HEREDITARILY INDECOMPOSABLE CONTINUA

By

ALBERT LEE CRAWFORD

Bachelor of Science in Education
Kansas State Teachers College
Emporia, Kansas
1962

Master of Arts
Kansas State Teachers College
Emporia, Kansas
1967

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
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Thesis Approved:

John Jobe
Thesis Adviser

William Marsden

Robert T. Alciatore

E. K. McLachlan

Forrest D. Whitfield

N. Durham
Dean of the Graduate College

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CHAPTER I

INTRODUCTION

The major purpose of this paper will be to expose the known examples of hereditarily indecomposable continua, the existence theorems of hereditarily indecomposable continua and the properties of such continua.

The second chapter of this paper develops the theory of topological dimension to the extent that will be needed for the rest of the paper.

Chapter III presents the basic properties of hereditarily indecomposable continua including three characterizations of these continua, one of which is original in this paper. Also in this chapter is a demonstration of the existence of hereditarily indecomposable contionu of all topological dimensions.

The forth chapter is a review of hereditarily indecomposable plane continua. The pseudo-arc and pseudo-circle are discussed in detail in this chapter. It is also proven that there exist an uncountable number of topologically distinct hereditarily indecomposable continua in the plane.

History

Brouwer (7), in 1910, was the first to describe an indecomposable continuum, that is, a continuum which could not be expressed as the union of two proper subcontinua. In 1922 Knaster (20) described a continuum that was hereditarily indecomposable. Moise (23), in 1948, described a hereditarily indecomposable continuum that was homeomorphic to each of its subcontinua. It was later proven by Bing (2) that the continuum of Knaster and the continuum of Moise were homeomorphic and homogeneous. In 1950 Bing (3) demonstrated the existence, for every n, of an n-1 dimensional hereditarily indecomposable continuum that separates E^n. Also he described a way that an infinite dimensional hereditarily indecomposable continuum could separate the Hilbert Cube.

In 1912 Poincare put forth an intuitive concept of the geometric meaning of dimension. A year later Brouwer constructed a precise and topologically invariant definition of dimension based on this intuitive concept. Brouwer's paper went unnoticed for several years until Menger and Urysohn, independent of Brouwer and of each other, recreated Brouwer's concept and made some improvements in the theory. They also made the concept of dimension the cornerstone of an extremely fruitful theory that brought unity to a large part of geometry. One of the basic beauties of the theory of dimension is that it provides a simple topological property that distinguishes the Euclidean n-spaces from one another (14, p. 3).

Definitions and Notation

This section will review the basic definitions and notation that will be used in the rest of the paper. If a term is used but not defined it will be assumed that it is defined in Hall and Spencer (11) or Moore (24).

Much of the work of this paper will deal with subsets of the

Euclidean n-spaces, E^n. The space E^n will be the collection of ordered n-tuples of real numbers with the norm defined as $\|x\| = \sqrt{\sum_1^n x_i^2}$, where $x = (x_1, x_2, \ldots, x_n)$. The surface of the unit sphere in E^n, $\{x \in E^n : \|x\| = 1\}$, will be denoted as S^{n-1}. At times E^n will be considered as a vector space with the scalar multiplication defined as $\gamma(x_1, x_2, \ldots, x_n) = (\gamma x_1, \gamma x_2, \ldots, \gamma x_n)$ and vector addition defined as $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$.

Since E^n is the topological product of n copies of the real line and since multiplication and addition are continuous on the real line it follows that scalar multiplication and vector addition are continuous on E^n.

The Hilbert space, H^N, will be considered to be the set of all real sequences $x = (x_1, x_2, \ldots)$ such that $\sum_1^\infty x_i^2 < \infty$. In the Hilbert space $\|x\| = \sqrt{\sum_1^\infty x_i^2}$ is the norm of x. The Hilbert space can be considered to be a vector space in much the same way as can E^n. At times it will be convenient to consider that $E^n \subset H^N$ by supposing that $(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$.

For notational convenience we will let $I^n = \{x \in E^n : |x_i| \leq 1,\ i = 1, 2, \ldots, n\}$ be the n-cube. Also, for notational convenience, let $I^\omega = \{x \in H^N : |x_i| \leq 1/i\}$ be the Hilbert cube. If E^n is considered to be a subspace of H^N then we will let $I^n = \{x \in E : |x_i| \leq 1/i\}$ be a subset of I^ω. Though these two definitions for I^n are not equivalent they do describe homeomorphic sets and will not cause confusion in their context. It shall be assumed that, unless otherwise stated, that all spaces will be subspaces of I^ω. This means that all spaces will be considered to be separable and metric (18, p. 125).

The notation d(x,y) will be used for the metric distance between x and y for some fixed, but usually undefined, metric. For E^n and H^N the distance will usually be the norm metric, where $d(x,y) = \|x-y\|$.

The following definitions will be used in this paper.

Definition 1.1 If H and K are two sets such that $\overline{H} \cap K = \emptyset$ and $\overline{K} \cap H = \emptyset$ then H and K are said to be separated sets.

If M is the union of two nonempty separated sets, H and K, then the notation $M = H \cup K$ sep. will be used.

Definition 1.2 If M is a space with $A \subset M$ and $B \subset M$ then A and B can be separated (in M) if and only if for some H and K, $M = H \cup K$ sep. with $A \subset H$ and $B \subset K$.

It will be said that A and B can be separated (in M) by L if and only if $A \cup B \subset M-L$ and A and B can be separated in M-L.

Definition 1.3 A space M is normal if and only if for any two closed disjoint subsets A and B of M there exist open subsets U and V of M such that $\overline{U} \cap \overline{V} = \emptyset$ with $A \subset U$ and $B \subset V$.

Definition 1.4 A space M is completely normal if and only if every subspace is normal.

Definition 1.5 A space M is a continuum if and only if M is a connected compact space.

Definition 1.6 A continuum M is indecomposable if and only if $M = H \cup K$, where H and K are continua, implies that either $M = H$ or $M = K$.

A continuum M is hereditarily indecomposable if and only if every subcontinua of M is indecomposable.

Definition 1.7 If M is a continuum and $p \in M$ then p is a cut-point of M if and only if $M-\{p\} = H \cup K$ sep.

Definition 1.8 An arc is a homeomorphic image of the closed unit interval $[0,1]$. The images of 0 and 1 are the end-points of an arc.

Basic Theorems

The theorems that are stated in this section are those elementary theorems that will be used in the rest of this paper. Some of these theorems will be proven. The source of the others are indicated by the references given at the end of their statements.

Theorem 1.9 The translation $f: E^n \to E^n$ defined as $f(x) = x + x_o$, where x_o is fixed, is a homeomorphism.

Proof: Since vector addition is continuous f is continuous. Also f^{-1}, defined as $f^{-1}(x) = x - x_o$, is continuous and hence f is a homeomorphism.

Theorem 1.10 The function $f: E^n \to E^n$ defined as $f(x) = \gamma x$, where $\gamma \neq 0$ is a fixed scalar, is a homeomorphism.

Proof: This follows from the continuity of scalar multiplication.

Theorem 1.11 If D is the boundary of an n-cube and S is the surface of an n-sphere in E^n then S and D are homeomorphic.

Proof: Both S and D are boundaries of closed n-cells. See the reference (31, p. 92).

Theorem 1.12 The set $S = S^n - \{p\}$, where $p = (0,0,\ldots,0,1) \in E^{n+1}$, is homeomorphic to E^n.

Proof: Define $f: S \to E^n$ as $f(x_1,\ldots,x_{n+1}) = (y_1,\ldots,y_n)$ where $y_i = x_i/(1-x_{n+1})$, $1 \le i \le n$.

Then $f^{-1}: E^n \to S$ can be given by $f^{-1}(y_1,\ldots,y_n) = (x_1,\ldots,x_{n+1})$ where $x_i = 2y_i/(\|y\|^2 + 1)$, $1 \le i \le n$, and $x_{n+1} = (\|y\|^2 - 1)/(\|y\|^2 + 1)$.

Since both f and f^{-1} are continuous it follows that f is a homeomorphism.

Theorem 1.13 If $b \in E^n$ and $U = \{x \in E^n: \|b-x\| < \gamma\}$ is an open sphere centered at b then there exists a homeomorphism f from $E^n-\{b\}$ onto itself such that $f(E^n-\bar{U}) = U-\{b\}$.

Proof: Define $g(x) = x - b$, $h(x) = (1/\gamma)x$ and $k(x) = x/\|x\|^2$. Then let $f = g^{-1}\circ h^{-1}\circ k\circ h\circ g$.

Theorem 1.14 The intersection of a collection of continua that are linearly ordered by set inclusion is a continuum (24, p. 14).

Theorem 1.15 If T is a component of the domain D relative to the continuum M and D is a proper subset of M then the boundary of D with respect to M contains a limit point of T (24, p. 18).

Theorem 1.16 A space M is a separable metric space if and only if M can be imbedded in the Hilbert cube (18, p. 125).

Theorem 1.17 If U is an open set in the separable metric space M then U is the countable union of closed subsets of M.

Proof: Let $x \in U$ and $\mathfrak{B}$ be a countable basis for M. Then M normal implies that there exists disjoint open sets U' and V' such that U' and V' contains M-U and $\{x\}$ respectively. The definition of basis implies that there exists $V \in \mathfrak{B}$ such that $x \in V \subset V'$. Since $V \cap U' = \emptyset$

and U' is open, $\overline{V} \cap U' = \emptyset$. Since U' contains M-U, $\overline{V} \subset U$. Therefore every point in U is contained in an element of $\mathfrak{B}$ whose closure is contained in U. Hence U is the union of those elements of $\mathfrak{B}$ whose closure is contained in U and hence $\mathfrak{B}$ being countable implies the desired result.

<u>Theorem 1.18</u> A space M is completely normal if and only if for any two separated sets A and B of M there exist disjoint open sets U and V of M such that $A \subset U$ and $B \subset V$. (11, p. 110).

CHAPTER II

INTRODUCTION TO DIMENSION THEORY

Definition and Characterizations of Dimension

One of the basic objectives for the concept of dimension is to present a topological property that distinguishes the various Euclidean n-spaces from one another. Since properties that involve the natural numbers suggest induction it is logical that dimension shall have an inductive definition. A logical starting point would be the dimension of the "smallest" possible set, the empty set.

Definition 2.1 The empty set has dimension -1.

If M is a space then the dimension of M is $\leq n$ (dim $M \leq n$) if and only if M has a basis $\mathfrak{B}$ such that if $A \in \mathfrak{B}$ then dim (bd A) $\leq n-1$.

If M is a space and dim $M \leq m$ for some m then dim $M = n$, where n is the least integer such that dim $M \leq n$. If for all n it is false that dim $M \leq n$ then dim $M = \infty$.

There are several sets of points whose dimension follows directly from the definition of dimension. Included among these are the following examples, stated as theorems.

Theorem 2.2 Every finite space has dimension zero.

Theorem 2.3 The dimension of the rationals as a subspace of the reals is zero.

Proof: The reals have a basis consisting of intervals whose endpoints are irrational. The boundaries of these intervals are empty relative to the rationals.

Theorem 2.4 The dimension of the irrationals as a subspace of the reals is zero.

Proof: The reals have a basis consisting of intervals whose endpoints are rational.

Theorem 2.5 If M is a connected space containing at least two points then dim M $\geq$ 1.

Proof: Let D be an open set of M such that D and M-D are non-empty. Then if $D' \subset D$, $D' \neq \emptyset$, is an open set the bd $D' \neq \emptyset$. Therefore M has no basis $\mathfrak{B}$ such that if $A \in \mathfrak{B}$, bd $A = \emptyset$. Hence dim M $\neq 0$. It is evident that dim M $\neq -1$. Therefore dim M $\geq$ 1.

Theorem 2.6 The dim $S^n \leq n$.

Proof: By induction, dim $S^0 \leq 0$ since S^0 is finite. Assume that dim $S^k \leq k$. Then S^{k+1} has a basis consisting of spherical open sets which have boundaries that are homeomorphic to S^k. (See theorem 1.12) Therefore dim $S^{k+1} \leq k+1$ by the inductive assumption.

The following theorem bounds the dimension of E^n above by n. However, the proof that dim $E^n = n$ will require theorems that occur in later sections of this paper.

Theorem 2.7 The dim $E^n \leq n$.

Proof: The space E^n has a basis consisting of spherical open sets. The boundaries of these open sets are homeomorphic to S^{n-1} since each spherical set is homeomorphic to the unit ball. Therefore the theorem follows from theorem 2.6 and the definition of dimension.

A result of this theorem and theorem 2.5 is that E has dimension one. This fact is stated in the following theorem.

Theorem 2.8 The dim E = 1.

Once the dimension of arbitrary spaces has been determined it is helpful to examine local dimension. For this it is necessary to have a definition of dimension of a space at a point in the space.

Definition 2.9 If M is a space and $p \in M$ then the dimension of M at p is n (dim M at p = n) where n is the least integer such that M has a basis $\mathcal{B}$ at p such that if $A \in \mathcal{B}$ then dim (bd A) $\leq$ n-1. If for all n it is false that dim M at p $\leq$ n then dim M at p = ∞.

Example 2.10 Let M be a closed interval I in E together with an isolated point $p \notin I$ in E. Then if $q \in I$, dim M at q = 1. Also dim M at p = 0.

It is apparent from the definitions that if M is a space then dim M = n where n is the least integer such that for every $p \in M$ dim M at p $\leq$ n. It should also be noted that since every space under consideration has a countable basis that the word "basis" in the definitions can be replaced by "countable basis". The dimension of a space is a topological property as will be shown in the following theorem.

Theorem 2.11 If M and N are homeomorphic spaces and dim M = n then dim N = n.

Proof: If $M = \emptyset$ then $N = \emptyset$ and dim M = dim N = -1. Suppose that the theorem holds for $n \leq k-1$. Then if dim M = k it follows that M has a basis $\mathfrak{B}$ such that if $A \in \mathfrak{B}$ then dim (bd A) $\leq k-1$. If f is a homeomorphism from M to N then the set $\mathfrak{B} = \{B : B = f(A), A \in \mathfrak{B}\}$ is a basis for N and if $B = f(A) \in \mathfrak{B}$ then bd B = f(bd A). Therefore dim (bd B) $\leq k-1$ by the inductive assumption since dim (bd A) $\leq k-1$. Hence dim N $\leq$ k by the definition of dimension. If dim N $<$ k then the inductive assumption would imply that dim M $<$ k. Therefore dim N = k. Hence the theorem follows by induction.

<u>Theorem 2.12</u> If M is a space, $K \subset M$ and dim M = n then dim K $\leq$ n.

Proof: By induction theorem 2.12 holds for n = -1. Hence if theorem 2.12 holds for k $<$ n then if dim M = n there is a basis $\mathfrak{B}$ of M such that if $A \in \mathfrak{B}$, dim (bd A) $<$ n. Then $\mathfrak{B} = \{B : B = K \cap A, A \in \mathfrak{B}\}$ is a basis for K and if $B = K \cap A \in \mathfrak{B}$ then bd A contains bd B relative to M which in turn contains bd B relative to K. Hence dim (bd B) $<$ n since dim (bd A) $<$ n. Therefore dim K $\leq$ n.

The following two theorems give characterizations of the dimension of spaces that will be useful in the proofs of several of the theorems that are to be found in the rest of this chapter.

<u>Theorem 2.13</u> A subspace M' of a space M has dimension $\leq$ n if and only if every point p of M' has a basis $\mathfrak{B}$ in M such that if $A \in \mathfrak{B}$, dim ($M' \cap$ bd A) $<$ n.

Proof: Suppose the condition holds and let $p \in M'$ and let $\mathfrak{B}$ be a basis of p in M such that if $A \in \mathfrak{B}$ then dim ($M' \cap$ bd A) $<$ n. Then $\mathfrak{B} = \{B : B = M' \cap A, A \in \mathfrak{B}\}$ is a basis of p in M' such that if $B \in \mathfrak{B}$ then dim (bd B rel M') $<$ n since if B = M' $\cap$ A then (bd B rel M') $\subset$ M' $\cap$ bd A. Hence for all p ϵ M' the dim M' at p $\leq$ n which implies that dim M' $\leq$ n.

Conversely, suppose that dim M' $\leq$ n and let p ϵ M'. Let U be an open subset of M containing p and let U' = M' $\cap$ U. Then the definition of dimension implies that there exists an open subset V' of M' containing p such that dim (bd V' rel M') $<$ n and V' $\subset$ U'. Since V' and M'-$\overline{V'}$ are separated sets the complete normality of M implies that there exist disjoint open sets A and B of M such that V' $\subset$ A and M'-$\overline{V'}$ $\subset$ B. Therefore, if V = A $\cap$ U $\subset$ U and x ϵ M' $\cap$ bd V then x $\notin$ V' since V' $\subset$ V. Thus M' $\cap$ bd V $\subset$ bd V' rel M', which implies that dim (M' $\cap$ bd V) $\leq$ dim (bd V' rel M') $<$ n. Therefore $\mathfrak{B}$ = {A: p ϵ A, A open in M, dim (M' $\cap$ bd A) $<$ n} is a basis of p in M that meets the necessary conditions, and the theorem is proven.

Theorem 2.14 If M is a space then dim M $\leq$ n if and only if for every p ϵ M and closed set K where K does not contain p, p can be separated from K by a closed set C where dim C $<$ n.

Proof: Suppose dim M $\leq$ n. Then if p ϵ M and K is a closed set not containing p there is an open set U containing p such that $\overline{U}$ $\subset$ M-K and dim (bd U) $<$ n. The bd U is closed and separates p from K.

Conversely, if the condition holds and p ϵ M and U is an open set containing p then there exists a closed set C such that dim C $<$ n. and M-C = A $\cup$ B, where p ϵ A, M-U $\subset$ B and A $\cap$ B = $\emptyset$. Since A is an open set, bd A $\subset$ C. Therefore, by theorem 2.12, dim (bd A) $\leq$ dim C $<$ n. Since A $\subset$ U it follows that M has a basis $\mathfrak{B}$ = {A:A is open and dim (bd A) $\leq$ n-1}. Hence dim M $\leq$ n.

Corollary 2.15 If M is a space then dim M = n if and only if dim $M \leq n$ and for some $p \in M$ and closed set C not containing p no closed set of dimension less than n-1 separates p from C.

Proof: To suppose the contrary contradicts theorem 2.14.

Union Theorems

One facet of the theory of dimension is the consideration of the dimension of the union of a collection of sets when the dimension of the elements of the collection are known. The first such theorem will concern the union of a countable collection of closed sets of dimension zero. However, to do this a lemma concerning the separation of closed sets will be needed.

Lemma 2.16 If M is a non-empty space then dim M = 0 if and only if any two disjoint closed subsets of M can be separated in M.

Proof: If any two disjoint closed subsets of M can be separated in M then dim M = 0 by theorem 2.14 since any point is a closed set.

If dim M = 0 let C and D be disjoint closed subsets of M. Then if $p \in M$ either $p \notin C$ or $p \notin D$. Hence by theorem 2.14 either p and C can be separated in M or p and D can be separated in M. Hence for each $p \in M$ there is a set U(p) containing p that is both open and closed in M such that either $U(p) \cap C = \emptyset$ or $U(p) \cap D = \emptyset$. Since M has a countable basis $\{U(p):p \in M\}$ covers M implies, by the Lindelof property, that there is a sequence $U_1, U_2, \ldots$ such that $U_i \in \{U(p);p \in M\}$ for each i and $\bigcup_1^\infty U_i = M$ (11, p. 107).

