The accuracy of the Ritz method depends largely on the proper choice of co-ordinate functions. These must be, of necessity, "admissible;" that is, they must satisfy the so-called "artificial boundary conditions." (See 27). These two types of boundary conditions are also known as "geometric boundary conditions" and "dynamic boundary conditions," respectively. In case of plates, the deflection and slope requirements constitute artificial boundary conditions, while the demand that the second or third derivatives vanish at the boundary is a natural boundary condition. From a practical consideration of the rate of convergence, it is desirable to satisfy the natural boundary conditions if possible. There is no other restriction as to the form of these functions. However, if these functions are orthogonal, a considerable simplification is achieved in the evaluation of energy expressions. For this reason the Fourier series play such a paramount role in the applications of the Ritz method in the theory of elasticity. The use of these series, of course, is limited to problems whose boundary conditions are in accord with the boundary values of the co-ordinate functions of a Fourier expansion.

It should be noted that the Ritz method does not provide the means for gauging the accuracy of the results obtained. While the accuracy is obviously increased by taking more terms, the only way to judge the convergence of the series is by comparing results obtained with increasing numbers of terms. However, this lengthy process was shortened by Trefftz (28) who, in 1935, supplemented the Rayleigh-Ritz method by establishing a bound of the characteristic-value problem. This permits one to enclose the solution between an upper and a lower limit-an important criterion for juding the accuracy of the solution. The physical interpretation of the lower limit solution was mentioned in Chapter I where it was applied for establishing the buckling criterion of a plate. The assumed displacement functions of the plate did not satisfy all the edge conditions, and consequently the analysis was carried out by the Lagrangean Multiplier method. (See10).

Thus the Ritz method is not confined to those functions alone which satisfy the boundary conditions, but may be extended to such problems where all the boundary conditions, are not satisfied by the assumed functions. In this case, the problem reduces to what is generally known as the "constrained extremum."

Determination of Co-ordinate (characteristic) Functions

Most of the work on vibrations of sandwich plates has been limited to simply-supported cases because for these edge conditions, co-ordinate functions are mere sine and cosine series which not only satisfy the boundary conditions, but in addition have orthogonal properties. Therefore, the integration of the energy expressions is simple and the final results can be expressed in closed form. For plates with arbitrary edge conditions, other means must be employed for obtaining the co-ordinate functions.

For the purpose of meeting the edge conditions of a rectangular plate in two perpendicular directions, x and y, the co-ordinate functions of two beams, one having the same edge conditions as those of the plate in one direction, say x, and the other having the same edge conditions as those of the plate in the perpendicular direction y, are introduced. If these co-ordinate functions are based upon the exact configuration of the vibration beams, the results obtained from the Rayleigh-Ritz method are highly accurate. Therefore, in search of co-ordinate functions, it is desirable to seek the solution of the differential equation of a vibrating beam with arbitrary edge conditions.

The governing differential equation for a freely vibrating beam of uniform cross section is given by

$$EI \frac{\partial^4 w}{\partial x^4} + \rho \frac{\partial^2 w}{\partial t^2} = 0, \qquad (2.22)$$

The general solution of this differential equation is

$$w = \left[A_1 \cosh\lambda x + A_2 \cos\lambda x + A_3 \sinh\lambda x + A_4 \sin\lambda x \right] \cdot \left[C \sin pt + D \cos pt \right] \qquad (2.3)$$

where $\lambda^4 = \rho \frac{p^2}{EI}$, λ being the shape parameter, and p the circular frequency. Since the boundary conditions are time-independent, they can only be reflected in the expression in the first bracket of (2.3)

$$A_1 \cosh \lambda x + A_2 \cos \lambda x + A_3 \sinh \lambda x + A_4 \sin \lambda x = X(x) \qquad (2.4)$$

Since $X(x)$ is a function of the co-ordinate x which determines the shape of the normal mode of vibration, it is called the "shape function" or "normal function." The constants A_i are determined by the conditions of restraint at both ends of the beam, resulting in a set of four linear homogeneous equations. The solution of such a system demands that the determinant of the coefficients vanish, which furnishes an equation for the shape parameter λ , the only unknown in the equation.

This equation is, in general, a transcendental equation, having an infinite number of roots λ_n; $(n = 1, 2, ...)$ which are referred to as the "characteristic values" of the parameter λ_n and define an infinite number of frequencies.

Substitution of one of the characteristic values λ_n into the normal function (2.4) yields four equations for the four constants A_{1n}, A_{2n}, A_{3n}, and A_{4n}. From equilibrium and compatibility considerations it follows that the rank of the determinant of the coefficients of these equations is three, so that only three independent equations in four unknowns are available. From these three equations the ratios $\overline{A}_{2n} = A_{2n}/A_{1n}$, $\overline{A}_{3n} = A_{3n}/A_{1n}$ and $\overline{A}_{4n} = A_{4n}/A_{1n}$ can be determined, and the solution (2.4) assumes the form

$$X_n = \left[A_{1n}\left(\cosh\lambda_n x + \overline{A}_{2n}\cos\lambda_n x + \overline{A}_{3n}\sinh\lambda_n x + \overline{A}_{4n}\sin\lambda_n x\right)\right] \qquad (2.6)$$

where A_{1n} remains an arbitrary constant.

The mode shapes can, therefore, only be found within an arbitrary constant by using the end conditions alone. X_n are called "characteristic functions" of the homogeneous differential equation (2.2) associated with the particular boundary condition of the case considered.

For a simply-supported beam, the equation (2.6) assumes a very simple form, (see Timoshenko 25). The normal function is

$$X_n = A_{1n}\,(\sin\lambda_n x)$$

where $\lambda_n = \frac{n\pi}{L}$ and the frequency is given by

$$p_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{EI/\rho}$$

For other edge conditions, the characteristic functions retain the same form as given in (2.6). Appendix (A) contains tables giving the values of the characteristic functions and their first three derivatives for each of the first five modes (n = 1,2,3,4,5) of three different types of beams, namely,

Table 1 ____ Clamped-Clamped beam 

Table 2 ____ Clamped-Free beam 

Table 3 ____ Clamped-Supported beam 

These tables have been prepared by Young and Felgar (29) at the University of Texas.

For further reference, equation (2.6) is used in the form

$$X_n(x) = \cosh \beta_n \frac{x}{a} - \cos \beta_n \frac{x}{a} - \alpha_n \left(\sinh \beta_n \frac{x}{a} - \sin \beta_n \frac{x}{a} \right)$$

where (β_n/a) is the shape parameter whose values can be directly read from Appendix (A).

The characteristic function for a free-free beam is the same as the second derivative of the characteristic function for a clamped-clamped beam, and the characteristic function for a free supported beam is the same as the second derivative of the characteristic function for a clamped supported beam. Therefore, the values of the functions for these two additional cases can be obtained directly from the tables and their first mode shapes correspond to n = 3. The only type of beam not included is the supported-supported beam for which the characteristic function is ordinary trigonometric sine function.

The orthogonal properties of the characteristic functions are discussed in detail in reference (27) and the results reproduced here:

$$\int_0^{\ell} X_m X_n \, dx = \ell \text{ for } n = m$$
$$= 0 \text{ for } n \neq m$$

The second derivatives of these functions are also orthogonal and satisfy the relations,

$$\int_0^{\ell}\left(\frac{\partial^2 X_m}{\partial x^2}\right)\left(\frac{\partial^2 X_n}{\partial x^2}\right) dx = \frac{(\beta_n)^4}{(\ell)^4}(\ell) \qquad \text{for } n = m$$

$$= 0, \qquad \text{for } n \neq m$$

With the exception of X_1 and X_2 for a free-free beam for which

$$\int_0^{\ell}\left(\frac{\partial^2 X_1}{\partial x^2}\right)^2 dx = \int_0^{\ell}\left(\frac{\partial^2 X_2}{\partial x^2}\right)^2 dx = 0$$

It should be noted that the integrals of the type

$$\int_0^{\ell} X_m \left(\frac{\partial^2 X_n}{\partial x^2}\right) dx \quad , \quad \int_0^{\ell}\left(\frac{\partial X_m}{\partial x}\right)\left(\frac{\partial X_n}{\partial x}\right) dx$$

do not in general possess orthogonal properties. The evaluation of such integrals for different end conditions is given in Appendix (A).

CHAPTER III

THEORETICAL ANALYSIS

In chapter II a method of analysis based upon the Rayleigh-Ritz approach which utilizes the energy criterion for the characteristic-value problems was presented. As an aid to understanding, the characteristic functions of a homogeneous beam with arbitrary end conditions were formulated to indicate their application to homogeneous plate problems. In this chapter these characteristic functions are employed in the Rayleigh-Ritz method for obtaining the natural frequencies for a freely vibrating sandwich plate with arbitrary edge conditions.

The coordinate system used in this analysis is shown in Fig. 2a. The axes are oriented such that the "transverse" and the "ribbon" directions of the core are parallel to the x and y axes respectively.

The analysis commences by establishing the strain and the kinetic energies of a vibrating plate. Owing to the peculiarities of the stress distributions, the energies in the core and the facings are derived separately.

<u>Core</u>:

In developing an expression for the strain energy in the core, the displacement functions u^c, v^c, and w^c in the x, y, and z directions are assumed in the following form:

$$u^c = \sum_m^\infty \sum_n^\infty X_{1m}(x)\; Y_{1n}(y)\; \theta_{mn}(z) f(t)$$

$$v^c = \sum_m^\infty \sum_n^\infty X_{2m}(x)\; Y_{2n}(y) \Phi_{mn}(z) f(t) \qquad \ldots(3.1)$$

$$w^c = \sum_m^\infty \sum_n^\infty X_{3m}(x)\; Y_{3n}(y) \Psi_{mn}(z) f(t)$$

where X_{im} are functions of x only; Y_{in} are functions of y only, $(i = 1,2,3)$; θ_{mn}, Φ_{mn}, and Ψ_{mn} are functions of z only; and f(t) is a harmonic function of time. The choice of the functions X_{im} and Y_{in} depend upon the edge conditions. In choosing these functions, use is made of the characteristic functions discussed previously and given in Appendix (A) for various edge conditions. For example, in the case of a rectangular plate clamped along the edge x=0 and free along the edges x=a, y=0 and y=b, the characteristic function for a clamped-free beam, given on page 108, should be used for X_{3m} and the characteristic function for a free-free beam, given on page 107, should be used for Y_{3m}. For consistency of notation, when one of the characteristic functions is used for X_{3m}, ℓ is replaced by a; if it is used for Y_{3m}, ℓ is replaced by b and x is changed to y.

The functions X_{im} and Y_{in}, $(i = 1,2)$ are then determined by using the tables in Appendix (A). To satisfy the compatibility conditions in the plane of the plate in addition to the "geometric" boundary conditions, the following replacements in the equations (3.1) become necessary:

X_{1m} is replaced by X'_{3m}

X_{2m} is replaced by X_{3m}

Y_{1n} is replaced by Y_{3n}

Y_{2n} is replaced by Y'_{3n}

where X'_{3m} and Y'_{3n} are the first derivatives of X_{3m} and Y_{3n} with respect to x and y respectively.

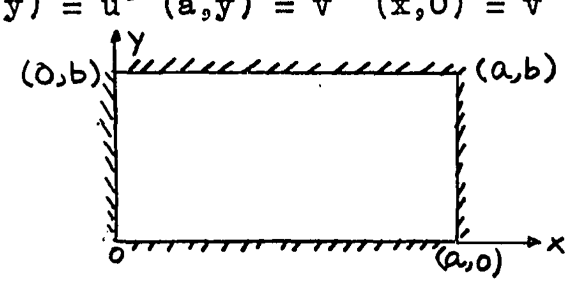
Making the above substitutions and discarding the subscript 3, the displacement functions (3.1) assume the form:

$$u^c = \sum_m^{\infty}\sum_n^{\infty} X_m(x)\, Y_n(y)\, \theta_{mn}(z)\, \sin\Omega_{mn}t$$
$$v^c = \sum_m^{\infty}\sum_n^{\infty} X_m(x)\, Y_n(y)\, \Phi_{mn}(z)\, \sin\Omega_{mn}t \qquad \ldots(3.2)$$
$$w^c = \sum_m^{\infty}\sum_n^{\infty} X_m(x)\, Y_n(y)\, \Psi_{mn}(z)\, \sin\Omega_{mn}t$$

where Ω_{mn} is the natural frequency of the m^{th} mode in the x direction and the n^{th} mode in the y direction.

To check the validity of the displacement functions (3.2), an example of a rectangular plate clamped along the four edges is considered. The edge conditions of such a plate with sides a and b are:

$$w^c\,(0,y) = w^c\,(a,y) = w^c\,(x,0) = w^c\,(x,b) = 0$$
$$\frac{\partial w^c}{\partial x}(0,y) = \frac{\partial w^c}{\partial x}(a,y) = \frac{\partial w^c}{\partial y}(x,0) = \frac{\partial w^c}{\partial y}(x,b) = 0 \quad \ldots(3.3)$$
$$u^c\,(0,y) = u^c\,(a,y) = v^c\,(x,0) = v^c\,(x,b) = 0$$



Using the characteristic functions for clamped-clamped beams, in both directions, and changing ℓ to 'a' for X_m and ℓ to 'b' for Y_m, it is obvious from Table 1 of the Appendix (A) that all the edge conditions in (3.3) are satisfied.

For the determination of Ψ_{mn} the displacement w^c is assumed to vary linearly over the depth of the core. That is, $\Psi_{mn}(z)$ in the third equation of (3.2) has the form:

$$\Psi_{mn}(z) = A_{mn} + z\, B_{mn} \qquad \ldots(3.4)$$

where A_{mn} and B_{mn} are parameters to be determined. For a core which is rigid in the thickness direction, $\Psi_{mn}(z)$ must be independent of z and consequently B_{mn} must vanish. For this reason, and with no loss in generality, equation (2-13) is re-written as:

$$\Psi_{mn}(z) = A_{mn} + z\, \frac{B_{mn}}{E_z^c} \qquad \ldots(3.5)$$

where E_z^c is the modulus of elasticity in the z-direction of the core.

The functions θ_{mn} and Φ_{mn} are determined in accordance with the assumption of uniform transverse shearing stresses in the core, i.e.,

$$\frac{\partial \tau_{xz}^c}{\partial z} = 0 \qquad \ldots(3.6)$$

Expressing the shearing stresses in terms of shearing strains yields

$$G_{xz}^c\left(\frac{\partial^2 U^c}{\partial z^2} + \frac{\partial^2 w^c}{\partial x \partial z}\right) = 0 \qquad \ldots(3.7).$$

Substitution of the displacement functions into (3-16) leads to

$$\theta''_{mn}(z) + \Psi'_{mn}(z) = 0 \qquad \ldots(3.8)$$

where θ''_{mn} and Ψ'_{mn} are the second, and the first derivatives of θ_{mn} and Ψ_{mn} with respect to z, respectively. Integrating equation (3-18) twice with respect to z, and making use of the relation (3-5), the following expression for θ_{mn} is obtained:

$$\theta_{mn}(z) = K_{mn} + z(F_{mn}-A_{mn}) - \frac{z^2}{2}\,\frac{B_{mn}}{E_z^c} \qquad \ldots(3.9)$$

where K_{mn} and F_{mn} are the constants of integration.

Thus the first of the equations (3-2) takes the form:*

$$u^c = \sum_m^{\infty} \sum_n^{\infty} X'_m(x)Y_n(y)\left[K_{mn}+z\left(\frac{F_{mn}}{G_{xz}}-A_{mn}\right)-\frac{z^2 B_{mn}}{2\,E_z^c}\right]\sin\Omega_{mn}t$$

$$\ldots(3.10).$$

In an analogous manner, v^c is given by

$$v^c = \sum\sum X_m(x)Y'_n(y)\left[L_{mn}+z\left(\frac{H_{mn}}{G_{xz}} - A_{mn}\right) - \frac{z^2 B_{mn}}{2\,E_z^c}\right]\sin\Omega_{mn}t$$

$$\ldots(3.11)$$

where L_{mn} and H_{mn} are constants of integration.

Since the parameters F_{mn} and H_{mn} in equations (3.10) and (3.11) are associated with shearing stresses, it will prove advantageous to represent them as $\frac{F_{mn}}{G^c_{xz}}$ and $\frac{H_{mn}}{G^c_{yz}}$, respectively.

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*See footnote on following page.

Re-writing equations (3.5), (3.10), and (3.11), the displacement functions in the x, y, and z directions are:

$$u^c = \sum_m^{\infty} \sum_n^{\infty} X_m'(x) Y_n(y) \left[K_{mn} + z\left(\frac{F_{mn}}{G^c_{xz}} - A_{mn}\right) - \frac{z^2}{2} \frac{B_{mn}}{E^c_z} \right] \cdot \left[\sin \Omega_{mn} t \right]$$

$$v^c = \sum_m^{\infty} \sum_n^{\infty} X_m(x) Y_n'(y) \left[L_{mn} + z\left(\frac{H_{mn}}{G^c_{yz}} - A_{mn}\right) - \frac{z^2}{2} \frac{B_{mn}}{E^c_z} \right] \cdot \left[\sin \Omega_{mn} t \right] \qquad \ldots(3.12)$$

$$w^c = \sum_m^{\infty} \sum_n^{\infty} X_m(x) Y_n(y) \left[A_{mn} + z \frac{B_{mn}}{E^c_z} \right] \sin \Omega_{mn} t$$

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*Employing the same assumptions as Ericksen, u^c in equation (3.10) reduces to:

$$u^c = X_m' Y_n \left[K_{mn} + z(F_{mn} - A_{mn}) \right]$$

$$= -X_m' Y_n A_{mn} \left(1 - \frac{F_{mn}}{A_{mn}} \right) \left[z - \frac{K_{mn}}{A_{mn} - F_{mn}} \right] \qquad \ldots(3.10a).$$

Observing that $X_m' \, Y_n \, A_{mn} = \frac{\partial w^c}{\partial x}$, equation (3.10a) becomes

$$u^c = -K(z-q) \frac{\partial w^c}{\partial x}, \text{ provided } \left(1 - \frac{F_{mn}}{A_{mn}} \right) = K \text{ and } \frac{K_{mn}}{A_{mn} - F_{mn}} = q$$

This is exactly the x displacement function assumed by Ericksen. Hence q may be regarded as an arbitrary parameter whose value can be determined by minimizing the energy.

Strain-Energy Considerations

The total strain energy of the system consists of

(1) strain energy of the core

(2) energy due to membrane strains in the facings and

(3) energy due to bending strains in the facings.

1. Strain Energy in the Core

The strain energy V^c in the core in accordance with the assumed state of stress is given by

$$V^c = \frac{1}{2}\int_V \left(\sigma_z^c \epsilon_z^c + \sigma_y^c \epsilon_y^c + \tau_{xz}^c \gamma_{xz}^c + \tau_{yz}^c \gamma_{yz}^c \right) dv \ldots (3.13)$$

Using the stress-strain relations derived in Appendix (B), equation (3.13) takes the form

$$V^c = \frac{1}{2}\int_V \left\{ \left(\frac{1}{1-\nu_z^c\nu_y^c}\right) \left[E_z^c(\epsilon_z^c)^2 + E_y^c(\epsilon_y^c)^2 + (E_z^c\nu_z^c + E_y\nu_y^c)\epsilon_y^c\epsilon_z^c \right] + \left[G_{xz}^c(\gamma_{xz}^c)^2 + G_{yz}^c(\gamma_{yz})^2 \right] \right\} dv \quad \ldots (3.14)$$

Denoting $\dfrac{E_z^c}{1-\nu_z^c\nu_y^c}$ by g_3^c, $\dfrac{E_y^c}{1-\nu_z^c\nu_y^c}$ by g_2^c, and $\dfrac{E_z^c\nu_z^c + E_y^c\nu_y^c}{1-\nu_z^c\nu_y^c}$ by g_{32}^c

and observing the relations,

$$\epsilon_z^c = \frac{\partial w^c}{\partial z}, \quad \epsilon_y^c = \epsilon_{By}^c = \left(z - \frac{c}{2}\right)\left(\frac{\partial^2 w^c}{\partial y^2}\right)$$

$$\gamma_{xz}^c = \frac{\partial u^c}{\partial z} + \frac{\partial w^c}{\partial x} \quad \text{and} \quad \gamma_{yz}^c = \frac{\partial v^c}{\partial z} + \frac{\partial w^c}{\partial y},$$

equation (3.14) becomes

$$V^c = \frac{1}{2}\int_0^a\int_0^b\int_0^c \Big[g_3^c \left(\frac{\partial w^c}{\partial z}\right)^2 + g_2^c \left[\left(z - \frac{c}{2}\right)\left(\frac{\partial^2 w^c}{\partial y^2}\right)^2\right.$$

$$+ g_{32}^c \left(\frac{\partial^2 w^c}{\partial y^2}\right)\left(\frac{\partial w^c}{\partial z}\right) \left[\left(z - \frac{c}{2}\right)\right]$$

$$G_{xz}^c \left(\frac{u^c}{z} + \frac{w^c}{x}\right) + G_{yz}^c \left(\frac{\partial v^c}{\partial z} + \frac{\partial w^c}{\partial y}\right)\Big\}\, dxdydz \quad \ldots\ldots(3.15)$$

The evaluation of the various quantities in equation (3.15) is accomplished in the following manner:

Let $I_3^c = \frac{1}{2} g_3^c \int_v \left(\frac{\partial w^c}{\partial z}\right)^2 dv$

$$I_2^c = \frac{1}{2} g_2^c \int_v \left[\left(z - \frac{c}{2}\right)\left(\frac{\partial^2 w^c}{\partial y^2}\right)\right] dv$$

$$I_{32}^c = \frac{1}{2} g_{32}^c \int_v \left[\left(z - \frac{c}{2}\right)\left(\frac{\partial^2 w^c}{\partial y^2}\right)\left(\frac{\partial w^c}{\partial z}\right)\right] dv$$

$$I_{44} = \frac{1}{2} G_{yz}^c \int_v \left[\frac{\partial v^c}{\partial z} + \frac{\partial w^c}{\partial y}\right]^2 dv$$

$$I_{55} = \frac{1}{2} G_{xz}^c \int_v \left[\frac{\partial u^c}{\partial z} + \frac{\partial w^c}{\partial x}\right]^2 dv.$$

With the displacement functions given in (3.12), the above integrals may be expressed as:

$$I_3^c = \frac{1}{2} g_3^c \int_0^a\int_0^b\int_0^c \left[\sum_m^\infty \sum_n^\infty X_m Y_n \left(\frac{B_{nm}}{E_z^c}\right)\right]^2 \sin^2\Omega_{mn}t \; dxdydz$$

Because of the orthogonal properties discussed previously

$$I_3^c = \frac{1}{2} g_3^c \,(a)(b)(c) \sum_m^\infty \sum_n^\infty \left(\frac{B_{mn}}{E_z^c}\right)^2 \sin^2\Omega_{mn}t$$

$$I_2^c = \frac{1}{2}\, g_2^c \int_0^a\int_0^b\int_0^c \left[\sum_m^\infty \sum_n^\infty X_m Y_n''\left(z - \frac{c}{2}\right)\left(A_{mn} + z\frac{B_{mn}}{E_z^c}\right)\right]^2 \sin^2\Omega_{mn}t \; dxdydz$$

$$= \frac{1}{2}\, g_2^c \int_0^c (a)(b) \sum_m \sum_n^\infty \left(\frac{\beta_n}{b}\right)^4 \left[\left(z - \frac{c}{2}\right)\left(A_{mn} + z\frac{B_{mn}}{E_z^c}\right)\right]^2 dz \; \sin^2\Omega_{mn}t$$

or

$$I_2^c = \frac{1}{2}\, g_2^c (a)(b)\, \frac{c^3}{12} \sum_m^\infty \sum_n^\infty \left(\frac{\beta_n}{b}\right)^4 \left[A_{mn}^2 + cA_{mn}\frac{B_{mn}}{E_z^c} + \frac{2}{5}c^2\left(\frac{B_{mn}}{E_z^c}\right)^2\right] \sin^2\Omega_{mn}t$$

Referring to Appendix (C), I_{32}^c, I_4^c amd I_5^c yield:

$$I_{32}^c = \frac{1}{2}\, g_{32}^c (a) \left[\sum_m^\infty \sum_n^\infty \int_0^c \left(\frac{B_{mn}}{E_z^c}\right)\left(A_{mn} + z\frac{B_{mn}}{E_z^c}\right)\left(z - \frac{c}{2}\right)(dz)\left(\frac{J_{nn}}{b}\right)\right.$$

$$\left. + \sum_m^\infty \sum_n^\infty \sum_{\substack{q \\ n \neq q}}^\infty \frac{J_{nq}}{b} \int_0^c \left(\frac{B_{mn}}{E_z^c}\right)\left(A_{mq} + z\frac{B_{mq}}{E_z^c}\right)\left(z - \frac{c}{2}\right) dz \; \sin^2\Omega_{mn}t\right.$$

where

$$\int_0^b Y_n Y_q'' \, dy = \frac{J_{nq}}{b}$$

$$I_{32}^c = \frac{1}{2}\, g_{32}^c (a) \frac{(c)^3}{12} \left[\sum_m^\infty \sum_n^\infty \frac{J_{nn}}{b}\left(\frac{B_{mn}}{E_z^c}\right)^2 + \sum_m^\infty \sum_n^\infty \sum_{\substack{q \\ n \neq q}}^\infty \frac{J_{nq}}{b}\left(\frac{B_{mn}}{E_z^c}\right)\left(\frac{B_{nq}}{E_z^c}\right)\right] \cdot \left[\sin^2\Omega_{mn}t\right]$$

$$I_4^c = \frac{1}{2}\, G_{yz}^c \int_0^a\int_0^b\int_0^c \left[\sum_m^\infty \sum_n^\infty X_m Y_n \left(\frac{H_{mn}}{G_{yz}^c}\right)\right]^2 dxdydz \; \sin^2\Omega_{mn}t$$

$$I_4^c = \frac{1}{2} G_{yz}^c (a)(c) \left[\sum_m^{\infty} \sum_n^{\infty} \frac{I_{nn}}{b} \left(\frac{H_{mn}}{G_{yz}^c} \right)^2 - \qquad \frac{I_{nq}}{b} \left(\frac{H_{mn}}{G_{yz}^c} \frac{H_{nq}}{G_{yz}^c} \right) \right] \sin^2 \Omega_{mn} t$$

where

$$\frac{I_{nq}}{b} = \int_0^b Y_n' Y_q' \, dy$$

$$I_5^c = \frac{1}{2} G_{xz}^c \int_0^a \int_0^b \int_0^c \left[\sum_m^{\infty} \sum_n^{\infty} X_m' Y_n \left(\frac{F_{mn}}{G_{xz}^c} \right) \right]^2 \sin^2 \Omega_{mn} t \, dx dy dz$$

$$I_5^c = \frac{1}{2} G_{xz}^c (b)(c) \left[\sum_m \sum_n \frac{I_{mm}}{a} \left(\frac{F_{mn}}{G_{xz}^c} \right)^2 \right.$$

$$\left. + \sum_m^{\infty} \sum_n^{\infty} \sum_{\substack{p \\ m \neq p}}^{\infty} \frac{I_{mp}}{a} \left(\frac{F_{mn}}{G_{xz}^c} \right) \left(\frac{F_{pn}}{G_{xz}^c} \right. \quad \sin \quad {}^2_{mn} \right.$$

where

$$\frac{I_{mp}}{a} = \int_0^a X_m' X_p' dx$$

Therefore total strain energy in the core is:-

$$V^c = \frac{1}{2} \left\{ g_3^c \; (abc) \sum_m^{\infty} \sum_n^{\infty} \left(\frac{B_{mn}}{E_z^c} \right)^2 \right.$$

$$+ g_2^c \; \frac{(abc^3)}{12} \sum_m^{\infty} \sum_n^{\infty} \left(\frac{\beta n}{b} \right)^4 \left[A_{mn}^2 + c \; A_{mn} \frac{B_{mn}}{E_z^c} + \frac{2}{5} c^2 \left(\frac{B_{mn}}{E_z^c} \right)^2 \right]$$

$$+ g_{32}^c \frac{(ac^3)}{12} \left[\sum_m^{\infty} \sum_n^{\infty} \frac{J_{nn}}{b} \left(\frac{B_{mn}}{E_z^c} \right)^2 + \sum_m^{\infty} \sum_n^{\infty} \sum_{\substack{q \\ n \neq q}}^{\infty} \frac{J_{nq}}{b} \; \frac{B_{mn}}{E_z^c} \; \frac{B_{nq}}{E_z^c} \right]$$

$$+ G_{yz}^c \; (ac) \left[\sum_m^{\infty} \sum_n^{\infty} \frac{I_{nn}}{b} \left(\frac{H_{mn}}{G_{yz}^c} \right)^2 + \sum_m^{\infty} \sum_n^{\infty} \sum_{\substack{q \\ n \neq q}}^{\infty} \frac{I_{nq}}{b} \left(\frac{H_{mn}}{G_{yz}} \right) \left(\frac{H_{nq}}{G_{yz}} \right) \right]$$

$$\left. + G_{xz}^c (bc) \left[\sum_m^{\infty} \sum_n^{\infty} \frac{I_{mm}}{a} \; \left(\frac{F_{mn}}{G_{xz}^c} \right)^2 + \sum_m^{\infty} \sum_n^{\infty} \sum_{\substack{p \\ m \neq p}}^{\infty} \frac{I_{mp}}{a} \left(\frac{F_{mn}}{G_{xz}^c} \right) \left(\frac{F_{pn}}{G_{xz}^c} \right) \right] \right\} \sin^2 \Omega_{mn} t$$

...(3.16)

2. Strain Energy in the Facings Due To Membrane Strains

The strains in the middle plane of the facings are referred to as membrane strains. Denoting the middle plane displacements of the upper facing f^a in the x,y, and z directions by u^a, v^a, and w^a, and the corresponding middle plane displacements of the lower facing f^b by u^b, v^b, and w^b, respectively, the following relations between the middle plane displacements of the facings and the displacements of the core exist:

$$
\begin{aligned}
u^a &= \left[u^c + \frac{1}{2} f^a \left(\frac{\partial w^c}{\partial x}\right)\right]_{z=0} \\
v^a &= \left[v^c + \frac{1}{2} f^a \left(\frac{\partial w^c}{\partial y}\right)\right]_{z=0} \\
w^a &= \left[w^c\right]_{z=0} \\
u^b &= \left[u^c - \frac{1}{2} f^b \left(\frac{\partial w^c}{\partial x}\right)\right]_{z=c} \\
v^b &= \left[v^b - \frac{1}{2} f^b \left(\frac{\partial w^c}{\partial y}\right)\right]_{z=c} \\
w^b &= \left[w^c\right]_{z=c}
\end{aligned}
\qquad \ldots(3.17)
$$

The relations (3.17) are based on the assumption that the facing displacements in the x and y directions vary linearly through the facing thicknesses and that the facing displacements in the z-direction are constant over the thicknesses. Using equations

(3.17), the resulting expressions for the linear and shearing strains in the upper facing are:

$$\epsilon^a_{mx} = \frac{\partial u^a}{\partial x} = \left[\frac{\partial u^c}{\partial x} + \frac{1}{2} f^a \left(\frac{\partial^2 w^c}{\partial x^2}\right)\right]_{z=0}$$

$$\epsilon^a_{my} = \frac{\partial v^a}{\partial y} = \left[\frac{\partial v^c}{\partial y} + \frac{1}{2} f^a \left(\frac{\partial^2 w^c}{\partial y^2}\right)\right]_{z=0}$$

$$\gamma^a_{mxy} = \frac{\partial u^a}{\partial y} + \frac{\partial v^a}{\partial x} \qquad \ldots(3.18)$$

$$= \left[\frac{\partial u^c}{\partial y} + f^a \left(\frac{\partial^2 w^c}{\partial x \partial y}\right) + \frac{\partial v^c}{\partial x}\right]_{z=0}$$

where ϵ^a_{mx} and ϵ^a_{my} represent the linear strains in the x and y directions, and γ^a_{mxy} represents the shearing strains.

