

LINEAR PROGRAMMING AS AN AID TO
FINANCIAL DECISION MAKING

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PREFACE

The purpose of this paper is to provide a description of the current applications of linear programming to problems in the field of finance. In addition, several points are made which clearly indicate that linear programming does hold a great deal of promise as being a useful tool for the financial manager.

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CHAPTER I

INTRODUCTION

The desire to be able to control their environment has driven men in business to seek new ways of understanding and coping with the many problems which face them every day. This desire has led to the recent emphasis placed on technical progress in the seemingly non-technical field of business, and as a result many new developments and applications of mathematical models have been made. One such development, linear programming, is the topic of this paper.

Scope and Purpose

The purpose of this paper is to describe linear programming and its specific uses in the field of finance. Although the tool itself is nearly 20 years old, applications in the area of finance have been limited. In describing linear programming and its uses in finance three different problems are presented. These three are seemingly the only types of problems in finance to which linear programming has been applied. The fact that only three models have been developed, however, does not limit the use of linear programming in the area of finance as much as it may seem to at first glance. The versatility of linear programming holds great promise of becoming the key factor to wide spread use of the tool in financial management. Versatility is demonstrated throughout the paper and, as shall be shown in Chapter III,

a great deal of it is obtained by the adaptability and addition of constraining equations.

Besides descriptions this paper attempts to analyze the problems and determine their usefulness in finance. In doing so some of the necessary and initial conditions are set forth in Chapter II. In addition, the particular drawbacks and limitations are pointed out.

In Chapter IV it becomes clear that linear programming holds supplementary power as a tool of analysis. The dual aspects are just the type of tools needed by the financial manager in developing and implementing rigorous plans and programs.

Financial Decision Making

The success of any financial manager is highly dependent upon the outcome of the decisions he makes throughout his career. In many situations the financial manager must make these decisions with few facts to use as guide posts. In the past the financial manager has had to rely upon intuition and rules of thumb which could be learned only by gaining many years of experience. This experience is costly yet, to a degree, necessary for good judgment in making decisions. In today's world, however, the fast pace set by business competition makes judgments based upon rules of thumb inaccurate and many times disappointing even when backed by a great deal of experience. Thus, good decision making is of premium value to any company that wishes to survive. As a consequence of this importance much emphasis has been placed upon the development of new tools for use in making better decisions. These tools are used as a supplement to experience and not as a substitute.

Involved with the decision is usually a great deal of uncertainty about the value of variables or the consequences of alternative strategies. This uncertainty is at the center of almost any business problem since we are not dealing with a pure science but one which involves human nature. "Both the importance and the generality of uncertainty require that it be built into any theory that purports to explain a large portion of man's behavior."¹

A business situation is often described as a situation of risk and/or uncertainty. There is, however, a distinction which should be made between the meaning of the two words, and several authors have pointed out such a difference. Farrar defines risk as a situation where the probabilities of alternative outcomes are known. He states that "to qualify as being a risk situation an experiment must be repetitive in nature and must possess a frequency distribution from which inferences can be made by objective statistical procedures."² For example, events such as a person's death, a fire, an automobile accident, or a storm are said to constitute risk. "Uncertainty in contrast is said to be present when the experiment in question cannot be carefully replicated by (or upon) other persons or at other times or places; that is when the situation is unique. Its frequency distribution, therefore, cannot be objectively specified."³

March and Simon agree with this definition. They say that risk assumes accurate knowledge of a probability distribution of the

¹Donald E. Farrar, The Investment Decision Under Uncertainty (Englewood Cliffs, New Jersey, 1962), p. 1.

²Ibid., p. 2.

³Ibid.

consequences of each alternative, whereas uncertainty assumes that the consequences of each alternative belong to some subset of all possible consequences but one cannot assign definite probabilities to the occurrence of particular consequences.⁴ In essence then, the decision maker would rather work with a situation of risk rather than uncertainty since some knowledge, short of being complete, is available for use.

Many problems face the financial decision maker and the amount of time involved in any one decision ranges from simple day to day problems to the long range corporate plans. In financial decisions covering long ranges of time, a greater possibility for uncertainty exists. Ordinarily more variables are apt to come into play during a long period; thus, there will be a greater chance of a unique situation arising. This is not to say that short run problems are without uncertainties for they may encounter many difficulties which cannot be known for certain.

Whether the problem be of a short or long run nature, the uncertainties are usually handled by assigning some type of probability distribution or a single assumed value to the unknown in order to facilitate the use of the decision tool. The predictions or results of the technique used will only be as good as the assumptions made.