Now let $V_1 = U_1$ and $V_i = U_i - \bigcup_1^{i-1} U_k$. Then if $p \in M$ let U_i be the first element of the sequence such that $p \in U_i$. Then $p \in V_j$ if and only if $i = j$. Hence:

(1) $\bigcup_{1}^{\infty} V_i = M$ and

(2) $V_i \cap V_j = \emptyset$ if $i \neq j$.

Since $V_i \subset U_i$ for each i

(3) either $V_i \cap C = \emptyset$ or $V_i \cap D = \emptyset$.

Since U_i is open and $\bigcup U_k$ is closed,

(4) $V_i = U_i - \bigcup_{1}^{i-1} U_k$ is open.

Let C' be the union of all V_i such that $V_i \cap D = \emptyset$ and let D' be the union of all other V_i.

Then $C' \cup D' = M$ by (1), $C' \cap D' = \emptyset$ by (2), $C \subset C'$ and $D \subset D'$ by (3) and both C' and D' are open by (4). Hence D and C are separated in M since $M = C' \cup D'$ sep.

Now we can prove the union theorem for sets of dimension zero.

Theorem 2.17 A space which is the countable union of closed subsets of itself each of which have dimension zero has dimension zero.

Proof: Suppose $M = \bigcup_{1}^{\infty} C_i$, where each C_i is a closed subset in M that has dimension zero. Let K and L be two disjoint closed sets in M.

Then $K \cap C_1$ and $L \cap C_1$ are disjoint closed subsets of the zero-dimensional set C_1. Hence, by lemma 2.14, there exists disjoint closed sets A_1 and B_1 in C_1 such that $K \cap C_1 \subset A_1$, $L \cap C_1 \subset B_1$, $A_1 \cap B_1 = \emptyset$ and $A_1 \cup B_1 = C_1$. Therefore $K \cup A_1$ and $L \cup B_1$ are disjoint closed subsets of M. Hence, by the normality of M, there exists open sets G_1 and H_1 in M such that $\overline{G_1} \cap \overline{H_1} = \emptyset$ and $K \cup A_1 \subset G_1$, $L \cup B_1 \subset H_1$. Hence $G_1 \cup H_1 \supset C_1$.

Suppose G_{i-1} and H_{i-1} are open sets in M such that $\overline{G_{i-1}} \cap \overline{H_{i-1}} = \emptyset$. Then by the same process as used above, using $\overline{G_{i-1}}$ instead of K and $\overline{H_{i-1}}$ instead of L, there exists open sets G_i and H_i such that $G_i \cup H_i \supset C_i$, $G_{i-1} \subset G_i$, $H_{i-1} \subset H_i$ and $G_i \cap H_i = \emptyset$.

Hence, if $G = \bigcup_i^{\infty} G_i$ and $H = \bigcup_i^{\infty} H_i$ then G and H are disjoint open sets such that $G \cup H \supset \bigcup_i^{\infty} C_i = M$, $K \subset G$ and $L \subset H$. Therefore K and L are separated in M and hence, by lemma 2.16, dim M = 0.

Theorem 2.18 Every countable set has dimension zero.

Proof: A countable set is the union of a countable collection of singleton sets, each of which is closed and has dimension zero.

It is of interest to determine the dimension of the union of two sets when the dimensions of each set is known. The following theorem provides an upper bound for the dimension of the union of two sets in terms of the dimensions of each of them.

Theorem 2.19 If M is a space of finite dimension and $M = A \cup B$ then $\dim M \leq 1 + \dim A + \dim B$.

Proof: By double induction. If dim A = -1 and dim B = -1 then $\dim M = -1 \leq 1 + \dim A + \dim B$ since $M = \emptyset$.

Now assume that theorem 2.17 holds for the following two cases:

1) $\dim A \leq m$ and $\dim B < n-1$

and 2) $\dim A < m-1$ and $\dim B \leq n$.

Suppose dim A = m and dim B = n and let $p \in A \cup B$. Then without loss of generality assume that $p \in A$.

Theorem 2.13 implies that if U is a neighborhood of p in M there exists a neighborhood $V \subset U$ such that $p \in V$ and $\dim (A \cap \text{bd } V) < m$. Since $B \cap \text{bd } V \subset B$ theorem 2.12 implies that $\dim (B \cap \text{bd } V) \leq \dim B = n$.

The bd $V = (A \cup B) \cap \text{bd } V = (A \cap \text{bd } V) \cup (B \cap \text{bd } V)$ and hence $\dim (\text{bd } V) = \dim [(A \cap \text{bd } V) \cup (B \cap \text{bd } V)] \leq 1 + (m-1) + n = n+m$ by the inductive assumption.

Therefore M has a basis $\mathfrak{B} = \{C: C$ is open in M and $\dim (\text{bd } C) \leq m+n\}$. Hence, $\dim M \leq m+n+1 = \dim A + \dim B + 1$.

The bound given by theorem 2.19 is the best possible, since $\dim E = 1 + \dim Ra + \dim Ir$. It is obvious that $\dim (A \cup B) \geq \max\{\dim A, \dim B\}$ by theorem 2.12. Between these bounds, however, no general conditions can apply.

The following union theorem for sets of dimension n is a generalization of theorem 2.17.

Theorem 2.20 A space which is the countable union of closed subsets of dimension $\leq n$ has dimension $\leq n$.

Before proving theorem 2.20 it is useful to prove a corollary. It should be carefully noted that the corollary for the case $n = k$ follows from the union theorem for sets of dimension $k-1$. The corollary for the case $n = k$ will then be used in the inductive proof of the union theorem for sets of dimension k.

Corollary 2.21 If a space has dimension $n \geq 0$ it is the union of a subspace of dimension $\leq n-1$ and a subspace of dimension zero.

Proof of corollary 2.21 Let $\dim M = n$. Then there exists a countable basis $\mathfrak{B}$ for M such that if $A \in \mathfrak{B}$, $\dim (\text{bd } A) < n$.

Let $K = \bigcup\{\text{bd } A: A \in \mathfrak{B}\}$. Then by the union theorem for sets of dimension $n-1$, $\dim K = n-1$.

If $A \in \mathfrak{B}$ then $(M-K) \cap \text{bd } A = \emptyset$, and hence, by theorem 2.13, dim (M-K) = 0. Therefore M is the union of K, a subspace of dimension $\leq$ n-1 and M-K, a subspace of dimension zero.

Proof of theorem 2.20 If n = -1 the theorem is thru. If n = 0 then theorem 2.20 becomes theorem 2.17 which has been proven.

Assume that theorem 2.20 is true for the case when n = k-1. Note again that corollary 2.21 now follows.

Let $M = \bigcup_1^\infty C_i$, where each C_i is closed and for each i, dim $C_i \leq k$.

Let $K_1 = C_1$ and $K_i = C_i - \bigcup_1^{i-1} C_j$, i = 2,3,4,.... Then if $p \in M$, let C_j be the first element of the sequence such that $p \in C_j$. Then $p \in K_i$ if and only if i = j. Therefore (1) $M = \bigcup_1^\infty K_i$ and (2) $K_i \cap K_j = \emptyset$ if $i \neq j$. Since $K_i \subset C_i$, (3) dim $K_i \leq k$.

Note that $M - \bigcup_1^{i-1} C_j$ is open in M and hence can be expressed as the union of a countable number of closed subsets of M since M is a metric space by theorem 1.17. Therefore, $K_i = C_i - \bigcup_1^{i-1} C_j = C_i \cap (M - \bigcup_1^{i-1} C_j)$ is the union of a countable number of closed sets in M.

By corollary 2.21 for the case n = k each $K_i = H_i \cup N_i$, where dim $H_i < k$ and dim $N_i \leq 0$, with = unless $K_i = \emptyset$. Let $H = \bigcup_1^\infty H_i$ and $N = \bigcup_1^\infty N_i$. Note that $H_i = H_i \cap K_i = \bigcup_{j=1}^\infty H_j \cap K_i = H \cap K_i$ since $H_j \cap K_i = \emptyset$ if $i \neq j$. Hence each H_i is the union of a countable number of closed sets of H since K_i is the union of a countable number of closed sets of N. Similarly, each N_i is the union of a countable number of closed subsets of N.

Since dim H < k, if $A \subset H_i$ then dim A < k and since dim $N \leq 0$ if $B \subset N_i$ then dim $B \leq 0$. Therefore H is the union of a countable number of closed sets of H, each of which has dimension < k which implies by the inductive assumption that dim H < k. Also N is the union of a countable number of closed sets of N each of which has dimension $\leq$ 0.

Therefore, by theorem 2.17, $\dim N \leq 0$.

Hence $\dim M = \dim (H \cup N) \leq 1 + \dim H + \dim N \leq k$ by theorem 2.19 and hence the theorem follows by induction.

An interesting result of this theorem is the following corollary.

<u>Corollary 2.22</u> If M is a space of dimension n then M is the union of n+1 subspaces of dimension zero.

Proof: By corollary 2.21, $M = M_1 \cup H_1$, where $\dim M_1 \leq n-1$ and $\dim H_1 = 0$. Similarly, $M_1 = M_2 \cup H_2$, where $\dim M_2 \leq n-2$ and $\dim H_2 = 0$. Repeating this n times, $M = M_n \cup H_n \cup H_{n-1} \cup \cdots \cup H_1$, where $\dim H_i = 0$ for each i and $\dim M_n = 0$.

Another interesting result of this theorem is that the dimension of a non-empty space remains uneffected by the addition of a single point. The following corollary generalized this fact.

<u>Corollary 2.23</u> If $M = A \cup B$, where B is closed in M, then $\dim M = \max\{\dim A, \dim B\}$.

Proof: Let $n = \max\{\dim A, \dim B\}$. Then M-B is open in M and hence is the union of a countable number of closed subsets of M each of which is a subset of A and hence has dimension $\leq n$. Hence M is the countable union of a closed subsets each of which have dimension $\leq n$. Therefore, by theorem 2.20, $\dim M \leq n$. Since either A or B has dimension n it follows that $\dim M = n$.

Separation Properties of Dimension

It is of interest to consider the dimensions of sets that separate a given space. This section will be concerned with such sets. One of the primary goals of this section will be to develop the tools necessary to prove that for each n, dim $E^n = n$. This proof will be done by considering separations of I^n, the Euclidean n-cube. The first two of the separation theorems concern the separation of two closed sets of a space relative to a given subspace.

<u>Theorem 2.24</u> If C and D are disjoint closed subsets of a space M and A is a subset of M of dimension zero then there exists a closed set B in M separating C and D such that $A \cap B = \emptyset$.

Proof: Since M is normal, there exists two open sets U and V such that $C \subset U$, $D \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$.

The disjoint sets $\overline{U} \cap A$ and $\overline{V} \cap A$ are closed in A and hence, by lemma 2.16, $A = C' \cup D'$, where $\overline{U} \cap A \subset C'$, $\overline{V} \cap A \subset D'$, $\overline{C'} \cap D' = \emptyset$ and $C' \cap \overline{D'} = \emptyset$.

Therefore $\overline{(C' \cup C)} \cap (D' \cup D) = (\overline{C'} \cup \overline{C}) \cap (D' \cup D) =$

$(\overline{C'} \cup C) \cap (D' \cup D) = (\overline{C'} \cap D') \cup (C \cap D') \cup (\overline{C'} \cap D) \cup (C \cap D) = \emptyset.$

since $\overline{C'} \subset \overline{U}$, $\overline{D'} \subset \overline{V}$, $C \cap D = \emptyset$ and C' and D' are separated sets.

Similarly, $(C' \cup C) \cap \overline{(D' \cup D)} = \emptyset$.

Therefore $C' \cup C$ and $D' \cup D$ are separated sets which implies, by the complete normality of M and theorem 1.18 that there exist open sets W and Y in M such that $W \cap Y = \emptyset$, $C' \cup C \subset W$, $D' \cup D \subset Y$ and $W \cap Y = \overline{(C' \cup C)} \cap \overline{(D' \cup D)}$.

Therefore B = bd W separates C and D and since $A \subset W \cup Y$, $A \cap B = \emptyset$.

Theorem 2.25 If C and D are disjoint closed subsets of a space M and A is a subset of M of dimension $\leq$ n then there exists a closed set B in M separating C and D with dim $(A \cap B) < n$.

Proof: If $n = 0$ then either $A = \emptyset$, in which case the dim $A \cap B = -1$, or dim $A = 0$, in which case the theorem is exactly theorem 2.24.

If $n > 0$ corollary 2.21 implies that $A = H \cup K$, where dim $H < n$ and dim $K = 0$. By theorem 2.24, C and D are separated in M by a closed set B such that $B \cap K = \emptyset$. Hence $B \cap A = B \cap H \subset H$ which implies that dim $B \cap A < n$.

This next characterization of dimension is an extension of the characterization of dimension zero given in lemma 2.16.

Corollary 2.26 A space M has dimension $\leq$ n if and only if any two closed disjoint subsets of M can be separated by a closed set of dimension $< n$.

Proof: If dim $M \leq n$ then the result follows from theorem 2.25 by letting $A = M$. If the conditions hold then any point can be separated from any closed set not containing it by a closed set of dimension $< n$. Hence, by theorem 2.14, dim $M \leq n$.

The next theorem gives a method for maximizing the dimension of a space. This will be one of the primary tools that will be used to prove the dimension of E^n.

Theorem 2.27 Let M be a space of dimension $< n$ and let C_i, C_i', $i = 1,2,3,\ldots,n$, be n pairs of closed subsets of M such that $C_i \cap C_i' = \emptyset$

for each i. Then there exist closed subsets of M, $B_1, B_2, \ldots, B_n$ such that B_i separates C_i from C_i' and $\bigcap_i^n B_i = \emptyset$.

Proof: From corollary 2.26 there exists a closed set B_1 separating C_1 and C_1' such that $\dim B_1 \leq n-2$. Suppose closed sets $B_1, B_2, \ldots, B_p$, $p < n$, have been defined so that $\dim (\bigcap_i^p B_i) \leq n-p-1$ and each B_i separates C_i and C_i'. Then there is a closed set B_{p+1} by theorem 2.25 such that $\dim (B_{p+1} \cap \bigcup_i^p B_i) \leq n-p-2$ and B_{p+1} separates C_{p+1} and C_{p+1}'. Hence a sequence $B_1, B_2, \ldots, B_n$ of closed sets have been defined so that each B_i separates C_i and C_i' and $\dim (\bigcap_i^n B_i) = -1$ which implies that $\cap B_i = \emptyset$.

The Dimension of E^n

The proof that $\dim E^n = n$ requires the use of the Brouwer Fixed Point Theorem. The proof of this theorem requires the use of concepts that would require a fairly large amount of development that would not add to the value of this paper. Therefore this famous theorem will be stated without proof. There are, however, several sources in which detailed proof may be found. Two of these are Dugundji's Topology (8) and Hurewicz and Wallman's Dimension Theory (14). The following is a statement of this theorem.

Theorem 2.28 If f is a continuous function from I^n to I^n then there exists an $x \in I^n$ such that $f(x) = x$.

Theorem 2.29 Let C_i be the face of I^n determined by the equation $x_i = 1$ and let C_i' be the opposite face determined by the equation $x_i = -1$ Then if for each i B_i is a closed subset of I^n separating C_i from C_i' it follows that $\bigcap_i^n B_i \neq \emptyset$.

Proof: Since B_i is closed and separated C_i and C_i' in I^n it follows that $I^n - B_i = U_i \cup U_i'$, where $C_i \subset U_i$, $C_i' \subset U_i'$, $U_i \cap U_i' = \emptyset$ and U_i and U_i' are open in I^n.

For each $x \in I^n$, let $V(x) = (v_1, v_2, \ldots, v_n) \in E^n$ where $|v_i| = d(x,B_i)$, with $d(x,B_i)$ being the metric distance between x and the closed set B_i using the usual metric on E^n, and with v_i being positive if $x \in U_i'$ negative if $x \in U_i$ and 0 if $x \in B_i$.

Define, for each $x \in I^n$, $f(x) = x + V(x)$.

Then let $x = (x_1, x_2, \ldots, x_n)$ and let $x \in U_i$. Now suppose that $d(x,C_i') < d(x,B_i)$ and let $D = \{y \in I^n\colon d(x,y) < d(x,B_i)\}$. Then D is a connected set, $D \cap B_i = \emptyset$ and $D \cap C_i' \neq \emptyset$. Therefore B_i does not separate x from C' in I^n which is a contradiction. Therefore $d(x,B_i) \leq d(x,C_i')$. But $d(x,C_i') = |-1-x_i| = |1+x_i|$ and hence $d(x,B_i) \leq |1+x_i|$.

Therefore, since $v_i = -d(x,B_i)$, the following is acquired;
$1 \geq x_i \geq x_i - d(x,B_i) = x_i + v_i \geq x_i - |1+x_i| \geq x_i - 1 - x_i = -1$. Thus $|x_i + v_i| \leq 1$.

In a similar maner, if $x_i \in U'$ then $|x_i + v_i| \leq 1$.

If $x \in B_i$ then $|x_i + v_i| \leq 1$ since $v_i = 0$.

Therefore for every $x \in I^n$ $f(x) \in I^n$.

To show that f is continuous it is sufficient to show that V is continuous since f is the sum of V and the identity function. For that it is sufficient to show that V_i is continuous for each i, where $V_i(x) = v_i$, the i-th. coordinate of V(x).

Let $x \in I^n$ and suppose that $x \notin B_i$. Then let $0 < \delta \leq d(x,B_i)$. Then if $d(x,y) < \delta$, $d(y,B_i) \leq \delta + d(x,B_i)$ and hence $d(y,B_i) - d(x,B_i) < \delta$. Since $\delta \leq d(x,B_i)$ then $y \in U_i$ if and only if $x \in U_i$ and $y \in U_i'$ if and only if $x \in U_i'$ by the argument that obtained the inequality

$d(x,B_i) \le d(x,C_i')$. Therefore $V_i(x)$ and $V_i(y)$ have the same sign and $|V_i(x) - V_i(y)| = |d(x,B_i) - d(y,B_i)| < \delta$. Therefore V_i is continuous at $x \notin B_i$.

If $x \in B_i$ then if $d(x,y) < \delta$ it follows that $d(y,B_i) < \delta + d(x,B_i) = \delta$. Therefore $|V_i(x) - V_i(y)| = |V_i(y)| < \delta$. Hence V_i is continuous at $x \in B_i$, and thus on all of I^n.

Since f is a continuous function from I^n to I^n theorem 2.28 implies that there exists an $x \in I^n$ such that $f(x) = x$. Hence there exists an $x \in I^n$ such that $f(x) = x + V(x) = x$ which implies that $V(x) = 0$ and thus that $V_i(x) = d(x,B_i) = 0$ for each i. But this means, since each B_i is closed, that $x \in B_i$ for each i. Hence $x \in \bigcap_1^n B_i \neq \emptyset$.

It is now possible to place a lower bound on the dimension of I^n which in turn places a lower bound on the dimension of E^n since $I^n \subset E^n$.

Lemma 2.30 The dim $I^n \ge n$.