The corresponding linear and shearing strains of the lower facings in terms of the core displacements are

$$\epsilon^b_{mx} = \left[\frac{\partial u^b}{\partial x} - \frac{1}{2} f^b \left(\frac{\partial^2 w^c}{\partial x^2}\right)\right]_{z=c}$$

$$\epsilon^b_{my} = \left[\frac{\partial v^b}{\partial y} - \frac{1}{2} f^b \left(\frac{\partial^2 w^c}{\partial y^2}\right)\right]_{z=c} \qquad \ldots(3.19)$$

$$\gamma^b_{mxy} = \left[\frac{\partial u^c}{\partial y} - f^b \left(\frac{\partial^2 w^c}{\partial y \partial x}\right) + \frac{\partial v^c}{\partial x}\right]_{z=c}$$

<u>Upper Facing.</u> The strain energy associated with the membrane strains in the upper facing is given by the expression:

$$V^a_m = \frac{1}{2} \int_V \left(\sigma^a_x \epsilon^a_{mx} + \sigma^a_y \epsilon^a_{my} + \tau^a_{xy} \gamma^a_{mxy}\right) dv \ldots\ldots\ldots\ldots(3.20)$$

Expressing stresses in terms of strains, equation (3.20) takes the form:

$$V_m^a = \frac{1}{2}\int_V\left[\left(\frac{1}{1-\nu_x^a\nu_y^a}\right)\left\{E_x^a(\epsilon_{mx}^a)^2 + E_y^a\,(\epsilon_{my}^a)^2\right.\right.$$

$$\left.\left.+\ (E_x^a\ {}^a_x + E_y\ {}^a_y)\epsilon_{mx}^a\epsilon_{my}^a\right\} + G_{xy}^a\left(\gamma_{mxy}^a\right)^2\right]dv \ldots\ldots\ldots\ldots\ldots (3.21)$$

Denoting

$\dfrac{E_x^a}{1-\nu_x^a\nu_y^a}$ by d_1^a, $\dfrac{E_y^a}{1-\nu_x^a\nu_y^a}$ by d_2^a, and $\dfrac{E_x^a\nu_x^a + E_y^a\nu_y^a}{1-\nu_x^a\nu_y^a}$ by d_{12}^a,

the equation (3.21) reduces to:

$$V_m^a = \frac{1}{2}\int_V\left[\,d_1^a(\epsilon_m^a)^2 + d_2^a(\epsilon_{my}^a)^2 + d_{12}^a(\epsilon_{mx}^a)(\epsilon_{my}^a) + G_{xy}^a(\gamma_{mxy}^a)^2\right]dv \quad \ldots(3.22)$$

The evaluation of the various integrals in (3.22) is accomplished in the following manner:

Let $I_{1m}^a = \frac{1}{2}\int_V d_1^a\ (\epsilon_{mx}^a)^2 dv$, $\qquad J_{2m}^a = \frac{1}{2}\int_V d_2^a\ (\epsilon_{my}^a)^2 dv$

$I_{12m}^a = \frac{1}{2}\int_V d_{12}^a(\epsilon_{mx}^a)(\epsilon_{my}^a)dv$, and $I_{3m}^a = \frac{1}{2}\int_V G_{xy}^a(\gamma_{mxy}^a)^2 dv$.

Substituting the relation (3.18), the above integrals take the form:

$$I_{1m}^a = \frac{1}{2}\,d_1^a \int_0^a\!\!\int_0^b\!\!\int_{-f^a}^0 \left[\sum_m^\infty\sum_n^\infty X_m''Y_n\left\{K_{mn} + z\left(\frac{F_{mn}}{G_{xz}^c} - A_{mn}\right)\right.\right.$$

$$\left.\left.-\ \frac{z^2}{2}\frac{B_{mn}}{E_z^c} + \frac{1}{2}\,f^a\left(A_{mn} + z\,\frac{B_{mn}}{E_z^c}\right)\right\}_{z=0}\right]^2 dxdydz\ \sin^2\Omega_{mn}t$$

OR $$I_{1m}^a = \frac{1}{2}\,d_1^a \int_0^a\!\!\int_0^b\!\!\int_{-f^a}^0 \left[\sum_m^\infty\sum_n^\infty X_m''Y_n\ \left(K_{mn} + \frac{1}{2}\,f^a A_{mn}\right)\right]^2 dxdydz\ \sin^2\Omega_{mn}t$$

Similarly,

$$I^a_{2m} = \frac{1}{2} d^a_2 \int_0^a\int_0^b\int_{-f^a}^0 \left[\sum_m^\infty\sum_n^\infty X_m Y_n''(L_{mn} + \frac{1}{2} f^a A_{mn})\right]^2 dxdydz \sin^2\Omega_{mn}t$$

$$I^a_{12m} = \frac{1}{2} d^a_{12} \int_0^a\int_0^b\int_{-f^a}^0 \left[\sum_m^\infty\sum_n^\infty X_m'' Y_n \left(K_{mn} + \frac{f^a}{2} A_{mn}\right)\right] \cdot$$

$$\left[\sum_m^\infty\sum_n^\infty X_m Y_n''\left(L_{mn} + \frac{f^a}{2} A_{mn}\right)\right] dxdydz \sin^2\Omega_{mn}t$$

$$I^a_{3m} = \frac{1}{2} G^a_{xy} \int_0^a\int_0^b\int_{-f^a}^0 \left[\sum_m^\infty\sum_n^\infty X_m' Y_n' (K_{mn}+L_{mn}+f^a A_{mn})\right]^2 dxdydz \sin^2\Omega_{mn}t$$

Employing the results of Appendix (C), the above integrals reduce to:

$$I^a_m = \frac{1}{2} d^a_1 abf^a \sum_m^\infty\sum_n^\infty \left(\frac{\beta_m}{a}\right)^4 \left[K_{mn} + \frac{f^a}{2} A_{mn}\right]^2 \sin^2\Omega_{mn}t \qquad \ldots(3\text{-}23)$$

$$I^a_{2m} = \frac{1}{2} d^a_2\, abf^a \sum_m^\infty\sum_n^\infty \left(\frac{\beta_n}{a}\right)^4 \left[L_{mn} + \frac{f^a}{2} A_{mn}\right]^2 \sin^2\Omega_{mn}t \qquad \ldots(3\text{-}24)$$

$$I^a_{12m} = \frac{1}{2} f^a d^a_{12}\Big[\sum\sum \frac{J_{mn}}{a}\,\frac{J_n}{b}\,(K_{mn} + \frac{f^a}{2} A_{mn})(L_{mn} + \frac{f^a}{2} A_{mn})$$

$$+\sum_{\substack{m \\ m\neq p}}^\infty\sum_n^\infty\sum_p^\infty \frac{J_{mp}}{a}\,\frac{J_{nn}}{b}\,(K_{mn}+ \frac{f^a}{2} A_{mn})(L_{pn}+ \frac{f^a}{2} A_{pn})$$

$$+\sum_m^\infty\sum_{\substack{n \\ n\neq q}}^\infty\sum_q^\infty \frac{J_{mm}}{a}\,\frac{J_{nq}}{b}\,(L_{mn}+ \frac{f^a}{2} A_{mn})(K_{mq}+ \frac{f^a}{2} A_{mq})$$

$$+\sum_m^\infty\sum_n^\infty\sum_p^\infty\sum_q^\infty \frac{J_{mp}}{a}\,\frac{J_{nq}}{b}\,(K_{pn}+ \frac{f^a}{2} A_{pn})(L_{mq}+ \frac{f^a}{2} A_{mq}) \sin^2 \quad {}_{mn}t$$

$m \neq p,\ n \neq q$ $\qquad \ldots(3\text{-}25)$

$$I^a_{3m} = \frac{1}{2} G^a_{xy} f^a \Big[\sum_m^\infty \sum_n^\infty \frac{I_{mn}}{a} \frac{I_{nn}}{b} (K_{mn}+L_{mn}+f^a A_{mn})^2$$

$$+\sum_{\substack{m\\m}}^\infty \sum_n^\infty \sum_{\substack{p\\p}}^\infty \frac{I_{mp}}{a} \frac{I_{nn}}{b} (K_{mn}+L_{mn}+f^a A_{mn})(K_{pn}+L_{pn}+f^a A_{pn})$$

$$+\sum_m^\infty \sum_{\substack{n\\n \neq q}}^\infty \sum_q^\infty \frac{I_{mm}}{a} \frac{I_{nq}}{b} (K_{mn}+L_{mn}+f^a A_{mn})(K_{mq}+L_{mq}+f^a A_{mq})$$

$$+\sum_{m \neq p,\, n \neq q}^\infty \sum_n^\infty \sum_p^\infty \sum_q^\infty \frac{I_{mp}}{a} \frac{I_n}{b} (K_{pn}+L_{pn}+f^a A_{pn})(K_{mq}+L_{mq}+f^a A_{mq})\Big] \Big[\sin^2 \Omega_{mn} t\Big] \quad \ldots(3\text{-}26)$$

In short, the total energy in the upper facing due to membrane strains can be represented by

$$V^a_M = (I^a_{1m} + I^a_{2m} + I^a_{12m} + I^a_{3m}) \quad \ldots(3.27)$$

where the expressions for I^a_{1m}, I^a_{2m}, I^a_{12m} and I^a_{3m} are given by equations (3.23), (3.24), (3.25), and (3.26), respectively.

<u>Lower Facing.</u> The energy associated with membrane strains in the lower facing f^b is :

$$V^b_M = \frac{1}{2} \int_v (\sigma^b_x \epsilon^b_{mx} + \sigma^b_y \epsilon^b_{my} + \tau^b_{mxy} \gamma^b_{mxy})\, dv \quad \ldots(3.28)$$

Employing stress-strain relations, equation (3.28) is written in the form:

$$V^b_m = \frac{1}{2} \int_v \Big[d^b_1 (\epsilon^b_{nx})^2 + d^b_2 (\epsilon^b_{my})^2 + d^b_{12} (\epsilon^b_{mx})(\epsilon^b_{my}) G^b_{xy} (\gamma^a_{mxy})^2 \Big] dv, \quad \ldots(3.29)$$

where

$$\frac{E^b_x}{1-\nu^b_x \nu^b_y} = d^b_1, \quad \frac{E^b_y}{1-\nu^b_x \nu^b_y} = d^b_2 \text{ and } \frac{E^b_x \nu^b_x + E^b_y \nu^b_y}{1-\nu^b_x \nu^b_y} = d^b_{12}$$

Let

$$I^b_{1m} = d^b_1 \int_v \frac{1}{2} d^b_1 (\epsilon^b_{mx})^2 dv, \qquad I^b_{2m} = \frac{1}{2}\int_v d^b_2 (\epsilon^b_{my})^2 dv,$$

$$I^b_{12m} = \frac{1}{2}\int_v d^b_{12} (\epsilon^b_{mx})(\epsilon^b_{my}) dv \text{ and } I^b_{3m} = \frac{1}{2}\int_v G^b_{xy} (\gamma^b_{mxy})^2 dv$$

Substituting the relations (3.19) for strain components, the above integrals are expressed as:

$$I^b_{1m} = \frac{1}{2} d^b_1 \int_0^a \int_0^b \int_c^{c+f} \Bigg[\sum_m^\infty \sum_n^\infty X''_m Y_n \Big\{K_{mn} + z\Big(\frac{F_{mn}}{G^c_{xz}} - A_{mn}\Big) - \frac{z^2}{2}\frac{B_{mn}}{E^c_z}$$

$$- \frac{f^b}{2}\Big(A_{mn} + z\frac{B_{mn}}{E^c_z}\Big)\Big\}_{z=c}\Bigg]^2 dxdydz \sin^2 \Omega_{mn} t$$

$$I^b_{1m} = \frac{1}{2} d^b_1 \int_0^a \int_0^b \int_c^{c+f^b} \Bigg[\sum_m^\infty \sum_n^\infty X''_m Y_n \Big\{K_{mn} + \frac{cF_{mn}}{G^c_{xz}} - \Big(c + \frac{f^b}{2}\Big) A_{mn}$$

$$- \frac{c}{2}(c+f^b) B_{mn}\Big\}\Bigg]^2 dxdydz \quad \sin^2 \Omega_{mn} t$$

$$I^b_{2m} = \frac{1}{2} d^b_2 \int_0^a \int_0^b \int_c^{c+f^b} \Bigg[\sum_m \sum_n X_m Y''_n \Big\{ L_{mn} + z\Big(\frac{H_{mn}}{G^c_{yz}} - A_{mn}\Big) - z^2 \frac{B_{mn}}{E^c_z}$$

$$\frac{1}{2} f^b \Big(A_{mn} + z\frac{B_{mn}}{E^c_z}\Big)\Big\}_{z=c}\Bigg]^2 dxdydz \sin^2 \Omega_{mn} t$$

$$I^b_{2m} = \frac{1}{2} d^b_2 \int_0^a \int_0^b \int_c^{c+f^b} \Bigg[\sum_m^\infty \sum_n^\infty X_m Y''_n \Big\{ L_{mn} + \frac{CH_{mn}}{G^c_{yz}} - \Big(c + \frac{f^b}{2}\Big) A_{mn}$$

$$- \frac{c}{2}(c+f^b) B_{mn}\Big\}\Bigg]^2 dx\, dy\, dz \sin^2 \Omega_{mn} t$$

$$I^{b}_{12m} = \frac{1}{2}\, d^{b}_{12} \int_{0}^{a}\int_{0}^{b}\int_{c}^{f^b}\left[\sum_{m}^{\infty}\sum_{n}^{\infty} X''_m Y_n \left\{ K_{mn} + \frac{CF_{mn}}{G^{c}_{xz}} - (c+\frac{f^b}{2})A_{mn} - \frac{c}{2}(c+f^{b})\frac{B_{mn}}{E^{c}_{z}}\right\} \cdot\right.$$

$$\left.\sum_{m}^{\infty}\sum_{n}^{\infty} X_m Y''_n \left\{ L_{mn} + c\,\frac{H_{mn}}{G^{c}_{yz}} - (c+\frac{f^b}{2})\, A_{mn} - \frac{c}{2}(c+f^{b})\,\frac{B_{mn}}{E^{c}_{z}}\right\}\right] \cdot$$

$$\left[dxdydz\right] \sin^{2}\Omega_{mn} t$$

$$I^{b}_{3m} = \frac{1}{2}G^{b}_{xy}\int_{0}^{a}\int_{0}^{b}\int_{c}^{c+f^b}\left[\sum_{m}^{\infty}\sum_{n}^{\infty} X'_m Y'_n \left\{ K_{mn} + C\,\frac{F_{mn}}{G^{c}_{xz}} + \frac{H_{mn}}{G^{c}_{yz}}\right.\right.$$

$$\left.\left. -2(c+\frac{f^b}{2})\, A_{mn} - c(c+f^{b})\,\frac{B_{mn}}{E^{c}_{z}}\right\}\right]^{2} dxdydz\ \sin^{2}\Omega_{mn} t$$

Referring to Appendix (C), the evaluation of the above integrals yields:

$$I^{b}_{1m} = \frac{1}{2}\, abd^{b}_{1}\, f^{b} \sum_{m}^{\infty}\sum_{n}^{\infty}\left(\frac{\beta_m}{a}\right)^{4}\left[\left(K_{mn} + c\,\frac{F_{mn}}{G^{c}_{xz}}\right)\right.$$

$$\left. - (c+\frac{f^b}{2})\, A_{mn} - \frac{c}{2}\,(c+f^{b})\,\frac{B_{mn}}{E^{c}_{z}}\right]^{2} \sin^{2}\Omega_{mn} \qquad \ldots(3\text{-}30)$$

$$I^{b}_{2m} = \frac{1}{2}\, d^{b}_{z}\ abf^{b} \sum_{m}^{\infty}\sum_{n}^{\infty}\left(\frac{\beta_n}{b}\right)^{4}\left[\left(L_{mn} + c\,\frac{H_{mn}}{G^{c}_{yz}}\right) - (c+\frac{f^b}{2})A_{mn}\right.$$

$$\left. - \quad c(c + f^{b})\,\frac{B_{mn}}{E^{c}_{z}}\right]^{2} \sin^{2}\Omega_{mn} t \qquad \ldots(3\text{-}31)$$

$$I^b_{12m} = \frac{1}{2} d^b_{12} f^b \Bigg[\sum_m^{\infty} \sum_n^{\infty} \frac{J_{mn}}{a} \frac{J_{nn}}{b} \Big\{ K_{mn} + c \frac{F_{mn}}{G^c_{xz}} - (c + \frac{f^b}{2}) A_{mn}$$

$$- \frac{c}{2}(c+f^b) \frac{B_{mn}}{E^c_z} \Big\} \Big\{ L_{mn} + c \frac{H_{mn}}{G^c_{yz}} - (x + \frac{f^b}{2}) A_{mn} - \frac{c}{2}(c+f^b) \frac{B_{mn}}{E^c_z}$$

$$+ \sum_m^{\infty} \sum_n^{\infty} \sum_p^{\infty} {}_{m \neq p} \frac{J_{mp}}{a} \frac{J_{nn}}{b} \quad K_{mn} + c \frac{F_{mn}}{G^c_{xz}} - (c + \frac{f^b}{2}) A_{mn} - \frac{c}{2}(c+f^b) \quad \frac{B_{mn}}{E^c_z} \Big\} \cdot$$

$$\Big\{ L_{pn} + c \frac{H_{pn}}{G^c_{yz}} - (c + \frac{f^b}{2}) \; A_{pn} - \frac{c}{2} \; (c+f^b) \; \frac{B_{mn}}{E^c_z} \Big\}$$

$$+ \sum_m^{\infty} \sum_n^{\infty} \sum_p^{\infty} \sum_q^{\infty} {}_{m \neq p, \; n \neq q} \frac{J_{mp}}{a} J_{nq} \Big\{ K_{pn} + c \frac{F_{pn}}{G^c_{xz}} - (c + \frac{f^b}{2}) A_{pn} - \frac{c}{z}(c+f^b) \frac{B_{pn}}{E^c_z} \Big\} \cdot$$

$$\Big\{ L_{mq} + c \frac{H_{mq}}{G^c_{yz}} - (c + \frac{f^b}{2}) A_{mq} - \frac{c}{2}(c+f^b) \frac{B_{mq}}{E^c_z} \Big\} \Bigg] \sin^2 \Omega_{mn} t \qquad \ldots(3\text{-}32)$$

$$I^b_{3m} = \frac{G^b_{xy}}{2} f^b \Bigg[\sum_m^{\infty} \sum_n^{\infty} \frac{I_{mm}}{a} \frac{I_{nn}}{b} \Big\{ K_{mn} + c \Big(\frac{F_{mn}}{G^c_{xz}} + \frac{H_{mn}}{G^c_{yz}} \Big)$$

$$- 2(c + \frac{f^b}{2}) A_{mn} - c(c+f^b) \frac{B_{mn}}{E^c_z} \Big\}^2 + \sum_m^{\infty} \sum_n^{\infty} \sum_p^{\infty} {}_{m \neq p} \frac{I_{mp}}{a} \frac{I_{nn}}{b} \Big\{ K_{mn} + L_{mn} +$$

$$c \Big(\frac{F_{mn}}{G^c_{xz}} + \frac{H_{mn}}{G^c_{yz}} \Big) - 2(c + \frac{f^b}{2}) A_{mn} - c(c+f^b) \frac{B_{mn}}{E^c_z} \Big\} \cdot$$

$$\Big\{ K_{pn} + L_{pn} + c \Big(\frac{F_{pn}}{G^c_{xz}} + \frac{H_{pn}}{G^c_{yz}} \Big) - 2 \; (c + \frac{f^b}{2}) A_{pn} - c(c+f^b) \frac{B_{pn}}{E^c_z} \Big\} \cdot$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{q}^{\infty}\frac{I_{mn}}{a}\ \frac{I_{nq}}{b}\left\{K_{mn}+L_{mn}+c\left(\frac{F_{mn}}{G^{c}_{xz}}\quad\frac{H_{mn}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{mn}-c\left(c+f^{b}\right)\frac{B_{mn}}{E^{c}_{z}}\right\}\cdot$$

$$\left\{K_{mq}+L_{mq}+c\left(\frac{F_{mq}}{G^{c}_{xz}}\quad\frac{H_{mq}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{mq}-c\left(c+f^{b}\right)\ \frac{B_{mq}}{E^{c}_{z}}\right\}$$

$$+\sum_{\substack{m \\ m\neq p,\ n\neq q}}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\sum_{q}^{\infty}\frac{I_{mp}}{a}\frac{I_{nq}}{b}\left\{L_{pn}+L_{pn}+c\left(\frac{F_{pn}}{G^{c}_{xz}}+\frac{H_{pn}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{pn}-c\left(c+f^{b}\right)\frac{B_{pn}}{E^{c}_{z}}\right\}\cdot$$

$$\left.\left\{K_{mq}+L_{mq}+c\left(\frac{F_{mq}}{G^{c}_{xz}}+\frac{H_{mq}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{mq}-c\left(c+f^{b}\right)\frac{B_{mq}}{E^{c}_{z}}\right\}\right]\sin^{2}\Omega_{mn}t$$

...(3.33)

The total energy of the lower facing due to membrane strains may be written as

$$V^{b}_{M}=\left(I^{b}_{1m}+I^{b}_{2m}+I^{b}_{12m}+I^{b}_{3m}\right) \qquad ...(3.34)$$

where the terms I^{b}_{1m}, I^{b}_{2m}, I^{b}_{12m}, and I^{b}_{3m} are given by equations (3.30), (3.31), (3.32), and (3.33), respectively.

3. Strain Energy in the Facings
Due To Bending Strains

The strains due to bending of the facings about their own middle planes can be determined from the slopes and curvatures of each facing. Since the strains due to bending vary linearly and are zero at the middle planes, the bending strain in the x-direction of the upper facing is:

$\epsilon^{a}_{Bx}=+z^{a}\left(\frac{\partial^{2}w^{a}}{\partial x^{2}}\right)_{z^{a}=0}$, where $z^{a}=0$ is the middle plane of the facing f^{a}.