Applications of new techniques such as the one discussed in this paper have been limited in the area of financial decision making. "Experience, rules of thumb, and the desire to retain the goodwill of the financial community have influenced financial decisions in the past

⁴James G. March and Herbert A. Simon, Organizations (New York, 1958), p. 137.

and will continue to influence them in the future."⁵ The future success of applications of new techniques such as linear programming will depend upon their validity and usefulness as compared with old methods used.

Uncertainty and the Use of Models

Models in finance, as in any other field, are conceptual schemes which specify relations existing among a set of financial variables. As was indicated above, there is usually a great deal of uncertainty existing as far as the value of financial variables are concerned. Building a model is no easy task since the model must be able to correctly specify the relationships among these variables which are many times uncertain. The purpose of a model is to impart some knowledge and insight into the workings of a complex problem and to provide guidance toward the optimal solution of that problem. The model presented in this paper (linear programming) does provide the knowledge and insight as well as some optimal solution given certain initial conditions. These initial conditions are the variable values and the relationships which must be specified between them. The validity of the linear programming model depends upon the degree to which it is descriptive of the actual problem being solved as well as the accuracy of the values of the variables used.

When a model is developed for use in financial decision making, it is assumed that the variables involved are either known or can be approximated by some type of probability distribution or some

⁵Harold Bierman, Jr. and Alan K. McAdams, "Financial Decisions and New Decision Tools," Financial Executive, May, 1964, p. 23.

estimating process. This is not so unrealistic since in the real situation a businessman who is familiar with his operations undoubtedly has some idea of the value of the variables with which he deals. He automatically makes his judgments on the basis of subjective probabilities.

Using these subjective probabilities in the model it is possible to come up with some estimate of the unknowns. In this way situations involving, at first glance, uncertainty can be reduced to situations of risk. Situations altered by use of subjective probabilities will not be true risk situations since each situation is unique and the probability is only an estimation.

CHAPTER II

USE OF LINEAR PROGRAMMING IN DECISION MAKING

Mathematical programming used in connection with decision making has the essential purposes of planning and/or controlling some aspect of the environment. These are the purposes for which it was developed and the reason for which it is used today.

The History and Purpose

Linear programming was developed in the 1940's by George B. Dantzig as a technique for planning the diversified activities of the United States Air Force.¹ It was applied to such problems as transportation, production scheduling, and contract awarding. It is basically a technique used in allocation problems where a number of alternative uses exist for a given number of factors of production or sources of funds as in the case of financial problems. The object is to maximize or minimize some objective function such as profits or costs. The main advantage of Dantzig's model over previous allocation models is that it allows the decision maker to take into consideration many different solutions to the objective function and determine which is optimal.

¹Robert Dorfman, Paul A. Samuelson and Robert M. Solow, Linear Programming and Economic Analysis (New York, 1958), pp. 3-5.

The solution of the linear programming problem by Dantzig has stimulated other developments in application to managerial problems.² These first problems were mostly concerned with production allocation and transportation such as may be found in any standard production management textbook.

Linear programming has also been applied to other areas such as accounting,³ economic theory,⁴ marketing,⁵ and finance⁶ which will be mentioned in the next section. In addition to being applied extensively to many areas, linear programming theory has been developed in many directions. These extensions of the theory such as nonlinear programming, integer programming, and dynamic programming give added flexibility to the use of programming methods.

Basic Concepts and Necessary Initial Conditions for Application

Linear programming may be defined mathematically as "the analysis of problems in which a linear function of a number of variables is to be maximized (or minimized) when those variables are subject to a number of restraints in the form of linear inequalities."⁷ In the real world

²Ibid., p. 4.

³Robert K. Jaedicke, "Improving Break-Even Analysis by Linear Programming Techniques," NAA Bulletin, XLII (1961), pp. 5-12.

⁴George B. Dantzig, "Programming Interdependent Activities II Mathematical Model," Econometrica, XVII (1949), pp. 200-211.

⁵Philip Kotler, "The Use of Mathematical Models in Marketing," Journal of Marketing, October, 1963, pp. 31-41.

⁶See Chapter III.

⁷Dorfman, p. 8.

problems do not often fit such exacting standards. They can, however, with a few simplifying assumptions be quantified in this manner.

Basically all linear programming problems are concerned with the maximization or minimization of some objective function which is linear in nature. When we speak of maximizing something we know that we have minimized its opposite or inverse. Maximization and minimization are therefore essentially the same problem.