Proof: Suppose dim $I^n < n$. Then theorem 2.27 implies that there exist n closed sets $B_1, B_2, \ldots, B_n$, such that each B_i separates C_i and C_i', where C_i and C_i' are as defined in 2.29, such that $\bigcap_1^n B_i = \emptyset$. But this contradicts theorem 2.29. Hence dim $I^n \ge n$.

Theorem 2.31 The dim $E^n = n$.

Proof: $I^n \subset E^n$ and hence by lemma 2.30 and theorem 2.12 dim $E^n \ge n$. By theorem 2.7, dim $E^n \le n$. Hence dim $E^n = n$.

Corollary 2.32 The dim $I^n = n$.

Proof: $I^n \subset E^n$ and hence dim $I^n \le$ dim $E^n = n$. Hence, by lemma 2.30, dim $I^n = n$.

Subspaces of E^n with Dimension n

It is apparent that if M is a subspace of E^n then dim M $\leq$ n. It is the purpose of this section to develop a necessary and sufficient condition for dim M to be n, namely that M must contain a non-empty open subset of E^n. To do this, however, it is necessary to develop some of the properties related to E^n, one being that for any two countable dense subsets of E^n there is a homeomorphism mapping one onto the other. The first few definitions and lemmas will be directed toward this task.

Definition 2.33 If $B = \{e^1, e^2, \ldots, e^n\}$ is a basis for E^n as a vector space and $x = \Sigma\gamma_i e^i$ then γ_i is the i-th coordinate of x relative to B. The components of the ordered n-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ are the coordinates of x relative to B.

If $x = (x_1, x_2, \ldots, x_n)$ then the coordinates shall be understood to be relative to the standard unit vectors, ε^i.

Definition 2.34 If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ then $x \circ y = \sum_1^n x_i y_i$.

Definition 2.35 If $H = \{x: a \circ x = \gamma\}$ where $a \neq (0,0,\ldots,0)$ is a fixed element of E^n and γ is a fixed real number then H is a hyperplane in E^n.

Theorem 2.36 If H is a hyperplane in E^n then H is homeomorphic to E^{n-1}. Furthermore, if H contains the origin then H is a sub-vector space of E^n with vector dimension n-1.

Proof: Suppose $H = \{x: a \circ x = \gamma\}$ and that H contains the origin.

Then $\gamma = a\ (0,0,\ldots,0) = 0$.

Let $x,y \in H$. Then for $\alpha,\beta \in E$, $a\circ(\alpha x+\beta y) = \sum_i^n a_i(\alpha x_i+\beta y_i) = \alpha\sum_i^n a_i x_i + \beta\sum_i^n a_i y_i = 0$. Therefore $\alpha x+\beta y \in H$ which implies that H is a sub-vector space of E^n.

Since $H' = \{x: a\ x = 1\} \neq \emptyset$ and $H' \cap H = \emptyset$ the vector dimension of H is less than n. Let $\{x^1,x^2,\ldots,x^k\}$, $k < n$, be a vector basis for H and let $x^n \in H'$.

Now suppose that $z \in E^n$ so that $a\circ z = \beta \neq 1$. Then if $y = [\beta/(\beta-1)]x^n - [1/(\beta-1)]z$ it follows that $a\circ y = a\circ\left[[\beta/(\beta-1)]x^n-[1/(\beta-1)]z\right] = [\beta/(\beta-1)]1 - [1/(\beta-1)]\beta = 0$ and $y \in H$. Therefore $z = \beta x^n + (1-\beta)y = \beta x^n + (1-\beta)\Sigma\gamma_i x^i$. Hence z is a linear combination of $\{x^1,\ldots,x^k,x^n\}$.

Now suppose that $a\circ z = 1$ and let $h \in E^n$ such that $a\circ h = 2$. By the previous paragraph h is a linear combination of $\{x^1,x^2,\ldots,x^k,x^n\}$. Then if $y = -h + 2z$ it follows that $a\circ y = a\circ(-h + 2z) = -2+2 = 0$ and hence $y \in H$. Since $z = (1/2)y + (1/2)h$ it follows that z is a linear combination of $\{x^1,x^2,\ldots,x^k,x^n\}$ since both h and y are.

Therefore $\{x^1,x^2,\ldots,x^k,x^n\}$ is a basis for E^n which implies that $k = n-1$ since a basis for E^n contains n elements. Hence the vector dimension of H is n-1.

The mapping f such that $f(x) = f(\sum_i^n \gamma_i x^i) = (\gamma_1,\gamma_2,\ldots,\gamma_{n-1})$ from H to E^{n-1} is a homeomorphism, since multiplication and addition are continuous in both directions.

Let $K = \{x: a\circ x = \gamma\}$ and let $b \in K$. Then the function f_b such that $f_b(x) = (x - b)$ is a homeomorphism from E^n to E^n since f_b is a translation. Let $y \in K$. Then $a\circ f_b(y) = a\circ(y - b) = \gamma-\gamma = 0$, and hence $f_b(y) \in H$.

Also if $z \in H$ then $a\circ f_b^{-1}(z) = a\circ(z + b) = 0 + \gamma = \gamma$ and $f_b^{-1}(z) \in K$.

Therefore $f_b(K) = H$ and K is homeomorphic to H which is homeomorphic to E^{n-1}.

Definition 2.37 A basis (coordinate system) in E^n is in general position with respect to a set $A \subset E^n$ if for each distinct pair $a^i, a^j \in A$, $a^i - a^j$ has all non-zero coordinates relative to the basis.

Lemma 2.38 If A is a countable subset of E^n then there exists a coordinate system in E^n that is in general position with respect to A.

Proof: Let A be a countable subset of E^n. If $n = 1$ then the lemma is true since each element of E has only one coordinate.

Assume that the lemma holds for $n = k-1$.

If $a^i, a^j \in A \subset E^k$ let $L_{ij} = \{x\colon x = \gamma(a^i - a^j),\ \gamma \in E\}$, and $H_{ij} = \{x\colon x \circ (a^i - a^j) = 0\}$.

Then L_{ij} is a line and is homeomorphic to E and H_{ij} is a hyperplane and is homeomorphic to E^{k-1}. Therefore $\dim L_{ij} = 1$ and $\dim H_{ij} = k-1$. Since both L_{ij} and H_{ij} are closed in E^k and the collection of L_{ij} and H_{ij} is countable then $\bigcup_{i,j}(L_{ij} \cup H_{ij}) = K$ has dimension $< k$ by theorem 2.20.

Since $\dim E^k = k$ there exists a point $e^k \in E^k - K$. Let $L = \{x\colon x = \gamma e^k\}$ and $H = \{x\colon x \circ e^k\} = 0$. Note that $e^k \in H$. Let $\{s^1, \ldots, s^{k-1}\}$ be a vector basis for the hyperplane H and let $s^k = e^k$. Then $\{s^1, \ldots, s^{k-1}, s^k\} = S$ is a vector basis for E^k.

Consider the projection map $P\colon E^k \to H$ defined as $P(x) = P(\Sigma\gamma_i s^i) = \Sigma\gamma_i s^i$. Now suppose there exists a pair $a^i, a^j \in A$ such that $a^i \neq a^j$ and $P(a^i) = P(a^j)$. Then $a^i = \Sigma\alpha_i s^i$ and $a^j = \Sigma\beta_i s^i$ have the same coordinates except for the k-th coordinates. Therefore $a^i - a^j = (\alpha_k - \beta_k)s^k = (\alpha_k - \beta_k)e^k$ which implies that $e^k = [1/(\alpha_k - \beta_k)](a^i - a^j) \in L_{ij}$ which is a contradiction since $e^k \notin L_{ij}$. Therefore P restricted to A is one-to-one.

By the inductive assumption, there is a basis $B' = \{e^1,\ldots,e^{k-1}\}$ for H such that the coordinate system of H relative to B' is in general position relative to P(A) since P(A) is countable. Let B = $\{e^1,\ldots,e^{k-1},e^k\}$ be a basis for E^k.

Let $a = \sum_1^k \alpha_i s^i = \sum_1^k \beta_i e^i$. To show that $\alpha_k = \beta_k$ consider $P(a) = \sum_1^{k-1} \alpha_i s^i = \sum_1^{k-1} \gamma_i e^i$. Then $a = \sum_1^{k-1} \alpha_i s^i + \alpha_k s^k = \sum_1^{k-1} \gamma_i e^i + \alpha_k s^k = \sum_1^{k-1} \gamma_i e^i + \alpha_k e^k = \sum_1^k \beta_i e^i$. Since coordinates relative to B are unique, $\alpha_k = \beta_k$. By similar reasoning $\gamma_i = \beta_i$ for $i < k$ and hence, for $i < k$, the i-th coordinate of a relative to B is the same as the i-th coordinate of P(a) relative to B'.

Hence if a,b are different elements of A then for $i < k$, a-b have non-zero i-th. coordinates relative to B since the coordinate system of H relative to B' is in general position relative to P(A) and P restricted to A is one-to-one.

Now suppose that a^i, a^j are different elements of A and that a^i, a^j have the same k-th coordinates relative to B. Then $a^i - a^j = \sum_1^k \alpha_i e^i - \sum_1^k \beta_i e^i = \sum_1^{k-1} (\alpha_i - \beta_i) e^i \in H$. Therefore $e^k \circ (a^i - a^j) = 0$ and $e^k \in H_{ij}$ which is a contradiction. Hence for any pair a,b different elements of A, a-b has a non-zero k-th coordinate relative to B.

Therefore the coordinate system of E^k relative to B is in general position relative to A.

<u>Definition 2.39</u> If $x \neq 0$ is a real number then $\mathrm{sgn}(x) = |x|/x$.

<u>Definition 2.40</u> If $X = \{x^1, x^2, \ldots\}$ and $Y = \{y^1, y^2, \ldots\}$ are two sequences (possibly finite of the same length) in E^n such that the co-ordinate system of E^n is in general position relative to both X and Y

then X and Y are similarly-placed if and only if for every pair i and j $\text{sgn}(x_k^i - x_k^j) = \text{sgn}(y_k^i - y_k^j)$ for $k = 1,2,\ldots,n$, where $x_k^i, x_k^j, y_k^i, y_k^j$ are the k-th coordinates of x^i, x^j, y^i, y^j.

Lemma 2.41 Let A and B be two countable dense sets in E^n and let the coordinate system be in general position with respect to A and B. Then A and B may be arranged into similarly-placed sequences.

Proof: Let $A = \{a^1, a^2, \ldots\}$ and $B = \{b^1, b^2, \ldots\}$ and as usual if $x \in E^n$ then x_i is the i-th coordinate of x.

Let $c^1 = a^1$, $d^1 = b^1$ and $d^2 = b^2$. Then if $\text{sgn}(d_i^1 - d_i^2) = -1$ let $D_i = \{\delta \in E\colon c_i^1 - \delta < 0\}$ and if $\text{sgn}(d_i^1 - d_i^2) = 1$ let $D_i = \{\delta \in E\colon c_i^1 - \delta > 0\}$. Since each D_i is open in E and A is dense in E^n it follows that there exists $a \in A$ such that for each $i = 1,2,\ldots,n$, $a_i \epsilon D_i$. Let c^2 be the first such element of A. Then (c^1, c^2) and (d^1, d^2) are similarly-placed.

Suppose $c^1, c^2, \ldots, c^{2j}$ and $d^1, d^2, \ldots, d^{2j}$ have been defined so that $(c^1, c^2, \ldots, c^{2j})$ and $(d^1, d^2, \ldots, d^{2j})$ are similarly-placed. Then let c^{2j+1} be the first element of A such that $c^{2j+1} \neq c^i$ for any $i < 2j+1$. Let d_k^M be the maximum element of the set of k-th coordinates of elements of $\{d^1, d^2, \ldots, d^{2j}\}$ such that $c_k^j < c_k^{2j+1}$, $i = 1,2,\ldots,2j$. If $c_k^j > c_k^{2j+1}$ for all $i = 1,2,\ldots,2j$ then choose d_k^M so that $d_k^M < d_k^i$ for all $i = 1,2,\ldots,2j$.

Now let d_k^m be the minimum element of the set of k-th coordinates of elements of $\{d^1, d^2, \ldots, d^{2j}\}$ such that $c_k^i > c_k^{2j+1}$, $i = 1,2,\ldots,2j$. If $c_k^i < c_k^{2j+1}$ for all i then choose d_k^m so that $d_k^m > d_k^i$ for all i.

Let us now show that $d_k^m > d_k^M$. If $c_k^i < c_k^{2j+1}$ or $c_k^i > c_k^{2j+1}$ for all i then it is obvious that $d_k^m > d_k^M$. Otherwise there exist c^q and c^p such that $c_k^q < c_k^{2j+1}$ and $c_k^p > c_k^{2j+1}$ and also that $d_k^q = d_k^M$ and $d_k^p = d_k^m$.

Since $(c^1,c^2,\ldots,c^{2j})$ and $(d^1,d^2,\ldots,d^{2j})$ are similarly-placed and $c_k^q - c_k^p > 0$ it follows that $d_k^q - d_k^p > 0$ and hence that $d_k^m > d_k^M$.

Since B is dense in E^n there exists a $b \in B$ such that $d_k^m < b_k < d_k^m$ for all $k = 1,2,\ldots,n$. Let d^{2j+1} be the first such element of B. Then $(c^1,c^2,\ldots,c^{2j}\ ,c^{2j+1})$ and $(d^1,d^2,\ldots,d^{2j}\ \ ,d^{2j+1})$ are similarly-placed.

Let d^{2j+2} be the first element of B not proveously choosen and pick c^{2j+2} to be the first element of A such that $(c^1,\ldots,c^{2j+2})$ and $(d^1,\ldots,d^{2j+2})$ are similarly-placed. Such an element of A can be found in a manner similar to that used to find d^{2j+1}.

Then by induction $C = \{c^1,c^2,\ldots\}$ and $D = \{d^1,d^2,\ldots\}$ have been defined. The sequences C and D are similarly-placed by the inductive definitions and are rearrangements of A and B since if $a^i \in A$, $b^i \in B$ then $a^i = c^j$ and $b^i = d^k$ where $j \leq 2i$ and $k \leq 2i$.

<u>Theorem 2.42</u> For any two countable dense subsets A and B of E^n there exists a homeomorphism of E^n on itself that maps A one-to-one onto B.

Proof; By lemma 2.38 it can be assumed that the coordinate system is in general position relative to both A and B. By lemma 2.39 it can be assumed that $A = \{a^1,a^2,\ldots\}$ and $B = \{b^1,b^2,\ldots\}$ are similarly-placed sequences.

Let $A_k = \{a_k^1,a_k^2,\ldots\}$ and $B_k = \{b_k^1,b_k^2,\ldots\}$. Define a function $f_k\colon A_k \to B_k$ by letting $f_k(a_k^i) = b_k^i$.

Since the coordinate system is in general position relative to both A and B, no two distinct elements of A_k are equal and no two distinct elements of B_k are equal. Therefore f_k is a one-to-one and onto function.

Let $r \in E-A_k$ and let $A_k^+ = \{a_k \in A_k: a_k > r\}$ and $A_k^- = \{a_k \in A_k: a_k < r\}$. Note that $A_k^+ \cup A_k^- = A_k$, which implies that $f(A_k^+ \cup A_k^-) = f(A_k^+) \cup f(A_k^-) = B_k^+ \cup B_k^- = B_k$. Since $A_k^+ \cap A_k^- = \emptyset$ it follows that $B_k^+ \cap B_k^- = \emptyset$.

Since A and B are similarly-placed it follows that $\text{sgn}(a_k^i - a_k^j) = \text{sgn}(b_k^i - b^j) = \text{sgn}(f(a_k^i) - f(a_k^j))$. Therefore $a_k^i < a_k^j$ if and only if $f(a_k^i) < f(a_k^j)$. Hence each element of B_k^- is less than each element of B_k^+.

Since B is dense in E^n it follows that B_k is dense in E which implies that the least upper bound of B_k^- equals the greast lower bound of B_k^+. Now define $f_k(r) = s = \text{lub } B_k^- = \text{glb } B_k^+$.

Now suppose that $s \in B_k$. Then there exists $a_k \in A_k$ such that $f_k(a_k) = s$. Without loss of generality, assume that $r < a_k$. Then A_k dense in E implies there is an $a_k^i \in A_k$ such that $r < a_k^i < a_k$, which implies that $b_k^i < s$. But $a_k^i \in A_k^+$ which implies that $b_k^i \in B_k^+$ and hence $s \geq b_k^i$ which is a contradiction. Therefore $s \in E-B$.

It should now be noted that $f_k(A_k^+) = B_k^+ = \{b_k \in B_k: b_k > s\}$ and that $f_k(A_k^-) = B_k^- = \{b_k \in B_k: b_k < s\}$. Therefore if $f_k^{-1}: B_k \to A_k$ is extended to include $s \in E-B_k$ in a manner similar to that used to extend f_k to include r it would follow that $f_k^{-1}(s) = r$ since $f_k^{-1}(B_k^+) = A_k^+$ and $f_k^{-1}(B_k^-) = A_k^-$. Therefore $f_k: E \to E$ is a one-to-one onto function.

It should also be noted that f_k is order perserving, that is $r_i < r_j$ if and only if $f_k(r_i) < f_k(r_j)$. Hence, if (r_i, r_j) and (s_i, s_j) are open intervals in E then $f_k[(r_i, r_j)] = (f_k(r_i), f_k(r_j))$ and $f_k^{-1}[(s_i, s_j)] = (f_k^{-1}(s_i), f_k^{-1}(s_j))$. Therefore f_k is both continuous and open and hence f_k is a homeomorphism from E to E.

If $x \in E^n$ let P_k be the projection map defined by $P_k(x) = x_k$.

Now let $f(x) = y$ where $y_k = f_k(P_k(x))$. Since f_k and P_k are continuous for each k it follows that f is continuous. Also if g is defined so that $g(y) = x$ where $x_k = f_k^{-1}(P_k(y))$ then g is continuous. Since $g(f(x)) = x$ and $f(g(x)) = x$ for all $x \in E^n$ it follows that f and g are inverse homeomorphisms. Also if $a^i \in A$ then $f_k(P_k(a^i)) = f_k(a_k^i) = b_k^i$ which implies that $f(a^i) = b^i$. Therefore $f(A) = B$ and f is the required homeomorphism.

This next lemma gives the dimension of E^n less those points that have rational coordinates. This lemma will be used to prove that it is necessary for a subset of E^n to have an open subset of E^n in order that it have dimension n.

<u>Lemma 2.43</u> If R_a^n is the set of points of E^n that have all rational coordinates then $E^n - R_a^n$ has dimension n-1.

Proof: If n = 1 the lemma is true since $E - R_a$ is the set of irrational numbers which has dimension zero.

Suppose the lemma is true for n = k-1.

Let $x \in E^k - R_a^k$. Then if D is an open set containing x there exist an open k-cube $C \subset D$ containing x of the form $C = \{y \in E^k: r_i < y_i < s_i, r_i, s_i \in R_a\}$.

Then each face F_i of C, determined by the equation $y_i = r_i$ or by the equation $y_i = s_i$, has the property that $R_a^k \cap F$ is dense in F. Therefore, if F is the boundary of C, which is the union of all of the faces of C, then $R_a^k \cap F$ is dense in F.

By theorem 1.11 and theorem 1.12 if $z \in F$ then $F - \{z\}$ is homeomorphic to E^{k-1} and hence, by theorem 2.42, there exists a homeomorphism $f: (F - \{z\}) \rightarrow E^{k-1}$ such that $f(F \cap R_a^k - \{z\}) = R_a^{k-1}$.