From Fig. 3(b)

$$z^a = z + \frac{f^a}{2}$$

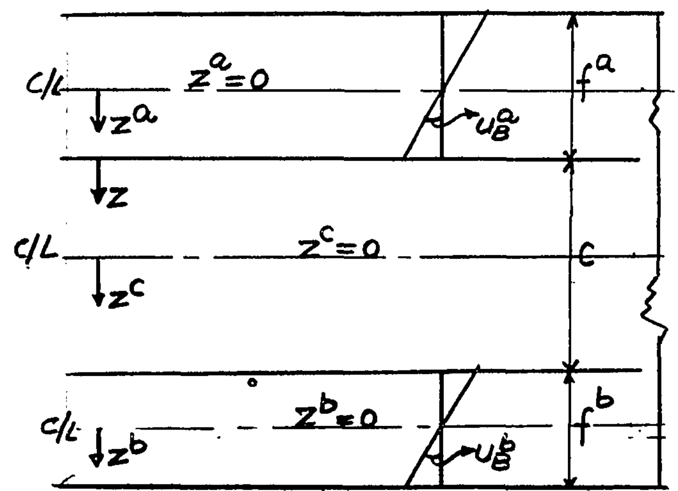

Fig. 3(b)

Therefore

$$\epsilon^a_{Bx} = +\left(z + \frac{f^a}{2}\right)\left(\frac{\partial^2 w^a}{\partial x^2}\right)_{z=-\frac{f^a}{2}}$$

Since

$$\left(w^a\right)_{z=\frac{f^a}{2}} = \left(\omega^c\right)_{z=0}$$

$$\epsilon^a_{Bx} = +\left(z + \frac{f^a}{2}\right)\left(\frac{\partial^2 w^c}{\partial x^2}\right)_{z=0} \qquad \ldots(3.34a)$$

Similarly, the bending strain of the upper facing in the y-direction is:

$$\epsilon^a_{By} = +((z + \frac{f^a}{2})\left(\frac{\partial^2 w^c}{\partial y^2}\right)_{z=0} \qquad \ldots(3.34b)$$

The shearing strains, ϵ^a_{Bxy} in the upper facing are now expressed in terms of the core displacements. Observing that $u^a_B = z^a \left(\frac{\partial w^a}{\partial x}\right)$, and $v^a_B = z^a \left(\frac{\partial w^a}{\partial y}\right)$, where u^a_B and v^a_B are the deformations in the upper facing in the x and y directions due to bending alone,

$$\gamma^a_{Bxy} = 2\left(z + \frac{f^a}{2}\right)\left(\frac{\partial^2 w^c}{\partial x \partial y}\right)_{z=0} \qquad \ldots(3.34c)$$

since

$$\gamma^a_{Bxy} = \left(\frac{\partial u^a_B}{\partial y} + \frac{\partial v^a_B}{\partial x}\right)$$

For the lower facing, f^b, the relation between z and z^b, where $z^b=0$ is the middle plane of the facing f^b, is:

$$z = (c + \frac{f^b}{2}) + z^b$$

or

$$z^b = (z - c - \frac{f^b}{2}).$$

Consequently, the strains are

$$\epsilon^b_{Bx} = (z - c - \frac{f^b}{2})(\frac{\partial^2 w^c}{\partial x^2})_{z=c} \qquad \ldots(3.35)$$

$$\epsilon^b_{By} = (z - c - \frac{f^b}{2})(\frac{\partial^2 w^c}{\partial y^2})_{z=c}$$

$$\gamma^b_{Bxy} = 2(z - c - \frac{f^b}{2})(\frac{\partial^2 w^c}{\partial x \partial y})_{z=c}$$

Upper facing. The strain energy in the upper facing due to bending strains is written as:

$$V^a_B = \frac{1}{2}\int_V \Big[d^a_1(\epsilon^a_{Bx})^2 + d^a_2(\epsilon^a_{By})^2 + d^a_{12}(\epsilon^a_{Bx})(\epsilon^a_{By}) + G^a_{xy}(\gamma^a_{Bxy})^2 \, dv$$

let

$$I^a_{1B} = \frac{1}{2}\int_V d^a_1(\epsilon^a_{Bx})^2 dv, \qquad \frac{1}{2}\int_V d^a_2(\epsilon^a_{By}) dv,$$

$$I^a_{12B} = \frac{1}{2}\int_V d^a_{12}(\epsilon^a_{Bx}\epsilon^a_{By}) dv \quad \text{and} \quad I^a_{3B} = \frac{1}{2}\int_V G^a_{xy}(\gamma^a_{Bxy})^2 dv$$

Substituting equations (3.34b), and (3.34c) into the above quantities, the following relations are obtained:

$$I^a_{1B} = \frac{1}{2} d^a_1 \int_0^a\int_0^b \int_{-f_1}^{0} \left[(z + \frac{f^a}{2}) \sum_m^{\infty}\sum_n^{\infty} X''_m Y_n (A_{mn} + z\frac{B_{mn}}{E^c_z})_{z=0}\right]^2 dxdydz \ \sin^2\Omega_{mn} t$$

$$= \int_0^a\int_0^b\int_{-f}^{0}\Big[\sum\sum X''_m Y_n A_{mn}(z + \frac{f^a}{2})\Big]^2 dxdydz \ \sin^2\Omega_{mn} t$$

$$I^a_{2B} = \frac{1}{2}d^a_2 \int_0^a \int_0^b \int_{-f_1}^0 \left[\sum_m^\infty \sum_n^\infty X_m Y''_n A_{mn} (z + \frac{f^a}{2})\right]^2 dxdydz \ \sin^2\Omega_{mn} t$$

$$I^a_{12B} = \frac{1}{2}d^a_{12} \int_0^a \int_0^b \int_{-f_1}^0 (z + \frac{f^a}{2})^2 \left[\sum_m^\infty \sum_n^\infty X''_m Y_n A_{mn}\right]\left[\sum^\infty \sum^\infty X_m Y''_n A_{mn}\right] dxdydz \ \sin^2\Omega_{mn} t$$

$$I^a_{3B} = \frac{1}{2}G^a_{xy} \int_0^a \int_0^b \int_{-f_1}^0 4(z + \frac{f^a}{2})^2 \left[\sum_m^\infty \sum_n^\infty X'_m Y'_n A_{mn}\right]^2 dxdydz \ \sin^2\Omega_{mn} t$$

or

$$I^a_{1B} = \frac{1}{2} d^a_1 ab (\frac{f^a}{12})^3 \sum_m^\infty \sum_n^\infty (\frac{\beta_m}{a})^4 A^2_{mn} \ \sin^2\Omega_{mn} t \qquad \ldots(3.36)$$

$$I^a_{2B} = \frac{1}{2} d^a_2 ab \ \frac{1}{12}(f^b)^3 \sum_m^\infty \sum_n^\infty (\frac{\beta_n}{b})^4 A^2_{mn} \ \sin^2\Omega_{mn} t \qquad \ldots(3.37)$$

$$I^a_{12B} = \frac{1}{2}d^a_{12} (\frac{1}{12})(f^a)^3 \Bigg[\sum_m^\infty \sum_n^\infty \frac{J_{mm}}{a} \frac{J_{nn}}{b} A^2_{mn} + \sum_{\substack{m \\ m \neq p}}^\infty \sum_n^\infty \sum_p^\infty \frac{J_{mp}}{a} \frac{J_{mn}}{b} A_{mn} A_{pn}$$
$$+ \sum_{\substack{m \\ n \neq q}}^\infty \sum_n^\infty \sum_q^\infty \frac{J_{mm}}{a} \frac{J_{nq}}{b} A_{mn} A_{mq} + \sum_{\substack{m \\ m \neq p, n \neq q}}^\infty \sum_n^\infty \sum_p^\infty \sum_q^\infty \frac{J_{mp}}{a} \frac{J_{nq}}{b} A_{pn} A_{mq} \ \sin^2\Omega_{mn} t \qquad \ldots(3.38)$$

$$I^a_{3B} = \frac{1}{2}G^a_{xy} (\frac{4}{12})(f^a)^3 \Bigg[\sum_m^\infty \sum_n^\infty \frac{I_{mm}}{a} \frac{I_{nn}}{b} A^2_{mn} + \sum_{\substack{m \\ m \neq p}}^\infty \sum_n^\infty \sum_p^\infty \frac{I_{mp}}{a} \frac{I_{nn}}{b} A_{mn} A_{pn}$$
$$+ \sum_{\substack{m \\ n \neq q}}^\infty \sum_n^\infty \sum_q^\infty \frac{I_{mm}}{a} \frac{I_{nq}}{b} A_{mn} A_{mq} + \sum_{\substack{m \\ m \neq p, n \neq q}}^\infty \sum_n^\infty \sum_p^\infty \sum_q^\infty \frac{I_{mp}}{a} \frac{I_{nq}}{b} A_{pn} A_{mq}\Bigg] \sin^2\Omega_{mn} t \qquad \ldots(3.39)$$

The total strain energy in the upper facing due to the bending strains is now given by:

$$V^a_B = I^a_{1B} + I^a_{2B} + I^a_{12B} + I^a_{3B} \qquad \ldots(3.40)$$

where I^a_{1B}, I^a_{2B}, I^a_{12B} and I^a_{3B} correspond to the equations (3.35), (3.37), (3.38) and (3.39) above

<u>Lower Facing.</u> The strain energy in the lower facing due to bending is:

$$V^b_B = \frac{1}{2}\int_V\left[d^b_1(\epsilon^b_{Bx})^2 + d^b_2(\epsilon^b_{By})^2 + d^b_{12}(\epsilon^b_{Bx})(\epsilon^b_{By}) + G^b_{Bxy}(\gamma^b_{Bxy})^2\right]dv \qquad \ldots(3.41)$$

Denoting

I^b_{1B} by $\frac{1}{2}\int_V d^b_1(\epsilon^b_{Bx})^2 dv$, $\qquad I^b_{2B}$ by $\frac{1}{2}\int_V d^b_2(\epsilon^b_{By})^2 dv$,

I^b_{12B} by $\frac{1}{2}\int_V d^b_{12}(\epsilon^b_{Bx})(\epsilon^b_{By}) dv$ and I^b_{3B} by $\int_V G^b_{xy}(\gamma^b_{Bxy})^2 dv$,

and using relations (3.35), the following expressions are obtained:

$$I^b_{1B} = \frac{1}{2}d^b_1 \int_0^a\int_0^b\int_c^{c+f^b}\left[(z-c-\frac{1}{2}f^b)\sum_m^\infty\sum_n^\infty X''_m Y_n (A_{mn} + c\frac{B_{mn}}{E^c_z})\right]^2 dxdydz\sin^2\Omega_{mn}t$$

$$I^b_{2B} = \frac{1}{2}d^b_2 \int_0^a\int_0^b\int_c^{c+f^b}\left[(z-c-\frac{1}{2}f^b)\sum_m^\infty\sum_n^\infty X_m Y''_n (A_{mn} + c\frac{B_{mn}}{E^c_z})\right]^2 dxdydz\sin^2\Omega_{mn}t$$

$$I^b_{12B} = \frac{1}{2}d^b_{12} \int_0^a\int_0^b\int_c^{c+f^b}\left[z-c-\frac{1}{2}f^b\right]\left[\sum_m^\infty\sum_n^\infty X''_m Y_n (A_{mn} + c\frac{B_{mn}}{E^c_z})\right] \cdot$$

$$\left[\sum_m^\infty\sum_n^\infty X_m Y''_n (A_{mn} + c\frac{B_{mn}}{E^c_z})\right] dxdydz\ \sin^2\Omega_{mn}t$$

$$I^b_{3B} = \frac{1}{2}G^b_{xy} \int_0^a\int_0^b\int_0^{c+f^b} 4(z-c-\frac{1}{2}f^b)^2\left[\sum_m^\infty\sum_n^\infty X'_m Y'_n (A_{mn} + c\frac{B_{mn}}{E^c_z})\right]^2 dxdydz\sin^2\Omega_{mn}t$$

with the results of Appendix (C), the above integrals yield:

$$I^b_{1B} = (\frac{1}{2}d^b_1)\frac{ab}{12}(f^b)^3\sum_m^\infty\sum_n^\infty(\frac{\beta_m}{a})^4(A_{mn} + c\frac{B_{mn}}{E^c_z})^2\sin^2\Omega_{mn}t \qquad \ldots(3.42)$$

$$I^b_{2B}=(\frac{1}{2}d^b_2)\frac{ab}{12}(f^b)^3\sum_m^\infty\sum_n^\infty(\frac{\beta n}{b})^4(A_{mn}+c\frac{B_{mn}}{E^c_z})^2\sin^2\Omega_{mn}t \qquad ...(3.43$$

$$I^b_{12B}=(\frac{1}{2}d^b_{12})\frac{1}{12}(f^b)^3\Big[\sum_m^\infty\sum_n^\infty\frac{J_{mm}}{a}\frac{J_{nn}}{b}(A_{mn}+c\frac{B_{mn}}{c})^2+$$

$$\sum_m^\infty\sum_n^\infty\sum_{p \atop m\neq p}^\infty\frac{J_{mp}}{a}\frac{J_{nn}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})(A_{pn}+c\frac{B_{pn}}{E^c_z})+$$

$$\sum_m^\infty\sum_n^\infty\sum_{q \atop n\neq q}^\infty\frac{J_{mm}}{a}\frac{J_{nq}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})(A_{nq}+c\frac{B_{mq}}{E^c_z})+$$

$$\sum_m^\infty\sum_n^\infty\sum_p^\infty\sum_{q \atop m\neq p,\, n\neq q}^\infty\frac{J_{mp}}{a}\frac{J_{nq}}{b}(A_{pn}+c\frac{B_{pn}}{E^c_z})(A_{mq}+c\frac{B_{nq}}{E^c_z})\sin^2\Omega_{mn}t \qquad ...(3.44)$$

$$I^b_{3B}=\frac{1}{2}G^b_{xy}(\frac{4}{12})(f^b)^3\Big[\sum_m^\infty\sum_n^\infty\frac{I_{mm}}{a}\frac{I_{nn}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})^2$$

$$+\sum_m^\infty\sum_n^\infty\sum_{p \atop m\neq p}^\infty\frac{I_{mp}}{a}\frac{I_{nn}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})(A_{pn}+c\frac{B_{pn}}{E^c_z})$$

$$+\sum_m^\infty\sum_n^\infty\sum_{q \atop n\neq q}^\infty\frac{I_{mm}}{a}\frac{I_{nq}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})(A_{mq}+c\frac{B_{mq}}{E^c_z})$$

$$+\sum_m^\infty\sum_n^\infty\sum_p^\infty\sum_{q \atop m\neq p,\, n\neq q}^\infty\frac{I_{mp}}{a}\frac{I_{nq}}{b}(A_{pn}+c\frac{B_{pn}}{E^c_z})(A_{mq}+c\frac{B_{mq}}{E^c_z})\Big]\sin^2\Omega_{mn}t \qquad ...(3.44a)$$

The strain energy in the lower facing due to bending strains is now expressed as:

$$V^b_B = I^b_{1B} + I^b_{2B} + I^b_{12B} + I^b_{3B} \qquad ...(3.45)$$

where I^b_{1B}, I^b_{2B}, I^b_{12B} and I^b_{3B} are given by equations (3.42), (3.43), (3.44) and (3.44a) respectively.

Total Strain-Energy

The total strain energy of the system is obtained by adding the results of equations (3.16), (3.27), (3.34), (3.40) and (3.45):

$$V_{t\ t} = \Bigg\{ \frac{1}{2}(g_3^c)(abc)\sum_m^{\infty}\sum_n^{\infty}\left(\frac{B_{mn}}{E_z^c}\right)^2$$

$$+g_2^c\left(\frac{abc^3}{12}\right)\sum_m^{\infty}\sum_n^{\infty}\left(\frac{\beta_n}{b}\right)^4\left[A_{mn}^2+cA_{mn}\frac{B_{mn}}{E_z^c}+\frac{2}{5}c^2\left(\frac{B_{mn}}{E_z^c}\right)^2\right]$$

$$+g_{32}^c\left[\frac{ac^3}{12}\sum_m^{\infty}\sum_n^{\infty}\frac{J_{nn}}{b}\left(\frac{B_{mn}}{E_z^c}\right)^2+\frac{ac^3}{12}\sum_m^{\infty}\sum_n^{\infty}\sum_{\substack{q\\ n\neq q}}^{\infty}\frac{J_{nq}}{b}\left(\frac{B_{mn}}{E_z^c}\right)\left(\frac{B_{mq}}{E_z^c}\right)\right]$$

$$+G_{yz}^c\left[ac\sum_m^{\infty}\sum_n^{\infty}\frac{I_{nn}}{b}\left(\frac{H_{mn}}{G_{yz}^c}\right)^2+ac\sum_m^{\infty}\sum_n^{\infty}\sum_{\substack{q\\ n\neq q}}^{\infty}\frac{I_{nq}}{b}\left(\frac{H_{mn}}{G_{yz}^c}\right)\left(\frac{H_{mq}}{E_z^c}\right)\right]$$

$$+G_{xz}^c\left[bc\sum_m^{\infty}\sum_n^{\infty}\frac{I_{mm}}{a}\left(\frac{F_{mn}}{G_{xz}^c}\right)^2+bc\sum_m^{\infty}\sum_n^{\infty}\sum_{\substack{p\\ m\neq p}}^{\infty}\frac{I_{mp}}{a}\left(\frac{F_{mn}}{G_{xz}^c}\right)\left(\frac{F_{pn}}{E_z^c}\right)\right]$$

$$+(d_1^a)abf^a\sum_{mn}^{\infty}\sum^{\infty}\left(\frac{\beta_m}{a}\right)^4\left[K_{mn}+\frac{1}{2}f^aA_{mn}\right]^2+d_2^aabf^a\sum_{mn}^{\infty}\sum^{\infty}\left(\frac{\beta_n}{b}\right)^4\left[L_{mn}+\frac{1}{2}f^aA_{mn}\right]^2$$

$$+d_{12}^af^a\Bigg[\sum_m^{\infty}\sum_n^{\infty}\frac{J_{mm}}{a}\,\frac{J_{nn}}{b}\left(K_{mn}+\frac{1}{2}f^aA_{mn}\right)\left(L_{mn}+\frac{1}{2}f^aA_{mn}\right)$$

$$+\sum_m^{\infty}\sum_n^{\infty}\sum_{\substack{p\\ m\neq p}}^{\infty}\frac{J_{mp}}{a}\,\frac{J_{nn}}{b}\left(K_{mn}+\frac{1}{2}f^aA_{mn}\right)\left(L_{pn}+\frac{1}{2}f^aA_{pn}\right)$$

$$+\sum_m^{\infty}\sum_n^{\infty}\sum_{\substack{q\\ n\neq q}}^{\infty}\frac{J_{mm}}{a}\,\frac{J_{nq}}{b}\left(L_{mn}+\frac{1}{2}f^aA_{mn}\right)\left(K_{mq}+\frac{1}{2}f^aA_{mq}\right)$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\sum_{q}^{\infty}\frac{J_{mp}}{a}\,\frac{J_{nq}}{b}\left(K_{pn}+\frac{1}{2}f^{a}A_{pn}\right)\left(L_{mq}+\frac{1}{2}f^{a}A_{mq}\right)$$

$m\neq p,\ n\neq q$

$$+G^{a}_{xy}f^{a}\left[\sum_{m}^{\infty}\sum_{n}^{\infty}\frac{I_{mm}}{a}\,\frac{I_{nn}}{b}\left(K_{mn}+L_{mn}+f^{a}_{1}A_{mn}\right)\right.$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\frac{I_{mp}}{a}\,\frac{I_{nn}}{b}\left(K_{mn}+L_{mn}+f^{a}A_{mn}\right)\left(K_{pn}+L_{pn}+f^{a}A_{pn}\right)$$

$m\neq p$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{q}^{\infty}\frac{I_{mm}}{a}\,\frac{I_{nq}}{b}\left(K_{mn}+L_{mn}+f^{a}A_{mn}\right)\left(K_{mq}+L_{mq}+f^{a}A_{mq}\right)$$

$n\neq q$

$$\left.+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\sum_{q}^{\infty}\frac{I_{mp}}{a}\,\frac{I_{nq}}{b}\left(K_{pn}+L_{pn}+f^{a}A_{pn}\right)\left(K_{mq}+L_{mq}+f^{a}A_{mq}\right)\right]$$

$m\neq p,\ n\neq q$

$$+d^{a}_{1}abf^{a}\sum_{m}^{\infty}\sum_{n}^{\infty}\left(\frac{\beta_m}{a}\right)^{4}\left[K_{mn}+c\frac{F_{mn}}{G^{c}_{xz}}\left(c+\frac{f^{b}}{2}\right)A_{mn}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{mn}}{E^{c}_{z}}\right]^{2}$$

$$+d^{b}_{2}abf^{b}\sum_{m}^{\infty}\sum_{n}^{\infty}\left(\frac{\beta_n}{b}\right)^{4}\left[L_{mn}+c\frac{H_{mn}}{G^{c}_{yz}}\left(c+\frac{1}{2}f^{b}\right)A_{mn}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{mn}}{E^{c}_{z}}\right]^{2}$$

$$+d^{b}_{12}f^{b}\left[\sum_{m}^{\infty}\sum_{n}^{\infty}\frac{J_{mm}}{a}\,\frac{J_{nn}}{b}\left\{K_{mn}+c\frac{F_{mn}}{G^{c}_{xz}}\left(c+\frac{f^{b}}{2}\right)A_{mn}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{mn}}{E^{c}_{z}}\right\}\cdot\right.$$

$$\left\{L_{mn}+c\frac{H_{mn}}{G^{c}_{yz}}\left(c+\frac{f^{b}}{2}\right)A_{mn}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{mn}}{E^{c}_{z}}\right\}$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\frac{J_{mp}}{a}\,\frac{J_{nn}}{b}\left\{K_{mn}+c\frac{F_{mn}}{G^{c}_{xz}}\left(c+\frac{f^{b}}{2}\right)A_{mn}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{mn}}{E^{c}_{z}}\right\}\cdot$$

$m\neq p$

$$\left\{L_{pn}+c\frac{H_{pn}}{G^{c}_{yz}}\left(c+\frac{f^{b}}{2}\right)A_{pn}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{pn}}{E^{c}_{z}}\right\}$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{q}^{\infty}\frac{J_{mm}}{a}\,\frac{J_{nq}}{b}\left\{L_{mn}+c\frac{H_{mn}}{G^{c}_{yz}}\left(c+\frac{f^{b}}{2}\right)A_{mn}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{pn}}{E^{c}_{z}}\right\}\cdot$$

$n\neq q$

$$\left\{K_{mq}+c\frac{F_{mq}}{G^{c}_{xz}}-\left(c+\frac{f^{b}}{2}\right)A_{mq}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{mq}}{E^{c}_{z}}\right\}$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\sum_{q}^{\infty}J_{mp}J_{nq}\left\{K_{pn}+c\frac{F_{pn}}{G^{c}_{xz}}-\left(c+\frac{f^{b}}{2}\right)A_{pn}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{pn}}{E^{c}_{z}}\right\}\cdot$$

$m\neq p, n\neq q$

$$\left\{L_{mq}+c\frac{H_{mq}}{c}-\left(c+\frac{f^{b}}{2}\right)A_{mq}-\frac{c}{2}\left(c+f^{b}\right)\frac{B_{mq}}{E^{c}_{z}}\right\}\Bigg]$$

$$+G^{b}_{xy}f^{b}\left[\sum_{m}^{\infty}\sum_{n}^{\infty}\frac{I_{mm}}{a}\,\frac{I_{nn}}{b}\left\{(K_{mn}+L_{mn})+c\left(\frac{F_{mn}}{G^{c}_{xz}}+\frac{H_{mn}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{mn}-c\left(c+f^{b}\right)\left(\frac{B_{mn}}{E^{c}_{z}}\right)\right\}^{2}\right.$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\frac{I_{mp}}{a}\,\frac{I_{nn}}{b}\left\{(K_{mn}+L_{mn})+c\left(\frac{F_{mn}}{G^{c}_{xz}}+\frac{H_{mn}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{mn}-c\left(c+f^{b}\right)\frac{B_{mn}}{E^{c}_{z}}\right\}\cdot$$

$m\neq p$

$$\left\{(K_{pn}+L_{pn})+c\frac{F_{pn}}{G^{c}_{xz}}+\frac{H_{pn}}{G^{c}_{xz}}-2\left(c+\frac{f^{b}}{2}\right)A_{pn}-c\left(c+f^{b}\right)\frac{B_{pn}}{E^{c}_{z}}\right\}$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{q}^{\infty}\frac{I_{mm}}{a}\,\frac{I_{nq}}{b}\left\{K_{mn}+L_{mn}+c\left(\frac{F_{mn}}{G^{c}_{xz}}\quad\frac{H_{mn}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{mn}-c\left(c+f^{b}\right)\frac{B_{mn}}{E^{c}_{z}}\right\}\cdot$$

$n\neq q$

$$\left\{K_{mq}+L_{mq}+c\frac{F_{mq}}{G^{c}_{xz}}+\frac{H_{mq}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{mq}-c\left(c+f^{b}\right)\frac{B_{mq}}{E^{c}_{z}}\right\}$$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\sum_{q}^{\infty}I_{mp}I_{nq}\left\{K_{pn}+L_{pn}+c\left(\frac{F_{pn}}{G^{c}_{xz}}+\frac{H_{pn}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{pn}-c\left(c+f^{b}\right)\frac{B_{pn}}{E^{c}_{z}}\right\}\cdot$$

$m\neq p, n\neq q$

$$\left\{K_{mq}+L_{mq}+c\left(\frac{F_{mq}}{G^{c}_{xz}}+\frac{H_{mq}}{G^{c}_{yz}}\right)-2\left(c+\frac{f^{b}}{2}\right)A_{mq}-c\left(c+f^{b}\right)\frac{B_{mq}}{E^{c}_{z}}\right]$$

$$+\frac{1}{12}\left(f^{a}\right)^{3}\left[abd^{a}_{1}\sum_{m}^{\infty}\sum_{n}^{\infty}\left(\frac{m}{a}\right)^{4}A^{2}_{mn}+abd^{a}_{2}\;mn\left(\frac{n}{b}\right)^{4}A^{2}_{mn}\right.$$

$$+d^{a}_{12}\sum_{m}^{\infty}\sum_{n}^{\infty}\frac{J_{mm}}{a}\,\frac{J_{nn}}{b}A^{2}_{mn}+\sum_{m}\sum_{n}\sum_{p}\frac{J_{mp}}{a}\,\frac{J_{nn}}{b}A_{mn}A_{pn}$$

$m\neq p$

$$+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{q}^{\infty}\frac{J_{mm}}{a}\,\frac{J_{nq}}{b}A_{mn}A_{mq}+\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\sum_{q}^{\infty}\frac{J_{pn}}{a}\,\frac{J_{nq}}{b}A_{pn}A_{mq}$$

$n\neq q$ $\qquad$ $m\neq p, n\neq q$

$$+4G^a_{xy}\sum_m^\infty\sum_n^\infty \frac{I_{mm}}{a}\frac{I_{nn}}{b}A^2_{mn}+\sum_m^\infty\sum_n^\infty\sum_p^\infty \frac{I_{mp}}{a}\frac{I_{nn}}{b}A_{mn}A_{pn}$$

$m \neq p$

$$+\sum_m^\infty\sum_n^\infty\sum_q^\infty \frac{I_{mm}}{a}\frac{I_{nq}}{b}A_{mn}A_{mq}+\sum_m^\infty\sum_n^\infty\sum_p^\infty\sum_q^\infty \frac{I_{mp}}{a}\frac{I_{nq}}{b}A_{pn}A_{mq}\Big]$$

$n \neq q$ $\qquad$ $m\ p, n\ q$

$$\frac{1}{12}(f^b)^3\Big[abd^b_1\sum_m^\infty\sum_n^\infty(\frac{m}{a})^4(A_{mn}+c\frac{B_{mn}}{E^c_z})^2+abd^b_2\sum_m^\infty\sum_n^\infty(\frac{n}{a})^4(A_{mn}+c\frac{m}{E^c_z})^2$$

$$+d^b_{12}\sum_m^\infty\sum_n^\infty \frac{J_{mm}}{a}\frac{J_{nn}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})^2$$

$$+\sum_m^\infty\sum_n^\infty\sum_p^\infty \frac{J_{mp}}{a}\frac{J_{nn}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})(A_{pn}+c\frac{B_{pn}}{E^c_z})$$

$m \neq p$

$$+\sum_m^\infty\sum_n^\infty\sum_q^\infty \frac{J_{mm}}{a}\frac{J_{nq}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})(A_{mq}+c\frac{B_{mq}}{E^c_z})$$

$n \neq q$

$$+4G^b_{xy}\sum_m^\infty\sum_n^\infty \frac{I_{mm}}{a}\frac{I_{nn}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})^2$$

$$+\sum_m^\infty\sum_n^\infty\sum_p^\infty \frac{I_{mp}}{a}\frac{I_{nn}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})(A_{pn}+c\frac{B_{pn}}{E^c_z})$$

$m \neq p$

$$+\sum_m^\infty\sum_n^\infty\sum_q^\infty \frac{I_{mm}}{a}\frac{I_{nq}}{b}(A_{mn}+c\frac{B_{mn}}{E^c_z})(A_{mq}+c\frac{B_{mq}}{E^c_z})$$

$n \neq q$

$$+\sum_m^\infty\sum_n^\infty\sum_p^\infty\sum_q^\infty \frac{I_{mp}}{a}\frac{I_{nq}}{b}(A_{pn}+c\frac{B_{pn}}{E^c_z})(A_{mq}+c\frac{B_{mq}}{E^c_z})\Big]\ \sin^2\Omega_{mn}t \quad ...(3.46)$$

$m \neq p, n \neq q$

Kinetic Energy

The total kinetic energy in the vibrating system consists of translational and rotational components. Since the aim of the present investigation centers on the low-frequency range, the rotational energy components are ignored because their inclusion will not significantly affect the accuracy of the results. Similarly, the energy components due to translations in the plane of plate are insignificant and can be neglected. Therefore, only the kinetic energy due to vertical translation is accounted for in the present analysis.