Also common to all linear programming problems are the so-called constraints which prevent us from unlimited profits or zero costs. Two types of constraints exist in a problem: one, the non-negative constraints placed on the variables involved; second, the constraints which are given with each problem. The second type will, of course, vary from problem to problem. These constraints must be put in linear form, usually as inequalities; and consequently, the real world relationships which they represent must approximate linearity. The inequalities are then changed to equalities by use of slack variables in order to facilitate the solution of problems.

A short example might help to explain some of the basic points just made. Assume that a company produces two products, both of which are in great demand; and they can sell as many of each as they can produce. Both products must be put through two processes, stamping and forming. The capacity on both the stamping and forming machines is 30,000 seconds per day. Product X takes 10 seconds on the forming and 15 seconds on the stamping machine to produce one unit. Each unit of Y, on the other hand, takes 20 seconds on the forming and 5 seconds on the stamping machine. Product X has a contribution to profit of two cents per unit, while Y has a contribution to profit of three cents per unit.

Using this information as a basis the problem may be set up in its usual form. The function $Z = .02X + .03Y$ is our objective function or the one which must be maximized since it represents profits. Since we cannot have negative production, we are constrained by the inequalities:

$$X \geq 0, Y \geq 0. \quad (2.1)$$

In addition, the time on each of the machines is limited to 30,000 seconds per day. In any one day the constraints limit production so that we have the inequalities:

$$10X + 20Y \leq 30,000 \quad (2.2)$$

$$15X + 5Y \leq 30,000. \quad (2.3)$$

These inequalities follow from the verbal statements above.

The requirements of the simplex method deem necessary the use of further variables known as slack variables. In this case they would represent unused machine time which may be zero or some positive value. Using such variables the two constraints above may be written as the equalities:

$$10X + 20Y + S_1 = 30,000 \quad (2.4)$$

$$15X + 5Y + S_2 = 30,000 \quad (2.5)$$

in which S_1 and S_2 are the slack variables. The non-negative requirements are also in effect for the two slack variables so that

$$S_1 \geq 0 \text{ and } S_2 \geq 0. \quad (2.6)$$

It is now possible to solve this problem for the maximum value of the objective function using the simplex method. The first tabular exposition of the problem is given in Table I, but the actual technique is not dwelled upon here.⁸ It shows that the optimum amount of X to be pro-

⁸See F. A. Ficken, The Simplex Method of Linear Programming (New York, 1961), pp. 21-35.

duced is 1800 units; and the optimum of Y, 600 units. Computer programs are available which with little difficulty can be used to solve linear programming problems.

TABLE I
INITIAL SIMPLEX TABLEAU FOR PRODUCTION PROBLEM

		.02	.03	0	0	
		X	Y	S ₁	S ₂	Solution
0	S ₁	10	20	1		30,000
0	S ₂	15	5		1	30,000

More descriptive of the actual process is the graphic solution to the problem. In Figure 1 we see graphed $10X + 20Y = 30,000$ which is line F, or the constraints imposed by the forming machine time limits.