Therefore $f(F-(R_a^k \cup z)) = E^{k-1}-R^{k-1}$ which implies that $\dim(F-(R_a^k \cup z)) = \dim(E^{k-1}-R^{k-1}) = k-2$ by the inductive assumption.

Therefore, by corollary 2.23, $\dim [(F-(R_a^k \cup z)) \cup z] = \dim(F-R_a^k)$ $= k-2$. since a space is not affected by the adjunction of a single point. Hence $\{C: C$ is open in E^k, $x \in C$, $\dim(\text{bd } C \cap (E^k-R_a^k)) = k-2\}$ is a basis for x. Thus by theorem 2.13 $\dim(E^k-R_a^k)$ at x is k-1 which implies that $\dim(E^k-R_a^k) = k-1$.

The theorem now follows by induction.

We can now show a necessary and sufficient condition for a subset of E^n to have dimension n.

<u>Theorem 2.44</u> A subspace N of E^n has dimension n if and only if N contains a non-empty open subset of E^n.

Proof: If N contains a non-empty open subset of E^n then N contains an open sphere in E that is homeomorphic to E and hence has dimension n. Therefore $\dim N \geq n$. Since $N \subset E^n$, $\dim N \leq n$ which implies that $\dim N = n$.

For the converse, suppose $\dim N = n$ and that N contains no non-empty open subset of E^n. Then E^n-N is dense in E^n which implies that there exists a countable subset A of E^n-N that is dense in E^n. (If β is a countable basis for E^n let A be the image of a choice function that assigns to each element B of β an element of $B \cap (E^n-N)$.) Let f be a homeomorphism from E^n to E^n that maps A one-to-one onto the set of points of E^n that have all rational coordinates. Note that f exists by theorem 2.42. Then $f(N) \subset E^n-R_a^n$ and f(N) has dimension n. But $\dim E^n-R_a^n = n-1$ which contradicts theorem 2.12 and thus M must contain a non-empty open subset of E^n and the theorem has been proven.

From this theorem a corollary concerning the dimension of the boundary of an open set is obtained.

Corollary 2.45 Let U be an open set in E^n that is neither empty nor dense in E^n. Then the dimension of the boundary of U is n-1.

Proof: Let B be the boundary of U. Since B contains no open subset of E^n it follows that the dim B $\neq$ n by theorem 2.44. Therefore dim B $<$ n. Now suppose that dim B $<$ n-1 and that U is bounded. Then if $x \in U$ there is an r such that $U' = \{y: d(x,y) < r\} \supset \bar{U}$. Therefore, if $x' \in E^n$ and $\delta > 0$ there is a homeomorphism from U' to $\{y: d(x',y) < \delta\} = U_\delta$ such that x is mapped to x'. Thus U_δ contains the image of U which has a boundary of dimension $<$ n-1. This would imply that the collection of open sets containing x' that have boundaries with dimension $<$ n-1 form a basis for E^n at x' and since x' is arbitrary, such sets form a basis for E^n. This then implies that dim $E^n < n$ which contradicts the fact that dim $E^n = n$. Therefore, dim B = n-1.

Now suppose that U is not bounded. Then there is a point $x \in E^n$ such that x is in the interior of E^n-U since U is not dense in E^n. This implies that there exists an open sphere U' centered at x such that $U' \cap \bar{U} = \emptyset$. Then by theorem 1.13 there exists a homeomorphism f from E^n-x onto itself that maps E^n-U' onto U'-x. Then $f(U) \subset U'$ is bounded and hence, by the first case, the dim bd f(U) = n-1. Since $f(E^n-U') = U'-x$, f(B) contains all of bd f(U) with the possible exception of x. If f(B) = bd f(U) then dim B = dim f(B) = dim f(U) = n-1. If f(B) = bd f(U) - x then dim B = dim f(B) = dim (bd f(U) - x) = dim bd f(U) = n-1 since the dimension of a set cannot be increased by the adjunction of a single point.

The last major theorem of this section states that any subspace of E^n that separates E^n has dimension of at least n-1. However, to prove this the following lemma is needed.

Lemma 2.46 The following three statements about a space M are equivalent:

(1) The space M can be separated by a subset D of dimension $\leq$ n.

(2) The space M contains an open set U which is neither empty nor dense in M whose boundary has dimension $\leq$ n.

(3) The space M = H $\cup$ K, where H and K are closed proper subsets of M and dim (H $\cap$ K) $\leq$ n.

Proof: Part I, (1) $\Rightarrow$ (2):

Since D separated M, M-D = A $\cup$ B sep. By the complete normality of M there exist open sets U and V of M such that A $\subset$ U, B $\subset$ V and U $\cap$ V = $\emptyset$. Therefore, U $\neq$ $\emptyset$ and M-$\overline{U}$ $\supset$ B $\neq$ $\emptyset$ hence U is neither empty nor dense in M. Also bd U $\subset$ D and hence dim bd U $\leq$ dim D $\leq$ n and hence (1) $\rightarrow$ (2).

Part II, (2) $\rightarrow$ (3):

Let U be such an open set. Then H = M-U and K = $\overline{U}$ are closed proper subsets of M and dim (H $\cap$ K) = dim bd U $\leq$ n. Thus (2) $\rightarrow$ (3).

Part III, (3) $\rightarrow$ (1):

Let H and K be such closed sets. Then H $\cap$ K has dimension $\leq$ n and separates M since M-(H $\cap$ K) = (K-H) $\cup$ (H-K) sep. Thus H $\cap$ K meets the necessary conditions to complete the proof.

Theorem 2.47 If N is a subspace of E^n and N separates E^n then dim N $\geq$ n-1.

Proof: Suppose E^n could be separated by a subspace whose dimension is $< n-1$. Then lemma 2.46 implies that there is a non-empty open subset of E^n that is not dense in E^n and has a boundary of dimension $<$ $n-1$. But this contradicts corollary 2.45. Hence the theorem is true.

From this theorem two corollaries are obtained. These corollaries concern the dimensions of sets that separate certain types of subspaces of E^n.

Corollary 2.48 If S is a connected open subset of E^n then S cannot be separated by a subset of S whose dimension is less than n-1.

Proof: Suppose D separates S and dim $D < n-1$. Then $S-D =$ $A \cup B$ sep. Since, by theorem 2.42, D contains no open set it follows that $\bar{A} \cup \bar{B} = S$. Since S is connected, $\bar{A} \cap \bar{B} \neq \emptyset$. Let $x \in \bar{A} \cap \bar{B}$. Then let U be a spherical neighborhood of x that is contained in S. Then U is homeomorphic to E^n.

But $U-(U \cap D) = (U \cap A) \cup (U \cap B)$ sep. which implies that $U \cap D$ separates U. But $\dim (U \cap D) \leq \dim D < n-1$ which contradicts theorem 2.47. Therefore the corollary follows.

Corollary 2.49 If S is a connected open subset of E^n then S cannot be separated by any subset of S of dimension $< n-1$.

Proof: Let D be a subset of $\bar{S}$ of dimension $< n-1$. Then since S is a connected open subset of E^n, corollary 2.48 implies that S-D is connected.

If $x \in (\text{bd } S)-D$ and R is any open set containing x then $R \cap S$ is a non-empty open subset of S. But $R \cap S \not\subset D$ since theorem 2.44 implies that D contains no open subset of E^n. Thus there exists

$y \in [(R \cap S)-D] \subset S-D$. This implies that x is a limit point of S-D which is connected. Therefore $S-D \subset \overline{S}-D \subset \overline{S-D}$ which implies that S-D is connected (11, p. 82). Thus the corollary follows.

The following theorem is a result, though not an obvious one, of Tietze's Extension Theorem (11, p. 80) which gives some conditions under which a function may be extended to a larger domain. While Tietze's Extension Theorem is a classical one it involves topics that are beyond the scope of this paper and hence will not be proven. The following theorem will, however, be useful in the next chapter in the discussion of the non-homogenity of hereditarily indecomposable continua of dimensions greater than one. This theorem is included in Chapter II since it is basicly a dimension theorem.

Theorem 2.50 If M is a compact set of dimension n then there exists a continuum in M that has dimension n at each of its points.

Infinite Dimensional Spaces

In this section two spaces, both of which have infinite dimension, will be discussed. The first, E^N, will be a vector space over E with a countable infinite vector basis $B = \{e^1, e^2, \ldots\}$. The topology on E^N can be generated by the norm $\|x\| = \sqrt{\sum_1^\infty |x_i|^2}$ where $x = \sum_1^\infty x_i e^i$, $x_i \in E$. It should be noted that for each x only a finite number of x_i are non-zero.

The second space to be considered, H^N, will be the Hilbert space, the set of all sequences $x = \{x_1, x_2, \ldots\}$ such that $\sum_1^\infty |x_i|^2 < \infty$. As is well known H^N is a vector space over E whose vector basis is uncountable. The topology on H^N can be generated by the norm $\|x\| = \sqrt{\sum_1^\infty |x_i|^2}$.

While these spaces will not be considered during the rest of this paper they do present interesting examples demonstrating the behavior of dimension on infinite dimensional topological spaces. Both of these spaces have infinite dimensions since each contains, in a homeomorphic sense, E^n for all finite n and hence it could not be true that they would have finite dimension.

<u>Theorem 2.51</u> The space E^N is the union of a countable number of subspaces each of which have zero dimension.

Proof: If E^n is considered to be the collection of points of E^N such that all but the first n coordinates are zero then $E^N = \bigcup^{\infty} E^n$. Since dim $E^n = n$ for each n, corollary 2.22 implies that for each n, $E^n = \bigcup_{i=1}^{\infty} A_i^n$, where each A_i^n has dimension zero. Therefore $E^N = \bigcup_{i,n} A_i^n$, and the theorem follows.

Before considering the dimensional properties of H^N let us note that I^n can be considered to be a subset of I^ω as was done in Chapter I and that the faces of $I^n \subset I^\omega$ are determined by the equations $y_i = 1/i$ or $y_i = -1/i$.

<u>Theorem 2.52</u> The space H^N is not the union of a countable number of spaces each of which have zero dimension.

Proof: Suppose the theorem were not true. Then $H^N = \bigcup A_i$, where for each i the dim $A_i = 0$. Then $D_i = A_i \cap I^\omega$ has dimension less than or equal to zero and $I^\omega = \bigcup D_i$.

Let C_i be the face of I^ω determined by the equation $x_i = 1/i$ and C' be the face determined by the equation $x_i = -1/i$. Then theorem 2.24 implies that there exist, for each i, a closed set B_i separating C_i from C_i' such that $B_i \cap D_i = \emptyset$.

Since, for each n, B_i separates $C_i \cap I^\omega$ and $C_i' \cap I^\omega$, theorem 2.29 says that, for each n, $\cap B_i \neq \emptyset$. For each n let $x^n \in \cap B_i$. If for an infinite number of n, there is an $x \in I^\omega$ such that $x = x^n$ then $x \in \cap B_i$ since $\cap B_i$ contains all but a finite number of x^n. If, on the other hand, there are an infinite number of distinct x^n then I^ω being compact implies that $\{x^1, x^2, \ldots\}$ has a limit point $x \in I^\omega$. Since for each n, $\cap B_i$ is closed and contains all but a finite number of x^n, $x \in \cap B_i$ which implies that $x \in \cap B_i$.

Since $\cup D_i = I^\omega$ there exists a D_i such that $x \in D_i$ which implies that $x \in D_i \cap B_i$. But this contradicts the fact that $D_i \cap B_i = \emptyset$. Hence the theorem follows.

An alternate definition of dimension can now be given that assigns different values of dimension to H^N and E^N. This definition assumes that dimension zero has been defined.

Definition 2.53 If M is a non-empty space then dim M = n, where n is the least cardinal such that M is the union of n+1 subspaces each of which have dimension zero.

Theorem 2.19 and corollary 2.22 implies that for finite dimension this definition and definition 2.1 are equivalent. This definition, however, assigns different values for E^N and H^N. Using this definition vector dimension and topological dimension behave in somewhat similar manners, even for the infinite cases.

CHAPTER III

HEREDITARILY INDECOMPOSABLE CONTINUA IN HIGHER DIMENSION

Definitions and Characterizations

Hereditarily indecomposable continua that separate Euclidean n-spaces provide examples of sets that have some of the properties of n-spheres but fail to behave in other "nice" ways. This section will provide the definition and characterizations of hereditarily indecomposable continua.

Definition 3.1 If M is a continuum then M is indecomposable if and only if M is not the union of two proper subcontinua. If M is not indecomposable then it is said to be decomposable. If M has the property that every subcontinuum is indecomposable then M is hereditarily indecomposable.

Definition 3.2 If M is a continuum and $m \in M$ then the composant of m in M, C_m, is the union of all proper subcontinua of M that contain M.

Theorem 3.3 If M is a continuum and C_p is the composant of p in M then $\overline{C_p} = M$.

Proof: If $C_p = M$ then the theorem is true since M is closed. Otherwise let $x \in M-C_p$. Let R be the open sphere about x with radius 1/n. Suppose that x is not a limit point of C_p. Then for some k and for all $n \geq k$, $R_n \cap C_p = \emptyset$.

For each $n \geq k$ let M_n be the component of $\overline{M-R_{n+1}}$ that contains p. If $q \in C_p$ then there exists a proper subcontinuum Q of M containing p and q. Since $Q \subset C_p \subset M-\overline{R_{n+1}}$ it follows that $Q \subset M_n$ which implies that $C_p \subset M_n \subset \overline{M_n}$. Note that $\overline{M_n}$ is a continuum and that $\overline{M_n} \cap \overline{R_{n+1}} = \emptyset$. Hence $\overline{M_n}$ is a proper subcontinuum of M containing p and $\overline{M_n} \subset C_p$. Hence $C_p = M_n = \overline{M_n}$.

But by theorem 1.15 there exists a point $r \in M_n \cap \text{bd}\,(M-\overline{R_{n+1}}) = \text{bd}\,\overline{R_{n+1}}$. Therefore $r \in C_p \cap \overline{R_{n+1}} = \emptyset$, which is a contradiction. Hence x is a limit point of C_p and $\overline{C_p} = M$.

Theorem 3.4 If M is an indecomposable continuum then its composants are pairwise disjoint.

Proof: Let C_p and C_q be composants of an indecomposable continuum M and suppose that $x \in C_p \cap C_q$. Then there exist proper subcontinua M_p and M_q of M such that $x,p \in M_p$ and $x,q \in M_q$. Since M is indecomposable $M_p \cup M_q \neq M$. Therefore $M_p \cup M_q$ is a proper subcontinuum of M containing p and q.

Suppose $y \in C_p$. Then there exists a proper subcontinuum M_y of M containing y and p. Since $(M_p \cup M_q) \cup M_y$ is a proper subcontinuum of M containing q and y it follows that $y \in C_q$ and that $C_p \subset C_q$.

Similarly $C_q \subset C_p$. Therefore $C_p = C_q$ and the theorem follows.

In order to prove the first characterization of hereditarily indecomposable continua it is necessary to prove the following lemma. It might be noted that this lemma is generalized by P. M. Swingle in his discussion of "Generalized Indecomposable Continua" (26).

Lemma 3.5 If M is a continuum that is not the union of three subcontinua none of which is contained in the union of the other two then either M is indecomposable or the union of two indecomposable continua.

Proof: Let M be such a continuum and suppose that M is not indecomposable. Then $M = H \cup K$ where H and K are proper subcontinua of M. Suppose $H' = \overline{H-K} = N_1 \cup N_2$ sep. If N_1 and N_2 are connected then $M = K \cup N_1 \cup N_2$ which contradicts the hypothesis since $K-H' \neq \emptyset$. If $N_2 = N_3 \cup N_4$ sep. then $H' = N_1 \cup N_3 \cup N_4$ sep.

Suppose that $K \cup N_1 = A \cup B$ sep. Since K is connected either $K \subset A$ or $K \subset B$. Without loss of generality let $K \subset A$ and thus $B \subset N_1$. Then $M = (A \cup N_3 \cup N_4) \cup B$ sep. which is a contradiction since B is separated from both N_3 and N_4. Therefore $K \cup N_1$ is connected. Similarly, $K \cup N_3$ and $K \cup N_4$ are connected. Hence $M = (K \cup N_1) \cup (K \cup N_3) \cup (K \cup N_4)$ which is a contradiction of the hypothesis since N_1, N_3 and N_4 are mutually exclussive. Therefore H' is connected and thus H' is a continuum.

Now suppose that H' is decomposable. Then $H' = H_1 \cup H_2$, where H_1 and H_2 are proper subcontinua of H'. Therefore neither H_1 nor H_2 contain H'K. Hence $M = H_1 \cup H_2 \cup K$ which is a contradiction. Therefore H' is indecomposable.

In a similar manner $K' = \overline{K-H}$ is an indecomposable continuum.

Now suppose that $P = M-(H' \cup K') \neq \emptyset$. Then if $\overline{P}$ is connected $M = H' \cup K' \cup \overline{P}$ which contradicts the hypothesis. Therefore $\overline{P} = P_1 \cup P_2$ sep.

It should be noted that either H' or K' intersect both P_1 and P_2 for otherwise M would fail to be connected. This fact can be verified by considering all other possible cases. Without loss of generality suppose that both P_1 and P_2 intersect H'. Then in a manner similar to that used above to show that $K \cup N_1$ is connected it follows that $H' \cup P_1$ and $H' \cup P_2$ are connected. Hence $M = K' \cup (H' \cup P_1) \cup (H' \cup P_2)$ which contradicts the hypothesis. Hence $P = \emptyset$ and $H' \cup K' = M$ and the lemma is proven.

The following characterization of hereditarily indecomposable continua is due to William R. Zame (33).

Theorem 3.6 If M is a continuum then M is hereditarily indecomposable if and only if each pair of subcontinua, H and K, of M has the property that H-K is connected.

Proof: Suppose M has two subcontinua H and K that fail to have the desired property. Then $H-K = A \cup B$ sep. Then $A \cup K$ and $B \cup K$ are connected by a process used in lemma 3.5. Therefore $H = (A \cup K) \cup (B \cup K)$ is decomposable and M is not hereditarily indecomposable.

Conversely, suppose that M is not hereditarily indecomposable. Then there is a continuum $N \subset M$ that is decomposable. Therefore $N = H \cup K$ where H and K are proper subcontinua of N. To show the existence of a pair of continua, one of which separates the other, several casses need to be considered.

Case I: The point set $H \cap K$ is connected. Then $N-(H \cap K) = (N-H) \cap (N-K)$ sep. which implies that N and $H \cap K$ is the desired pair.

If $H \cap K$ is not connected there exist two components C and D of $H \cap K$. By the reference (24, p. 15) there exists a continuum $H' \subset H$ that intersects both C and D so that no subcontinuum of H' intersects both C and D. Note that H' is not a subset of $H \cap K$. For the rest of the discussion let $c \in C \cap H'$ and $d \in D \cap H'$.

Case II: The point set $H \cap K$ is not connected and $H' = N_1 \cup N_2 \cup N_3$ where each N_i is a subcontinuum of H' and no N_i is a subset of the union of the other two.