Considering the kinetic energy dT of a differential element of the sandwich plate, the energy due to vertical translation is:

$$dT = \frac{1}{2}\left[(dm^a)(\dot{w}^a)^2 + (dm^b)(\dot{w}^b)^2 + (dm^c)(\dot{w}^c)^2\right] \qquad \ldots(3.47)$$

where the superscrips a, b, and c refer to the upper facing, the lower facing, and the core. If, ρ^a, ρ^b and ρ^c are the mass densities per unit volume of the upper facing, lower facing and the core, and considering $w^a=(w^c)_{z=0}$ and $w^b=(w^c)_{z=c}$, equation (3.47) produces

$$dT = \frac{1}{2}\left[\rho^a\left\{(\dot{w}^c)_{z=0}\right\}^2 dv^a + \rho^b\left\{(\dot{w}^c)_{z=c}\right\}^2 dv^b + \rho^c\left\{(\dot{w}^c)\right\}^2 dv^c\right] \qquad \ldots(3.48)$$

By means of the displacement functions (3.12), the terms of the above equations can be expressed as:

$$\rho^a\left[(\dot{w}^c)_{z=0}\right]^2 dv^a = \rho^a\Omega^2_{mn}\left[\sum^{\infty}\sum^{\infty} X_m Y_n A_{mn}\right]^2 dv^a \cos^2\Omega_{mn}t$$

$$\rho^b\left[(\dot{w}^c)_{z=c}\right]^2 dv^b = \rho^b\Omega^2_{mn}\left[\sum_m^{\infty}\sum_n^{\infty} X_m Y_n \left(A_{mn} + c\frac{B_{mn}}{E^c_z}\right)\right]^2 dv^b \cos^2\Omega_{mn}t$$

$$\rho^c\left[(\dot{w}^c)\right]^2 dv^c = \rho^c\Omega^2_{mn}\left[\sum_m^{\infty}\sum_n^{\infty} X_m Y_n \left(A_{mn} + z\frac{B_{mn}}{E^c_z}\right)\right]^2 dv^c \cos^2\Omega_{mn}t$$

The total kinetic energy 'T' due to vertical translation of the vibrating plate is, therefore, given by:

$$T = \frac{1}{2}\Omega^2_{mn}\left\{\int_0^a\int_0^b\int_{-f^a}^0 \rho^a\left[\sum_m^\infty\sum_n^\infty X_m Y_n A_{mn}\right]^2 dxdydz\right.$$

$$+\int_0^a\int_0^b\int_c^{c+f^b}\rho^b\left[\sum_m^\infty\sum_n^\infty X_m Y_n\left(A_{mn}+c\frac{B_{mn}}{E_z^c}\right)\right]^2 dxdydz$$

$$\left.+\int_0^a\int_0^b\int_0^c\rho^c\left[\sum_m^\infty\sum_n^\infty X_m Y_n\left(A_{mn}+z\,\frac{B_{mn}}{E_z^c}\right)\right]^2 dxdydz\right\}\cos^2\Omega_{mn}t \qquad \ldots(3.49)$$

Observing the orthogonal properties of X_m and Y_n, and carrying out the indicated integrations, equation (3.49) yields:

$$T = \frac{1}{2}\Omega^2_{mn}(a)(b)\left\{\sum_m^\infty\sum_n^\infty \rho^a f^a A^2_{mn} + \sum_m^\infty\sum_n^\infty \rho^b f^b\left(A_{mn}+c\frac{B_{mn}}{E_z^c}\right)\right.$$

$$\left.+\sum_m^\infty\sum_n^\infty\rho^c\left[c\;A^2_{mn}+cA_{mn}\frac{B_{mn}}{E_z^c}+\frac{1}{3}c^2\left(\frac{B_{mn}}{E_z^c}\right)^2\right]\right\}\cos^2\Omega_{mn}t \qquad \ldots(3.50)$$

Frequency Criterion

The vibrating plate is assumed to be a conservative system, so that the variation of the total energy associated with the arbitrary displacements must vanish. These displacements are defined by the parameters A_{mn}, B_{mn}, F_{mn}, H_{mn}, K_{mn} and L_{mn}; thus the parameters A_{mn}, ..., L_{mn} must be chosen so as to make $(V_{max}-T_{max}) = 0$. This leads to the following set of equations:

$$\frac{\partial}{A_{mn}}(V_{max}-T_{max}) = 0$$

$$\frac{\partial}{B_{mn}}(V_{max}-T_{max}) = 0$$

$$\frac{\partial}{F_{mn}}(V_{max}-T_{max}) = 0$$

$$\frac{\partial}{H_{mn}}(V_{max}-T_{max}) = 0$$

$$\frac{\partial}{K_{mn}}(V_{max}-T_{max}) = 0$$

$$\frac{\partial}{L_{mn}}(V_{max}-T_{max}) = 0 \qquad \ldots(3.51)$$

When the expressions for V_{max} and T_{max}, given by the equations (3.45) and (3.50) are substituted into the equations (3.51), a system of six equations containing the parameters $A_{mn} \ldots L_{mn}$ is obtained. It is observed that these six equations contain series of the form,

$$\sum_{p}^{\infty} A_{pn}, \ldots\ldots, \ldots, \qquad \sum_{p}^{\infty} L_{pn} \ldots\ldots \qquad \ldots(3.52).$$

For a practical solution, only a finite number of terms in these series need by considered.

The convergence of these series depends largely upon the proper choice of the coordinate functions. Only the first term of these series is considered here because (1) the present analysis employs the coordinate functions which describe the mode configuration exactly, and (2) because of the prime interest in the low-frequency range. This reduces the system of equations (3.51) to the following six equations in six parameters, A_{mn}, B_{mn}, F_{mn}, H_{mn}, K_{mn}, and L_{mn}.

$$\begin{bmatrix} C_{11}-\Omega^2_{mn}\lambda_{11} & C_{12}-\Omega^2_{mn}\lambda_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12}-\Omega^2_{mn}\lambda_{12} & C_{22}-\Omega_{mn}\lambda_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} A_{mn} \\ B_{mn} \\ F_{mn} \\ H_{mn} \\ K_{mn} \\ L_{mn} \end{bmatrix} = 0 \qquad \ldots(3.52)$$

where,

$$C_{11} = \Big\{ \frac{2}{3}(f^a)^3 \Big[\Big(\frac{\beta m}{a}\Big)^4 d_1^a + \Big(\frac{\beta n}{b}\Big)^4 \Big\{ \frac{g_2^c}{4}\Big(\frac{c}{f^a}\Big)^3 + d_2^a \Big\} + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^a + G_{xy}^a \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} \Big]$$

$$+ 2f^b \Big[\Big(c+\frac{f^b}{2}\Big)^2 + \frac{1}{12}(f^b)^2 \Big] \Big[\Big(\frac{\beta m}{a}\Big)^4 d_1^b + \Big(\frac{\beta n}{b}\Big)^4 d_2^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \Big] \Big\}$$

$$C_{12} = \Big\{ \frac{cf^b}{g_3^c} \Big[\Big(c+\frac{f^b}{2}\Big)(c+f^b) + \Big(\frac{f^b}{6}\Big)^2 \Big] \Big[\Big(\frac{\beta m}{a}\Big)^4 d_1^b + \Big(\frac{\beta n}{b}\Big)^4 d_2^b$$

$$+ \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \Big] + \frac{g_2^c}{g_3^c}\Big(\frac{\beta n}{b}\Big)^4 \frac{c^4}{12} \Big\}$$

$$C_{13} = -\Big\{ \frac{cf^b}{G_{xz}^c}\Big(c+\frac{f^b}{2}\Big) \Big[2\Big(\frac{\beta m}{a}\Big)^4 d_1^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^a + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \Big] \Big\}$$

$$C_{14} = -\Big\{ \frac{cf^b}{G_{yz}}\Big(c+\frac{f^b}{2}\Big) \Big[2\Big(\frac{\beta n}{b}\Big)^4 d_2^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \Big] \Big\}$$

$$C_{15} = \Big\{ \frac{1}{2}(f^a)^2 \Big[2\Big(\frac{\beta m}{a}\Big)^4 d_1^a + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2}(d_{12}^a) + 4G_{xy}^a \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \Big]$$

$$- f^b\Big(c+\frac{f^b}{2}\Big) \Big[2\Big(\frac{\beta m}{a}\Big)^4 d_1^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \Big] \Big\}$$

$$c_{16} = \left\{ \frac{1}{2}(f^a)^2 \left[2\left(\frac{\beta n}{b}\right)^4 d_2^a + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^a + 4G_{xy}^a \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] \right.$$

$$\left. - f^b\left(c + \frac{f^b}{2}\right) \left[2\left(\frac{\beta n}{b}\right)^4 d_2^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] \right\}$$

$$c_{22} = \left\{ \frac{c^2 f^b}{6(g_3^c)^2} \left[3(c+f^b)^2 + (f^b)^2 \right] \left[\left(\frac{\beta m}{a}\right)^4 d_1^b + \left(\frac{\beta n}{b}\right)^4 d_2^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b \right. \right.$$

$$\left. \left. + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] + \frac{1}{g_3^c} \left[2c + \frac{c^5}{15}\left(\frac{\beta n}{b}\right)^4 \frac{g_2^c}{g_3^c} + \frac{c^3}{6}\frac{J_{nn}}{b^2}\frac{g_{32}^c}{g_3^c} \right] \right\}$$

$$c_{23} = -\left\{ \frac{1}{(g_3^c)} \frac{f^b c^2}{G_{xz}^c} \frac{1}{2}(c+f^b) \left[2\left(\frac{\beta m}{a}\right)^4 d_1^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] \right\}$$

$$c_{24} = -\left\{ \frac{1}{(g_3^c)} \frac{f^b c^2}{G_{yz}^c} \frac{1}{2}(c+f^b) \left[2\left(\frac{\beta n}{b}\right)^4 d_2^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] \right\}$$

$$c_{25} = -\left\{ \frac{f^b c}{2(g_3^c)} (c+f^b) \left[2\left(\frac{\beta m}{a}\right)^4 d_1^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] \right\}$$

$$c_{26} = -\left\{ \frac{f^b c}{2(g_3^c)} (c+f^b) \left[2\left(\frac{\beta n}{b}\right)^4 d_2^b + \frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 4G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] \right\}$$

$$c_{33} = \left\{ 2\frac{I_{mm}}{a^2}\frac{c}{G_{xz}^c} + 2\frac{c^2 f^b}{(G_{xz}^c)^2} \left[\left(\frac{\beta n}{a}\right)^4 d_1^b + G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] \right\}$$

$$c_{34} = \left\{ \frac{f^b c^2}{2(G_{yz}^c)} \frac{1}{(G_{xz}^c)} \left[\frac{J_{mm}}{a^2}\frac{J_{nn}}{b^2} d_{12}^b + 2G_{xy}^b \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} \right] \right\}$$

$$c_{35} = \left\{ \frac{2f^b c}{(G_{xz}^c)} \left[\left(\frac{\beta n}{a}\right)^d d_1^b + \frac{I_{mm}}{a^2}\frac{I_{nn}}{b^2} G_{xy}^b \right] \right\}$$

$$c_{36} = \frac{f^b c}{G^c_{xz}} \left[\frac{J_{mm}}{a^2} \frac{J_{nn}}{b^2} d^b_{12} + 2G^b_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right]$$

$$c_{44} = 2\frac{I_{nn}}{b^2} \frac{c}{G^c_{yz}} + \frac{2f^b c^2}{(G^c_{yz})^2} \left[\left(\frac{\beta n}{b}\right)^4 d^b_2 + G^b_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right]$$

$$c_{45} = \frac{c f^b}{G^c_{yz}} \left[\frac{J_{mm}}{a^2} \frac{J_{nn}}{b^2} d^b_{12} + 2G^b_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right]$$

$$c_{55} = 2f^a \left[\left(\frac{\beta n}{a}\right)^4 d^a_1 + G^a_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right] + 2f^b \left[\left(\frac{\beta m}{a}\right)^4 d^b_1 + G^b_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right]$$

$$c_{56} = f^a \left[\frac{J_{mm}}{a^2} \frac{J_{nn}}{b^2} d^a_{12} + 2G^a_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right] + f^b \left[\frac{J_{mm}}{a^2} \frac{J_{nn}}{b^2} d^b_{12} \right.$$

$$\left. + 2G^b_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right]$$

$$c_{66} = 2f^a \left[\left(\frac{\beta n}{b}\right)^4 d^a_1 + G^a_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right] + 2f^b \left[\left(\frac{\beta n}{b}\right)^4 d^b_1 + G^b_{xy} \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} \right]$$

$$c_{46} = (2f^b c)/G^c_{yz} \left[\left(\frac{\beta n}{b}\right)^4 d^b_2 + \frac{I_{mm}}{a^2} \frac{I_{nn}}{b^2} G^b_{xy} \right]$$

$$\lambda_{11} = 2\left[\rho^c c + \rho^a f^a + \rho^b f^b \right]$$

$$\lambda_{12} = \frac{1}{g^c_3} \left[\rho^c c^2 + 2\rho^b c f^b \right]$$

$$22 = \frac{1}{(g^c_3)^2} \left[\frac{2}{3} \rho^c c^3 + 2\rho^b \, (c^2 f^b) \right]$$

For a non-trivial solution of the system (3.52) the determinant of the coefficients A_{mn}, B_{mn}, ... L_{mn} must vanish. This condition of a vanishing coefficient determinant constitutes the frequency criterion.

Hence

$$\begin{bmatrix} c_{11}-\Omega^2_{mn}\lambda_{11} & c_{12}-\Omega^2_{mn}\lambda_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12}-\Omega^2_{mn}\lambda_{12} & c_{22}-\Omega^2_{mn}\lambda_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} = 0 \qquad \ldots(3.53)$$

The evaluation of the determinant yields a quadratic in Ω^2_{mn}. That is, for each mode shape which is characterized by m-half waves in the x-direction and n-half waves in the y-direction, there are two values of Ω^2_{mn}. These two frequencies have the following significance:

Suppose $(\Omega^2_{mn})_1$ and $(\Omega^2_{mn})_2$ are the two frequencies for the m-n mode with $(\Omega^2_{mn})_1$ being the smaller of the two. This lower frequency $(\Omega^2_{mn})_1$ is associated with that type of motion in which the two facings and the core all move in phase. This corresponds to the motion of the neutral plane and thus $(\Omega^2_{mn})_1$ is the frequency of the normal mode of vibration. The practical interest of the engineer centers on the lower frequency, $(\Omega^2_{mn})_1$, since for sandwich construction of the usual proportions and physical properties, the value of $(\Omega^2_{mn})_2$ is so much larger than $(\Omega^2_{mn})_1$ that the corresponding vibration mode is hardly encountered. The larger frequency $(\Omega^2_{mn})_2$ is associated with the "face-wrinkling" of the sandwich plate. The mode shape given by $(\Omega^2_{mn})_2$ may be attributed to a motion in which the two parts of sandwich, above and below the neutral plane, move independently.

CHAPTER IV

APPLICATIONS AND DISCUSSION

In Chapter III, a theoretical analysis of sandwich plates was developed without restricting the edge conditions of the plates. The analysis led to the frequency criterion in terms of the determinantal equation (3.53). In the present chapter this equation is employed to predict the frequencies for,

(1). sandwich beams

(2). solid plates

(3). general cases of sandwich plates.

In spite of the simplified form of the frequency equation (3.53), the resulting quadratic in Ω^2_{mn} is too complicated for formulating results in parametric form. However, before attempting a numerical evaluation of the quadratic, the analysis is validated by comparing the known frequencies of solid beams and plates, with those of sandwich structures reduced to homogeneous and isotropic beams and plates.

1. Sandwich Beam

In reducing the sandwich plate to a sandwich beam, the parameters H_{mn} and I_{mn} vanish and the system of equations (3.52) assumes the form

$$\begin{bmatrix} C_{11}-\Omega^2_m\lambda_{11} & C_{12}-\Omega^2_m\lambda_{12} & C_{13} & C_{15} \\ C_{12}-\Omega^2_m\lambda_{12} & C_{22}-\Omega^2_m\lambda_{22} & C_{23} & C_{25} \\ C_{13} & C_{23} & C_{33} & C_{35} \\ C_{15} & C_{25} & C_{35} & C_{45} \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ F_m \\ K_m \end{bmatrix} = 0 \quad \ldots(4.1)$$

The coefficients C_{ij}, λ_{ij}, and other derivations pertaining to sandwich beams with arbitrary edge conditions are given in Appendix (D). The frequency equation (4.1) is applied here to a simply-supported beam with isotropic facings. The following variations in the physical properties are considered.

Case (a):

$$E^c_z \longrightarrow \infty \qquad\qquad G^c_{xz} = \text{Finite}$$

By means of this condition the effect of shear strains in the core can be determined. With B_m approaching zero, equation (4.1) yields the following frequencies:

$$\Omega^2_m = \left(\frac{m\pi}{a}\right)^4 \frac{E}{(1-\nu^2)\left[\rho^c c+\rho(f^a+f^b)\right]}\left[I_F + \frac{I_T}{1+\dfrac{m^2 f^a f^b c\,\pi^2 E}{(f^a+f^b)a^2(1-\nu^2)G^c_{xz}}}\right] \quad \ldots(4.2)$$

Where m is the mode number, and $I_F + I_T$ are the moments of inertia of the spaced facings about the neutral plane.

As G^c_{xz} goes to infinity, the frequencies of vibration of a sandwich beam are the same as those of a homogeneous beam whose moment of inertia is equal to the moment of inertia of the spaced facings of the sandwich.

The expression

$$c\left(\frac{\pi}{a}\right)^2 \frac{f^a f^b}{(f^a+f^b)(1-\nu^2)} \frac{E}{G^c_{xz}} = s \qquad \ldots(4.3)$$

represents the shear effect in the core. For small values of m, the factor I_F in equation (4.2) is small compared to $\frac{I_T}{1+m^2 s}$.

The factor I_F represents the contribution of the flexural stiffness of the individual facings to the over all stiffness of the sandwich.

The shear effect can best be illustrated by means of an example for which

$$f^a = f^b = \left(\frac{1}{10}\right)'', \qquad \left(\frac{\pi}{a}\right) = \frac{1}{10}; \qquad c = 1''$$

$$\frac{E}{G^c_{xz}(1-\nu^2)} = \frac{10^5}{4203}$$

with these substitutions the expression (4.2) takes the form:

$$\Omega^2_m = \left(\frac{m\pi}{a}\right)^4 \frac{E}{(1-\nu^2)\left[\rho^c c+\rho\left(\frac{2}{10}\right)\right]} \left\{\frac{1}{6000} + \frac{121}{2000\left[1+m^2(0.0119)\right]}\right\} \qquad \ldots(4.4)$$

When the expression inside the brackets was evaluated for m = 1, 3, and 5, the following results were obtained:

mode number	$G^c_{xz} \to \infty$	G^c_{xz}=finite	difference
m = 1	0.0605	0.0599	1.18%
m = 3	0.0606	0.0543	10.60%
m = 5	0.0606	0.047	28.72%

It is seen that even in low frequency range the effect of shear is significant and becomes more pronounced with the increasing mode numbers.

If the core and one of the facing thicknesses are assumed to be zero, then the frequencies obtained from (4.2) are those of a simply-supported solid beam and are given by,

$$\Omega^2_m = \left(\frac{m\pi}{a}\right)^4 \frac{E \quad I_F}{s(1-\nu^2)} \qquad \ldots(4.5)$$

where I_F is the moment of inertia of the facing about the neutral plane.

Case (b): $G^c_{xz} \rightarrow \infty$, $\qquad E^c_z$ = finite

This condition demonstrates the influence of the core elasticity in the thickness direction. Even under the assumption of $G^c_{xz} \rightarrow \infty$, the resulting quadratic equation in (Ω^2_m) is too involved for a parametric representation. However, the results of this case are fully discussed in Appendix (D). For each mode, two values of the frequency are obtained. The mode corresponding to the smaller frequency is the normal mode and that corresponding to the larger frequency is the face-wrinkling mode. The mode shapes of the fundamental frequencies are shown in Appendix (D).

The influence of core elasticity in the thickness direction is shown by the following comparison:

mode number	$E^c_z \rightarrow \infty$	E^c_z = finite	difference
m = 1	104.820×10^4	104.485×10^4	0.32%
m = 2	167.712×10^5	167.045×10^5	0.40%
m = 3	849.042×10^5	843.845×10^5	0.61%

Obviously, from the comparison of Case (a) and Case (b), the effect of elasticity in the core is negligible in comparison to the transverse shear effects.

2. Homogeneous and Isotropic Plates

The reduction of a sandwich plate to a homogeneous and isotropic solid plate involves the following simplifications:

(1). The core is considered rigid, necessitating that E_z^c approach infinity, so that the elasticity parameter of the core $\frac{B_{mn}}{E_z^c}$ goes to zero.

(2). The transverse shear effects in the core are considered insignificant, requiring that G_{xz}^c and G_{yz}^c approach infinity, so that the parameters $\frac{F_{mn}}{G_{xz}}$ and $\frac{H_{mn}}{G_{yz}}$ become zero.

With these modifications, the frequency criterion (3.53) reduces to:

$$\begin{bmatrix} c_{11} - \Omega^2_{mn}\lambda_{11} & c_{15} & c_{16} \\ c_{15} & c_{55} & c_{56} \\ c_{16} & c_{56} & c_{66} \end{bmatrix} = 0 \qquad \text{...(4.6)}$$

For homogeneous and isotropic facings of the same material and equal thicknesses

$$f^a = f^b = f; \qquad d_1^a = d_2^a = d_1^b = d_2^b = \frac{E}{(1-\nu^2)}$$

$$G_{xy}^a = G_{xy}^b = G = \frac{E}{2(1+\nu)}$$

$$d_{12}^a = d_{12}^b = \frac{2\nu E}{(1-\nu^2)}$$

Considering a square plate of length "a" with all edges clamped, and introducing the above substitutions, equation (4.6) yields the following value for the fundamental frequency:

$$\Omega^2_{11} = \frac{E}{(1-\nu^2)a^4} \frac{(\beta_1^4 + I_{11}^2)}{(\rho^c c+2\rho f\ )} \left[\frac{4}{3}\ f^3+12f^2c+fc^2 \right] \qquad \ldots(4.7)$$

$$= \frac{E}{12(1-\nu^2)a^4} \frac{2(\beta_1^4 + I_{11}^2)}{(\rho^c c + 2\rho f)} \left[2f^3 + f(c+f)^2 \right]$$

$$= \frac{2D_s}{a^4(\rho^c c+2\rho f)} (\beta_1^4 + I_{11}^2) \qquad \ldots(4.8)$$

where $D_s = \frac{E}{12(1-\nu^2)a^4} \left[2f^3 + f(c+f)^2 \right]$,

and is referred to as the "flexural rigidity" of the sandwich plate.

As "c" goes to zero, the two facings approach each other. Assuming that the two facings move as a unit, the problem reduces to that of a solid, homogeneous and isotropic plate whose thickness is equal to 2f.

With this substitution,

$$\Omega^2_{11} = \frac{E}{12(1-\nu^2)a^4} \left[\frac{2\beta_1^4 + 2I_{11}^2}{2\rho f} \right] 8f^3$$

$$= \frac{E}{12(1-\nu^2)a^4\rho(2f)} \left[2\beta_1^4 + 2I_{11}^2 \right] (2f)^3$$

Let $h = 2f$

$$\Omega^2_{11} = \frac{Eh^3}{12(1-\nu^2)\rho a^4} \left[\frac{2\beta_1^4 + 2I_{11}^2}{h} \right] \qquad \ldots(4.9)$$

or

$$\Omega_{11} = \sqrt{\frac{D}{\rho a^4 h} \left[2(\beta_1^4 + I_{11}^2) \right]} \qquad \ldots(4.10)$$

where D is the flexural rigidity of the two combined facings. For a plate clamped along four edges, β_1 and I_{11} are given in Appendix (A). Substitution of these values into equation (4.10) yields

$$\Omega_{11} = 36.10\sqrt{D/(\rho ha^4)} \qquad \text{...(4.11)}$$

Considering that for a beam $\nu \to o$, $\beta_n \to o$, and $I_{mn} \to o$, the frequency for the fundamental mode of a beam can be directly derived from (4.9). As shown in Appendix (D), the higher frequencies for beams with arbitrary edge conditions are given by,

$$\Omega_m^2 = \frac{Eh^3}{12\, a^4 h} \beta_m^4 \qquad \text{...(4.12)}$$

Using the appropriate values of β_m for the prescribed boundary conditions from Appendix (A), the exact solutions of solid beams with arbitrary edge conditions are obtained.

Stanisic (33) investigated the problem of free vibrations of a homogeneous square plate, clamped along the edges by means of Galarkin's method. The value of the fundamental frequency obtained by this approach is

$$\Omega_{11} = 36.11\sqrt{D/\rho ha^4} \qquad \text{...(4.13)}$$

Young (27), solving the same problem by the Ritz method, achieved better accuracy by considering 36 terms of the resulting series. His value for the fundamental frequency is

$$\Omega_{11} = 35.99\sqrt{D/\rho ha^4} \qquad \text{...(4.14)}$$

which differs by 0.66% and 0.666% from those given by the present analysis and that of Stanisic, respectively. For a square plate clamped along the edges, the frequencies obtained from equation (4.6) for the first three modes, which correspond to (m=n=1), (m=2, n=1; m=1, n=2) and (m=n=2), are tabulated here together with the frequencies given by Stanisic and Young.

Table 4.1

	1st	2nd	3rd
Stanisic	36.11	73.73	108.85
Present Analysis	36.10	73.72	108.85
Young	35.99	73.41	108.27

The results of the present analysis compare satisfactorily with those of Stanisic and Young, in the low frequency range. However, they deviate steadily from Young's results as the mode number is increased. To achieve better accuracy for frequencies at higher modes of vibration, additional terms of the series in equation (3.46) must be considered.

3. Applications to Sandwich Plates

In the preceding sections the general frequency criterion (3.53) for the vibrations of sandwich plates was applied to sandwich beams, as well as to solid beams and plates. The resulting frequencies involved constants of the form β_m, β_n, and I_{mn}. (See equation 4.10). The values of these constants depend upon the edge conditions and can be taken directly from Appendix (A). Therefore, the frequency analyses were performed without specifying the edge conditions prior to the final step of the computations, thereby establishing a unified approach for the vibrations of plates with different edge conditions.

In this section the frequency equation (3.53) is applied to the general case of sandwich plates. The evaluation of the frequency equation is considered for a plate with arbitrary edge conditions for the following cases:

(1) A square plate with facings of equal thicknesses, but of different materials.

  (a) (m,n) = (1,1); (2,1); (1,2); (2,2); (3,3).

  (b) Negligible bending stiffness in the ribbon direction of the core, (m,n) = (1,1); (2,1); (1,2).

(2) A plate with varying aspect ratio, and facings of unequal thicknesses and of different material: (m,n) = (1,1).

(3) A square plate with a variable "core-facing thickness ratio" and facings of equal thicknesses and of the same material, (m,n) = (1,1).

  (a) With a rigid core in the thickness direction.

  (b) With a non-rigid core in the thickness direction.

  (c) With a core having an infinite elasticity modulus in the thickness direction and infinite transverse shear moduli.

(4) A square plate with facings of the same material, and having variable "facing thickness-ratio."

Since the evaluation of the determinantal equation (3.53) is too involved for parametric representation, numerical values are directly substituted into the frequency equation. The evaluation is based upon the physical properties of aluminum honeycomb core and aluminum facings taken from the data prepared by Kunzi (34). These properties are shown in Appendix (E). An example of a sandwich plate clamped along the four edges is considered in the light of the above cases. The computer solutions for the frequencies are given in the following tables.

Table 4.1(a)

$k_1 = \frac{b}{a} = 1.0$; $k_2 = \frac{f^b}{f^a} = 1.0$; $k_3 = \frac{c}{f^a} = 32.0$ $a = 36''$, $f^a = 0.016''$				
n \ m	1	2	3	
1	1125.5290	2233.7490		
2	2283.4660	3284.9990		
3			6366.5610	

Table 4.1(b)

Same data as in Table 4.1(a) excepting $g_2^{\xi} \longrightarrow 0$ $g_{32}^{\xi} \longrightarrow 0$				
n \ m	1	2	3	
1	1125.4960	2233.7340		
2	2283.3440	3284.9150		
3			6366.394	

Table 4.2

$k_2 = 2$ $K_3 = 32.0$ $(m,n) = (1,1)$					
K_1	.5	1	.15	2	2.5
Ω_{11}	2645.7530	993.619	744.694	676.994	650.555

Table 4.3(a)

$K_1 = 1.0$; $K_2 = 1.0$, $\mu = \eta = \gamma = 1$; $a = 36''$; $f = 0.016''$, $(m,n) = (1,1)$					
K_3	16	32	48	64	80
Ω_{11}	678.902	1144.912	1519.073	1833.645	2104.042

Table 4.3(b)

Same data as in Table 4.3 (a) but $E_z^c \rightarrow \infty$					
K_3	16	32	48	64	80
Ω_{11}	682.7140	1152.0940	1529.3060	1846.797	2122.097

Table 4.3(c)

Same data as in Table 4.3(b) but $G_{xz}^c \rightarrow \infty$, $G_{yz}^c \rightarrow \infty$				
K_3	0	32	48	64
Ω_{11}	55.8820	1171.4070	1567.5300	1907.9750

Table 4.4

$k_1 = 1$, $k_3 = 32$, $\mu = \eta = \gamma = 1$, $(m,n = 1)$					
K_2	0.5	1	1.25	1.5	2
Ω_{11}	1001.942	1144.912	1167.923	1180.165	1192

From tables 4.1(a) and 4.1(b), it is observed that the effect of the bending stiffness in the ribbon direction of the core is insignificant. Therefore, the assumption of an antiplane stress distribution in the core should be considered accurate enough for problems involving sandwich construction. For the fundamental mode, the effect of elasticity, as seen from Table 4.3(a), increases steadily for various core thicknesses, but deviates less than one percent when compared to the rigid core as shown in Table 4.3(b). A comparison between Table 4.3(a) and 4.3(c) shows the differences in frequencies due to shear effects. Even for the fundamental mode, with a variable core-facing thickness ratio, the frequency values for a core with finite shear moduli are approximately 3.3% lower than those obtained from a core with infinite shear moduli. However, in the analysis of the sandwich beam, it was shown that this difference increases considerably for higher modes. Therefore, in case of plates, significant differences can be expected at higher modes.