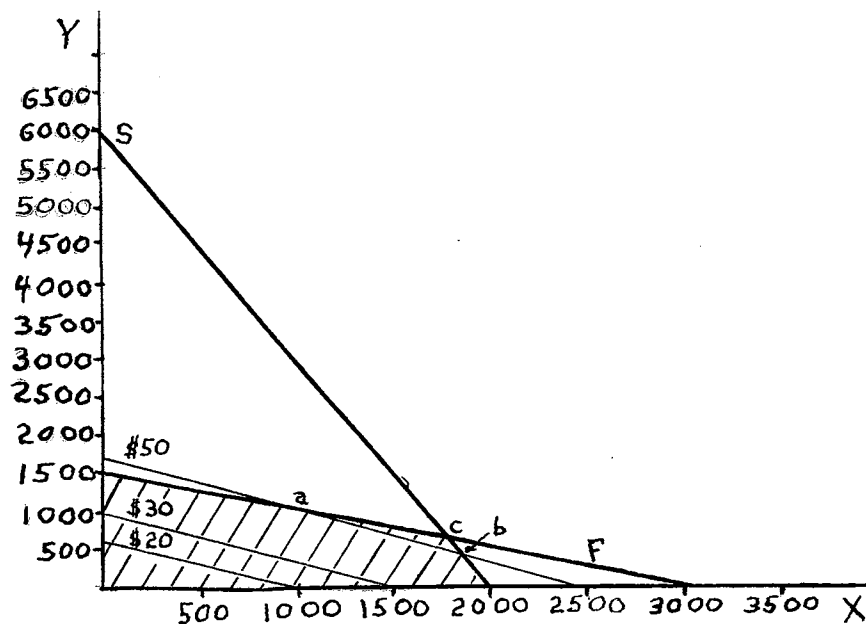


Figure 1. Graphical Solution to Production Problem

In addition $15X + 5Y = 30,000$ is line S or the constraint imposed by the stamping machine. Each point in the graph above the X axis and to the right of the Y axis represents some combination of production of X and Y. However, since line S represents the limit on the stamping machine, any combination above that line is not feasible. The same holds true for line F. This leaves only the shaded region as possible combinations of production under the given constraints. The idea is to find which of the infinite number of combinations in the feasible region will obtain maximum profits.

To begin the graphical analysis some arbitrary value of the objective function is chosen in order to be able to graph the line. Choosing \$20, \$30 and \$50 the lines in Figure 1 are obtained by graphing the equations,

$$.02X + .03Y = 20 \quad (2.7)$$

$$.02X + .03Y = 30 \quad (2.8)$$

$$.02X + .03Y = 50. \quad (2.9)$$

Two things of importance can be noticed concerning these lines: first, each is parallel to the others; second, each moves further out and to the right when profits are increased. All points on both the \$20 and \$30 lines are in the feasible region and all points on the segment ab of the \$50 line are feasible. If the line is moved parallel and outward to the right, it is easily seen that the highest line, which corresponds to the maximum profits, is the line which just touches point c. Any lines past this one do not contain any feasible solutions. Reading the values of point c from the graph, X equals 1800 units and Y equals 600 units. Solving our objective function we have:

$$.02(1800) + .03(600) = 36 + 18 = \$54. \quad (2.10)$$

The problem dealt with here involved two variables and was relatively simple in nature. Linear programming is not, however, limited to the two-dimensional case. Its real power lies in its ability to handle large complex problems. The graphical solution as shown here is of no use once more than three variables become involved. These more sophisticated problems can be handled by the simplex method and as was mentioned previously computer programs are already available for solving problems adaptable to the linear programming model.

Business Applications

As was mentioned previously the requirement of linearity in programming problems restrict somewhat the application of this model to business problems. In the short run, however, the functional relationships can many times be approximated by a straight line. For example in finance, interest rates vary widely over time. In the short run, however, they are usually assumed to remain constant. Consequently, linear programming is sometimes best applied to only short run problems.

In order to get an idea of the type of problems being handled by linear programming in the business world, a short synopsis will now be given for problems in the areas of accounting, economics and marketing. The problems in finance will be dealt with in Chapter III.

Linear programming has been applied to problems involving break-even analysis in accounting.⁹ Its value lies in the ability to extend the analysis and lend a better understanding of complex problems.

⁹Jaedicke.

Essentially the technique allows the consideration of a multi-product firm with many different scarce resources or factors where break-even analysis as used previously was unable to cope with these problems. In such problems the objective function becomes the marginal contribution to profit of each product and constraints are involved with limits placed on production and demand.

The second area of application is economics. It has been basically discussed as an aid to economic theory. The objective function becomes some welfare function of the economy, constrained by a limited amount of resources such as raw materials, manpower, or facilities. In this role linear programming becomes a theoretical tool or "...a convenient way of idealizing the production and profit-maximizing side of a model designed for answering abstract economic questions."¹⁰

The third area to be mentioned here is that of marketing. In this area linear programming becomes useful in determining the optimal distribution system for a company. This has come to be known as the transportation model which minimizes total costs given specific shipping requirements and associated costs. In addition, linear programming has become useful in allocating advertising expenditures to available media.¹¹

Much more could be said about these applications; however, since they are not the main area of concern here, the subject will not be continued at this point. The author is quite aware of the fact that

¹⁰Dorfman, p. 36. (For general discussion, see Chapter 13.)

¹¹Kotler, pp. 35-37.

because these problems have been covered in the literature does not imply they have been used in actual practice.

CHAPTER III

APPLICATIONS TO PROBLEMS IN FINANCE

The object of this chapter is to expose in detail some of the phases of financial management which can be dealt with by the linear programming technique. The three problems presented in this chapter are examples of linear programming applications to financial problems. They proceed in complexity, the first being the least complex of the three. The first problem is concerned with allocating warehouse capacities in such a way as to maximize profits, and the second attempts to allocate available funds to competing proposals. These first two problems then deal with single aspects of business. The third problem on the other hand attempts to encompass all phases of the business enterprise which entail planning transactions flows encountered by the business firm. These transactions flows include such things as buying securities, purchasing materials, paying dividends and other variables over which management can exercise some discretion.