If $c \in N_i$ and $d \in N_j$ then $i \neq j$ since no proper subcontinuum of H' intersects both C and D. Without loss of generality, suppose $c \in N_1$ and $d \in N_2$. If $N_1 \cap N_2 \neq \emptyset$ then $N_1 \cup N_2$ would be a proper subcontinuum of H' intersecting both C and D. Hence $N_1 \cap N_2 = \emptyset$ and $H'-N_3 = (N_1-N_3) \cup (N_2-N_3)$ sep. and H' and N_3 is the desired pair.

If H' is not the union of three continua, no one of which is contained in the union of the other two lemma 3.5 implies H' is either indecomposable or the union of two indecomposable continua.

Case III: The point set $H \cap K$ is not connected and H' is indecomposable. Consider C_c and C_d to be the composants of H' containing c and d. Since no proper subcontinuum of H' contains both c and d theorem 3.4 implies that $C_c \cap C_d = \emptyset$. Also since $\bar{C_c} = H'$ by theorem 3.3 C_c is not a subset of K and there is an $x \in C_c-K$. Similarly there exists $y \in C_d-K$. Therefore there are proper subcontinua V and W of H' so that $x,c \in V \subset C_c$ and $y,d \in W \subset C_d$. Since $V \cap W = \emptyset$ and $K \cup V \cup W$ is a continuum, $K \cup V \cup W$ and K is the desired pair as K separates $K \cup V \cup W$.

Case IV: The point set $H \cap K$ is not connected and H' is the union of T and T' where T and T' are indecomposable proper subcontinua of H', and both $T-T'$ and $T'-T$ are connected. Without loss of generality let $T \cap C \neq \emptyset$. Then $T \cap D = \emptyset$ for otherwise T would not be a proper subcontinuum of H'. Hence $(H' \cap D) \subset T'$. Similarly $T' \cap C = \emptyset$ and $(H' \cap C) \subset T$.

If $T \subset K \cup T'$ then $T-T' \subset K$ which implies that $T-T' \subset C$ since $T-T'$ is connected which implies that $\overline{T-T'} \subset C$ since C is closed. But $T' \cap \overline{T-T'} \neq \emptyset$ and hence $T' \cap C \neq \emptyset$ which is a contradiction. Therefore $T-(K \cup T') \neq \emptyset$. Similarly $T'-(K \cup T) \neq \emptyset$.

Now let C_c and C_d to be composants of T and T' respectively containing c and d. Then there exist $x \in C_c \cap (T-(K \cup T'))$ and $y \in C_d \cap (T'-(K \cup T))$ since every open set of a continuum intersects every composant of the continuum. Also there exist proper subcontinua $V \subset T$ and $W \subset T'$ such that $x,c \in V$ and $y,c \in W$. Therefore, as in case III, the desired pair is $K \cup V \cup W$ and K.

Case V: In case IV either T-T' or T'-T fails to be connected, in which case T and T' is the desired pair.

As all possibilities have been exausted the theorem is proven.

The next characterization of a hereditarily indecomposable continuum is stated in terms of the following defined property Q. The definition of property Q is motivated by a theorem by John Jobe. (15)

Definition 3.7 Let S be a separable metric space and M a continuum in S. Then M has property Q in S if and only if for every continuum N in S such that $N \cap M \neq \emptyset$, $N-M \neq \emptyset$ and $M-N \neq \emptyset$, then there exists a point $p \in M \cap N$ such that p is a limit point of both M-N and N-M. A continuum M is S has property Q hereditarily in S if and only if each subcontinuum of M has property Q in S.

Theorem 3.8 Let T be a separable metric space and M a continuum in T. Then M is hereditarily indecomposable if and only if for every function f and separable metric space S such that f imbeds M in S, then f(M) has property Q hereditarily in S.

Proof: Assume the condition of the theorem. Let M be a continuum in a separable metric space S and suppose that M is not hereditarily indecomposable. This implies the existence of a decomposable subcontinuum M' of M. The definition of decomposable implies that $M' = H \cup K$ where H and K are proper subcontinua of M'. Thus $M'-K = H-K$ is a non-empty point set. Choose a point h in H-K. Note that since S is a separable metric space then $S \times S$ is also a separable metric space. Define $f: M' \to S \times S$ such that $f(m) = (m,h)$ for each $m \in M'$. Also define $g: M' \to S \times S$ such that $g(m) = (h,m)$ for each $m \in M'$. Then both f and g imbed M' in $S \times S$ and hence both $f(M') = M' \times \{h\} = M_1$ and $g(M') = \{h\} \times M' = M_2$ are homeomorphic to M' and therefore are continua in $S \times S$. Let $H_1 = f(H)$ and $K_1 = f(K)$. Then $M_1 = f(M') = f(H) \cup f(K) = H_1 \cup K_1 \subset f(M)$ is decomposable with $(h,h) \in H_1 - K_1$ since $h \in H-K$. Also the definitions of f and g imply that $M_1 \cap M_2 = \{(h,h)\} = H_1 \cap M_2$ since $H \subset M'$. Since $h \notin K$ then $K_1 \cap M_2 = \emptyset$ and since $(h,h) \in H_1 \cap M_2$ then $H_1 \cup M_2$ is a continuum in $S \times S$.

The point set $M_1-(H_1 \cup M_2) = (M_1-H_1)-M_2 = (K_1-H_1)-M_2 = K_1-H_1$ since $M_1 = H_1 \cup K_1$ and $M_2 \cap K_1 = \emptyset$. It is noted that $M_1-(H_1 \cup M_2) \neq \emptyset$ since $K_1-H_1 \neq \emptyset$ and that $\overline{M_1-(H_1 \cup M_2)} = \overline{K_1-H_1} \subset K_1$ since K_1 is a closed set.

The point set $(H_1 \cup M_2)-M_1 = (H_1-M_1) \cup (M_2-M_1) = M_2-\{(h,h)\}$ since $H_1 \subset M_1$ and $M_2 \cap M_1 = \{(h,h)\}$. It is noted that $(H_1 \cup M_2)-M_1 \neq \emptyset$ since M_2 is non-degenerate and that $\overline{(H_1 \cup M_2)-M_1} = \overline{M_2-\{(h,h)\}} = M_2$ since (h,h) is a limit point of M_2 and M_2 is closed.

Since $M_2 \cap K_1 = \emptyset$, it follows that $\overline{M_1-(H_1 \cup M_2)} \cap \overline{(H_1 \cup M_2)-M_1} \subset K_1 \cap M_2 = \emptyset$. Therefore, by the definition of property Q in $S \times S$, $f(M') = M_1$ does not have property Q in $S \times S$ and thus, $f(M)$ does not have property Q hereditarily in $S \times S$. This is a contradiction of the condition and thus the sufficiency part of the theorem is proved.

Conversely, suppose M is a hereditarily indecomposable continuum in the separable metric space S and let $N \subset S$ be a continuum such that $N \cap M \neq \emptyset$, $N-M \neq \emptyset$ and $M-N \neq \emptyset$.

Suppose that $\overline{M-N} \cap \overline{N-M} = \emptyset$. Let T be a component of M-N. Then theorem 1.15 implies there exists $p \in \text{bd}\ (M-N) \cap \overline{T}$. (Note: Boundaries are relative to $M \cup N$.) Since $T \subset M$ and M is closed it follows that $\overline{T} \subset M$. Since M-N is open relative to $M \cup N$ and $p \in \text{bd}\ (M-N)$ it follows that $p \notin M-N$. Hence $p \in N$.

Note that $p \in \overline{N-M}$ since $\overline{N-M} \cap \overline{M-N} = \emptyset$. Therefore $N-(\overline{N-M}) \subset M \cap N$ is a domain relative to N that contains p. Let L be the component of $N-(\overline{N-M})$ that contains p. Then theorem 1.15 implies that there exists $q \in \text{bd}\ (N-M) \cap L$. Note that $q \notin T$ for if it were then $q \in \overline{M-N} \cap \overline{N-M}$ since $T \subset \overline{M-N}$ and this would be a contradiction. Also note that $L \subset M$ and hence $\overline{L} \subset M$ since M is closed.

Therefore both $\overline{T}$ and $\overline{L}$ are subcontinua of M containing p. Hence $\overline{T} \cup \overline{L}$ is a subcontinuum of M. Since $T \subset M-N$ and $\overline{L} \subset N$ it follows that $\overline{T}-\overline{L} \neq \emptyset$. Also $q \in \overline{L}-\overline{T} \neq \emptyset$. Therefore $\overline{T} \cup \overline{L}$ is a decomposable subcontinuum of M which contradicts M being hereditarily indecomposable. Hence $\overline{M-N} \cap \overline{N-M} \neq \emptyset$ and there exists a point p such that p is a limit point of both M-N and N-M. Hence M has property Q in S.

Now assume that M is a hereditarily indecomposable continuum in a separable metric space S. Let f be any function that imbedds M in a separable metric space S' and consider f(M). Since f is a homeomorphism then f(M) is a hereditarily indecomposable continuum in S'. Let M' be any subcontinuum of f(M). Then the above paragraph implies that M' has property Q in S' and hence f(M) has property Q hereditarily in S' and the theorem is proven.

It was thought that in theorem 3.8 the condition "for every function f and space S such that f imbeds M in S, f(M) has property Q hereditarily in S" could be replaced by the condition "M has property Q hereditarily in T." To see that this cannot be done the following example exhibits a space T and a decomposable continuum M in T such that M has property Q hereditarily in T. Thus, this example compliments the statement of theorem 3.8.

<u>Example 3.9</u> Let S_1 and S_2 be two pseudo-arcs in the plane constructed from (-1,0) to (0,0) and (0,0) to (1,0) respectively such that $S_1 \quad S_2 = \{(0,0)\}$. Let p = (0,0). Let T be the subspace of the plane such that $T = S_1 \cup S_2$. Let H and K be non-degenerate proper subcontinua of S_1 and S_2 respectively such that $H \cap K = \{p\}$. Then $M = H \cup K$ is a decomposable compact continuum that has property Q hereditarily in T.

<u>Verification</u> Let N be any subcontinuum of T such that $N \cap M \neq \emptyset$, $N-M \neq \emptyset$, and $M-N \neq \emptyset$. If $N \subset S_1$ or $N \subset S_2$ then the reference (15) implies that N is a hereditarily indecomposable continuum and thus theorem 3.8 implies that N has property Q in T. Therefore, there exists a point $p \in M \cap N$ such that p is a limit point of both M-N and N-M.

Now consider $N \cap (S - \{p\}) \neq \emptyset$ and $N \cap (S - \{p\}) \neq \emptyset$. Let $H_1 = S_1 \cap N$ and $K_1 = S_2 \cap N$ and thus $N = H_1 \cup K_1$. If it is noted that $N-\{p\} = (H_1-\{p\}) \cup (K_1-\{p\})$ sep. then reference (16) implies that H_1 and K_1 are non-degenerate subcontinua of S_1 and S_2 respectively. The reference (15) implies that both H_1 and K_1 are pseudo-arcs. The definition of N and the fact that the pseudo-arc is hereditarily indecomposable imply that either (a) $H \subset H_1$, $K_1 \subset K$, $H_1 - H \neq \emptyset$, and $K - K_1 \neq \emptyset$ or (b) $H_1 \subset H$, $K \subset K_1$, $H - H_1 \neq \emptyset$, and $K_1 - K \neq \emptyset$. The consideration of each possibility is similar so without loss of generality case (a) is considered.

First note that reference (23) implies that H and K_1 are indecomposable subcontinua of H_1 and K respectively. The point p is a limit point of $H_1 - H \subset N - M$ and $K - K_1 \subset M - N$ since indecomposable subcontinua are continua of condensation. Since $p \in M \cap N$ it has been verified that M has property Q in T.

Now if M' is a subcontinuum of M, then either (a) $M' \subset S_1$ or (b) $M' \subset S_2$ or (c) $M' - S_1 \neq \emptyset$ and $M' - S_2 \neq \emptyset$. In cases (a) and (b) M' is hereditarily indecomposable and hence has property Q in T. In case (c) M' has property Q in T by the method used to show that M has property Q in T. Therefore, M has property Q in T hereditarily.

Since M is decomposable the example is verified.

In 1942 J. L. Kelley published a paper on "The Hyperspaces of a Continuum" (19) in which he proved that if a hereditarily indecomposable continuum of dimension greater than one existed that an infinite dimensional hereditarily indecomposable continuum also existed. In this paper Kelly also gave a characterization of indecomposable and hereditarily indecomposable continua. These characterizations will be stated here without proof since their value to this paper is historic rather than mathematical and since their proofs involve ideas not in the mainstream of this paper.

Definition 3.10 If M is a continuum then C(M) is the collection of all subcontinua of M with the topology generated by the metric defined as follows: If $H,K \in C(M)$ then $d'(H,K) = \sup\{x : x = d(H,k), k \in K \text{ or } x = d(h,K), h \in H\}$.

Theorem 3.11 If M is a continuum then in order that M be indecomposable it is necessary and sufficient that $C(M) - \{M\}$ fail to be arcwise connected.

Theorem 3.12 If M is a continuum then M is hereditarily indecomposable if and only if C(M) contains a unique arc between every pair of its elements.

Existence

This section of this chapter will be concerned with the existence of hereditarily indecomposable continua of all dimensions, including the infinite dimension. These continua will be the intersection of increasingly "crooked" domains of E^n under the following definitions of crooked. Much of the material of this chapter is based on a paper, "Higher Dimensional Hereditarily Indecomposable Continua", by R. H. Bing (3).

Definition 3.13 An arc xy is ϵ-crooked if for each pair of points a and b in xy there exist points c and d in the subarc ab such that c is between a and d, $d(a,d) < \epsilon$ and $d(b,c) < \epsilon$.

Definition 3.14 A domain D is ϵ-crooked if every arc contained in D is ϵ-crooked.

Definition 3.15 If hk is an arc with endpoints h and k and H and K are two sets containing h and k respectively then hk is ϵ-crooked with respect to H and K if and only if there exist points r and s in hk such that r is between h and s, $d(s,H) < \epsilon$ and $d(r,K) < \epsilon$.

Definition 3.16 If D is a domain and H and K are two sets then D is ϵ-crooked with respect to H and K if and only if every arc in D with endpoints in H and K respectively is ϵ-crooked with respect to H and K.

These definitions can now be used to show the existence of ϵ-crooked domains that are contained in arbitrary bounded domains.

Theorem 3.17 If D is a bounded domain in E^n (or I^ω) that separates the point b from the point c, $\epsilon > 0$, and H and K are two non-empty sets in E^n (or I^ω) then there exists a domain G such that G is ϵ-crooked with respect to H and K and $\overline{G} \subset D$ and G separates b from c.

Proof: Let $K' = \{x: x \in K \text{ and } d(x,H) \geq \epsilon\}$ and $E^n - D = B \cup C$ sep. where $b \in B$ and $c \in C$. Also let $\delta = d(B,C)$. Note that $\delta > 0$ since D is bounded and either B or C is compact. (11, p. 91)

Then if $K' = \emptyset$ let $E = \{x: \delta/4 < d(x,B) < 3\delta/4\}$. Then $\overline{G} \cap (B \cup C) = \emptyset$ and hence $\overline{G} \subset D$. Also $b \in B \subset \{x: d(x,B) < \delta/4\}$ and $c \in C \subset \{x: d(x,B) > 3\delta/4\}$ and thus G separates b from c. Also if M is an arc in G from a point h in H to a point k in K then an r can be choosen so that $d(r,k) < \epsilon$ and an s can be choosen so that s is between r and k and $d(s,k) < \epsilon - d(k,H)$ since $d(k,H) < \epsilon$. Therefore $d(s,H) \leq d(s,k) + d(k,H) < \epsilon$ and G is ϵ-crooked with respect to H and K.

Now suppose that $K' \neq \emptyset$ and let $V = \bigcup_1^5 V_i$ where

$$V_1 = \{x: d(x,B) = 3\delta/4,\ d(x,K') \geq \epsilon/2\},$$

$$V_2 = \{x: \delta/2 \leq d(x,B) \leq 3\delta/4,\ d(x,K') = \epsilon/2\},$$

$$V_3 = \{x: d(x,B) = \delta/2,\ d(x,H \cup K') \geq \epsilon/2\},$$

$$V_4 = \{x: \delta/4 \leq d(x,B) \leq \delta/2,\ d(x,H) = \epsilon/2\},$$

and $V_5 = \{x: d(x,B) = \delta/4,\ d(x,H) \geq \epsilon/2\}$.

Consider $F = \{x: d(x,B) \leq \delta/4\} \cup \{x: d(x,B) \leq 3\delta/4,\ d(x,H) \leq \epsilon/2\} \cup \{x: \delta/2 \leq d(x,B) \leq 3\delta/4,\ d(x,K') \geq \epsilon/2 = B_1 \cup B_2 \cup B_3$. Note that F is closed since it is the union of closed sets. Also note that $B \subset \text{int } B_1 \subset \text{int } F$ and that $F \cap C = \emptyset$ since $d(F,C) \geq \delta/4$. Therefore bd F separates B from C.

Now let $x \in \text{bd } B_1 \cup \text{bd } B_2 \cup \text{bd } B_3$. If $x \in \text{bd } B_1$ then $d(x,B) = \delta/4$. Thus if $d(x,H) \geq \epsilon/2$ then $x \in V_5 \subset V$. If $d(x,H) < \epsilon/2$ then $x \in \text{int } B_2 \subset \text{int } F$. Similarly if $x \in \text{bd } B_2$ then $d(x,H) = \epsilon/2$. Then $x \in V_4 \subset V$ or $x \in \text{int } (B_1 \cup B_3) \subset \text{int } F$. If $x \in \text{bd } B_3$ then $x \in V_1 \cup V_2 \cup V_3 \subset V$ or $x \in \text{int } B_2 \subset \text{int } F$. In all cases $x \in V$ or $x \in \text{int } F$. Therefore, since $\text{bd } F \subset \text{bd } B_1 \cup \text{bd } B_2 \cup \text{bd } B_3$, $\text{bd } F \subset V$. Thus V separates B from C since bd F separates B from C and $d(V, B \cup C) = \delta/4 > 0$.

Now let $\gamma = \min\{\epsilon/4, \delta/16\}$ and let $G_i = \{x: d(x,V_i) < \gamma\}$. Then $G = \bigcup_1^5 G_i$ is an open set containing V. Note that if $x \in G_2$ then $d(x,K') \leq d(x,V_2) + d(V_2,K') \leq \epsilon/4 + \epsilon/2 < \epsilon$ and that similarly if $x \in G_4$ then $d(x,H) < \epsilon$. Also observe that $d(G_1 \cup G_2 \cup G_3, G_5) \geq \delta/8$ and hence G_4 separates $(G_1 \cup G_2 \cup G_3)-G_4$ from G_5-G_4 in G and that in a similar manner G_2 separates G_1-G_2 from $(G_3 \cup G_4 \cup G_5)-G_2$ in G. Also note that if $x \in H \cap G$ then $x \in G_1$ and if $x \in K' \cap G$ then $x \in G_5$. Note too that $d(G, B \cup C) \geq 3\delta/16$ and hence $\bar{G} \cap (B \cup C) = \emptyset$ which implies that $\bar{G} \subset D$

and that G separates B from C since $V \subset G$.