To demonstrate the applicability of the method to the plates with other edge conditions, another example of a plate clamped along one edge and free along the remaining three edges is considered here. This example considers a square plate whose upper facing, core, and lower facing are made of aluminum, aluminum honeycomb, and of steel, respectively. The frequencies for the first three mode numbers are given in the following table.

Table 4.5

Data same as in Table 4.1(a)			
n \ m	1	2	3
1	112.0	696.622	1907.943
2	298.6040	1004.1090	2196.7030
3	907.0405	1789.4040	3041.9550

To establish the validity of the results, of the above case, the sandwich plate is reduced to a solid, homogeneous and isotropic plate as before. Assuming $c \rightarrow 0$, the elements C_{15} and C_{16} of the equation (4.6) reduce to zero and the resulting frequency is given by

$$\Omega^2_{mn} = \frac{C_{11}}{\lambda_{11}}$$

where

$$C_{11} = \frac{2}{3}f^3\left[\left(\frac{\beta m}{a}\right)^4 d + \left(\frac{\beta n}{a}\right)^4 d + \frac{J_{mm}}{a^2}\frac{J_{nn}}{a^2}d_{12} + 4G_{xy}\frac{I_{mm}}{a2}\frac{I_{nn}}{a2}\right]$$

$$+2f\left[\left(\frac{f}{2}\right)^2 + \frac{1}{12}f^2\right]\left[\left(\frac{\beta m}{a}\right)^4 d + \left(\frac{\beta n}{a}\right)^4 d + \frac{J_{mm}}{a^2}\frac{J_{nn}}{a^2}d_{12} + 4G_{xy}\frac{I_{mm}}{a^2}\frac{I_{mm}}{a^2}\right]$$

and

$$\lambda_{11} = 4\rho f$$

Introducing the values of β_m, I_{mm}, J_{mm} for a clamped-free beam and the values of β_n, I_{nn}, J_{nn} for a free-free beam, the frequency for the fundamental mode (m,n = 1,1) is given by,

$$\Omega^2_{11} = \frac{8f^3}{6(4\rho f)} \left[12 \cdot 362\right] \frac{E}{(1-\nu^2)a^4}$$

Let $2f = h$,

therefore,

$$\Omega^2_{11} = \frac{Eh^3}{12(1-\nu^2)} \left[12 \cdot 362\right] \frac{1}{\rho h a^4}$$

or

$$\Omega_{11} = 3.515 \sqrt{D/\rho a^4 h} \qquad \ldots(4.15)$$

The frequency obtained by Young (27) for a square cantilever plate is

$$\Omega_{11} = 3.494 \sqrt{D/\rho a^4 h} \qquad \ldots(4.16)$$

which is approximately $\frac{1}{2}$% lower than the one given by the present analysis.

CHAPTER V

CONCLUSIONS

A unified approach to the vibrations of sandwich plates with arbitrary edge conditions was presented by utilizing the Rayleigh-Ritz energy method. The theoretical analysis was based upon more rigorous assumptions than commonly found in the literature. The validity was established by reducing the general frequency criterion of sandwich plates to cases of sandwich beams, solid beams and solid plates. The results of the present theory were in close agreement with the known values of the above cases.

The present analysis, in addition to being applicable to various edge conditions of plates, is not limited to the low frequency range. By considering more terms of the infinite series of the resulting frequency equation, the analysis can be applied to high frequency ranges as well.

The effects of various parameters in the analysis were brought out by an examination of special cases of sandwich beams, as well as by tabulated values pertaining to sandwich plates. It was shown that even in the low frequency range, the assumption of finite shear moduli in the core yields results which deviate considerably from those obtained by assuming infinite core moduli. In higher frequency ranges, this deviation will become greater.

The frequencies obtained by considering the elasticity of the core in the thickness direction did not show significant deviations from those of the rigid core for the fundamental mode. However, one would expect an increase in this deviation at higher modes. A formulation of actual variation of the displacement in the thickness direction of the core must be developed for a definite conclusion. Such a law can be established by means of experimental investigations.

The theory was developed in such a manner that different laws of variation regarding the transverse shearing stresses, and the strains in the thickness direction of the core, can be easily introduced in the analysis. The use of digital computers is mandatory for accurate prediction of frequency analysis of sandwich plates, particularly for the higher frequency ranges.

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NOMENCLATURE

All quantities with superscripts a, b, and c refer to upper facing, lower facing, and core, respectively.

a,b = sides of a rectangular plate

f^a = thickness of the upper facing

f^b = thickness of the lower facing

c = thickness of the core

u^c = core-displacement in the x-direction

v^c = core-displacement in the y-direction

w^c = core-displacement in the z-direction

E_x^a = modulus of elasticity of upper facing in the x-direction

E_y^a = modulus of elasticity of upper facing in the y-direction

E_z^c = modulus of elasticity of the core in the z-direction

E_y^c = modulus of elasticity of the core in the y-direction

^{a} = Poisson's ratio for the upper facing

^{b} = Poisson's ratio for the lower facing

^{c} = Poisson's ratio for the core

$$d_1^a = \frac{E_x^a}{1-\nu_x^a \nu_y^a}$$

$$d_2^a = \frac{E_x^a}{1-\nu_x^a \nu_y^a}$$

$$d_{12}^{a} = \frac{E_x^a \nu_x^a + E_y^a \nu_y^a}{1-\nu_x^a \nu_y^a}$$

$$g_2^c = \frac{E_y^c}{1-\nu_y^c \nu_z^c}$$

$$g_3^c = \frac{E_z^c}{1-\nu_y^c \nu_z^c}$$

$$g_{32}^c = \frac{E_z^c \nu_z^c + \nu_y E_y^c}{1-\nu_y^c \nu_z^c}$$

G_{xy}^a = shear modulus of upper facing

G_{xy}^b = shear modulus of the lower facing

G_{xz}^c = transverse shear modulus of the core in x-z

G_{yz}^c = transverse shear modulus of the core in y-z

C_{ij} = elements of a determinant

S_{ij} = elastic constants of an orthotropic body

ρ^a = mass density per unit volume of the upper facing

ρ^b = mass density per unit volume of the lower facing

ρ^c = mass density per unit volume of the core

Ω_{mn} = circular frequency of a plate for m^{th} mode in the x-direction and n^{th} mode in the y-direction

A_{mn}, B_{mn}, F_{mn}, H_{mn}, K_{mn}, L_{mn} = arbitrary parameters

X_m = function of x only

Y_n = function of y only

Θ_{mn}, Φ_{mn}, Ψ_{mn} = functions of z only

ϵ_{mx}^a = membrane strains of the upper facing in the x-direction

ϵ^{a}_{my} = membrane strains of the upper facing in the y-direction

ϵ^{b}_{mx} = membrane strains of the lower facing in the x-direction

ϵ^{b}_{my} = membrane strains of the lower facing in the y-direction

γ^{a}_{mxy} = shearing strains in the upper facing associated with the membrane strains

γ^{b}_{mxy} = shearing strains in the lower facing associated with the membrane strains

ϵ^{a}_{Bx} = bending strains of the upper facing in the x-direction

ϵ^{a}_{By} = bending strains of the upper facing in the y-direction

ϵ^{b}_{Bx} = bending strains of the lower facing in the x-direction

ϵ^{b}_{By} = bending strains of the lower facing in the y-direction

γ^{a}_{Bxy} = shearing strains in the upper facing associated with the bending strains

γ^{b}_{Bxy} = shearing strains in the lower facing associated with the bending strains

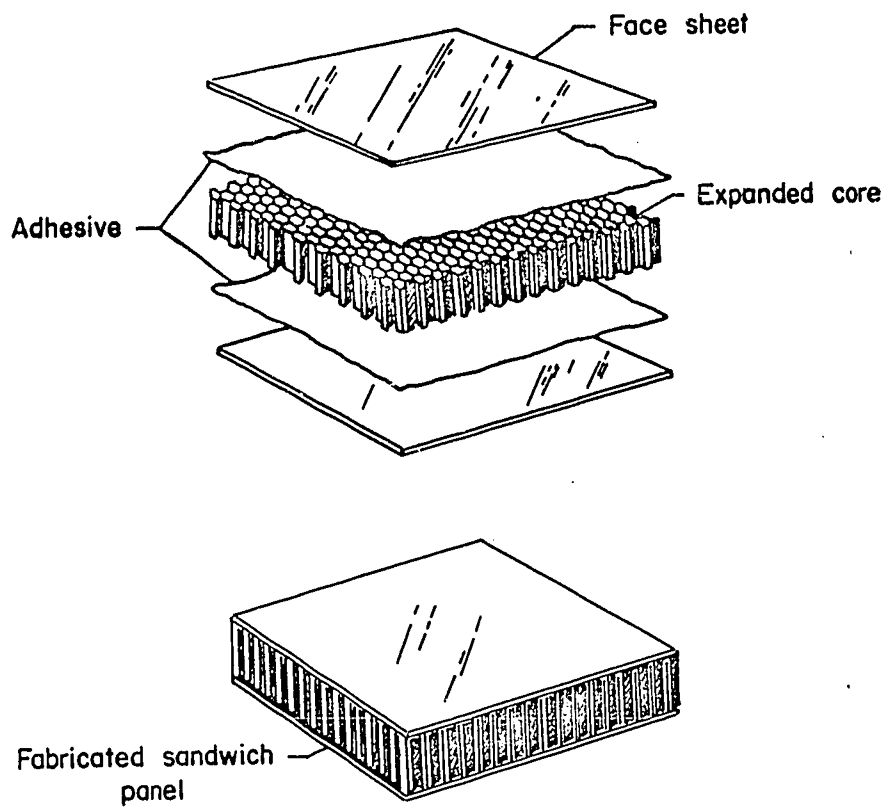
ϕ_n = characteristic function of a beam for the n^{th} mode

β_n, α_n = beam parameters for the n^{th} mode

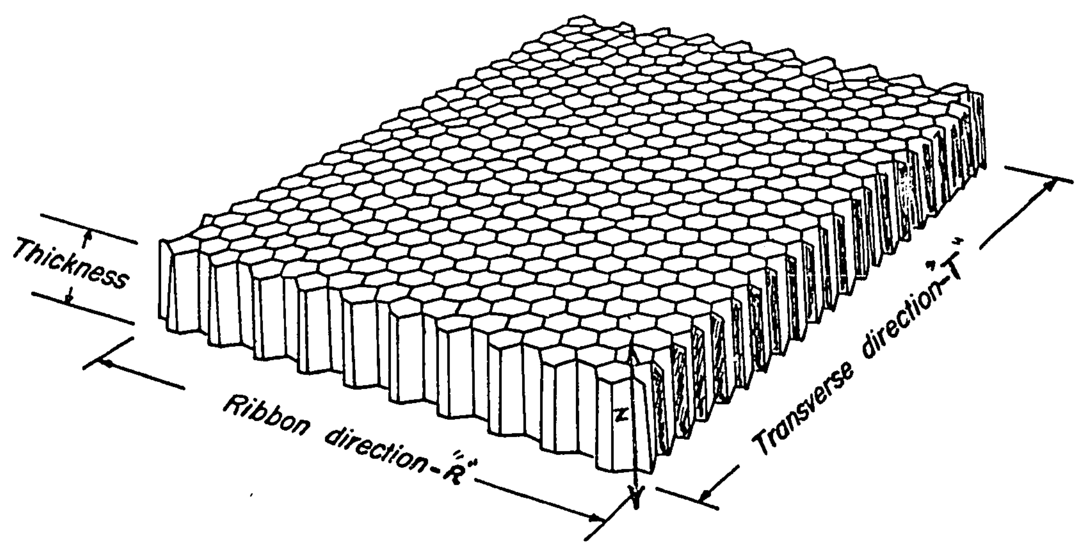
$\mu = d_1^b / d_1^a$

$\eta = \dfrac{\rho^b}{\rho^a}$

$\gamma = \dfrac{\nu^b}{\nu^a}$



(a) Primary parts.



(b) Core notation.

Figure 1.- Sandwich structure.

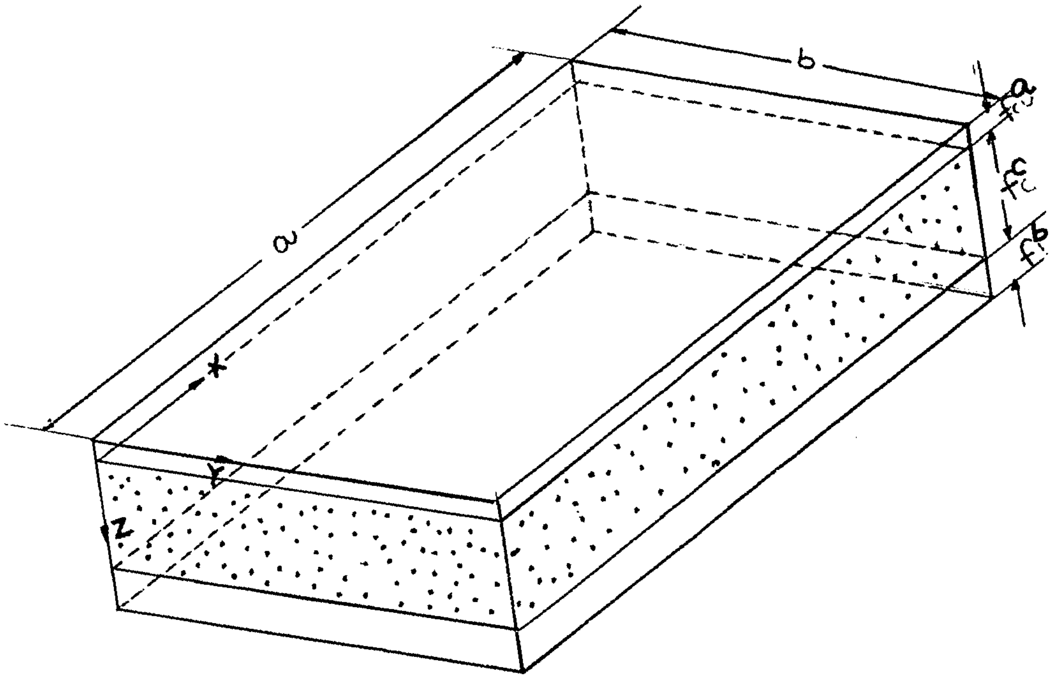


Fig. 2(a) - sandwich panel

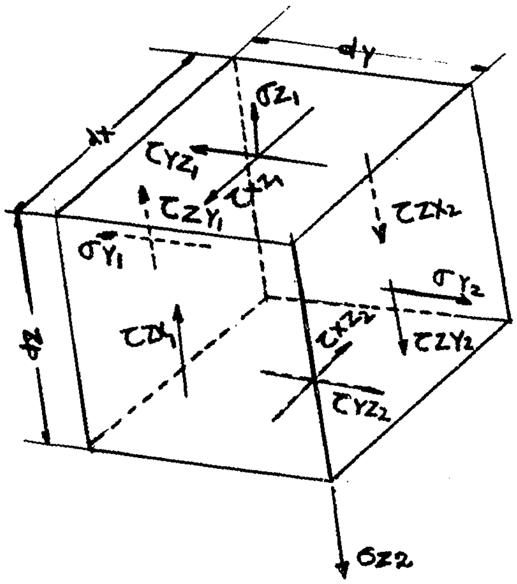


Fig. 2(b) core element

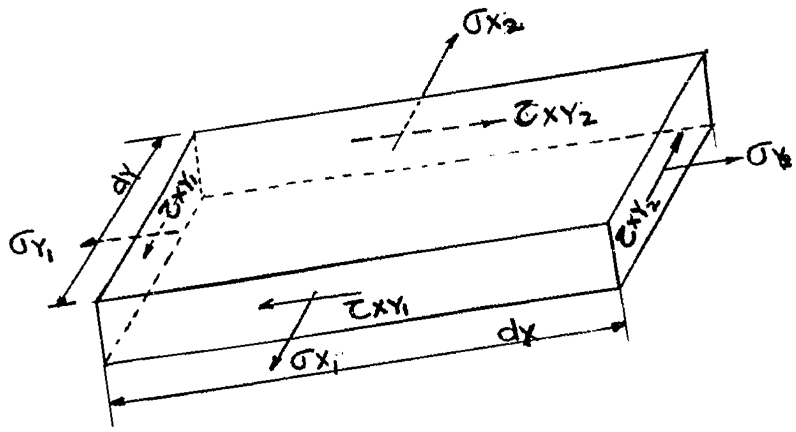


Fig. 2(c) facing element

APPENDIX A

CHARACTERISTIC FUNCTIONS

Appendix A provides tables of the characteristic functions and their derivatives, for nearly all common types of beams. These functions are tabulated at intervals of the arguement, corresponding to $\frac{1}{50}$ of the beam lenght: These tables have been taken from the University of Texas Publication, series No. 44. See (29).

Following the tables is a summary of data which includes the mathematical expressions for the characteristic functions, and the numerical values of the beam constants α_n and β_n.

The appendix concludes by giving the numerical values of the integrals of the type,

$$\frac{I_{mp}}{a} = \int_0^a \phi_m' \phi_p' \, dx$$

$$\frac{J_{mp}}{a} = \int_0^a \phi_m \phi_p''$$

where all subscripts range from 1 to 5. These values are given in the tables 4 through 7.

Table 1

Characteristic Functions and Derivatives
Clamped-Clamped Beam
First Mode

$\frac{x}{\ell}$	ϕ_1	$\phi_1' = \frac{1}{\beta_1}\frac{d\phi_1}{dx}$	$\phi_1'' = \frac{1}{\beta_1^2}\frac{d^2\phi_1}{dx^2}$	$\phi_1''' = \frac{1}{\beta_1^3}\frac{d^3\phi_1}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 1.96500
0.02	0.00867	0.18041	1.81412	- 1.96473
0.04	0.03358	0.34324	1.62832	- 1.96285
0.06	0.07306	0.48850	1.44284	- 1.95792
0.08	0.12545	0.61624	1.25802	- 1.94862
0.10	0.18910	0.72655	1.07433	- 1.93383
0.12	0.26237	0.81956	0.89234	- 1.91254
0.14	0.34363	0.89546	0.71270	- 1.88393
0.16	0.43126	0.95451	0.53615	- 1.84732
0.18	0.52370	0.99702	0.36346	- 1.80219
0.20	0.61939	1.02342	0.19545	- 1.74814
0.22	0.71684	1.03418	0.03300	- 1.68494
0.24	0.81459	1.02986	- 0.12305	- 1.61250
0.26	0.91124	1.01113	- 0.27180	- 1.53085
0.28	1.00546	0.97870	- 0.41240	- 1.44017
0.30	1.09600	0.93338	- 0.54401	- 1.34074
0.32	1.18168	0.87608	- 0.66581	- 1.23296
0.34	1.26141	0.80774	- 0.77704	- 1.11735
0.36	1.33419	0.72992	- 0.87699	- 0.99452
0.38	1.39913	0.64219	- 0.96500	- 0.86516
0.40	1.45545	0.54723	- 1.04050	- 0.73007
0.42	1.50247	0.44574	- 1.10297	- 0.59008
0.44	1.53962	0.33397	- 1.15202	- 0.44611
0.46	1.56647	0.22821	- 1.18728	- 0.29911
0.48	1.58271	0.11478	- 1.20854	- 0.15007
0.50	1.58815	0.00000	- 1.21565	0.00000
0.52	1.58271	- 0.11478	- 1.20854	0.15007
0.54	1.56647	- 0.22821	- 1.18728	0.29911
0.56	1.53962	- 0.33397	- 1.15202	0.44611
0.58	1.50247	- 0.44574	- 1.10297	0.59008
0.60	1.45545	- 0.54723	- 1.04050	0.73007
0.62	1.39913	- 0.64219	- 0.96500	0.86516
0.64	1.33419	- 0.72992	- 0.87699	0.99452
0.66	1.26141	- 0.80774	- 0.77704	1.11735
0.68	1.18168	- 0.87608	- 0.66581	1.23296
0.70	1.09600	- 0.93338	- 0.54401	1.34074
0.72	1.00546	- 0.97870	- 0.41240	1.44017
0.74	0.91124	- 1.01113	- 0.27180	1.53085
0.76	0.81459	- 1.02986	- 0.12305	1.61250
0.78	0.71684	- 1.03418	0.03300	1.68494
0.80	0.61939	- 1.02342	0.19545	1.74814
0.82	0.52370	- 0.99702	0.36346	1.80219
0.84	0.43126	- 0.95451	0.53615	1.84732
0.86	0.34363	- 0.89546	0.71270	1.88393
0.88	0.26237	- 0.81956	0.89234	1.91254
0.90	0.18910	- 0.72655	1.07433	1.93383
0.92	0.12545	- 0.61624	1.25802	1.94862
0.94	0.07306	- 0.48850	1.44284	1.95792
0.96	0.03358	- 0.34324	1.62832	1.96285
0.98	0.00867	- 0.18041	1.81412	1.96473
1.00	0.00000	0.00000	2.00000	1.96500

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
Second Mode

$\frac{x}{\ell}$	ϕ_2	$\phi_2' = \frac{1}{\beta_2}\frac{d\phi_2}{dx}$	$\phi_2'' = \frac{1}{\beta_2^2}\frac{d^2\phi_2}{dx^2}$	$\phi_2''' = \frac{1}{\beta_2^3}\frac{d^3\phi_2}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00155
0.02	0.02338	0.28944	1.68568	- 2.00031
0.04	0.08834	0.52955	1.37202	- 1.99205
0.06	0.18715	0.72055	1.06061	- 1.97030
0.08	0.31214	0.86296	0.75386	- 1.93186
0.10	0.45573	0.95776	0.45486	- 1.87176
0.12	0.61058	1.00644	0.16713	- 1.78813
0.14	0.76958	1.01105	- 0.10554	- 1.67975
0.16	0.92602	0.97427	- 0.35923	- 1.54652
0.18	1.07363	0.89940	- 0.59010	- 1.38933
0.20	1.20674	0.79030	- 0.79450	- 1.21002
0.22	1.32032	0.65138	- 0.96918	- 1.01127
0.24	1.41005	0.48755	- 1.11133	- 0.79651
0.26	1.47245	0.30410	- 1.21876	- 0.56977
0.28	1.50485	0.10660	- 1.28991	- 0.33555
0.30	1.50550	- 0.09916	- 1.32402	- 0.09872
0.32	1.47357	- 0.30736	- 1.32106	0.13566
0.34	1.40914	- 0.51224	- 1.28181	0.36246
0.36	1.31314	- 0.70819	- 1.20786	0.57665
0.38	1.18740	- 0.88997	- 1.10157	0.77340
0.40	1.03457	- 1.05271	- 0.96605	0.94823
0.42	0.85794	- 1.19209	- 0.80507	1.09714
0.44	0.66150	- 1.30448	- 0.62296	1.21670
0.46	0.44973	- 1.38693	- 0.42456	1.30414
0.48	0.22751	- 1.43728	- 0.21508	1.35744
0.50	0.00000	- 1.45420	0.00000	1.37532
0.52	- 0.22751	- 1.43728	0.21508	1.35744
0.54	- 0.44973	- 1.38693	0.42456	1.30414
0.56	- 0.66150	- 1.30448	0.62296	1.21670
0.58	- 0.85794	- 1.19209	0.80507	1.09714
0.60	- 1.03457	- 1.05271	0.96605	0.94823
0.62	- 1.18740	- 0.88997	1.10157	0.77340
0.64	- 1.31314	- 0.70819	1.20786	0.57665
0.66	- 1.40914	- 0.51224	1.28181	0.36246
0.68	- 1.47357	- 0.30736	1.32106	0.13566
0.70	- 1.50550	- 0.09916	1.32402	- 0.09872
0.72	- 1.50485	0.10660	1.28991	- 0.33555
0.74	- 1.47245	0.30410	1.21876	- 0.56977
0.76	- 1.41005	0.48755	1.11133	- 0.79651
0.78	- 1.32032	0.65138	0.96918	- 1.01127
0.80	- 1.20674	0.79030	0.79450	- 1.21002
0.82	- 1.07363	0.89940	0.59010	- 1.38933
0.84	- 0.92602	0.97427	0.35923	- 1.54652
0.86	- 0.76958	1.01105	0.10554	- 1.67975
0.88	- 0.61058	1.00644	- 0.16713	- 1.78813
0.90	- 0.45573	0.95776	- 0.45486	- 1.87176
0.92	- 0.31214	0.86296	- 0.75386	- 1.93186
0.94	- 0.18715	0.72055	- 1.06061	- 1.97030
0.96	- 0.08834	0.52955	- 1.37202	- 1.99205
0.98	- 0.02338	0.28944	- 1.68568	- 2.00031
1.00	0.00000	0.00000	- 2.00000	- 2.00155

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
Third Mode

$\frac{x}{l}$	ϕ_3	$\phi_3' = \frac{1}{\beta_3}\frac{d\phi_3}{dx}$	$\phi_3'' = \frac{1}{\beta_3^2}\frac{d^2\phi_3}{dx^2}$	$\phi_3''' = \frac{1}{\beta_3^3}\frac{d^3\phi_3}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 1.99993
0.02	0.04481	0.39147	1.56038	- 1.99658
0.04	0.16510	0.68646	1.12323	- 1.97469
0.06	0.33975	0.88609	0.69428	- 1.91998
0.08	0.54804	0.99303	0.28189	- 1.82280
0.10	0.77005	1.01202	- 0.10393	- 1.67795
0.12	0.98720	0.95006	- 0.45252	- 1.48447
0.14	1.18265	0.81649	- 0.75348	- 1.24535
0.16	1.34190	0.62285	- 0.99738	- 0.96698
0.18	1.45317	0.38256	- 1.17657	- 0.65867
0.20	1.50782	0.11050	- 1.28572	- 0.33199
0.22	1.50059	- 0.17759	- 1.32220	- 0.00005
0.24	1.42971	- 0.46573	- 1.28637	0.32333
0.26	1.29690	- 0.73833	- 1.18165	0.62425
0.28	1.10719	- 0.98087	- 1.01443	0.88956
0.30	0.86864	- 1.18057	- 0.79387	1.10762
0.32	0.59186	- 1.32694	- 0.53145	1.26880
0.34	0.28949	- 1.41222	- 0.24051	1.36606
0.36	- 0.02445	- 1.43171	0.06438	1.39529
0.38	- 0.33528	- 1.38399	0.36811	1.35554
0.40	- 0.62837	- 1.27099	0.65569	1.24912
0.42	- 0.88987	- 1.09782	0.91301	1.08148
0.44	- 1.10739	- 0.87257	1.12747	0.86096
0.46	- 1.27060	- 0.60586	1.28860	0.59842
0.48	- 1.37174	- 0.31031	1.38852	0.30669
0.50	- 1.40600	0.00000	1.42238	0.00000
0.52	- 1.37174	0.31031	1.38852	- 0.30669
0.54	- 1.27060	0.60586	1.28860	- 0.59842
0.56	- 1.10739	0.87257	1.12747	- 0.86096
0.58	- 0.88987	1.09782	0.91301	- 1.08148
0.60	- 0.62837	1.27099	0.65569	- 1.24912
0.62	- 0.33528	1.38399	0.36811	- 1.35554
0.64	- 0.02445	1.43171	0.06438	- 1.39529
0.66	0.28949	1.41222	- 0.24051	- 1.36606
0.68	0.59186	1.32694	- 0.53145	- 1.26880
0.70	0.86864	1.18057	- 0.79387	- 1.10762
0.72	1.10719	0.98087	- 1.01443	- 0.88956
0.74	1.29690	0.73833	- 1.18165	- 0.62425
0.76	1.42971	0.46573	- 1.28637	- 0.32333
0.78	1.50059	0.17759	- 1.32220	0.00005
0.80	1.50782	- 0.11050	- 1.28572	0.33199
0.82	1.45317	- 0.38256	- 1.17657	0.65867
0.84	1.34190	- 0.62285	- 0.99738	0.96698
0.86	1.18265	- 0.81649	- 0.75348	1.24535
0.88	0.98720	- 0.95006	- 0.45252	1.48447
0.90	0.77005	- 1.01202	- 0.10393	1.67795
0.92	0.54804	- 0.99303	0.28189	1.82280
0.94	0.33975	- 0.88609	0.69428	1.91998
0.96	0.16510	- 0.68646	1.12323	1.97469
0.98	0.04481	- 0.39147	1.56038	1.99658
1.00	0.00000	0.00000	2.00000	1.99993