To show that G is ϵ-crooked relative to H and K let hk be an arc in G such that $h \in H$ and $k \in K$. Then if $k \notin K'$ an r and s can be choosen as was done in the case when $K' = \emptyset$. If $k \in K'$ then $k \in G_5$ and $h \in G_1$. Since G_4 separates G_1 from $G_5 \cap K'$ in G and hk is a connected subset of G there exists an $s \in hk \cap G_4$. Similarly, since G_2 separates G_5 from $G_1 \cap H$ in G there exists an $r \in hk \cap G_2$. Since G_4 separates G_2 from $G_5 \cap K'$ in G it follows that s is between r and k in hk. Also $d(s,H) < \epsilon$ and $d(r,K) \leq d(r,K') < \epsilon$ and hence hk is ϵ-crooked with respect to H and K. Therefore G is ϵ-crooked with respect to H and K and the theorem is proven.

Before showing the existence of connected, ϵ-crooked domains that separate E^n (or I^ω) a discussion of a property of domains in E^n (and I^ω) is needed.

Definition 3.18 A connected space M is unicoherent if and only if $M = A \cup B$, where A and B are closed connected subsets of M, implies that $A \cap B$ is connected.

Whyburn (30, p. 225-228) shows that E^n is unicoherent for all n. It should be noted that the proof presented by Whyburn also holds for I^ω. The ideas used by Whyburn's proof include topics from homotopy theory and as a result the proof that E^n and I^ω are unicoherent will not be included in this paper. However the following fact about E^n (and I^ω) will be proven by assuming the unicoherence of E^n and I^ω.

Lemma 3.19 If D is a bounded domain in E^n or I^ω that separates points b and c then some component of D also separates b and c.

Proof: Let E^n-D = B $\cup$ C sep., b $\in$ B and c $\in$ C. Also let d(B,C) = $\delta > 0$, since either B or C is compact. Let B' be the component of {x: d(x,B) $\leq \delta/4$} that contains b and let C' be the component of {x: d(x,C) $\leq \delta/4$} that contains c. Then since both B' and C' are closed D' = E^n-(B' $\cup$ C') is a domain. It should be noted that if A is a component of D' then A is itself a domain (24, p. 86) and that A has limit points in either B' or C', for otherwise E^n = A $\cup$ ((D'-A) $\cup$ B' $\cup$ C') sep. Let H be the union of B' with all of the components of D' that have limit points in B' but not in C'. Then H is a closed and connected set. Let K be the union of C' with all of the components of D' that have limit points in C'. Then K is connected, H $\cup$ K = E^n and H $\cap$ K = $\emptyset$.

Since E^n is unicoherent H $\cap$ $\overline{K}$ = bd H is connected. Also since b $\in$ int B' $\subset$ int H and c $\in$ C' $\subset$ K it follows that bd H separates b from c. Furthermore, bd H $\subset$ bd B' $\subset$ D which implies that some component D'' of D contains bd H. Then D'' separates b from c and the lemma is true.

The following theorem of ϵ-crooked domains can now be proven.

Theorem 3.20 If D is a bounded domain in E^n or I^ω that separates the point b from the point c and if $\epsilon > 0$ then there exists a connected ϵ-crooked domain G such that $\overline{G} \subset$ D and G separates b from c.

Proof: Since D is bounded, $\overline{D}$ is bounded and hence compact. therefore there exists a finite open covering $\mathfrak{B}$ of D = D_o of spheres of diameter less that $\epsilon/2$. Let $\{(g_1,g_1'),(g_2,g_2'),\ldots,(g_n,g_n')\}$ be the finite collection of all pairs of elements of $\mathfrak{B}$.

Now suppose D_{i-1}, $1 \leq i \leq n$, has been defined so as to meet the hypothesis of theorem 3.17. Then theorem 3.17 implies that there exists a domain D_i such that D_i is $\epsilon/2$-crooked with respect to g_i and g_i', $\overline{D_i} \subset D_{i-1}$ and D_i separates b from c.

Let A be an arc in D_n and let $x,y \in A$. Then there is a pair g_i, g_i' such that, without loss of generality, $x \in g_i$ and $y \in g_i'$. Since the arc $xy \subset A$ is contained in D_i by the inductive definition of D_n there exists points r and s in arc xy such that r is between x and s and $d(s,g\ ) < \epsilon/2$ and $d(r,g') < \epsilon/2$. Since the diameters of g_i and g_i' are less than $\epsilon/2$, the triangle inequality implies that $d(s,x) < \epsilon$ and $d(r,y) < \epsilon$. Therefore D_n is ϵ-crooked.

Also $\overline{D_n} \subset D$ and D_n separates b from c. Lemma 3.7 implies that some component G of D_n separates b from c. Since $\overline{G} \subset D$ and G is a domain theorem 3.20 follows.

From the existence of ϵ-crooked domains as given in theorem 3.20 it is now possible to show the existence of hereditarily indecomposable continua of all dimensions, including infinite dimensional hereditarily indecomposable continua.

<u>Theorem 3.21</u> If H and K are mutually exclusive continua in E^n (in I^ω) then there exists a hereditarily indecomposable continua of dimension n-1 (of infinite dimension) that separates H from K.

Proof: Let $h \in H$ and $k \in K$. Since H and K are bounded there exists an open sphere D such that $H \cup K \subset D$. Let $S_0 = D-(H \cup K)$. Then S_0 is a bounded domain that separates H from K and hence h from k.

Suppose S_i, $0 \leq i < j$, has been defined so that $\overline{S_i} \subset S_{i-1}$, S_i is $1/i$-crooked, $i \geq 1$, and each S_i is a connected bounded domain, $i \geq 1$, and each $S\ $ separates h from k.

Then theorem 3.20 implies that there exists a $1/j$-crooked domain

S_j that separates h from k so that $\overline{S_j} \subset S_{j-1}$.

Let $C = \bigcap_1^\infty \overline{S_i}$. Since C is the intersection of a nest of continua theorem 1.14 implies that C is a continuum.

Now suppose there is a continuum $M \subset C$ such that M is decomposable. Then $M = A \cup B$ where A and B are proper subcontinua of M. Let $p \in B-A$ and $q \in A-B$ and choose n so that $d(p,A) > 2/n$ and $d(q,B) > 2/n$. Let O_A be the union of all open spheres that are subsets of S_n, have centers in A and have radii less than 1/n. Similarly, let O_B be the union of all open spheres that are subsets of S_n, have centers in B and have radii less than 1/n. Then O_A and O_B are connected open sets containing A and B respectively and $d(q,O_A) > 1/n$ and $d(p,O_B) > 1/n$. Let $x \in A \cap B \neq \emptyset$. Then $x \in O_A \cap O_B$ which is a connected domain. Since a connected domain is arcwise connected and since no arc separates a connected domain let px be an arc in O_B and xq be an arc in $(O_A - px) \cup \{x\}$. Then $px \cup xq = pxq$ is an arc in $O_A \cup O_B$. Since $pxq \subset S_n$, which is 1/n-crooked, there exist points r and s such that r is between p and s in pxq and $d(p,s) < 1/n$ and $d(r,q) < 1/n$. Since $d(p,s) < 1/n$ and $d(p,xq) \geq d(p,O_A) > 1/n$ it follows that $s \notin xq$. Therefore $s \in px$ and since r is between q and s, $r \in px$. Therefore $d(r,q) \geq d(px,q) \geq d(O_B,q) > 1/n$ which is a contradiction and hence there does not exist a decomposable subcontinuum of C. Therefore C is hereditarily indecomposable.

To show that C separates H from K suppose that h and k lie in the same component P of $E^n - C$ (or $I^\omega - C$). Then, since P is open and connected there exists an arc $hk \subset P$. For every $i \geq 1$, $\overline{S_i} \cap hk \neq \emptyset$ since S_i separates H from K. Therefore $C \cap P \subset C \cap hk = \bigcap_1^\infty (\overline{S_i} \cap hk) \neq \emptyset$ since the intersection of a nest of non-empty compact sets is non-empty. Since this is a contradiction h and k lie in distinct components of E^n-C (or I^ω-C). Therefore if P is the component of E^n-C (or I^ω-C) containing h then $k \notin P$ and $E^n - C = [(E^n - C) - P] \cup P$ sep. with $K \subset (E^n - C) - P$ and $H \subset P$. Hence C separates H from K.

If $C \subset E^n$ then dim $C \neq n$ for otherwise C would contain an open set of E^n by theorem 2.44 which would imply that C would contain a decomposable continuum. Also dim $C \geq n-1$ by theorem 2.47 since C separates E^n. Therefore dim $C = n-1$.

If $C \subset I^\omega$ let K^n, $n \geq 2$, be a sub-vector space of H^N, the Hilbert space, containing h and k. Then K^n is homeomorphic to E^n and $K^n \cap I^\omega$ is the closure of a connected open set in K^n. Since C separates $K^n \cap I^\omega$ corollary 2.49 implies that dim $C \geq n-2$ for all n. Therefore dim $C = \infty$.

Hence the existence of hereditarily indecomposable continua of all dimensions has been proven.

<u>Corollary 3.22</u> If $\{S_i\}$ is a nested sequence of domains such that for each i, S_i is 1/i-crooked then every continuum in $\bigcap_i S_i$ is hereditarily indecomposable.

Proof: This was included in the proof of theorem 3.21 by showing that C contained no decomposable continuum.

<u>Theorem 3.23</u> There exist bases for E^n and I^ω such that each element in the bases has as its boundary a hereditarily indecomposable continuum.

Proof: Let $p \in E^n$, let $\epsilon > 0$, and let $B = \{x: d(x,p) < \epsilon\}$. Then the domain B-{p} separates p from bd B. Hence theorem 3.20 implies the existence of a hereditarily indecomposable continuum $H \subset B$-{p} that separated p from bd B. Let C be the component of E^n-H that contains p. Let D be the union of all components of E^n-H that do not contain p. Then $\overline{C}$ and $D \cup H$ are closed and connected sets and $E^n = \overline{C} \cup (D \cup H)$. Therefore the unicoherence of E^n implies that $\overline{C} \cap (D \cup H) = K$ is connected. Since K is closed and $K \subset H$ it follows that K is hereditarily indecomposable. Also K = bd C $\subset$ B. Thus for every point in E^n there exists arbitrarily small open sets that have as their boundaries hereditarily indecomposable continua and the theorem holds for E^n. This proof also holds for I^ω.

Separating Properties

As was noted in the previous section, in E^n there exist, for every pair of disjoint continua H and K in E^n, a hereditarily indecomposable continua separating H and K. This can be generalized to the point that if A and B are disjoint compact sets in E^n, or in any space, then there exists a closed set, each of whose components are hereditarily indecomposable, that separates A from B in E^n. To do this the following lemma will be useful.

Lemma 3.24 If $A_1, A_2, \ldots, A_n$ is a finite sequence of pairwise disjoint compact subsets of the separable metric space S then the space S' whose points are $A_1, A_2, \ldots, A_n$ and those points in $S-(A_1 \cup \ldots \cup A_n)$ is a separable metric space under the metric for S' that is such that the distance between x and y in S' is their distance in S as subsets of S.

Proof: Since the sets $A_1, A_2, \ldots, A_n$ are pairwise disjoint and compact, if $x, y \in S'$ then $d(x,y) > 0$ if $x \neq y$ and hence S' is a metric space. If K is a countable dense set in S then $K-(A_1 \cup \ldots \cup A_n)$ is a countably dense set in S' and S' is separable. Therefore the lemma holds.

<u>Theorem 3.25</u> If A and B are disjoint compact subsets of a separable metric space S then there exists a closed set H in S that separates A from B in S such that each component of H is hereditarily indecomposable.

Proof: Since, by lemma 3.24, the space S' determined by considering A and B as points is separable and metric then theorem 1.16 implies there exists a function $g:S' \to I^{\omega}$ that imbeds S' in I^{ω}. Also since $h:S \to S'$, where $h(m) = m$ if $m \in S-(A \cup B)$, $h(m) = A$ if $m \in A$ and $h(m) = B$ if $m \in B$, is an open continuous function (18, p. 94) it follows that $f = g\ h$ is an open continuous function from S to $f(S) \subset I^{\omega}$ that maps A and B to the single points a and b in I^{ω} and that f is one-to-one on $S-(A \cup B)$.

Theorem 3.21 implies there exists a hereditarily indecomposable continuum H' in I^{ω} that separates a from b in I^{ω}. Let $I^{\omega}-H' = P' \cup Q'$ sep., where $a \in P'$ and $b \in Q'$. Then let $H = f^{-1}(H')$, $P = f^{-1}(P')$ and $Q = f^{-1}(Q')$. Then H is closed since the inverse of closed sets under continuous functions are closed. Also if x is a limit point of P then f(x) is a limit point of P' which implies that $f(x) \notin Q'$ since P' and Q' are separated sets. Therefore $x \notin Q$ and no limit point of P is in Q. Similarly no limit point of Q is in P. Therefore P and Q are separated sets. Also since f is one-to-one on H it follows that S-H = P Q sep. Furthermore, if D is a component of H then f(D) is a subcontinuum of H' since $f:H \to f(H) \subset H'$ is a homeomorphism. Therefore f(D) is hereditarily indecomposable and hence D is hereditarily indecomposable. Hence the theorem follows.

Homogeneity

Definition 3.26 A space M is homogeneous if and only if for each pair of points x and y in M there exists a homeomorphism f from M into M such that $f(x) = y$.

Though the pseudo-arc is a homogeneous hereditarily indecomposable continuum no hereditarily indecomposable continuum of dimension greater than one is homogeneous. This will be shown by showing that every hereditarily indecomposable continuum of dimension n contains a subcontinuum of dimension n-1 and that every hereditarily indecomposable continuum of dimension n contains a point such that if a non-degenerate subcontinuum contains that point then that subcontinuum is also of dimension n. It should be noted, however, that this method will not apply for infinite dimensions as there exist infinite dimensional hereditarily indecomposable continua that contain no non-degenerate finite dimensional continua (32). This will be discussed in the next section.

Theorem 3.27 If M is an n dimensional continuum then M contains a subcontinuum of dimension k for all $k \leq n$.

Proof: The definition of dimension implies there exists a point $p \in M$ such that some domain D containing p has a boundary (relative to M) of dimension n-1.

Since bd D is compact theorem 2.50 implies that bd D contains a continuum H that has dimension n-1 at each of its points. Therefore H is a subcontinuum of M that has dimension $n-1$.

By repeated application of this proof there exists a subcontinuum of M of every dimension less than n and the theorem follows.

Theorem 3.28 If M is a hereditarily indecomposable continuum of finite dimension then there exists $p \in M$ such that if N is a non-degenerate subcontinuum of M containing p then $\dim M = \dim N$.

Proof: Let $M = M$ and suppose that for $i < k$, M_i has been defined so that for each i, $0 < i < k$, M_i is a subcontinuum of M_{i-1}, the diameter of each M_i is less than $1/i$ and $\dim M_i = \dim M$.

Since M_{k-1} is compact there exists a finite open covering $G_1, \ldots, G_n$ of M_{k-1} such that each element of $G_1, \ldots, G_n$ has diameter less than $1/k$. Then $M_{k-1} = \bigcup^{n}(M_{k-1} \cap G_i)$ is the union of a finite number of compact sets of diameter less than $1/k$. Thus theorem 2.18 implies that for some j, $M_{k-1} \cap G_j$ has the dimension of M_{k-1}. Since $M_{k-1} \cap G_j$ is compact theorem 2.50 implies that some continuum M_k in $M_{k-1} \cap G_j$ has the same dimension as M_{k-1}. Hence $\dim M_k = \dim M$ and the diameter of M_k is less than $1/k$ and $M_k \subset M_{k-1}$.

Thus a sequence $\{M_i\}$ of continua has been defined with the desired properties.

Since the sequence is nested and each element is compact there exists a point $p \in \bigcap_i^{\infty} M_i$. Suppose N is a non-degenerate subcontinuum of M containing p and let $q \in N$ such that $q \neq p$. Then for some n, $1/n < d(q,p)$. Since $p \in M_n$ and the diameter of M_n is less than $1/n$ it follows that $q \notin M$. Since $M_n \cap N \neq \emptyset$, either $M_n \subset N$ or $N \subset M_n$, for otherwise $M_n \cup N$ would be a decomposable subcontinuum of M. Since $q \in M_n - N$, $M_n \subset N$. Therefore theorem 2.12 implies that $\dim M = \dim M_n$

$\leq$ dim N $\leq$ dim M since $M_n \subset N \subset M$. Hence dim N = dim M and the theorem follows.

Theorem 3.29 No hereditarily indecomposable continuum of dimension greater than one is homogeneous.

Proof: Suppose that M is a homogeneous hereditarily indecomposable continuum and that the dim M = n > 1. Then theorem 3.27 implies that M contains a continuum H of dimension n-1. Note that dim H > 0 and hence H is nondegenerate. Also theorem 3.28 implies that there exists a point p ϵ M such that if N is a subcontinuum of M containing p then dim N = n. Let q ϵ H. Then M being homogeneous implies there exists a homeomorphism f:M $\rightarrow$ M such that f(q) = p. Therefore f(H) is a subcontinuum of M containing p and dim f(H) = n since f is a homeomorphism. But theorem 2.11 implies that dim f(H) = dim H = n-1 which is a contradiction. Therefore no hereditarily indecomposable continuum of dimension greater than one is homogeneous.

HID Continua

In 1926 L. A. Tumarkin (27) asked if there might be a non-degenerate continuum such that each of its subcontinua were infinite or zero dimensional. Van Heemert, in 1946, claimed to prove that all infinite dimensional continua contained one dimensional continua (28). This proof was in error as D. W. Henderson demonstrated, in 1965, the existence of a continuum each of whose subcontinua were either infinite dimensional or degenerate (12). Such a continuum is a HID continuum as defined below. It is the purpose of this section to present some of the properties of such continua. Most of the properties are a result of J. M. Yohe (32).

Definition 3.30 If M is an infinite dimensional compact set such that M contains no n dimensional compact sets for $0 < n < \infty$ then M is said to be hereditarily infinite dimensional (HID).

Theorem 3.31 Let M be a HID continuum and let $p \in M$. Then dim M at $p = 1$ or dim M at $p = \infty$.

Proof: Since M is connected and nondegenerate the definition of dimension implies that dim M at $p > 0$, for otherwise M could be separated by the empty set.

Suppose dim M at $p = n$ where $1 < n < \infty$. Then there exists a neighborhood U (in M) of p such that dim (bd U) $= n-1 > 0$. Since bd U is a compact subset of M this contradicts M being HID. Thus dim M at $p = 1$ or dim M at $p = \infty$.

Theorem 3.32 Every HID continuum in I^{ω} contains uncountably many mutually exclussive hereditarily indecomposable HID continua.

Proof: By theorem 3.23, I^{ω} has a basis consisting of neighborhoods whose boundaries are hereditarily indecomposable continua.

Let M be a HID continuum in I^{ω}. Since dim $M = \infty$ there exists $p \in M$ such that dim M at $p \neq 1$. Theorem 3.31 implies that dim M at p is ∞. Let $\epsilon > 0$ be choosen so that for each α, $0 < \alpha < \epsilon$, the set $S_{\alpha} = \{x \in M: d(x,p) < \alpha\}$ has a boundary in M of dimension greater than 0. This can be done by the definition of dimension. Therefore dim (bd S in M) $= \infty$, for otherwise M would contain a compact set of positive dimension. Since not every component of bd S_{α} in M is a singleton, for this would imply that dim (bd S_{α} in M) $= 0$ (10, p. 22), let N be a non-degenerate component of bd S_{α} in M. Then dim $N = \infty$ since dim N $\neq$ 0. Hence N is a HID continuum and there exists $q \in N$ such that dim N at $q = \infty$ and there exists a $\delta > 0$ such that if $V \subset \{x \in N: d(x,q) < \delta\}$ is an open set relative to N then bd V in N has infinite dimension. Let U be an open set in I^ω such that bd U is a hereditarily indecomposable continuum and $U \subset \{x \in I^\omega: d(x,q) < \delta\}$. Then dim $(N \cap \text{bd } U) = \infty$ and $N \cap \text{bd } U$ contains a component N_α of infinite dimension. Since $N_\alpha \subset \text{bd } U$, N is a hereditarily indecomposable HID continuum. Also since if $x \in N_\alpha$ then $d(x,p) = \alpha$, it follows that if $\alpha \neq \beta$ then $N_\alpha \cap N_\beta = \emptyset$. Therefore, the theorem follows.

<u>Definition 3.33</u> A compact n dimensional space, $n > 0$, is called an n-dimensional Cantor-manifold if it cannot be separated be a closed subset of dimension $< n-1$. A compact infinite dimensional space is an infinite dimensional Cantor-manifold if it cannot be separated by a closed subset of finite dimension.

Note that every Cantor-manifold is connected since it cannot be separated by the empty set. Also an n-dimensional Cantor-manifold has dimension n at each of its points and an infinite dimensional Cantor-manifold is infinite dimensional at each of its points.

L. A. Tumarkin (27) has shown that every HID space contains an infinite dimensional Cantor-manifold. Therefore each of the hereditarily HID continua in the space M of theorem 3.32 contains an infinite dimensional Cantor-manifold and the following corollary to theorem 3.32 is a result.

<u>Corollary 3.34</u> Every HID continuum contains an uncountable number of mutually exclussive hereditarily indecomposable HID Cantor-manifolds.

The next theorem divides HID continua into "manifold components" and gives some insight into the structure of HID continua.

Theorem 3.35 Let M be a HID continuum. Then $M = \cup M_p$ where each M_p is a maximal HID Cantor-manifold in M containing p.

Also if $p,q \in M$ then either $M_p = M_q$ or $\dim (M_p \cap M_q) \leq 0$.

Proof: Let $p \in M$ and $\mathcal{E}_p$ be the collection of all Cantor-manifolds containing p. Since $\{p\} \in \mathcal{E}_p$, $\mathcal{E}_p \neq \emptyset$.

Let $\{C_\alpha\}$ be a nest in $\mathcal{E}_p$ and let $C = \cup C_\alpha$. Then if $C = \{p\}$ then C is a 0-dimensional Cantor-manifold and an upper bound for $\{C_\alpha\}$.

Otherwise, if $x,y \in C$ then there exists C_α such that $x,y \in C_\alpha$. Since C_α is a non-degenerate subcontinuum of M it follows that $\dim C > 0$. Therefore $\dim C_\alpha = \infty$ and C_α is an infinite dimensional Cantor-manifold. Therefore x and y cannot be separated in C_α by a subset of finite dimension and hence x and y cannot be separated in C by a closed subset of finite dimension.

Hence no closed set of finite dimension separates C which implies that no closed set of finite dimension separates $\overline{C}$. Therefore $\overline{C}$ is a Cantor-manifold containing p and $\overline{C}$ is an upperbound for $\{C_\alpha\}$. Hence, by Zorn's lemma (18, p. 33), there exists a maximal Cantor-manifold M_p containing p.

Let $p,q \in M$ and suppose that $M_p \cap M_q \neq \emptyset$. Then if $M_p \subset M_q$ or $M_q \subset M_p$ it follows that $M_p = M_q$ since M_p and M_q are maximal. Otherwise $M_p \cup M_q$ is not a Cantor-manifold since M_p and M_q are maximal. Also neither M_p nor M_q is degenerate since $M_p \cap M_q \neq \emptyset$. Therefore $M_p \cup M_q$ can be separated by a closed subset Z of finite dimension. Since M is HID it follows that $\dim Z = 0$. Since Z separates neither M_p nor M_q it it follows that $(M_p \cup M_q)-Z = A \cup B$ sep. with $M_p-Z \subset A$ and $M_q-Z \subset B$. Therefore $M_p \cap M_q \subset Z$ and $\dim (M_p \cap M_q) \leq \dim Z = 0$. Hence $\dim (M_p \cap M_q)$ is zero and the theorem is proven.

The next three theorems will be stated without proof. The first is a result of the Baire Category Theorem (14, p. 160), the second is an indirect result of the first and the third is a result of function theory (32, p. 181).

Theorem 3.36 If M is a compact space and M is the countable union of HID compact spaces then M is itself a HID space.

Theorem 3.37 There exist an uncountable number of topologically different HID continua.

Theorem 3.38 There exist hereditarily indecomposable HID continua that can be separated by sets of dimension zero.

The following questions were raised by J. M. Yohe (32) in his paper on hereditarily infinite dimensional spaces in 1969. To the author's knowledge these questions are yet to be answered.

1. Does there exist a homogeneous HID space?

2. Do there exist uncountable many topologically different hereditarily indecomposable HID Cantor-manifolds?

3. Do there exist uncountable many topologically different HID Cantor-manifolds?

4. If M is an HID continuum and N is an HID Cantor-manifold in M and if M is decomposed as in theorem 3.32, is it necessarily true that $N \subset M_p$ for some $p \in M$?

CHAPTER IV

CHAINABLE CONTINUA IN THE PLANE

Introduction

In chapter III the properties and existence of hereditarily indecomposable continua of all dimensions was discussed. This chapter will be a discussion of chainable hereditarily indecomposable continua that lie in the plane. It will not be the purpose of this chapter to prove in detail the properties of the continua that will be mentioned, but rather to give the reader an understanding of the scope of the literature that concerns itself with such continua. This chapter will be a review of the thesis of McKellips (21) along with an updating of information discovered since his paper was completed. The material in this chapter will be carefully referenced in order that the reader might examine in detail the concepts involved. In this chapter all sets will be considered to be in the plane unless otherwise stated.

There are two types of chainable hereditarily indecomposable continua in the plane, the pseudo-arc and the pseudo-circle. These can be combined in order that an uncountable number of topologically distinct hereditarily indecomposable continua can be found in the plane. The pseudo-arc and the pseudo-circle will be discussed with their properties stated and referenced. That there exist uncountably many topologically distinct hereditarily indecomposable continua in the plane will be proven in some detail.

The following definitions will be used for chainable.

Definition 4.1 A finite collection of domains $D = \{d_1, \ldots, d_n\}$ is called a linear chain if and only if $d_i \cap d_j \neq \emptyset$ if and only if $|i-j| \leq 1$, $i,j = 1,2,\ldots,n$. If p and q are points belonging to d_1 and d_n respectively then D is called a linear chain from p to q.

Definition 4.2 A finite collection of domains $D = \{d_1, \ldots, d_n\}$ is called a circular chain if and only if $d_i \cap d_j \neq \emptyset$ if and only if $|i-j| \leq 1$, $i,j = 1,2,\ldots,n$, except that $d_1 \cap d_n \neq \emptyset$.

Definition 4.3 A continuum M is said to be linearly (circularly) chainable if and only if for every positive number ϵ there is a linear (circular) chain D such that $M \subset \bigcup\{d\colon d \in D\}$ and for each $d \in D$ the diameter of D is less that ϵ. If a continuum is refered to as chainable it will be either linearly or circularly chainable.

Definition 4.4 If D is a chain then each element of D is called a link of D. If D and E are chains then E is a subchain of D if and only if each link of E is a link of D. If E is a linear chain then E will be denoted by D(i,j) where d_i and d_j are the end links of E.

The following definition will be of key importance in the definitions of the pseudo-arc and the pseudo-circle.

Definition 4.5 The linear chain $E = \{e_1, e_2, \ldots, e_n\}$ is crooked in the linear chain $D = \{d_1, d_2, \ldots, d_m\}$ if and only if:

(1) $\bigcup_1^m d_i$ contains $\bigcup_1^n e_i$.

(2) For every subchain E(i,j) of E such that $e_i \cap d_h \neq \emptyset$, $e_j \cap d_k \neq \emptyset$, where $|h-k| > 2$, the following conditions hold:

(a) $E(i,j)$ is the union of three chains $E(i,r)$, $E(r,s)$ and $E(s,j)$ such that $(s-r)(j-i) > 0$. ($i < r < s < j$ or $j < s < r < i$),

(b) e_r is a subset of a link of $D(h,k)$ adjacent to d_k,

(c) e_s is a subset of a link of $D(h,k)$ adjacent to d_h.

The Pseudo-arc

In 1922 Knaster (20) described a hereditarily indecomposable continuum in the plane. At that time Knaster thought that his continuum was homogeneous though he could not demonstrate that this was so. In 1948 Moise (23) gave an example of a linearly chainable continuum, which he called a pseudo-arc, that was indecomposable and homeomorphic to each of its subcontinua. Later in 1948 R. H. Bing (1) demonstrated that the pseudo-arc was homogeneous. In 1951 Bing (2) proved that any linearly chainable non-degenerate hereditarily indecomposable continuum is homeomorphic to the pseudo-arc of Moise. This then showed that the continuum of Knaster was a pseudo-arc. In 1951 F. B. Jones (15) proved that every homogeneous bounded plane continuum that does not separate the plane was indecomposable. By using this result and by adding linearly chainable to the hypothesis Bing (5) proved that every linearly chainable homogeneous plane continuum was a pseudo-arc.

The following definition of pseudo-arc is similar to the one used by Moise.

Definition 4.6 Let S be a compact set and let p and q be distinct points of S. Let $D_1, D_2, \ldots$ be a sequence of linear chains from p to q such that:

(1) For each i, D_{i+1} is crooked in D_i,

(2) For each i, each link of D_i has diameter less than $1/i$,

(3) For each i, the closure of each link of D_{i+1} is contained in D_i^* where D_i^* is the union of the links of D_i,

(4) For each i, each link of D_i is connected.

Let $M = \bigcap D_i^*$. Then M is called a pseudo-arc from p to q.

Before proving that the pseudo-arc is a hereditarily indecomposable continuum the following theorem should be considered. This theorem relates the concept of crooked chains with the concept of crooked domains.

<u>Theorem 4.7</u> If D and E are linear chains such that each link of D has diameter less than $\epsilon/3$ and E is crooked in D then E*, the union of the links of E, is an ϵ-crooked domain.

Proof: Let $D = \{d_1, d_2, \ldots, d_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ be such chains and let A be an arc in E*. Then let $x, y \in A$. Suppose that $x \in e_i \cap d_h$ and $y \in e_j \cap d_k$.

Then if $|h-k| < 2$, $d_h \cap d_k \neq \emptyset$ and the diameter of $d_h \cup d_k$ is less than $2\epsilon/3$. By choosing $m \in$ (subarc xy) $\cap\ d_k$ and $n \in$ (subarc my) $\cap\ d_h$ it follows that n is between m and y, $d(m,y) < \epsilon$ and $d(n,x) < \epsilon$.

If $|h-k| = 2$ then choose integer g so that g is between h and k. Then $d_h \cup d_g \cup d_k$ has diameter less than ϵ since $d_h \cap d_g \neq \emptyset$ and $d_g \cap d_k \neq \emptyset$. Hence choose $m \in$ (subarc xy) $\cap\ d_h$ and $n \in$ (subarc my) $\cap\ d_k$ and it follows that n is between m and y, $d(m,y) < \epsilon$ and $d(n,x) < \epsilon$.

If $|h-k| > 2$ the definition of crooked implies that $E(i,j)$ is the union of three subchains, $E(i,r)$, $E(r,s)$ and $E(s,j)$ such that s is between r and j, e_r is a subset of a link of D(h,k) adjacent to d_k and e_s is a subset of a link of D(h,k) adjacent to d_h. Since $x \in e_i$ and $y \in e_j$ the definition of a chain implies that the subarc xy intersects each element of E(i,j). Therefore choose m $\in$ (subarc xy) $\cap$ e_s. Similarly the subarc my intersects each element of E(i,s) = E(i,r) $\cup$ E(r,s). Therefore choose n $\in$ (subarc my) $\cap$ e_r. Then n is between m and y and $n,x \in d_{h-1} \cup d_h \cup d_{h+1}$ and $d(n,x) < \epsilon$. Also $m,y \in d_{k-1} \cup d_k \cup d_{k+1}$ and $d(m,y) < \epsilon$.

Therefore A is ϵ-crooked which implies that E* is ϵ-crooked.

It can now be shown that a pseudo-arc is a non-degenerate hereditarily indecomposable continuum.

<u>Theorem 4.8</u> If M is a pseudo-arc then M is a hereditarily indecomposable non-degenerate continuum.

Proof: Let $M = \bigcap_1^\infty D_i^*$ where each D_i meets the definition of pseudo-arc. Then, for each i, since each link of D_i is connected the definition of chain implies that D* is connected. Also the definition of pseudo-arc implies that $\overline{D_i^*} \subset D_{i+1}^*$. Thus M = $\overline{D_i^*}$ is the intersection of a nest of continua and hence M is a continuum. Since D_{3i+1} is crooked in D_{3i} and each link of D_{3i} has diameter less than 1/3i theorem 4.7 implies that D_{3i+1}^* is 1/i-crooked. Hence corollary 3.22 implies that $M = \bigcap_1^\infty D_i^* = \bigcap_1^\infty D_{3i+1}^*$ is hereditarily indecomposable.

Since both p and q are elements of M it follows that M is non-degenerate and the theorem follows.

While it will not be shown that the pseudo-arc is homogeneous it is suggested that the reader should look at the proof by McKellips (22) for further information and insight. In the opening paragraph of this section two characterizations of the pseudo-arc were suggested. These will now be formally stated without proof.

Theorem 4.9 A set M is a pseudo-arc if and only if M is a linearly chainable non-degenerate hereditarily indecomposable continuum. (2).

Theorem 4.10 A set M is a pseudo-arc if and only if M is a linearly chainable nondegenerate homogeneous continuum. (5).

The Pseudo-circle

After the pseudo-arc was found to be homogeneous, Bing, Jones and others became interested in characterizing homogeneous continua in the plane. In 1951 Bing (2) published a paper in which he described a circularly chainable hereditarily indecomposable continuum that separated the plane. This continuum became known as a pseudo-circle and Bing posed two questions about the pseudo-circle. He asked if it were topologically equivalent to all other circularly chainable hereditarily indecomposable continua that separate the plane and if it were homogeneous. Lawrence Fearnley in 1969 proved that the pseudo-circle was unique with respect to Bing's first question (10). Later that same year Fearnley also proved that the pseudo-circle was not homogeneous (11).

The following definition will be used in defining a pseudo-circle.

Definition 4.11 If D and E are two circular chains such that each link of E is contained in some link of D then E is crooked in D if and only if D_1 is a proper subchain of D and E_1 is a subchain of E such that E_1^* D_1^* implies that E_1 is crooked in D_1.

<u>Theorem 4.12</u> If D and E are circular chains such that E is crooked in D and the diameter of each link of D is less than $\epsilon/6$ then E* is an ϵ-crooked domain.

Proof: Let $D = \{d_1, d_2, \ldots, d_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ be two such chains. Let A be an arc in E* containing the points x and y. Let $x \in d_h$ and $y \in d_k$.

Then if d_h and d_k are adjacent in D to each other or to the same link of D then A is ϵ-crooked relative to x and y by the process used in theorem 4.7.

Otherwise, let z be the first point from x to y such that $z \in (\text{subarc } xy) \cap (d_{k-1} \cup d_k \cup d_{k+1})$ where consideration is given if $k = 1$ or $k = n$. Note that $(\text{subarc } xz) \cap (d_{k-1} \cup d_k \cup d_{k+1}) = \emptyset$. The definition of chain implies that there exists a subchain E_1 of E such that subarc xz intersects each link of E_1 and $(\text{subarc } xz) \subseteq E^*$.

Suppose $E_1^* \cap d_k \neq \emptyset$. Then there exists $e \in E_1$ such that $e \cap d_k \neq \emptyset$. The definition of crooked implies that $e \cap (d_{k-1} \cup d_k \cup d_{k+1})$ which implies that $(\text{subarc } xz) \cap (d_{k-1} \cup d_k \cup d_{k+1}) \neq \emptyset$ which is a contradiction. Therefore $E_1^* \cap d_k = \emptyset$.

Hence if $D_1 = D - \{d_k\}$ then $E_1^* \subset D_1^*$ and hence E_1 is crooked in D_1. Therefore theorem 4.7 implies that E_1^* is an $\epsilon/2$-crooked domain. Hence, in subarc xz, there exists an r and s such that s is between x and r, $d(x,r) < \epsilon/2$ and $d(s,z) < \epsilon/2$. Since $d(z,y) < \epsilon/3$ it follows that $d(s,y) < \epsilon$. Hence A is ϵ-crooked relative to x and y which implies that E* is an ϵ-crooked domain.

Definition 4.13 Let $D_1, D_2, \ldots$ be a sequence of circular chains such that

(1) each link of D_i is a connected domain with diameter less than $1/i$,

(2) the closure of each link of D_{i+1} is contained in a link of D_i,

(3) each D_{i+1} is crooked in D_i,

(4) each D_i^* separates the plane.

Then $M = \bigcap D_i^*$ is a pseudo-circle.

Theorem 4.14 If M is a pseudo-circle then M is a hereditarily indecomposable continuum that separates the plane.

Proof: If M is a pseudo-circle then it follows from theorem 4.13 that M is a hereditarily indecomposable continuum in much the same way that theorem 4.8 followed from theorem 4.7.

That M separates the plane follows from the fact that each D_i^* separates the plane. See the proof of theorem 3.21.

While it will not be shown that the pseudo-circle is unique with respect to Bing's first question nor will it be shown that the pseudo-circle fails to be homogeneous, the papers of Fearnley as described at the first of this section do give rise to the following characterizations of the pseudo-circle. These characterizations will be stated without proof.

Theorem 4.15 If M is a set then M is a pseudo-circle if and only if M is a non-homogeneous chainable continuum (10).

Theorem 4.16 If M is a set then M is a pseudo-circle if and only if M is a chainable continuum that separates the plane. (9).

Homogeneous Continua in the Plane

After a non-degenerate homogeneous continuum different than a simple closed curve was discovered the big problem in this area was to characterize homogeneous continua in the plane. Though no such general characterization as yet exists it is now known that there exist exactly three topologically distinct nondegenerate homogeneous chainable continua. Historically, this characterization was arrived at by the following path.

In 1920 Knaster and Kuratowski (21) presented the problem: Is every non-degenerate homogeneous bounded plane continuum a simple closed curve? In 1922 Knaster (20) gave his example of a hereditarily indecomposable continuum which he suggested might be homogeneous. In 1937 Zenon Waraszkiewicz (29) claimed to have proven that the only non-degenerate homogeneous plane continuum was the simple closed curve. His proof was in error as Bing (4) proved in 1948 that the pseudo-arc was homogeneous. In 1951 Jones (16) proved that all non-degenerate homogeneous plane continuum that does not separate the plane was indecomposable. By extending this Bing (5) proved that all nondegenerate homogeneous chainable continua that failed to separate the plane was the pseudo-arc. In 1954 Bing and Jones (6) discovered a homogeneous decomposable continuum that separates the plane. This example was called a "circle of pseudo-arcs", which are topologically unique. (For a detailed discription of a circle of pseudo-arcs see the thesis of McKellips (22, p. 61).) Jones (17) proved in 1955 that the only decomposable homogeneous continua are the simple closed curve and the circle of pseudo-arcs. This left only indecomposable chainable continua that separate the plane open to the question as to its homogenity. When Fearnly (10) proved that such continua were not homogeneous it followed that the following three continua are the only topologically distinct nondegenerate homogeneous chainable plane continua.