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
Fourth Mode

$\frac{x}{\ell}$	ϕ_4	$\phi_4' = \frac{1}{\beta_4}\frac{d\phi_4}{dx}$	$\phi_4'' = \frac{1}{\beta_4^2}\frac{d^2\phi_4}{dx^2}$	$\phi_4''' = \frac{1}{\beta_4^3}\frac{d^3\phi_4}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.07241	0.48557	1.43502	- 1.99300
0.04	0.25958	0.81207	0.87658	- 1.94824
0.06	0.51697	0.98325	0.33937	- 1.83960
0.08	0.80177	1.00789	- 0.15633	- 1.65333
0.10	1.07449	0.90088	- 0.58802	- 1.38736
0.12	1.30078	0.68345	- 0.93412	- 1.05012
0.14	1.45308	0.38242	- 1.17673	- 0.65879
0.16	1.51208	0.02894	- 1.30380	- 0.23725
0.18	1.46765	- 0.34351	- 1.31068	0.18649
0.20	1.31923	- 0.70122	- 1.20092	0.58286
0.22	1.07550	- 1.01271	- 0.98634	0.92349
0.24	0.75348	- 1.25091	- 0.68630	1.18364
0.26	0.37700	- 1.39515	- 0.32640	1.34442
0.28	- 0.02537	- 1.43265	0.06348	1.39439
0.30	- 0.42268	- 1.35944	0.45136	1.33056
0.32	- 0.78413	- 1.18058	0.80569	1.15876
0.34	- 1.08159	- 0.90972	1.09776	0.89319
0.36	- 1.29186	- 0.56793	1.30395	0.55537
0.38	- 1.39858	- 0.18205	1.40755	0.17245
0.40	- 1.39351	0.21753	1.40010	- 0.22494
0.42	- 1.27726	0.59923	1.28198	- 0.60506
0.44	- 1.05920	0.93289	1.06244	- 0.93759
0.46	- 0.75676	1.19208	0.75879	- 1.19604
0.48	- 0.39407	1.35629	0.39504	- 1.35983
0.50	0.00000	1.41251	0.00000	- 1.41592
0.52	0.39407	1.35629	- 0.39504	- 1.35983
0.54	0.75676	1.19208	- 0.75879	- 1.19604
0.56	1.05920	0.93289	- 1.06244	- 0.93759
0.58	1.27726	0.59923	- 1.28198	- 0.60506
0.60	1.39351	0.21753	- 1.40010	- 0.22494
0.62	1.39858	- 0.18205	- 1.40755	0.17245
0.64	1.29186	- 0.56793	- 1.30395	0.55537
0.66	1.08159	- 0.90972	- 1.09776	0.89319
0.68	0.78413	- 1.18058	- 0.80569	1.15876
0.70	0.42268	- 1.35944	- 0.45136	1.33056
0.72	0.02537	- 1.43265	- 0.06348	1.39439
0.74	- 0.37700	- 1.39515	0.32640	1.34442
0.76	- 0.75348	- 1.25091	0.68630	1.18364
0.78	- 1.07550	- 1.01271	0.98634	0.92349
0.80	- 1.31923	- 0.70122	1.20092	0.58286
0.82	- 1.46765	- 0.34351	1.31068	0.18649
0.84	- 1.51208	0.02894	1.30380	- 0.23725
0.86	- 1.45308	0.38242	1.17673	- 0.65879
0.88	- 1.30078	0.68345	0.93412	- 1.05012
0.90	- 1.07449	0.90088	0.58802	- 1.38736
0.92	- 0.80177	1.00789	0.15633	- 1.65333
0.94	- 0.51697	0.98325	- 0.33937	- 1.83960
0.96	- 0.25958	0.81207	- 0.87658	- 1.94824
0.98	- 0.07241	0.48557	- 1.43502	- 1.99300
1.00	0.00000	0.00000	- 2.00000	- 2.00000

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
Fifth Mode

$\frac{x}{\ell}$	ϕ_5	$\phi_5' = \frac{1}{\beta_5}\frac{d\phi_5}{dx}$	$\phi_5'' = \frac{1}{\beta_5^2}\frac{d^2\phi_5}{dx^2}$	$\phi_5''' = \frac{1}{\beta_5^3}\frac{d^3\phi_5}{dx^3}$
0.00	0.00000	0.00000	2.00000	-2.00000
0.02	0.10567	0.57181	1.30996	-1.98743
0.04	0.36791	0.90694	0.63409	-1.90894
0.06	0.70632	1.01517	0.00291	-1.72440
0.08	1.04591	0.91867	-0.54391	-1.42067
0.10	1.32178	0.65359	-0.96646	-1.00891
0.12	1.48381	0.26880	-1.23231	-0.52030
0.14	1.50043	-0.17781	-1.32242	-0.00021
0.16	1.36090	-0.62465	-1.23490	0.49865
0.18	1.07551	-1.01269	-0.98632	0.92351
0.20	0.67360	-1.29164	-0.61048	1.22851
0.22	0.09959	-1.42540	-0.15491	1.38072
0.24	-0.29269	-1.39597	0.32432	1.36434
0.26	-0.74658	-1.20525	0.76897	1.18287
0.28	-1.10952	-0.87470	1.12538	0.85886
0.30	-1.33938	-0.44262	1.35061	0.43141
0.32	-1.40954	0.04046	1.41749	-0.04838
0.34	-1.31208	0.51781	1.31772	-0.52341
0.36	-1.05881	0.93326	1.06282	-0.93721
0.38	-0.67987	1.23790	0.68273	-1.24067
0.40	-0.22021	1.39584	0.22226	-1.39777
0.42	0.26575	1.38850	-0.26425	-1.38983
0.44	0.72046	1.21684	-0.71933	-1.21771
0.46	1.09011	0.90119	-1.08923	-0.90172
0.48	1.33098	0.47892	-1.33023	-0.47917
0.50	1.41457	0.00000	-1.41386	0.00000
0.52	1.33098	-0.47892	-1.33023	0.47917
0.54	1.09011	-0.90119	-1.08923	0.90172
0.56	0.72046	-1.21684	-0.71933	1.21771
0.58	0.26575	-1.38850	-0.26425	1.38983
0.60	-0.22021	-1.39584	0.22226	1.39777
0.62	-0.67987	-1.23790	0.68273	1.24067
0.64	-1.05881	-0.93326	1.06282	0.93721
0.66	-1.31208	-0.51781	1.31772	0.52341
0.68	-1.40954	-0.04046	1.41749	0.04838
0.70	-1.33938	0.44262	1.35061	-0.43141
0.72	-1.10952	0.87470	1.12538	-0.85886
0.74	-0.74658	1.20525	0.76897	-1.18287
0.76	-0.29269	1.39597	0.32432	-1.36434
0.78	0.19959	1.42540	-0.15491	-1.38072
0.80	0.67360	1.29164	-0.61048	-1.22851
0.82	1.07551	1.01269	-0.98632	-0.92351
0.84	1.36090	0.62465	-1.23490	-0.49865
0.86	1.50043	0.17781	-1.32242	0.00021
0.88	1.48381	-0.26880	-1.23231	0.52030
0.90	1.32178	-0.65359	-0.96646	1.00891
0.92	1.04591	-0.91867	-0.54391	1.42067
0.94	0.70632	-1.01517	0.00291	1.72440
0.96	0.36791	-0.90694	0.63409	1.90894
0.98	0.10567	-0.57181	1.30996	1.98743
1.00	0.00000	0.00000	2.00000	2.00000

Table 2

Characteristic Functions and Derivatives
Clamped-Free Beam
First Mode

$\frac{x}{L}$	ϕ_1	$\phi_1' = \frac{1}{\beta_1}\frac{d\phi_1}{dx}$	$\phi_1'' = \frac{1}{\beta_1^2}\frac{d^2\phi_1}{dx^2}$	$\phi_1''' = \frac{1}{\beta_1^3}\frac{d^3\phi_1}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 1.46819
0.02	0.00139	0.07397	1.94494	- 1.46817
0.04	0.00552	0.14588	1.88988	- 1.46805
0.06	0.01231	0.21572	1.83483	- 1.46773
0.08	0.02168	0.28350	1.77980	- 1.46710
0.10	0.03355	0.34921	1.72480	- 1.46607
0.12	0.04784	0.41286	1.66985	- 1.46455
0.14	0.06449	0.47446	1.61496	- 1.46245
0.16	0.08340	0.53400	1.56016	- 1.45968
0.18	0.10452	0.59148	1.50549	- 1.45617
0.20	0.12774	0.64692	1.45096	- 1.45182
0.22	0.15301	0.70031	1.39660	- 1.44656
0.24	0.18024	0.75167	1.34247	- 1.44032
0.26	0.20936	0.80100	1.28859	- 1.43302
0.28	0.24030	0.84832	1.23500	- 1.42459
0.30	0.27297	0.89364	1.18175	- 1.41497
0.32	0.30730	0.93696	1.12889	- 1.40410
0.34	0.34322	0.97831	1.07646	- 1.39191
0.36	0.38065	1.01771	1.02451	- 1.37834
0.38	0.41952	1.05516	0.97309	- 1.36334
0.40	0.45977	1.09070	0.92227	- 1.34685
0.42	0.50131	1.12435	0.87209	- 1.32884
0.44	0.54408	1.15612	0.82262	- 1.30924
0.46	0.58800	1.18606	0.77392	- 1.28801
0.48	0.63301	1.21418	0.72603	- 1.26512
0.50	0.67905	1.24052	0.67905	- 1.24052
0.52	0.72603	1.26512	0.63301	- 1.21418
0.54	0.77392	1.28801	0.58800	- 1.18606
0.56	0.82262	1.30924	0.54408	- 1.15612
0.58	0.87209	1.32884	0.50131	- 1.12435
0.60	0.92227	1.34685	0.45977	- 1.09070
0.62	0.97309	1.36334	0.41952	- 1.05516
0.64	1.02451	1.37834	0.38065	- 1.01771
0.66	1.07646	1.39191	0.34322	- 0.97831
0.68	1.12889	1.40410	0.30730	- 0.93696
0.70	1.18175	1.41497	0.27297	- 0.89364
0.72	1.23500	1.42459	0.24030	- 0.84832
0.74	1.28859	1.43302	0.20936	- 0.80100
0.76	1.34247	1.44032	0.18024	- 0.75167
0.78	1.39660	1.44656	0.15301	- 0.70031
0.80	1.45096	1.45182	0.12774	- 0.64692
0.82	1.50549	1.45617	0.10452	- 0.59148
0.84	1.56016	1.45968	0.08340	- 0.53400
0.86	1.61496	1.46245	0.06449	- 0.47446
0.88	1.66985	1.46455	0.04784	- 0.41286
0.90	1.72480	1.46607	0.03355	- 0.34921
0.92	1.77980	1.46710	0.02168	- 0.28350
0.94	1.83483	1.46773	0.01231	- 0.21572
0.96	1.88988	1.46805	0.00552	- 0.14588
0.98	1.94494	1.46817	0.00139	- 0.07397
1.00	2.00000	1.46819	0.00000	0.00000

Table 2

Characteristic Functions and Derivatives
Clamped-Free Beam
Second Mode

$\frac{x}{l}$	ϕ_2	$\phi_2' = \frac{1}{\beta_2}\frac{d\phi_2}{dx}$	$\phi_2'' = \frac{1}{\beta_2^2}\frac{d^2\phi_2}{dx^2}$	$\phi_2''' = \frac{1}{\beta_2^3}\frac{d^3\phi_2}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.03693
0.02	0.00853	0.17879	1.80877	- 2.03666
0.04	0.03301	0.33962	1.61764	- 2.03483
0.06	0.07174	0.48253	1.42680	- 2.03002
0.08	0.12305	0.60754	1.23660	- 2.02097
0.10	0.18526	0.71475	1.04750	- 2.00658
0.12	0.25670	0.80428	0.86004	- 1.98590
0.14	0.33573	0.87631	0.67484	- 1.95814
0.16	0.42070	0.93108	0.49261	- 1.92267
0.18	0.51002	0.96892	0.31409	- 1.87901
0.20	0.60211	0.99020	0.14007	- 1.82682
0.22	0.69544	0.99539	- 0.02865	- 1.76592
0.24	0.78852	0.98502	- 0.19123	- 1.69625
0.26	0.87992	0.95970	- 0.34687	- 1.61791
0.28	0.96827	0.92013	- 0.49475	- 1.53113
0.30	1.05227	0.86707	- 0.63410	- 1.43624
0.32	1.13068	0.80136	- 0.76419	- 1.33373
0.34	1.20236	0.72389	- 0.88431	- 1.22416
0.36	1.26626	0.63565	- 0.99384	- 1.10821
0.38	1.32141	0.53764	- 1.09222	- 0.98667
0.40	1.36694	0.43094	- 1.17895	- 0.86040
0.42	1.40209	0.31665	- 1.25365	- 0.73034
0.44	1.42619	0.19593	- 1.31600	- 0.59748
0.46	1.43871	0.06995	- 1.36578	- 0.46291
0.48	1.43920	- 0.06012	- 1.40289	- 0.32772
0.50	1.42733	- 0.19307	- 1.42733	- 0.19307
0.52	1.40289	- 0.32772	- 1.43920	- 0.06012
0.54	1.36578	- 0.46291	- 1.43871	0.06995
0.56	1.31600	- 0.59748	- 1.42619	0.19593
0.58	1.25365	- 0.73034	- 1.40209	0.31665
0.60	1.17895	- 0.86040	- 1.36694	0.43094
0.62	1.09222	- 0.98667	- 1.32141	0.53764
0.64	0.99384	- 1.10821	- 1.26626	0.63565
0.66	0.88431	- 1.22416	- 1.20236	0.72389
0.68	0.76419	- 1.33373	- 1.13068	0.80136
0.70	0.63410	- 1.43624	- 1.05227	0.86707
0.72	0.49475	- 1.53113	- 0.96827	0.92013
0.74	0.34687	- 1.61791	- 0.87992	0.95970
0.76	0.19123	- 1.69625	- 0.78852	0.98502
0.78	0.02865	- 1.76592	- 0.69544	0.99539
0.80	- 0.14007	- 1.82682	- 0.60211	0.99020
0.82	- 0.31409	- 1.87901	- 0.51002	0.96892
0.84	- 0.49261	- 1.92267	- 0.42070	0.93108
0.86	- 0.67484	- 1.95814	- 0.33573	0.87631
0.88	- 0.86004	- 1.98590	- 0.25670	0.80428
0.90	- 1.04750	- 2.00658	- 0.18526	0.71475
0.92	- 1.23660	- 2.02097	- 0.12305	0.60754
0.94	- 1.42680	- 2.03002	- 0.07174	0.48253
0.96	- 1.61764	- 2.03483	- 0.03301	0.33962
0.98	- 1.80877	- 2.03666	- 0.00853	0.17879
1.00	- 2.00000	- 2.03693	0.00000	0.00000

TABLE 2

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-FREE BEAM
Third Mode

$\frac{x}{l}$	ϕ_3	$\phi_3' = \frac{1}{\beta_3}\frac{d\phi_3}{dx}$	$\phi_3'' = \frac{1}{\beta_3^2}\frac{d^2\phi_3}{dx^2}$	$\phi_3''' = \frac{1}{\beta_3^3}\frac{d^3\phi_3}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 1.99845
0.02	0.02339	0.28953	1.68610	- 1.99721
0.04	0.08839	0.52979	1.37287	- 1.98892
0.06	0.18727	0.72099	1.06189	- 1.96766
0.08	0.31238	0.86367	0.75558	- 1.92871
0.10	0.45614	0.95879	0.45702	- 1.86854
0.12	0.61120	1.00785	0.16974	- 1.78480
0.14	0.77049	1.01291	- 0.10245	- 1.67629
0.16	0.92728	0.97665	- 0.35563	- 1.54286
0.18	1.07535	0.90237	- 0.58594	- 1.38540
0.20	1.20901	0.79394	- 0.78975	- 1.20575
0.22	1.32324	0.65580	- 0.96375	- 1.00656
0.24	1.41376	0.49285	- 1.10515	- 0.79124
0.26	1.47707	0.31040	- 1.21172	- 0.56380
0.28	1.51056	0.11405	- 1.28189	- 0.32872
0.30	1.51248	- 0.09041	- 1.31485	- 0.09085
0.32	1.48203	- 0.29711	- 1.31055	0.14479
0.34	1.41931	- 0.50026	- 1.26974	0.37310
0.36	1.32534	- 0.69422	- 1.19398	0.58908
0.38	1.20196	- 0.87368	- 1.08556	0.78797
0.40	1.05185	- 1.03374	- 0.94753	0.96533
0.42	0.87841	- 1.17003	- 0.78359	1.11723
0.44	0.68568	- 1.27881	- 0.59802	1.24030
0.46	0.47822	- 1.35704	- 0.39555	1.33188
0.48	0.26103	- 1.40247	- 0.18130	1.39004
0.50	0.03937	- 1.41366	0.03937	1.41366
0.52	- 0.18130	- 1.39004	0.26103	1.40247
0.54	- 0.39555	- 1.33188	0.47822	1.35704
0.56	- 0.59802	- 1.24030	0.68568	1.27881
0.58	- 0.78359	- 1.11723	0.87841	1.17003
0.60	- 0.94753	- 0.96533	1.05185	1.03374
0.62	- 1.08556	- 0.78797	1.20196	0.87368
0.64	- 1.19398	- 0.58908	1.32534	0.69422
0.66	- 1.26974	- 0.37310	1.41931	0.50026
0.68	- 1.31055	- 0.14479	1.48203	0.29711
0.70	- 1.31485	0.09085	1.51248	0.09041
0.72	- 1.28189	0.32872	1.51056	- 0.11405
0.74	- 1.21172	0.56380	1.47707	- 0.31040
0.76	- 1.10515	0.79124	1.41376	- 0.49285
0.78	- 0.96375	1.00656	1.32324	- 0.65580
0.80	- 0.78975	1.20575	1.20901	- 0.79394
0.82	- 0.58594	1.38540	1.07535	- 0.90237
0.84	- 0.35563	1.54236	0.92728	- 0.97665
0.86	- 0.10245	1.67629	0.77049	- 1.01291
0.88	0.16974	1.78480	0.61120	- 1.00785
0.90	0.45702	1.86854	0.45614	- 0.95879
0.92	0.75558	1.92871	0.31238	- 0.86367
0.94	1.06189	1.96766	0.18727	- 0.72099
0.96	1.37287	1.98892	0.08829	- 0.52979
0.98	1.68610	1.99721	0.02339	- 0.28953
1.00	2.00000	1.99845	0.00000	0.00000

TABLE 2

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-FREE BEAM
Fourth Mode

$\frac{x}{l}$	ϕ_4	$\phi_4' = \frac{1}{\beta_4}\frac{d\phi_4}{dx}$	$\phi_4'' = \frac{1}{\beta_4^2}\frac{d^2\phi_4}{dx^2}$	$\phi_4''' = \frac{1}{\beta_4^3}\frac{d^3\phi_4}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00007
0.02	0.04482	0.39147	1.56035	- 1.99672
0.04	0.16510	0.68645	1.12317	- 1.97482
0.06	0.33974	0.88606	0.69420	- 1.92012
0.08	0.54801	0.99298	0.28179	- 1.82294
0.10	0.77002	1.01194	- 0.10407	- 1.67809
0.12	0.98714	0.94994	- 0.45270	- 1.48463
0.14	1.18256	0.81633	- 0.75368	- 1.24552
0.16	1.34177	0.62264	- 0.99762	- 0.96717
0.18	1.45299	0.38230	- 1.17687	- 0.65891
0.20	1.50758	0.11017	- 1.28608	- 0.33228
0.22	1.50027	- 0.17801	- 1.32262	- 0.00038
0.24	1.42928	- 0.46624	- 1.28688	0.32290
0.26	1.29634	- 0.73895	- 1.18226	0.62370
0.28	1.10648	- 0.98164	- 1.01518	0.83888
0.30	0.86774	- 1.18154	- 0.79478	1.10676
0.32	0.59073	- 1.32813	- 0.53258	1.26772
0.34	0.28808	- 1.41368	- 0.24191	1.36469
0.36	- 0.02621	- 1.43351	0.06264	1.39357
0.38	- 0.33748	- 1.38622	0.36594	1.35339
0.40	- 0.63112	- 1.27376	0.65299	1.24643
0.42	- 0.89330	- 1.10126	0.90964	1.07812
0.44	- 1.11166	- 0.87683	1.12327	0.85675
0.46	- 1.27592	- 0.61115	1.28336	0.59315
0.48	- 1.37836	- 0.31690	1.38199	0.30011
0.50	- 1.41424	- 0.00819	1.41424	- 0.00819
0.52	- 1.38199	0.30012	1.37836	- 0.31690
0.54	- 1.28336	0.59316	1.27592	- 0.61115
0.56	- 1.12327	0.85675	1.11166	- 0.87684
0.58	- 0.90964	1.07812	0.89330	- 1.10126
0.60	- 0.65299	1.24643	0.63112	- 1.27376
0.62	- 0.36594	1.35339	0.33748	- 1.38622
0.64	- 0.06264	1.39357	0.02621	- 1.43351
0.66	0.24191	1.36469	- 0.28808	- 1.41368
0.68	0.53258	1.26772	- 0.59073	- 1.32813
0.70	0.79478	1.10676	- 0.86774	- 1.18153
0.72	1.01518	0.88888	- 1.10648	- 0.98164
0.74	1.18226	0.62370	- 1.29634	- 0.73895
0.76	1.28688	0.32290	- 1.42928	- 0.46624
0.78	1.32262	- 0.00039	- 1.50027	- 0.17801
0.80	1.28608	- 0.33228	- 1.50758	0.11017
0.82	1.17687	- 0.65890	- 1.45299	0.38230
0.84	0.99762	- 0.96717	- 1.34177	0.62264
0.86	0.75368	- 1.24552	- 1.18256	0.81633
0.88	0.45270	- 1.48463	- 0.98714	0.94994
0.90	0.10407	- 1.67809	- 0.77002	1.01194
0.92	- 0.28179	- 1.82294	- 0.54801	0.99298
0.94	- 0.69420	- 1.92012	- 0.33974	0.88606
0.96	- 1.12317	- 1.97482	- 0.16510	0.68645
0.98	- 1.56035	- 1.99672	- 0.04482	0.39147
1.00	- 2.00000	- 2.00007	0.00000	0.00000

Table 2

Characteristic Functions and Derivatives
Clamped-Free Beam
Fifth Mode

$\frac{x}{\ell}$	ϕ_5	$\phi_5' = \frac{1}{\beta_5}\frac{d\phi_5}{dx}$	$\phi_5'' = \frac{1}{\beta_5^2}\frac{d^2\phi_5}{dx^2}$	$\phi_5''' = \frac{1}{\beta_5^3}\frac{d^3\phi_5}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.07241	0.48557	1.43502	- 1.99300
0.04	0.25958	0.81207	0.87658	- 1.94824
0.06	0.51697	0.98325	0.33937	- 1.83959
0.08	0.80177	1.00789	- 0.15633	- 1.65332
0.10	1.07449	0.90089	- 0.58801	- 1.38736
0.12	1.30078	0.68346	- 0.93411	- 1.05011
0.14	1.45309	0.38243	- 1.17672	- 0.65878
0.16	1.51209	0.02895	- 1.30378	- 0.23723
0.18	1.46767	- 0.34348	- 1.31066	0.18651
0.20	1.31925	- 0.70119	- 1.20090	0.58289
0.22	1.07553	- 1.01267	- 0.98631	0.92352
0.24	0.75353	- 1.25086	- 0.68626	1.18368
0.26	0.37706	- 1.39509	- 0.32634	1.34448
0.28	- 0.02529	- 1.43257	0.06355	1.39446
0.30	- 0.42257	- 1.35934	0.45146	1.33065
0.32	- 0.78399	- 1.18045	0.80582	1.15889
0.34	- 1.08140	- 0.90954	1.09793	0.89337
0.36	- 1.29162	- 0.56770	1.30418	0.55561
0.38	- 1.39826	- 0.18174	1.40786	0.17276
0.40	- 1.39310	0.21794	1.40051	- 0.22452
0.42	- 1.27670	0.59978	1.28253	- 0.60450
0.44	- 1.05846	0.93361	1.06317	- 0.93686
0.46	- 0.75579	1.19304	0.75976	- 1.19508
0.48	- 0.39278	1.35757	0.39632	- 1.35855
0.50	0.00170	1.41421	0.00170	- 1.41421
0.52	0.39632	1.35855	- 0.39278	- 1.35757
0.54	0.75976	1.19508	- 0.75579	- 1.19304
0.56	1.06317	0.93686	- 1.05846	- 0.93361
0.58	1.28253	0.60450	- 1.27670	- 0.59978
0.60	1.40051	0.22452	- 1.39310	- 0.21794
0.62	1.40786	- 0.17276	- 1.39826	0.18174
0.64	1.30418	- 0.55561	- 1.29162	0.56770
0.66	1.09793	- 0.89337	- 1.08140	0.90954
0.68	0.80582	- 1.15889	- 0.78399	1.18045
0.70	0.45146	- 1.33065	- 0.42257	1.35934
0.72	0.06355	- 1.39446	- 0.02529	1.43257
0.74	- 0.32634	- 1.34448	0.37706	1.39509
0.76	- 0.68626	- 1.18368	0.75353	1.25086
0.78	- 0.98631	- 0.92352	1.07553	1.01267
0.80	- 1.20090	- 0.58289	1.31925	0.70119
0.82	- 1.31066	- 0.18651	1.46767	0.34348
0.84	- 1.30378	0.23723	1.51209	- 0.02895
0.86	- 1.17672	0.65878	1.45309	- 0.38243
0.88	- 0.93411	1.05011	1.30078	- 0.68346
0.90	- 0.58801	1.38736	1.07449	- 0.90089
0.92	- 0.15633	1.65332	0.80177	- 1.00789
0.94	0.33937	1.83959	0.51697	- 0.98325
0.96	0.87658	1.94824	0.25958	- 0.81207
0.98	1.43502	1.99300	0.07241	- 0.48557
1.00	2.00000	2.00000	0.00000	0.00000

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
First Mode

$\frac{x}{l}$	ϕ_1	$\phi_1' = \frac{1}{\beta_1}\frac{d\phi_1}{dx}$	$\phi_1'' = \frac{1}{\beta_1^2}\frac{d^2\phi_1}{dx^2}$	$\phi_1''' = \frac{1}{\beta_1^3}\frac{d^3\phi_1}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00155
0.02	0.00600	0.15089	1.84282	- 2.00140
0.04	0.02338	0.28944	1.68568	- 2.00031
0.06	0.05114	0.41566	1.52869	- 1.99745
0.08	0.08834	0.52955	1.37202	- 1.99203
0.10	0.13400	0.63116	1.21590	- 1.98336
0.12	0.18715	0.72055	1.06060	- 1.97079
0.14	0.24685	0.79778	0.90647	- 1.95379
0.16	0.31214	0.86296	0.75386	- 1.93187
0.18	0.38208	0.91623	0.60318	- 1.90464
0.20	0.45574	0.95776	0.45486	- 1.87177
0.22	0.53221	0.98775	0.30935	- 1.83299
0.24	0.61058	1.00643	0.16712	- 1.78812
0.26	0.68999	1.01410	0.02866	- 1.73706
0.28	0.76958	1.01105	- 0.10554	- 1.67975
0.30	0.84852	0.99764	- 0.23500	- 1.61620
0.32	0.92601	0.97427	- 0.35923	- 1.54652
0.34	1.00129	0.94137	- 0.47775	- 1.47082
0.36	1.07363	0.89940	- 0.59009	- 1.38932
0.38	1.14233	0.84886	- 0.69582	- 1.30229
0.40	1.20675	0.79029	- 0.79450	- 1.21002
0.42	1.26626	0.72427	- 0.88574	- 1.11288
0.44	1.32032	0.65138	- 0.96918	- 1.01128
0.46	1.36841	0.57226	- 1.04447	- 0.90566
0.48	1.41006	0.48755	- 1.11133	- 0.79652
0.50	1.44486	0.39794	- 1.16950	- 0.68437
0.52	1.47245	0.30410	- 1.21875	- 0.56977
0.54	1.49253	0.20675	- 1.25894	- 0.45330
0.56	1.50485	0.10661	- 1.28992	- 0.33555
0.58	1.50922	0.00440	- 1.31162	- 0.21715
0.60	1.50550	- 0.09916	- 1.32402	- 0.09872
0.62	1.49363	- 0.20332	- 1.32714	0.01910
0.64	1.47357	- 0.30736	- 1.32106	0.13566
0.66	1.44537	- 0.41057	- 1.30588	0.25033
0.68	1.40913	- 0.51224	- 1.28180	0.36247
0.70	1.36498	- 0.61167	- 1.24904	0.47145
0.72	1.31313	- 0.70820	- 1.20786	0.57666
0.74	1.25384	- 0.80117	- 1.15858	0.67750
0.76	1.18741	- 0.88996	- 1.10157	0.77340
0.78	1.11418	- 0.97400	- 1.03725	0.86382
0.80	1.03457	- 1.05270	- 0.96606	0.94823
0.82	0.94899	- 1.12556	- 0.88849	1.02616
0.84	0.85795	- 1.19210	- 0.80507	1.09714
0.86	0.76194	- 1.25187	- 0.71636	1.16078
0.88	0.66151	- 1.30448	- 0.62295	1.21670
0.90	0.55724	- 1.34960	- 0.52547	1.26458
0.92	0.44974	- 1.38693	- 0.42455	1.30414
0.94	0.33962	- 1.41621	- 0.32086	1.33515
0.96	0.22752	- 1.43727	- 0.21507	1.35743
0.98	0.11410	- 1.44996	- 0.10789	1.37085
1.00	0.00000	- 1.45420	0.00000	1.37533