(1) Simple closed curves

(2) Pseudo-arcs

(3) Circles of pseudo-arcs

While it is not known if there might exist homogeneous continua in the plane that are not chainable it is this writer's feeling that no homogeneous non'chainable continua exist in the plane.

Other Hereditarily Indecomposable Continua

It will be the purpose of this section to demonstrate the existence of an uncountable number of topologically distinct hereditarily indecomposable continua in the plane. These continua are not, of course, chainable as the pseudo-arc and the pseudo-circle are the only chainable hereditarily indecomposable continua. These continua will be, however, combinations of the pseudo-arc and pseudo-circle.

Before begining the major task of this section some theorems from general topology will be stated.

<u>Theorem 4.17</u> If $M_1, M_2, \ldots$ is a sequence (finite or infinite) of disjoint compact sets that do not separate the plane such that for some point m every open neighborhood of m contains all but a finite number of the sets $\{M_i\}$ then the space determined by considering each M_i as a point is homeomorphic to the plane.

Proof: See Ottinger's thesis (25, p. 7).

Definition 4.18 If M is a set such that M is homeomorphic to the closed unit disk in the plane then M is a closed 2-cell.

Theorem 4.19 If J is a simple closed curve and M is a compact set that does not separate the plane contained in the interior of J then $B = J \cup$ (interior of J) and $B' = J \cup$ (interior of J) with M considered to be a point are closed 2-cells.

Proof: See Wilder, page 94 (31, p. 94).

Theorem 4.20 If C^2 is the closed unit disk, S^1 is the unit circle, $x,y \in C^2 - S^1$ and $g:S^1 \to S^1$ is a homeomorphism then there exists a homeomorphism $f:C^2 \to C^2$ such that f restricted to S^1 is g and $f(x) = y$.

Proof: Let S be $S^1 \times I$, where I is the closed unit interval, with $S^1 \times \{0\}$ identified as a single point. Define $h:S \to C^2$ by letting $h(s,r) = rg(s) + (1-r)y$ and define $k:S \to C^2$ by letting $k(s,r) = rs + (1-r)x$. Note that multiplication by r and 1-r is scalar multiplication and that the addition is vector addition.

Since g is continuous and scalar multiplication and vector addition are continuous it follows that h is continuous.

If $h(s_1,r_1) = h(s_2,r_2)$, $r \neq 0$, then both $g(s_1)$ and $g(s_2)$ lie on the ray from y through $h(s_1,r_1)$. Therefore $g(s_1) = g(s_2)$ since that ray intersects S^1 at only one point. Therefore $s_1 = s_2$ since g is one-to-one.

Also $h(s_1,r_1) = h(s_2,r_2)$ implies that $r_1(g(s_1)-y) + y = r_2(g(s_2)-y) + y$ which implies that $r_1 = r_2$. Therefore h is one-to-one since $r_1 = 0$ implies that $r_2 = 0$.

If $z \in C^2$, $z = y$, then there exists $s \in S^1$ such that z is on the line segment from s to y. Therefore, for some r, $0 < r \leq 1$, $z = rs + (1-r)y = h(g^{-1}(s),r)$. Hence h is onto.

Since S is compact, C^2 is Hausdorff and h is a one-to-one continuous outo mapping the reference (14, p. 76) implies that h is a homeomorphism.

In a similar manner it follows that k is also a homeomorphism.

Let $f = h \circ k^{-1}$. Then if $s \in S^1$, $f(s) = h \circ k^{-1}(s) = h(s,1) = g(s)$. Also $f(x) = h(k^{-1}(x)) = h(S^1 \times \{0\}) = y$. Hence f meets the required conditions.

<u>Theorem 4.21</u> Let $M_1, M_2, \ldots$ be a sequence of compact sets that do not separate the plane such that for each n, M_n is contained in the interior of the compact set B_n. Suppose also that for each n, bd $B_n = J_n$ is a simple closed curve and that the set $\{B_n\}$ is pairwise disjoint. Furthermore, suppose that for some $b \in E^2$ every neighborhood of b contains all but a finite number of B_n. Let $x_1, x_2, \ldots$ be a sequence of points such that $x_n \in$ int B_n for each n. Then there exists a continuous map $f: E^2 \to E^2$ such that for each n, $f(M_n) = x_n$ and on $E^2 - \bigcup_1^\infty M_n$, f is a homeomorphism.

Proof: Theorem 4.19 implies that B_n and $B_n' = B_n$ with M_n considered as a point are both homeomorphic to C^2, where C^2 is the closed unit disk. Let $g: B_n \to C^2$ and $h: B_n' \to C^2$ be homeomorphisms. It should be noted that $g(J_n) = h(J_n) = S^1$. (29, p. 31) Then theorem 4.20 implies there exists a homeomorphism $k: C^2 \to C^2$ such that k restricted to S^1 is $g \circ h^{-1}$ and $k(h(M_n)) = g(x_n)$.

If $x \in \bigcup_1^\infty B_n'$ let $f(x) = (g^{-1} \circ k \circ h)(x)$. If $x \notin \bigcup_1^\infty B_n'$ let $f(x) = x$.

Since $B = \overline{\bigcup B'_n} = \bigcup_1^{\infty} B'_n \cup \{b\}$ it follows that f is continuous of $B-\{b\}$ since $f = g^{-1} \circ k \circ h$ is continuous on each B'_n. If $\{y_1, y_2, \ldots\} \subset B$ is a sequence converging to b and U is a neighborhood of b then U contains $\bigcup\{B_n : n > N\}$ for some N. Then, for some N', if $n > N'$ then either $y_n = b$ or $y_n \in \bigcup\{B'_n : n > N\}$. In either case $f(y_n) \in \bigcup\{B_n : n > N\} \cup \{x\} \subset U$. Therefore f is continuous on B at b and hence f is continuous on B.

If $y \in E^2$-int B then $y \in E^2-(B-\{b\})$ or $y \in J_n$ for some n. If $y \in E^2-(B-\{b\})$ then $f(y) = y$. If $y \in J_n$ then $f(y) = (g^{-1} \circ k \circ h)(y) = (g^{-1} \circ g \circ h^{-1} \circ h)(y) = y$. Therefore f is continuous on E^2-int B since it is the identity function and hence f is continuous on E^2.

In a similar manner f^{-1} is continuous and hence f is a homeomorphism.

Since $f(M_n) = (g^{-1} \circ k \circ h)(M_n) = x_n$ the theorem follows.

Definition 4.22 A point p is accessible from a set D if and only if there exists an arc A with one of its endpoints p such that $A-\{p\} \subset D$.

Lemma 4.23 If M is a hereditarily indecomposable continuum then there exists $p \in M$ such that p is not accessible from the complement of any nondegenerate subcontinuum of M containing p. That is, for every nondegenerate subcontinuum N of M containing p, p is not accessible from E^2-N.

Proof: Let L_1 and L_2 be two parallel lines each of which separate M. Then L_1 separates the plane into two components U_1 and U_2. Hence if C_m is a composant of M there exist points x and y in $C_m \cap U_1$ and $C_m \cap U_2$ respectively. Therefore there is a proper subcontinuum M' of M containing x and y. Also $M' \subset C_m$ and $M' \cap L_1 \neq \emptyset$ since L_1 separates

M'. Therefore $C_m \cap L_1 \neq \emptyset$. Similarly $C_m \cap L_2 \neq \emptyset$. Hence there exists a proper subcontinuum of M contained in C_m that intersects both L_1 and L_2. Let M_m be an irreducible continuum from L_1 to L_2 contained in C_m (23, p. 15).

Then, since there exist an uncountable number of mutually exclussive composants of M (23, p. 59), the collection $\{M_m\}$ is uncountable. Let K be the union of L_1, L_2 and the closure of the union of the elements of $\{M_m\}$.

Note that neither L_1 nor L_2 separates an element of $\{M_m\}$ for if this were so a proper subcontinuum of some M_m that intersects both L_1 and L_2 could be found by a process used to determine each M_m. This would contradict M_m being irreducible and hence each M_m lies on or between L_1 and L_2. It should also be noted that each M_m separates the closed strip K' between L_1 and L_2.

Let D be a component of E^2-K that lies between L_1 and L_2. Suppose $\overline{D}$ intersects three elements of $\{M_m\}$, M_1, M_2 and M_3. Let x_1, x_2 and x_3 be points in $M_1 \cap L_1, M_2 \cap L_1$ and $M_3 \cap L_1$ with x_2 between x_1 and x_3 on L_1. Since $K' - M_2 = V_1 \cup V_3$ sep. with $x_1 \in V_1$ and $x_3 \in V_3$ it follows that $M_1 \subset V_1$ and $M_3 \subset V_3$. Also either $D \subset V_1$ or $D \subset V_3$. In either case $\overline{D}$ fails to intersect both M_1 and M_3 which is a contradiction. Hence no component of E^2-K that lies between L_1 and L_3 has boundary points in more than two elements of $\{M_m\}$.

Since the number of components of E^2-K are countable there exists $M_k \in \{M_m\}$ that does not intersect the closure of any component of E^2-K that lies between L_1 and L_2. Let M_1 be a nondegenerate subcontinuum of M_k that does not intersect either L_1 or L_2 and let A be an arc from the complement of M to a point $p \in M_1$. Suppose $A - \{p\} \subset E^2 - M$. Then some subarc xp lies between, but not on, L_1 and L_2. Therefore (subarc xp)-$\{p\} \subset E^2$-K, which implies that subarc xp is contained in $\overline{D}$, where D is a component of E^2-K that lies between L_1 and L_2, which in turn implies that $M_1 \subset M_k$ intersects $\overline{D}$ which is a contradiction. Therefore A-$\{p\} \not\subset E^2$-M and no point of M_1 is accessible from E^2-M. Hence every hereditarily indecomposable continuum M contains a non-degenerate subcontinuum containing no point accessible from the complement of M.

As was done in the proof of theorem 3.28, M contains a non-degenerate subcontinuum M' with diameter less than 1 and M' contains a subcontinuum M_1 such that no point of M_1 is accessible from E^2-M. Suppose that M_i has been defined with diameter less than 1/i. Then there exists a subcontinuum M_{i+1} of M_i with diameter less than 1/(i+1) such that no point of M_{i+1} is accessible from the complement of M_i. Let $p \in \cap M_i$. Then if N is a subcontinuum of M containing p there exists i such that $M_i \subset N$. Since $p \in M_{i+1}$, p is not accessible from the complement of M_i and hence p is not accessible from the complement of N. Thus the lemma holds.

<u>Lemma 4.24</u> If M is a hereditarily indecomposable continuum, D is a domain intersecting M and C is a composant of M then there exists $p \in C \cap D$ such that p is not accessible from the complement of any subcontinuum of M containing p.

Proof: Since C is dense in M, there exists $q \in C \cap D$. Let U be an open sphere containing q such that $\overline{U} \subset D$ and $\overline{U}$ does not contain M. Let N be an irreducible subcontinuum of M from q to bd U. Suppose for some $x \in N$, $x \notin \overline{U}$. Then bd U separates N and there exists a point q' in the composant of N that contains q such that $q' \in N-\overline{U}$. Hence N contains a proper subcontinuum N' that contains both q and q'. Since bd U separates N', (bd U) $\cap$ N' $\neq \emptyset$ which contradicts N being irreducible from q to bd U. Therefore $N \subset \overline{U}$ which implies that $N \subset D \cap C$. Lemma 4.23 implies that there exists a point $p \in N$ such that p is not accessible from the complement of any subcontinuum of N containing p. Such a p meets the conditions of this lemma.

It can now be shown that there exists an uncountable number of topologically distinct hereditarily indecomposable continua in the plane. This will be done by associating with every subsequence of the sequence of positive integers a hereditarily indecomposable continuum in such a way that given two such sequences the continua associated with them would be topologically distinct. Since there are an uncountable number of subsequences of the positive integers this would produce the desired result.

Theorem 4.25 There are an uncountable number of topologically distinct hereditarily indecomposable continua in the plane.

Proof: Let M be a pseudo-arc in the plane, let $x \in M$ and let $x_1, x_2, \ldots$ be a sequence of points of M that converge to x. Let B_1 be a closed disk centered at x_1 such that $B_1 \cap \{x_i : i > 1\} = \emptyset$. Suppose the closed disks B_i centered at x_i, $i < n$, have been defined so that they are pair-wise disjoint and that $(\bigcup_1^{n-1} B_i) \cap \{x_i : i \geq n\} = \emptyset$. Then let B_n be a closed disk centered at x_n so that $B_n \cap (\bigcup_1^{n-1} B_i \cup \{x : i > n\}) = \emptyset$.

Let $C_1, C_2, \ldots$ be a sequence of distinct composants of M and let $P = \{p_1, p_2, \ldots\}$ be a strictly increasing sequence of positive integers. Then for each n, if $\sum_1^{j-1} p_i < n \leq \sum_1^{j} p_i$ let $y_n \in \text{int } B_n \cap C_j$ such that y_n is not accessible from the complement of any subcontinuum of M containing

y_n. Such a y_n exists by lemma 4.24. (Note: $\sum_{1}^{\infty} p_i = 0$.) It should be noted that each C_j contains exactly p_j elements of the sequence $\{y_n\}$.

For each n let M_n be a pseudo-circle Y_n togeather with the union of all of the bounded complementary domains of Y_n (there exists only one bounded complementary domain of a pseudo-circle, though this has not been shown) such that $M_n \subset \text{int } B_n$. Note that bd $M_n = Y_n$. (2, p. 48) The definition of M_n implies that M_n fails to separate the plane so that theorem 4.21 implies that there exists a continuous map $f: E^2 \to E^2$ so that $f(M_n) = \{y_n\}$ for each n and on $E^2 - M_n$, f is a homeomorphism.

Let $M_P = f^{-1}(M) - \bigcup_{1}^{\infty}(\text{int } M_n)$. It should be observed that M_P is, loosely speaking, M with the points y_n replaced with the pseudo-circles Y_n.

Suppose H is a non-degenerate subcontinuum of M_P such that for some n, H intersects but does not contain Y_n. Then the reference (24, p. 86) implies that there exists $q \in Y_n - H$ such that q is accessible from $E^2-(H \cup Y_n)$. Therefore there exists an arc $A \subset E^2-(H \cup Y_n-\{q\})$ with end-point q. Note that some subarc yq of A lies in $f^{-1}(B_n)$ and that f(subarc yq) = arc $y'y_n$ is an arc in E^2 with one end-point y_n, since f is a homeomorphism on $B_n - M_n$, and that $H \cap$ arc $y'y_n = \{y_n\}$. This implies that y_n is accessible from the complement of f(H) which implies that f(H) is degenerate since y_n is not accessible from the complement of a non-degenerate subcontinuum of M containing y_n. Therefore $H \subset Y_n$ and H is hereditarily indecomposable.

Now suppose H is a non-degenerate subcontinuum of M_P such that for all n, either $Y_n \subset H$ or $Y_n \cap H = \emptyset$. Suppose that $H = K_1 \cup K_2$ is the union of two proper subcontinua of H. Since H is decomposable it follows that $H \not\subset Y$ for any n. Therefore f(H) is a non-degenerate subcontinuum of M. Also it follows that $f(H) = f(K_1) \cup f(K_2)$ is the union of two continua. Since $f(H)$ is indecomposable either $f(H) = f(K_1)$ or $f(H) = f(K_2)$. Without loss of generality suppose $f(H) = f(K_1)$. Then by the previous paragraph, since $f(H) = f(K_1)$ is nondegenerate, if $y_n \in f(H) = f(K_1)$ then $Y_n \subset H$ and $Y_n \subset K_1$. Also, since f is one-to-one on $M_P - \bigcup_{i}^{\infty} Y_i$, it follows that $H = K_1$. This contradicts K_1 being a proper subcontinuum of H. Therefore H is indecomposable and it follows that M_P is hereditarily indecomposable.

It should be noted that for any set $N \subset M$, N is a proper subcontinuum of M if and only if $f^{-1}(N)$ is a proper subcontinuum of M_P. Therefore C is a composant of M if and only if $f^{-1}(C)$ is a composant of M_P. Therefore M_P is a hereditarily indecomposable continuum with exactly one composant containing exactly p_i pseudo-circles for each i and also with no other composants containing a pseudo-circle.

Therefore if P and P' are two strictly increasing sequences of integers with $P \neq P'$ then, without loss of generality, there exists $b \in P - P'$. Then M_P contains a composant with exactly b pseudo-circles while $M_{P'}$ has no such composant. Therefore M_P and $M_{P'}$ are topologically distinct and the theorem is proven.

CHAPTER V

SUMMARY

It has been the purpose of this paper to present to the reader an introduction to the theory of dimension and also to present a detailed account of the current literature involved with the topic of hereditarily indecomposable continua.

Chapter II discussed those elements of the theory of dimension that would be applied in the later chapters. An effort was made to write all of the proofs in this chapter so as not to omit any steps that would be difficult for a person who had just finished his first course in topology.

Chapter III presents a discription of the existence of hereditarily indecomposable continua of all dimensions. Included in this chapter are the statements of all known characterizations of hereditarily indecomposable continua, including one that had previousely been undiscovered. Also in this chapter is a statement of the ideas involved with hereditarily infinite dimensional compact spaces.

In Chapter IV the paper limits itself to the plane with a discussion of the pseudo-arc and the pseudo-circle. Much of this chapter is a review and an updating of the thesis of Terral McKellips (22). The two most important developments of this chapter are the relating of the ideas of crooked domains and crooked chains and the proof of the existence of an uncountable number of topologically distinct hereditarily indecomposable continua in the plane.

Related to this paper are several topics that might prove interesting for further study and research. Among these would be the study of the properties of the pseudo-arc or the pseudo-circle and the study of hereditarily infinite dimensional spaces.

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VITA

Albert Lee Crawford

Candidate for the Degree of

Doctor of Education

Dissertation: HEREDITARILY INDECOMPOSABLE CONTINUA

Major Field: Higher Education

Biographical:

Personal Data: Born in Larned, Kansas, August 24, 1941, the son of Woodrow E. and Leona Crawford.

Education: Graduated from Pomona Rural High School, Pomona, Kansas, in May, 1959; received the Bachelor of Science in Education degree from Kansas State Teachers College, Emporia, Kansas, in May, 1962, with teaching fields in mathematics and the social sciences; received the Master of Arts degree from Kansas State Teachers College, Emporia, Kansas, in May, 1967, with a major in mathematics; completed requirements for the Doctor of Education degree at Oklahoma State University in July, 1970.

Professional Experience: Taught as a mathematics teacher at St. George, Kansas, High School, 1962-1964; taught as a mathematics teacher at Frankfort, Kansas, High School, 1964-1966; taught as a mathematics teacher at Ottawa, Kansas, Junior High School, 1966-1967; taught as a graduate assistant at Oklahoma State University, 1967-1970.