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
Second Mode

$\frac{x}{l}$	ϕ_2	$\phi_2' = \frac{1}{\beta_2}\frac{d\phi_2}{dx}$	$\phi_2'' = \frac{1}{\beta_2^2}\frac{d^2\phi_2}{dx^2}$	$\phi_2''' = \frac{1}{\beta_2^3}\frac{d^3\phi_2}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.01904	0.26276	1.71729	- 1.99910
0.04	0.07241	0.48557	1.43502	- 1.99300
0.06	0.15446	0.66857	1.15424	- 1.97727
0.08	0.25958	0.81207	0.87658	- 1.94824
0.10	0.38223	0.91666	0.60415	- 1.90305
0.12	0.51697	0.98325	0.33937	- 1.83960
0.14	0.65851	1.01310	0.08494	- 1.75656
0.16	0.80176	1.00789	- 0.15633	- 1.65333
0.18	0.94192	0.96966	- 0.38158	- 1.53001
0.20	1.07449	0.90088	- 0.58802	- 1.38736
0.22	1.19534	0.80441	- 0.77300	- 1.22676
0.24	1.30078	0.68345	- 0.93412	- 1.05012
0.26	1.38759	0.54152	- 1.06927	- 0.85985
0.28	1.45308	0.38242	- 1.17673	- 0.65879
0.30	1.49510	0.21017	- 1.25518	- 0.45011
0.32	1.51208	0.02894	- 1.30380	- 0.23724
0.34	1.50305	- 0.15704	- 1.32224	- 0.02381
0.36	1.46765	- 0.34350	- 1.31068	0.18649
0.38	1.40611	- 0.52625	- 1.26983	0.38993
0.40	1.31923	- 0.70122	- 1.20092	0.58286
0.42	1.20839	- 0.86456	- 1.10569	0.76180
0.44	1.07550	- 1.01270	- 0.98634	0.92349
0.46	0.92292	- 1.14243	- 0.84553	1.06496
0.48	0.75348	- 1.25090	- 0.68631	1.18364
0.50	0.57035	- 1.33577	- 0.51204	1.27736
0.52	0.37700	- 1.39515	- 0.32640	1.34442
0.54	0.17715	- 1.42770	- 0.13323	1.38365
0.56	- 0.02536	- 1.43265	0.06348	1.39438
0.58	- 0.22661	- 1.40978	0.25968	1.37654
0.60	- 0.42268	- 1.35944	0.45136	1.33056
0.62	- 0.60973	- 1.28256	0.63460	1.25745
0.64	- 0.78413	- 1.18058	0.80569	1.15876
0.66	- 0.94244	- 1.05549	0.96112	1.03650
0.68	- 1.08158	- 0.90972	1.09776	0.89319
0.70	- 1.19882	- 0.74612	1.21281	0.73172
0.72	- 1.29186	- 0.56793	1.30395	0.55537
0.74	- 1.35888	- 0.37866	1.36930	0.36769
0.76	- 1.39858	- 0.18205	1.40755	0.17245
0.78	- 1.41019	0.01800	1.41789	- 0.02643
0.80	- 1.39351	0.21752	1.40010	- 0.22494
0.82	- 1.34890	0.41256	1.35450	- 0.41912
0.84	- 1.27726	0.59923	1.28198	- 0.60506
0.86	- 1.18004	0.77383	1.18399	- 0.77904
0.88	- 1.05919	0.93288	1.06244	- 0.93759
0.90	- 0.91715	1.07323	0.91976	- 1.07752
0.92	- 0.75676	1.19208	0.75879	- 1.19604
0.94	- 0.58122	1.28706	0.58271	- 1.29078
0.96	- 0.39406	1.35629	0.39504	- 1.35983
0.98	- 0.19902	1.39839	0.19951	- 1.40183
1.00	0.00000	1.41251	0.00000	- 1.41592

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
Third Mode

$\frac{x}{L}$	ϕ_3	$\phi_3' = \frac{1}{\beta_3}\frac{d\phi_3}{dx}$	$\phi_3'' = \frac{1}{\beta_3^2}\frac{d^2\phi_3}{dx^2}$	$\phi_3''' = \frac{1}{\beta_3^3}\frac{d^3\phi_3}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.03866	0.36672	1.59173	- 1.99731
0.04	0.14410	0.65020	1.18532	- 1.97961
0.06	0.29879	0.85122	0.78508	- 1.93509
0.08	0.48626	0.97168	0.39742	- 1.85535
0.10	0.69037	1.01491	0.03009	- 1.73537
0.12	0.89584	0.98593	- 0.30845	- 1.57331
0.14	1.08857	0.89148	- 0.60968	- 1.37037
0.16	1.25604	0.74002	- 0.86560	- 1.13046
0.18	1.38759	0.54152	- 1.06927	- 0.85985
0.20	1.47476	0.30725	- 1.21523	- 0.56678
0.22	1.51147	0.04939	- 1.29988	- 0.26098
0.24	1.49419	- 0.21934	- 1.32168	0.04683
0.26	1.42202	- 0.48616	- 1.28137	0.34551
0.28	1.29662	- 0.73864	- 1.18195	0.62397
0.30	1.12212	- 0.96520	- 1.02863	0.87171
0.32	0.90489	- 1.15556	- 0.82867	1.07934
0.34	0.65324	- 1.30107	- 0.59110	1.23893
0.36	0.37703	- 1.39512	- 0.32637	1.34445
0.38	0.08727	- 1.43330	- 0.04596	1.39199
0.40	- 0.20439	- 1.41364	0.23807	1.37996
0.42	- 0.48616	- 1.33665	0.51362	1.30919
0.44	- 0.74658	- 1.20525	0.76897	1.18287
0.46	- 0.97504	- 1.02471	0.99330	1.00646
0.48	- 1.16223	- 0.80234	1.17711	0.78746
0.50	- 1.30050	- 0.54726	1.31263	0.53513
0.52	- 1.38422	- 0.26994	1.39411	0.26005
0.54	- 1.41001	0.01818	1.41807	- 0.02624
0.56	- 1.37687	0.30522	1.38344	- 0.31179
0.58	- 1.28624	0.57929	1.29160	- 0.58465
0.60	- 1.14194	0.82907	1.14631	- 0.83344
0.62	- 0.95000	1.04422	0.95356	- 1.04778
0.64	- 0.71844	1.21582	0.72134	- 1.21873
0.66	- 0.45691	1.33678	0.45927	- 1.33915
0.68	- 0.17628	1.40210	0.17821	- 1.40403
0.70	0.11174	1.40906	- 0.11017	- 1.41064
0.72	0.39519	1.35742	- 0.39391	- 1.35870
0.74	0.66227	1.24931	- 0.66123	- 1.25036
0.76	0.90188	1.08924	- 0.90103	- 1.09010
0.78	1.10404	0.88387	- 1.10335	- 0.88458
0.80	1.26035	0.64175	- 1.25980	- 0.64233
0.82	1.36432	0.37294	- 1.36386	- 0.37341
0.84	1.41160	0.08860	- 1.41124	- 0.08900
0.86	1.40025	- 0.19943	- 1.39996	0.19910
0.88	1.33072	- 0.47918	- 1.33049	0.47891
0.90	1.20590	- 0.73904	- 1.20573	0.73881
0.92	1.03093	- 0.96820	- 1.03085	0.96800
0.94	0.81323	- 1.15713	- 0.81313	1.15695
0.96	0.56168	- 1.29798	- 0.56162	1.29782
0.98	0.28680	- 1.38490	- 0.28677	1.38476
1.00	0.00000	- 1.41429	0.00000	1.41414

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
Fourth Mode

$\frac{x}{l}$	ϕ_4	$\phi_4' = \frac{1}{\beta_4}\frac{d\phi_4}{dx}$	$\phi_4'' = \frac{1}{\beta_4^2}\frac{d^2\phi_4}{dx^2}$	$\phi_4''' = \frac{1}{\beta_4^3}\frac{d^3\phi_4}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.06496	0.46278	1.46633	- 1.99408
0.04	0.23451	0.78357	0.93792	- 1.95600
0.06	0.47104	0.96521	0.42662	- 1.86287
0.08	0.73820	1.01441	- 0.05091	- 1.70171
0.10	1.00204	0.94270	- 0.47581	- 1.46893
0.12	1.23237	0.76664	- 0.82947	- 1.16955
0.14	1.40407	0.50751	- 1.09559	- 0.81599
0.16	1.49825	0.19041	- 1.26206	- 0.42660
0.18	1.50306	- 0.15704	- 1.32223	- 0.02380
0.20	1.41422	- 0.50624	- 1.27577	0.36779
0.22	1.23502	- 0.82944	- 1.12901	0.72343
0.24	0.97582	- 1.10140	- 0.89466	1.02024
0.26	0.65324	- 1.30107	- 0.59110	1.23893
0.28	0.28879	- 1.41295	- 0.24121	1.36537
0.30	- 0.09274	- 1.42807	0.12917	1.39164
0.32	- 0.46510	- 1.34455	0.49299	1.31666
0.34	- 0.80250	- 1.16772	0.82386	1.14636
0.36	- 1.08150	- 0.90963	1.09785	0.89328
0.38	- 1.28266	- 0.58823	1.29518	0.57571
0.40	- 1.39201	- 0.22602	1.40160	0.21644
0.42	- 1.40200	0.15152	1.40934	- 0.15886
0.44	- 1.31209	0.51780	1.31771	- 0.52342
0.46	- 1.12877	0.84697	1.13308	- 0.85127
0.48	- 0.86513	1.11580	0.86843	- 1.11910
0.50	- 0.53994	1.30530	0.54246	- 1.30782
0.52	- 0.17628	1.40210	0.17821	- 1.40403
0.54	0.20000	1.39937	- 0.19853	- 1.40084
0.56	0.56222	1.29734	- 0.56109	- 1.29847
0.58	0.88466	1.10326	- 0.88379	- 1.10413
0.60	1.14445	0.83092	- 1.14379	- 0.83159
0.62	1.32317	0.49963	- 1.32266	- 0.50014
0.64	1.40813	0.13289	- 1.40774	- 0.13328
0.66	1.39330	- 0.24329	- 1.39301	0.24299
0.68	1.27973	- 0.60226	- 1.27950	0.60203
0.70	1.07546	- 0.91854	- 1.07529	0.91837
0.72	0.79497	- 1.16974	- 0.79484	1.16960
0.74	0.45814	- 1.33802	- 0.45804	1.33792
0.76	0.08884	- 1.41146	- 0.08876	1.41138
0.78	- 0.28676	- 1.38486	0.28682	1.38480
0.80	- 0.64202	- 1.26010	0.64206	1.26005
0.82	- 0.95176	- 1.04602	0.95180	1.04598
0.84	- 1.19405	- 0.75779	1.19407	0.75776
0.86	- 1.35168	- 0.41585	1.35170	0.41583
0.88	- 1.41351	- 0.04443	1.41352	0.04441
0.90	- 1.37513	0.33014	1.37514	- 0.33015
0.92	- 1.23928	0.68130	1.23929	- 0.68131
0.94	- 1.01558	0.98416	1.01559	- 0.98418
0.96	- 0.71989	1.21727	0.71990	- 1.21728
0.98	- 0.37317	1.36409	0.37318	- 1.36409
1.00	0.00000	1.41421	0.00000	- 1.41422

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
Fifth Mode

$\frac{x}{l}$	ϕ_5	$\phi_5' = \frac{1}{\beta_5}\frac{d\phi_5}{dx}$	$\phi_5'' = \frac{1}{\beta_5^2}\frac{d^2\phi_5}{dx^2}$	$\phi_5''' = \frac{1}{\beta_5^3}\frac{d^3\phi_5}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.09685	0.55098	1.34119	- 1.98902
0.04	0.33974	0.88607	0.69424	- 1.92005
0.06	0.65851	1.01311	0.08494	- 1.75656
0.08	0.98717	0.95000	- 0.45262	- 1.48455
0.10	1.26755	0.72628	- 0.88320	- 1.11064
0.12	1.45308	0.38243	- 1.17672	- 0.65879
0.14	1.51200	- 0.03274	- 1.31329	- 0.16597
0.16	1.42950	- 0.46599	- 1.28662	0.32312
0.18	1.20840	- 0.86454	- 1.10567	0.76182
0.20	0.86819	- 1.18105	- 0.79432	1.10719
0.22	0.44239	- 1.37825	- 0.38928	1.32514
0.24	- 0.02533	- 1.43261	0.06352	1.39442
0.26	- 0.48616	- 1.33665	0.51362	1.30919
0.28	- 0.89158	- 1.09954	0.91132	1.07980
0.30	- 1.19872	- 0.74602	1.21291	0.73183
0.32	- 1.37505	- 0.31360	1.38526	0.30340
0.34	- 1.40200	0.15152	1.40934	- 0.15886
0.36	- 1.27698	0.59950	1.28226	- 0.60478
0.38	- 1.01369	0.98227	1.01748	- 0.98607
0.40	- 0.64067	1.25871	0.64340	- 1.26144
0.42	- 0.09828	1.39912	0.20024	- 1.40109
0.44	0.26570	1.38846	- 0.26429	- 1.38987
0.46	0.70119	1.22792	- 0.70018	- 1.22894
0.48	1.06118	0.93487	- 1.06045	- 0.93560
0.50	1.30682	0.54093	- 1.30630	- 0.54146
0.52	1.41161	0.08861	- 1.41124	- 0.08899
0.54	1.36423	- 0.37331	- 1.36395	0.37304
0.56	1.16977	- 0.79500	- 1.16957	0.79481
0.58	0.84919	- 1.13100	- 0.84905	1.13086
0.60	0.43706	- 1.34505	- 0.43696	1.34495
0.62	- 0.02218	- 1.41408	0.02225	1.41400
0.64	- 0.47902	- 1.33063	0.47907	1.33058
0.66	- 0.88421	- 1.10371	0.88425	1.10368
0.68	- 1.19405	- 0.75779	1.19407	0.75776
0.70	- 1.37513	- 0.33015	1.37515	0.33013
0.72	- 1.40793	0.13308	1.40794	- 0.13310
0.74	- 1.28892	0.58196	1.28892	- 0.58197
0.76	- 1.03091	0.96809	1.03092	- 0.96810
0.78	- 0.66175	1.24983	0.66176	- 1.24984
0.80	- 0.22123	1.39680	0.22123	- 1.39680
0.82	0.24314	1.39315	- 0.24314	- 1.39316
0.84	0.68130	1.23928	- 0.68130	- 1.23929
0.86	1.04600	0.95178	- 1.04600	- 0.95178
0.88	1.29790	0.56165	- 1.29790	- 0.56165
0.90	1.40985	0.11096	- 1.40985	- 0.11096
0.92	1.36978	- 0.35170	- 1.36978	0.35170
0.94	1.18201	- 0.77644	- 1.18201	0.77644
0.96	0.86678	- 1.11745	- 0.86678	1.11745
0.98	0.45809	- 1.33797	- 0.45809	1.33797
1.00	0.00000	- 1.41421	0.00000	1.41421

Data For Beams With Various Edge Conditions

(1) Clamped-Clamped Beam

<u>Characteristic Function</u>

$$\phi_n = \cosh\beta_n\frac{x}{l} - \cos\beta_n\frac{x}{l} - \alpha_n(\sinh\beta_n\frac{x}{l} - \sin\beta_n\frac{x}{l})$$

where β_n and α_n are given in the following;

n	n	α n
1	4•730041	0•98250
2	7•853205	1•000777
3	10•995608	0•999966
4	14•137166	1•000001
5	17•278760	1•000000
n>5	$(2n+1)\frac{\pi}{2}$	1•000000

(2) Free-Free Beam

<u>Characteristic Function</u>

The characteristic function for a free-free beam is the same as the second derivative of a clamped-clamped beam; that is,

$$\frac{1}{\beta_n^2}\phi_n = \cosh\beta_n\frac{x}{l} + \cos\beta_n\frac{x}{l} - \alpha_n(\sinh\beta_n\frac{x}{l} + \sin\beta_n\frac{x}{l})$$

The values of α_n amd β_n are shown on the following page.

n	β_n	α_n
1	0	
2	0	
3	4·730041	0·982502
4	7·853205	1·000777
5	10·995608	0·999966
6	14·137166	1·000001
7	17·27876	1·000000
n > 7	$(2n-3)\frac{\pi}{2}$	1·000000

(3) Clamped-Free Beam

<u>Characteristic Function</u>

$$\phi_n = \cosh\beta_n\frac{x}{l} - \cos\beta_n\frac{x}{l} - d_n(\sinh\beta_n\frac{x}{l} - \sin\beta_n\frac{x}{l}$$

n	β_n	α_n
1	1 875104	0·7340955
2	4 694091	1·081847
3	7 854757	0·999225
4	10 995541	1·000034
5	14 137168	0·999999
> 5	$(2n-1)\frac{\pi}{2}$	1·000000

(4) CLAMPED-SUPPORTED BEAM

Characteristic Function

$$\phi_n = \cosh\beta_n\frac{x}{l} - \cos\beta_n\frac{x}{l} - n\left(\sinh\beta_n\frac{x}{l} - \sin\beta_n\frac{x}{l}\right)$$

n	β_n	α_n
1	3.926602	1.000777
2	7.068583	1.000001
3	10.210176	1.000000
4	13.351769	1.000000
5	16.493361	1.000000
>5	$(4n+1)\frac{\pi}{4}$	1.000000

(5) FREE-SUPPORTED BEAM

Characteristic Function

The characteristic function of a free-supported beam is the same as the second derivative of a clamped-supported beam, that is,

$$\phi_n / (\beta_x)^2 \pm \cosh\beta_n\frac{x}{l} + \cos\beta_n\frac{x}{l} - \alpha_n\left(\sin\beta_n\frac{x}{l} + \sin\beta_n\frac{x}{l}\right)$$

The constants β_n and α_n are obtained from the data of clamped-supported beam with the exception that $\beta_n = 0$, for n = 1 and 2. For $n \gg 3$, the values of β_n and α_n for the present case correspond to $n \gg 1$ for case (4).

(6) SIMPLE-SUPPORTED BEAM

Characteristic Function

$$\phi_n = \sin\beta_n\frac{x}{l}$$

where, $\beta_n = (n\pi)$

INTEGRALS OF CHARACTERISTIC FUNCTIONS FOR VARIOUS MODES

Table (4) Clamped-Clamped Beam

Values of $\frac{I_{mp}}{l} = \int_0^l \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_p}{\partial x} dx$

p \ m	1	2	3	4	5
1	12.30262	0	-9.73079	0	-7.61544
2	0	46.05012	0	-17.12892	0
3	-9.73079	0	98.90480	0	-24.34987
4	0	-17.12892	0	171.58566	0
5	-7.61544	0	-24.34987	0	263.99798

Note: $\frac{J_{mp}}{l} = \int_0^l \phi_m \frac{\partial^2 \phi_p}{\partial x^2} dx = -l \int_0^l \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_p}{\partial x} dx$

Table (5) Clamped-Free Beam

Values of $I_{mp} = +l \int_0^l \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_p}{\partial x} dx$

p \ m	1	2	3	4	5
1	4.64778	-7.37987	3.94151	-6.59339	4.59198
2	-7.37987	32.41735	-22.35243	13.58245	-22.83952
3	3.94151	-22.35243	77.29889	-35.64827	20.16203
4	-6.59339	13.58245	-35.64827	149.90185	-48.71964
5	4.59198	-22.83952	20.16203	-48.71964	228.13325

Values of $\frac{J_{mp}}{l} = \int_0^l \phi_m \frac{\partial^2 \phi_p}{\partial x^2} dx$

p \ m	1	2	3	4	5
1	0.85824	-11.74322	27.45315	-37.39025	51.95662
2	1.87385	-13.29425	-9.04222	30.40119	-33.70907
3	1.56451	3.22933	-45.90425	-8.33537	36.38656
4	1.08737	5.54065	4.25360	-98.91821	-7.82895
5	0.91404	3.71642	11.23264	4.73605	-171.58466

Table (6) Free-Free Beam

Values of $\frac{I_{mp}}{l} = \int_0^l \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_p}{\partial x} dx$

p \ m	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	12.000	0	13.856	0	13.856	0
3	0	0	49.481	0	35.378	0	36.608
4	0	13.856	0	108.925	0	57.589	0
5	0	0	35.378	0	186.867	0	78.101
6	0	13.856	0	57.589	0	284.683	0
7	0	0	36.608	0	78.101	0	402.228

Values of $\frac{J_{mp}}{l} = \int_0^l \phi_m \frac{\partial^2 \phi_p}{\partial x^2} dx$ (see following page)

p \ m	1	2	3	4	5	6	7
1	0	0	18.58910	0	43.98096	0	69.11504
2	0	0	0	40.59448	0	84.08889	0
3	0	0	-12.30262	0	52.58440	0	101.62255
4	0	0	0	-46.05012	0	55.50868	0
5	0	0	1.80069	0	-98.9048	0	60.12891
6	0	0	0	5.28566	0	-171.585	0
7	0	0	0.57069	0	9.86075	0	-263.9979

Table (7) Supported-Supported Beam

Values of

$$\frac{I_{mp}}{\ell} = \int_0^\ell \frac{\partial \phi_m}{\partial x}\frac{\partial \phi_p}{\partial x}\, dx$$

p \ m	1	2	3	4	5
1	$\frac{\pi^2}{2}$	0	0	0	0
2	0	$\frac{4\pi^2}{2}$	0	0	0
3	0	0	$\frac{9\pi^2}{2}$	0	0
4	0	0	0	$16\frac{\pi^2}{2}$	0
5	0	0	0	0	$25\frac{\pi^2}{2}$

Note $$\frac{J_{mp}}{\ell} = \int_0^\ell \phi_m \frac{\partial^2 \phi_p}{\partial x^2}\, dx = -\int_0^\ell \frac{\partial \phi_m}{\partial x}\frac{\partial \phi_p}{\partial x}\, dx$$

APPENDIX B

STRESS-STRAIN RELATIONS

The derivations of the stress-strain relations for the facings and the core are based upon the states of stresses as shown in Figs. (2c) and (2b).

The general expression for the stress-strain relations of an orthotropic body may be written as: (See (30) and (31).

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ S_{14} & S_{24} & S_{34} & S_{44} & 0 & 0 \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \quad \ldots(B.1)$$

<u>Facings</u>

For facings, the zero components of stress and strain are σ_z, τ_{yz}, τ_{xz}, and ϵ_z. Introducing the zero values for these components, equations (B.1) yield:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad \ldots(B.2)$$

solving equations (B.2) for ϵ_x, ϵ_y, and γ_{xy}, the strain-stress relations take the form:

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_y}{E_y} & 0 \\ -\frac{\nu_x}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \ldots(B.3)$$

provided $\frac{S_{11}S_{22}-S^2_{12}}{S_{22}} = E_x$; $\quad \frac{S_{11}S_{22}-S^2_{12}}{S_{11}} = E_y$

$\frac{S_{12}}{S_{11}} = \nu_x \quad \frac{S_{12}}{S_{22}} = \nu_y$ and $S_{66} = G_{xy}$.

Conversely,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E_x}{1-\nu_x\nu_y} & \frac{\nu_x E_x}{1-\nu_x\nu_y} & 0 \\ \frac{\nu_y E_y}{1-\nu_x\nu_y} & \frac{E_y}{1-\nu_x\nu_y} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad \ldots(B.4)$$

Lang (34) has shown that $G_{xy} = \sqrt{\frac{E_x E_y}{2(1+\nu)}}$ where $\nu = \sqrt{\nu_x \nu_y}$

Core

The stress-strain relations of the core are based upon the assumption that σ_x^c and τ_{xy}^c are zero. When zero values are substituted for σ_x^c and τ_{xy}^c in equations (B.1), the following relations are established.

$$\begin{bmatrix} \epsilon_y^c \\ \epsilon_z^c \\ \gamma_{yz}^c \\ \gamma_{xz}^c \end{bmatrix} = \begin{bmatrix} \frac{1}{E_y^c} & \frac{-\nu_y^c}{E_z^c} & 0 & 0 \\ \frac{-\nu_z^c}{E_y^c} & \frac{1}{E_z^c} & 0 & 0 \\ 0 & 0 & \frac{1}{G_{yz}^c} & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xz}^c} \end{bmatrix} \begin{bmatrix} \sigma_y^c \\ \sigma_z^c \\ \tau_{yz}^c \\ \tau_{xz}^c \end{bmatrix} \qquad \ldots (B.5)$$

where $\frac{\Delta}{S_{11}S_{33}-S_{13}^2} = E_y^c$, $\qquad \frac{\Delta}{S_{11}S_{22}-S_{12}^2} = E_z^c$,

$\frac{S_{11}S_{23}-S_{12}S_{13}}{S_{11}S_{33}-S_{13}^2} = \nu_z^c$, $\qquad \frac{S_{11}S_{23}-S_{12}S_{13}}{S_{11}S_{22}-S_{12}^2} = \nu_y^c$,

$S_{44} = G_{yz}^c$, $\qquad S_{55} = G_{xz}^c$ and

$$\Delta = S_{11}(S_{22}S_{33}-S_{23}) + S_{12}(S_{13}S_{23}-S_{12}S_{33}) + S_{13}(S_{12}S_{23}-S_{13}S_{22}).$$

Inversion of (B.5) yields:

$$\begin{bmatrix} \sigma_y^c \\ \sigma_z^c \\ \tau_{yz}^c \\ \tau_{xz}^c \end{bmatrix} = \begin{bmatrix} \frac{E_y^c}{1-\nu_y^c\nu_z^c} & \frac{{}_y{}^cE_y^c}{1-\nu_y^c\nu_z^c} & 0 & 0 \\ \frac{\nu_z^cE_z^c}{1-\nu_y^c\,\nu_z^c} & \frac{E_z^c}{1-\nu_y^c\,\nu_z^c} & 0 & 0 \\ 0 & 0 & G_{yz}^c & 0 \\ 0 & 0 & 0 & G_{xz}^c \end{bmatrix} \begin{bmatrix} \epsilon_y^c \\ \epsilon_z^c \\ \gamma_{yz}^c \\ \gamma_{xz}^c \end{bmatrix}$$

APPENDIX C

EVALUATION OF INTEGRALS OF INFINITE SERIES

In the present appendix, necessary expressions for the evaluation of the integrals of infinite series are developed. These expressions are employed in the course of analysis in Chapter III. As a first example, consider the integral

$$I_1 = \left[\int_0^a \int_0^b \int_0^c \sum_m^\infty \sum_n^\infty X_m'(x) Y_n'(y) \;\; \xi_{mn}(z) \right] \left[\sum_m^\infty \sum_n^\infty X_m'(x) Y_n'(y) \;\; \eta_{mn}(z) \right] (dxdydz)$$

Let the summation be extended to m = n = 2

$$I_1 = \int_0^a \int_0^b \int_0^c \left[X_1'(Y_1'\xi_{11} + Y_2'\xi_{12}) + X_2'(Y_1'\xi_{21} + Y_2'\xi_{22}) \right] \cdot$$

$$\left[X_1'(Y_1'\eta_{11} - Y_2'\eta_{12}) + X_2'(Y_1'\eta_{21} + Y_2'\eta_{22}) \right] dxdydz$$

$$= \int_0^a \int_0^b \int_0^c \Big[X_1'^2 Y_1'^2 \xi_{11}\eta_{11} + X_1'^2 Y_2'^2 \xi_{12}\eta_{12} + X_2'^2 Y_1'^2 \xi_{21}\eta_{21} + X_2'^2 Y_2'^2 \xi_{22}\eta_{22}$$

$$+ X_1' X_2' (Y_1'^2 \xi_{21}\eta_{11} + Y_2'^2 \xi_{21}\eta_{12}) + X_2' X_1' (Y_1'^2 \xi_{11}\eta_{21} + Y_2'^2 \xi_{12}\eta_{22}$$

$$+ Y_1' Y_2' (X_1'^2 \xi_{11}\eta_{12} + X_2'^2 \xi_{21}\eta_{22}) + Y_2' Y_1' (X_1'^2 \xi_{21}\eta_{11} + X_2'^2 \xi_{22}\eta_{21})$$

$$+ X_1' X_1' (Y_1' Y_2' \xi_{22}\eta_{11} + Y_1' Y_2' \xi_{21}\eta_{12} + Y_1' Y_2' \xi_{12}\eta_{21} + Y_1' Y_2' \xi_{11}\eta_{22} \Big] dxdydz$$

$$= \sum_m^2 \sum_n^2 \int_0^a \int_0^b \int_0^c (X_m')^2 (Y_n')^2 \zeta_{mn} \eta_{mn} dv + \sum_m^2 \sum_n^2 \sum_{p, m \neq p}^2 \int_0^a \int_0^b \int_0^c (X_m' X_p' (Y_n')^2 \zeta_{mn} \eta_{pn}) \cdot (dxdydz)$$

$$+ \sum_m^2 \sum_n^2 \sum_{q, n \neq q}^2 \int_0^a \int_0^b \int_0^c (X_m')^2 Y_n' Y_q' \zeta_{nq} \eta_{mn} \, dx \, dy \, dz$$

$$+ \sum_m^2 \sum_n^2 \sum_p^2 \sum_{q, m \neq p, n \neq q}^2 \int_0^a \int_0^b \int_0^c X_m' X_p' Y_n' Y_q' \zeta_{pn} \eta_{mq} \, dx \, dy dz \qquad \ldots (C.1)$$

Denoting $\int_0^a X_m' X_p' dx$ by $\frac{I_{mp}}{a}$, and $\int_0^b Y_n' Y_q' dy$ by $\frac{I_{nq}}{b}$

equation (C.1) reduces to:

$$I_1 = \sum_m^2 \sum_n^2 \int_0^c \frac{I_{mm}}{a} \frac{I_{nn}}{b} \zeta_{mn} \eta_{mn} \, dz + \sum_m^2 \sum_n^2 \sum_{p, m \neq p}^2 \int_0^c \frac{I_{mp}}{a} \frac{I_{nn}}{b} \zeta_{mn} \eta_{pn} \, dz$$

$$+ \sum_m^2 \sum_n^2 \sum_{q, n \neq q}^2 \int_0^c \frac{I_{mm}}{a} \frac{I_{nq}}{b} \zeta_{pn} \eta_{mn} + \sum_m^2 \sum_n^2 \sum_p^2 \sum_{q, m \neq p, n \neq q}^2 {}^c \frac{I_{mp}}{a} \frac{I_{nq}}{b} \, pn \; mq \; dz \qquad \ldots (C.2)$$

If $\zeta_{mn} = \eta_{mn} = Z_{mn}$, it follows that

$$I_1 = \int_0^a \int_0^b \int_0^c \left[\sum_m^\infty \sum_n^\infty X_m' Y_n' Z_{mn} \right]^2 dx \, dy \, dz$$

$$= \sum_m^\infty \sum_n^\infty \int_0^c \frac{I_{mm}}{a} \frac{I_{nn}}{b} Z_{mn}^2 dz + \sum_m^\infty \sum_n^\infty \sum_{p, m \neq p}^\infty \int_0^c \frac{I_{mp}}{a} \frac{I_{nn}}{b} Z_{mn} Z_{pn} dz$$

$$+\sum_{m}^{\infty}\sum_{\substack{n\\n\neq q}}^{\infty}\sum_{q}^{\infty}\int_0^c \frac{I_{mn}}{a}\frac{I_{nq}}{b} Z_{mn}Z_{mq}\, dz + \sum_{\substack{m\\m\neq p,\, n\neq q}}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty}\sum_{q}^{\infty}\int_0^c \frac{I_{mp}}{a}\frac{I_{nq}}{b} Z_{pn}Z_{mq}\, dz \qquad \ldots(C.3)$$

Let

$$I_2 = \int_0^a\int_0^b\int_0^c \left[\sum_m^{\infty}\sum_n^{\infty} X_m Y_n' \xi_{mn}\right]\cdot\left[\sum_m^{\infty}\sum_n^{\infty} X_m Y_n' \eta_{mn}\right] dz$$

Since $\int_0^a X_i X_j dx = a$ for $i = j$

$= 0$ for $i \neq j$,

terms which are multiples of $X_1 X_2$ vanish in the expansion of I_1 on the previous page.

$$I_2 = \sum_m^{\infty}\sum_n^{\infty}\int_0^a\int_0^b\int_0^c (X_m)^2 (Y_n')^2 \xi_{mn}\eta_{mn}\, dx\, dy dz$$
$$+ \sum^{\infty}\sum^{\infty}\sum^{\infty}\int_0^a\int_0^b\int_0^c (X_m)\, Y_n' Y_q' \xi_{mq}\eta_{mn}\, dxdydz \qquad \ldots(C.4)$$

If $\xi_{mn} = \eta_{mn} = Z_{mn}$, then

$$I_2 = \int_0^a\int_0^b\int_0^c \left[\sum_m^{\infty}\sum_n^{\infty} X_m Y_n' Z_{mn}\right]^2 dxdydz$$
$$= a\sum_m\sum_n\int_0^c \frac{I_{nn}}{b} Z_{mn}^2\, dz + a\sum_m^{\infty}\sum_{\substack{n\\n\neq q}}^{\infty}\sum_q^{\infty}\int_0^c \frac{I_{nq}}{b} Z_{mn} Z_{nq}\, dz \qquad \ldots(C.5)$$

Similarly,

$$I_3 = \int_0^a\int_0^b\int_0^c \left[\sum^{\infty}\sum^{\infty} X_m' Y_n\, X_m Y_n\, Z_{mn}\right]^2 dxdydz$$
$$= b\sum_m\sum_n\int_0^c I_{mn} Z_{mn}^2\, dz + b\sum_{\substack{m\\m\neq p}}^{\infty}\sum_n^{\infty}\sum_p^{\infty}\int_0^c I_{mp} Z_{mn} Z_{pn}\, dz \qquad \ldots(C.6)$$

Next consider

$$I_4 = \int_0^a\int_0^b\int_0^c \left[\sum_m^{\infty}\sum_n^{\infty} X_m'' Y_n \xi_{mn}\right]\cdot\left[\sum_m^{\infty}\sum_n^{\infty} X_m Y_n'' \eta_{mn}\right] dx\, dy\, dz$$

This case is directly obtainable from I_1 by replacing X_m by X_m^4 and Y_n by Y_n^4. Thus,

$$I_4 = \sum_m^\infty \sum_n^\infty \int_0^a\int_0^b\int_0^c X''_m X_m Y''_n Y_n \xi_{mn} \eta_{mn}\, dx\, dy\, dz$$

$$+ \sum_m^\infty \sum_n^\infty \sum_{\substack{p \\ m \neq p}}^\infty \int_0^a\int_0^b\int_0^c X''_m X_p Y_n \xi_{mn} \eta_{pn}\, dx\, dy\, dz$$

$$+ \sum_m^\infty \sum_n^\infty \sum_{\substack{q \\ n \neq q}}^\infty \int_0^a\int_0^b\int_0^c X''_m X_m Y''_n Y_q \xi_{mq} \eta_{mn}\, dx\, dy\, dz$$

$$+ \sum_m^\infty \sum_n^\infty \sum_p^\infty \sum_{\substack{q \\ m \neq p,\, n \neq q}}^\infty \int_0^a\int_0^b\int_0^c X''_m X_p Y''_n Y_q \xi_{mq} \eta_{pn}\, dx\, dy\, dz \qquad \ldots(C.7)$$

Denoting,

$$\int_0^a X''_m X_p\, dx \text{ by } \frac{J_{mp}}{a} \quad \text{and} \quad \int_0^b Y''_n Y_q\, dy \text{ by } \frac{J_{nq}}{b},$$

I_4 takes the form:

$$I_4 = \sum_m^\infty \sum_n^\infty \int_0^c \frac{J_{mn}}{a} \frac{J_{nn}}{b} \xi_{mn} \eta_{mn}\, dz + \sum_m^\infty \sum_n^\infty \sum_{\substack{p \\ m \neq p}}^\infty \int_0^c \frac{J_{mp}}{a} \frac{J_{nn}}{b} \xi_{mn} \eta_{pn}\, dz$$

$$+ \sum_m^\infty \sum_n^\infty \sum_{\substack{q \\ n \neq q}}^\infty \int_0^c \frac{J_{mn}}{a} \frac{J_{nq}}{b} \xi_{mq} \eta_{mn}\, dz + \sum_m \sum_n \sum_p \sum_{\substack{q \\ m \neq p,\, n \neq q}} \int_0^c \frac{J_{mp}}{a} \frac{J_{nq}}{b} \xi_{mq} \eta_{pn}\, dz \qquad \ldots(C.8)$$

If X''_m is replaced by X_m, I_4 becomes

$$I_5 = \int_0^a\int_0^b\int_0^c \left[\sum_m^\infty \sum_n^\infty X_m Y_n \xi_{mn}\right] \cdot \left[\sum_m^\infty \sum_n^\infty X_m Y''_n \eta_{mn}\right] dx\, dy\, dz$$

$$= a \sum_m^\infty \sum_n^\infty \int_0^c \frac{J_{nn}}{b} \xi_{mn} \eta_{mn}\, dz + a \sum_m^\infty \sum_n^\infty \sum_{\substack{q \\ n \neq q}}^\infty \int_0^c \frac{J_{nq}}{b} \xi_{mq} \eta_{mn}\, dz \qquad \ldots(C.9)$$

Similarly,

$$I_6 = \int_0^a\int_0^b\int_0^c \sum_m^\infty \sum_n^\infty \left[X''_m Y_n \xi_{mn}\right] \left[\sum_m^\infty \sum_n^\infty X_m Y_n \eta_{mn}\right] dx\, dy\, dz$$

$$= b \sum_m^\infty \sum_n^\infty \int_0^c \frac{J_{mn}}{a} \xi_{mn} \eta_{mn}\, dz + b \sum_m^\infty \sum_n^\infty \sum_{\substack{p \\ m \neq p}}^\infty \int_0^c \frac{J_{mp}}{a} \xi_{mn} \eta_{pn}\, dz \qquad \ldots(C.10)$$

APPENDIX D

APPLICATION TO SANDWICH BEAMS

In this appendix, frequency criteria for sandwich beams are developed from the general frequency equation (3.53). In applying this equation to the cases of beams, the parameters H_{mn} and L_{mn} approach zero and the resulting determinantal equation assumes the form:

$$\begin{bmatrix} c_{11}-\Omega_m^2\lambda_{11} & c_{12}-\Omega_m^2\lambda_{12} & c_{13} & c_{15} & A_m \\ c_{12}-\Omega_m^2\lambda_{12} & c_{22}-\Omega_m^2\lambda_{22} & c_{23} & c_{25} & B_m \\ c_{13} & c_{23} & c_{33} & c_{35} & F_m \\ c_{15} & c_{25} & c_{35} & c_{55} & K_m \end{bmatrix} = 0 \quad \text{---D.1}$$

The constants C_{ij} and λ_{ij} in (D.1) are derived from the elements of the determinant of the equation (3.53) by making β_n, I_{nn}, J_{mm}, and J_{nn} all approach zero. Under these conditions, the constants of (D.1) become:

$$c_{11} = \left\{ \frac{2}{3}(f^a)^3 d_1^a + 2(f^b)\left[(c+\frac{f^b}{2})^2 + \frac{1}{12}(f^b)^2\, d_1^b\right](\frac{\beta m}{a})^4 \right\}$$

$$c_{12} = \frac{c(f^b)}{E_z^c}\left[(c+\frac{f^b}{2})(c+f^b) + \frac{1}{6}(f^b)^2\right] d_1^b\, (\frac{\beta m}{a})^4$$

$$c_{13} = \frac{-(cf^b)}{G^c_{xz}} \quad (c+\frac{1}{2}f^b) \quad 2\, d^b_1 \left(\frac{\beta m}{a}\right)^4$$

$$c_{15} = \left[\frac{1}{2}(f^a)^2 d^a_1 - f^b (c+\frac{1}{2}f^b)(d^b_1)\right] 2\left(\frac{\beta m}{a}\right)^4$$

$$c_{22} = \frac{f^b c^2}{6(E^c_z)^2}\left\{\left[3(c+f^b)^2+(f^b)^2\right] + 12\frac{E^c_z}{cf^b}\frac{1}{d^a_1(\frac{\beta m}{a})^4}\right\} d^b_1\left(\frac{\beta m}{a}\right)^4$$

$$c_{23} = -\frac{1}{E^c_z}\cdot\frac{1}{G^c_{xz}}\left(\frac{f^b c^2}{2}\right)(c+f^b)\left[2\left(\frac{\beta m}{a}\right)^4 d^b_1\right]$$

$$c_{25} = -\frac{1}{E^c_z} f^b \frac{c}{2}(c+f^b)\left[2\left(\frac{m}{a}\right)^4 d^b_1\right]$$

$$c_{33} = \left[\frac{I_{mm}}{a^2} c \frac{G^c_{xz}}{(\frac{\beta m}{a})^4 d^b_1} + 2\, c^2 f^b\right]\left[2\left(\frac{\beta m}{a}\right)^4 \frac{d^b_1}{(G^c_{xz})^2}\right]$$

$$c_{35} = \frac{f^b c}{G^c_{xz}}\left[2\left(\frac{\beta m}{a}\right)^4 d^a_1\right]$$

$$c_{55} = (f^a d^a_1 = f^b d^b_1)\left[2\left(\frac{\beta m}{a}\right)^4\right]$$

$$\lambda_{11} = 2\left[\rho^c c + \rho^a f^a + \rho^b f^b\right]$$

$$\lambda_{12} = \frac{1}{E^c_z}\left[\rho^c c^2 + 2\rho c\right]$$

$$\lambda_{22} = \frac{1}{(E^c_z)^2}\left[\frac{2}{3}\rho^c c^3 + 2\rho^b c^2 f^b\right]$$

The solution of (D.1) is too complicated for a parametric representation. However, a number of special cases can be developed from this equation by considering the following values of the physical constants.

Case (a) $E_z^c \rightarrow \infty$, G_{xz}^c = finite.

Under this condition, the frequency equation becomes

$$\Omega_m^2 = \frac{1}{\lambda_{11}(C_{33}C_{55}-C_{35}^2)}\left[C_{33}(C_{11}C_{55}-C_{15})-C_{11}C_{35}-C_{55}C_{13}+2C_{13}C_{15}C_{35}\right] \quad \text{...(D.2)}$$

Assuming that the facings are of the same material, so that $d_1^a = d_1^b = d$, and $\rho^a = \rho^b = \rho$, the substitution of the constants C_{ij} and λ_{ij} in (D.2) Yields:

$$\Omega_m^2 = \left(\frac{\beta_m}{a}\right)^4 d \left[\frac{f^a f^b}{f^a+f^b}\left(c+\frac{f^a+f^b}{2}\right)^2 + \frac{(f^a)^3}{12} + \frac{(f^b)^3}{12}\right] \left[1 + \left(\frac{\beta_m}{a}\right)^4 \frac{d\,c\,f^a f^b}{G_{xz}^c(f^a+f^b)}\left(\frac{a^2}{I_{mm}}\right)\right] \div$$

$$\left[\rho_c^c + \rho(f^a+f^b)\right]\left[1 + \frac{f^a f^b c\left(\frac{\beta_m}{a}\right)^4 d}{(f^a+f^b)\left(\frac{I_{mm}}{a}\right)\frac{1}{a}G_{xz}^c}\right]$$

$$\Omega_m^2 = \frac{\left(\frac{\beta_m}{a}\right)^4 d}{\rho^c c + (f^a+f^b)}\left[\frac{1}{12}(f^a)^3 + \frac{1}{12}(f^b)^3 + \frac{\frac{f^a f^b}{(f^a+f^b)}\left(c+\frac{f^a+f^b}{2}\right)^2}{1+\frac{f^a f^b c\left(\frac{\beta_m}{a}\right)^4 d}{(f^a+f^b)\left(\frac{I_{mm}}{a^2}\right)G_{xz}^c}}\right] \quad \text{...(D.3)}$$

The values of I_{mm} and β_m depend upon the edge condition of beams. For simply-supported edges,

$$\frac{I_{mm}}{a^2} = \left(\frac{m\pi}{a}\right)^2; \quad \left(\frac{\beta_m}{a}\right)^4 = \left(\frac{m\pi}{a}\right)^4 \qquad \text{(see Appendix (A).}$$

With these substitutions, equation (D.3) yields:

$$\Omega_m^2 = \frac{\left(\frac{\beta_m}{a}\right)^4 d}{\left[\rho^c c + \rho(f^a+f^b)\right]} \left[\frac{1}{12}(f^a)^3 + \frac{1}{12}(f^b)^3 + \frac{\frac{f^a f^b}{(f^a+f^b)}\left(c + \frac{f^a+f^b}{2}\right)}{1 + \frac{f^a f^b c \left(\frac{\beta_m}{a}\right)^4 d}{(f^a+f^b)\left(\frac{I_{mm}}{a^2}\right) G^c_{xz}}} \right]$$

...(D.3)

The values of I_{mm} and β_m depend upon the edge condition of beams. For simply-supported edges,

$$\frac{I_{mm}}{a^2} = \frac{(m\pi)^2}{a^2}, \quad \frac{(\beta_m)^4}{a^4} = \frac{(m\pi)^4}{a^4};$$ See Appendix (A).

With these substitutions, equation (D.3) yields:

$$\Omega_m^2 = \frac{\frac{(m\pi)^4}{a^4} d}{\left[\rho^c c + \rho(f^a+f^b)\right]} \left[I_F + \frac{I_T}{1 + \frac{m^2 f^a f^b c\, \pi^2 d}{(f^a+f^b)\, a^2 G^c_{xz}}} \right]$$

...(D.4)

where

$$I_F = \left(\frac{1}{12}\right)(f^a)^3 + \left(\frac{1}{12}\right)(f^b)^3$$

$$I_T = \frac{f^a f^b}{(f^a + f^b)} \left(c + \frac{f^a+f^b}{2}\right)^2$$

Case (b) $G^c_{xz} \to \infty$, E^c_z = finite

Introducing these values in the system (D.1), the parameter F_m vanishes and the resulting frequency equation takes the form:

$$\Omega_m^4 \left[\lambda_{11}\lambda_{22} - \lambda_{12}^2\right]$$
$$-\frac{\Omega_m^2}{C_{55}} \left[\lambda_{11}(C_{22}C_{55} - C_{25}^2) + \lambda_{22}(C_{11}C_{55} - C_{15}^2) + 2\lambda_{12}(C_{25}C_{15} - C_{12}C_{55}) \right]$$

$$+ \frac{1}{C_{55}} \left[C_{22}(C_{22}C_{22}-C_{15}^2) - C_{12}^2 C_{55} - C_{25}^2 C_{11} + 2C_{15}C_{25}C_{12} \right] = 0$$

When the values of C_{ij} and λ_{ij} are substituted in (D.1), a lengthy and a complicated second degree equation in Ω^2_m results. To determine the effect of finite modulus of the core in thickness direction, the following values are directly substituted in (D.5):

$$f^b = 2f^a; \quad c = 10f^a; \quad \rho^c = \mu\rho^a; \quad \rho^b = \rho^a = \rho.$$

$$d = d_1^b = d_1^a; \quad \rho = \frac{0.0975}{(32.2)(12)} \frac{\text{#-sec}^2}{\text{in}^2}; \quad d = \frac{(10.6)(10)^{10}}{8911} \text{ psi}$$

$$E_z^c = \frac{1}{3} \times 10^{-3} d_1^a; \quad \left(\frac{\pi}{a}\right) = \frac{1}{10}; \quad f^a = \left(\frac{1}{10}\right); \quad \mu = \left(\frac{1}{10}\right)$$

For the fundamental mode, the two values of Ω^2_m are:

$$(\Omega_1^2)_1 = 104.485 \times 10^4$$

$$(\Omega_1^2)_2 = 401 \times 10^7 \qquad \text{...(D.6)}$$

The first of these equations (D.6) is the usual frequency of the fundamental normal mode of vibration and the second value of frequency is attributed to the face-wrinkling mode. (See Appendix E). Due to the excessively large value of $(\Omega^2_1)_2$ relative to $(\Omega^2_1)_1$, the mode corresponding to $(\Omega^2_1)_2$ is seldom realized. However, for very short panels with weak cores the possibility of the face wrinkling mode cannot be overlooked.

Mode Shapes: Having found the frequencies $(\Omega^2_1)_1$ and $(\Omega^2_1)_2$, the corresponding mode shapes are determined by eliminating the parameter K_m from the system (D.1).

Thus,

$$\frac{A_m}{B_m} = -\left[\frac{C_{55}(C_{12}-\Omega_m^2\lambda_{12})-C_{15}C_{15}}{C_{55}(C_{11}-\Omega_m^2\lambda_{11})-C_{15}^2}\right] = -\left[\frac{C_{55}(C_{22}-\Omega_m^2\lambda_{12})-C_{25}^2}{C_{55}(C_{12}-\Omega_m^2\lambda_{12})-C_{15}C_{25}}\right]$$

Substituting $(\Omega^2_1)_1$ and $(\Omega^2_1)_2$, (D.7) becomes

$$\left(\frac{A_1}{B_1}\right)_1 = 166.67 \text{ and } \left(\frac{A_1}{B_1}\right)_2 = -0.659$$

Introducing these ratios in the displacement function,

$$w^c = \sum_m^\infty \sin\frac{m\pi x}{a}\ (A_m + z\,B_m)\sin\Omega_m t,$$

the following results are obtained:-

$$w^c = a_1\ (1+0.006z)\sin\frac{\pi x}{a}, \text{corresponding to } (\Omega^2_1)_1 \quad \ldots(D.8)$$

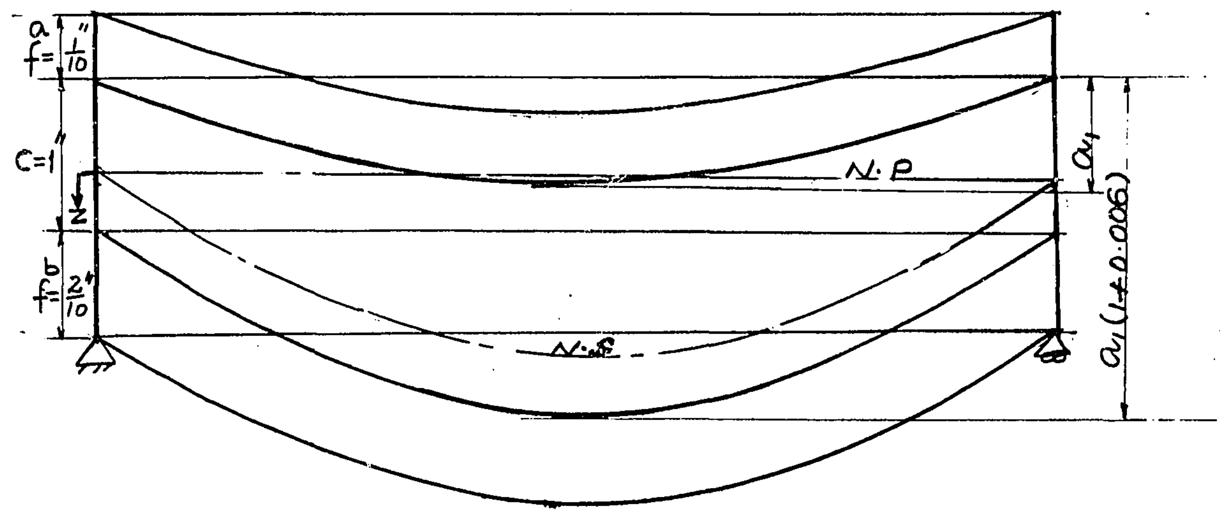
and

$$w^c = a_2\ (z-0.659)\sin\frac{\pi x}{a}, \text{corresponding to } (\Omega^2_1)_2 \quad \ldots(D.9)$$

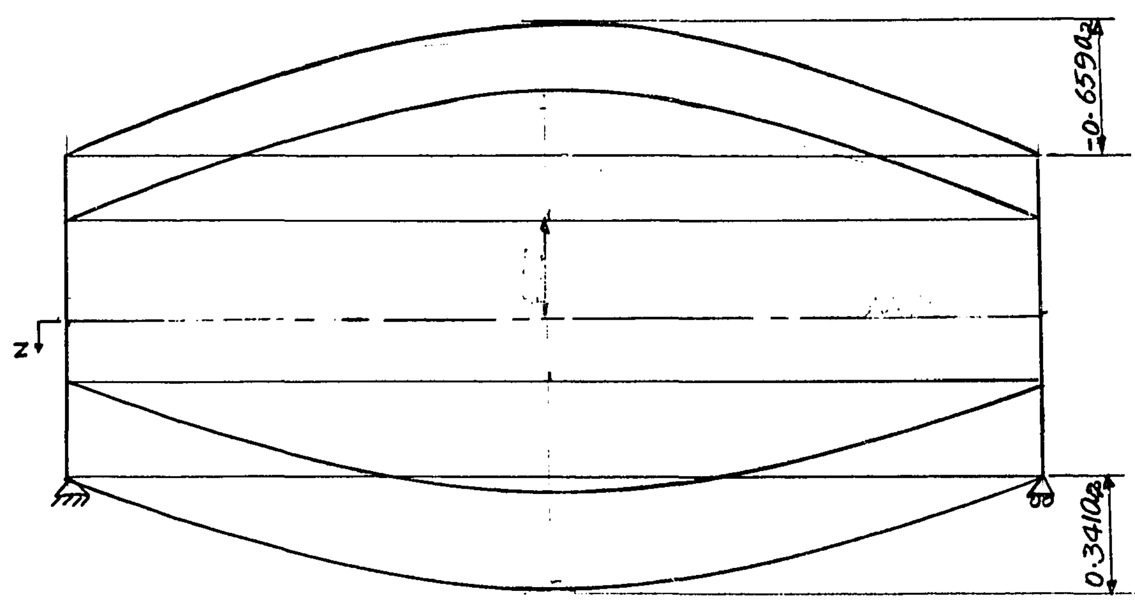
where,

a_1 and a_2 are constants.

These equations are plotted on the following page. Thus (D.8) is the motion of the neutral plane in which both facing and the core move in phase. (D.9) exhibits a type of motion in which neutral plane stays stationary, but the parts of sandwich above and below the neutral plane move out of phase.



Fundamental mode shape corresponding to (D.8)



Fundamental mode shape corresponding to (D.9)

APPENDIX E

This appendix contains numerical values of the physical constants which are taken from reference (34). For convenience, the thicknesses, and the physical constants of the core and of the lower facing are expressed in terms of those of the upper facing by means of parameters. The resulting frequencies of sandwich plates tabulated on pages (77-80) are obtained by varying these parameters in the general frequency equation (3.53). The upper facing f^a, and the lower facing f^b, and the core, are assumed to be of aluminum, steel, and aluminum honey comb, respectively. For a rectangular sandwich plate of sides 'a' and 'b', let

$$b = k_1 a; \quad f^b = k_2 f^a; \quad c = k_3 f^a$$

$$d_2^b = d_1^b = d_1^a = d_2^a; \quad \text{where } d_1^a = \frac{E_x^a}{1-(\nu^a)^2}$$

$$\nu^b = \rho \nu^a; \quad d_{12}^b = \frac{2\nu^b E^b}{1-\nu_x^b \nu_y^b} = 2\nu^a \rho \, d_1^a \mu$$

$$G_{xy}^a = \frac{1}{2}(1-\nu^a)\, d_1^a; \qquad G_{xy}^b = \frac{1}{2}\mu(1-\rho\nu^a)\, d_1^a$$

$$g_3^c = \delta_3 d_1^a; \quad g_2^c = \delta_2 z\, d_1^a; \quad g_{32}^c = \delta_1 d_1^a$$

$$G_{zy}^c = \delta_4 d_1^a; \qquad G_{xz}^c = \delta_5 d_1^a$$

$\rho^b = \eta \rho^a$; $\qquad \rho^c = \zeta^a$

$f^a = 0.016''$; $\qquad a = 36''$

$E^a = (10.6)\times 10^6$ psi; $E^b = 30\times 10^6$ psi; $\nu^a = 0.33$; $\nu^b = 0.29$

$g_3^c = 5\times 10^5$ psi; $g_2^c = 5\times 10^2$ psi; $g_{32}^c = 5\times 10^3$ psi

$\rho^c = \dfrac{9.05}{(32.2)(12)^4} \dfrac{\#\text{-sec}^2}{\text{in}^4}$; $\rho^b = \dfrac{0.284}{(32.2)(12)} \dfrac{\#\text{-sec}^2}{\text{in}^4}$ and

$\rho^a = \dfrac{0.0975}{(32.2)(12)} \dfrac{\#\text{-sec}^2}{\text{in}^4}$