Charnes, Cooper, and Miller Model¹

The first application to be considered here involves the maximization of net returns in light of operating constraints determined by the

¹A. Charnes, W. W. Cooper, and M. H. Miller, "Application of Linear Programming to Financial Budgeting and the Costing of Funds," Journal of Business, XXXII (1959), pp. 20-46.

firm's warehouse capacities. The initial conditions involve the structure of the firm's assets which include a warehouse and an initial stock of inventory. The problem involves n periods which constitute its planning horizon. Specifically, the constants and variables involved are:

- B = the fixed warehouse capacity
- A = the initial stock of inventory in the warehouse
- X_j = the amount to be purchased in period j
- Y_j = the amount to be sold in period j
- P_j = The sales price per unit in period j
- C_j = the purchase price per unit in period j

Certain simplifying assumptions which are adhered to underlie the model. First of all, it is assumed that the warehouse capacity and initial inventory are given and known. In the next chapter greater insights will be gained in the understanding of how linear programming can aid in evaluating objectives in view of changes in warehouse capacity.

A second assumption is that the sales price in any one period, P_j , is known and is relatively stable throughout period j . In addition, the purchase price per unit, C_j , is also assumed to be known and stable throughout period j . The last assumption is that the firm sells in a market of perfect competition so that in any one period the potential sales may be of any size. This assumption could be dropped and another constraint added on the number of sales made; however, for the sake of simplicity, sales will be viewed as unlimited.

The problem then boils down to determining the optimal values of X_j and Y_j given P_j and C_j for each time period involved. In doing this the financial manager is constrained by two major things. First, the

cumulative sales at the beginning of any one period cannot be greater than the initial stock of inventory plus the cumulative amount of purchases during those periods. This may be written as the inequality:

$$\sum_{j=1}^i Y_j \leq A + \sum_{j=1}^{i-1} X_j \quad (3.1)$$

where $i = 1, 2, 3, \dots, n$.

Secondly, the firm cannot store any more than its available capacity will allow. This capacity is equal to its fixed warehouse capacity minus the initial stock plus the amount sold during each period. This constitutes its net available capacity which can be expressed mathematically as the inequality:

$$\sum_{j=1}^i X_j \leq B - A + \sum_{j=1}^i Y_j \quad (3.2)$$

where $i = 1, 2, 3, \dots, n$.

The object is to maximize net returns which for any period is given by sales price per unit times the amount purchased all in period one. This can be expressed by:

$$Z = P_1 Y_1 - C_1 X_1 \quad (3.3)$$

for the first period and this is what we want maximized in each period.

Written for all periods this equation would be:

$$\text{maximize } Z = \sum_{j=1}^n P_j Y_j - \sum_{j=1}^n C_j X_j \quad (3.4)$$

Equations 3.1 and 3.2 above can be rewritten in a more convenient form as:

$$\sum_{j=1}^i X_j + \sum_{j=1}^i Y_j \leq A \quad (3.5)$$

$$\sum_{j=1}^i X_j - \sum_{j=1}^i Y_j \leq B - A \tag{3.6}$$

$i = 1, 2, 3, \dots, n$

and finally the non-negative constraints required are,

$$X_j, Y_j \geq 0 \quad j = 1, 2, 3, \dots, n \tag{3.7}$$

since sales and purchases cannot be negative.

As indicated above the constraints on sales and purchases are each a set of constraints. These sets consist of one equation for each period involved. If equation 3.5 is written explicitly for $i=1, 2, \dots, n$, the following expressions are obtained:

$$\begin{array}{rcl} & +Y_1 & \leq A \\ -X_1 & +Y_1+Y_2 & \leq A \\ -X_1-X_2 & +Y_1+Y_2+Y_3 & \leq A \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ -X_1-X_2-\dots-X_{n-1} & +Y_1+Y_2+\dots+Y_n & \leq A \end{array} \tag{3.8}$$

In addition equation 3.6 can be written as,

$$\begin{array}{rcl} X_1 & -Y_1 & \leq B - A \\ X_1+X_2 & -Y_1-Y_2 & \leq B - A \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ X_1+X_2+\dots+X_n & -Y_1-Y_2-\dots-Y_n & \leq B - A \end{array} \tag{3.9}$$

The problem then is reduced to the linear programming structure and can be solved for optimum values of X_j and Y_j . The initial simplex matrix for the problem using a planning horizon of four periods is given in Table II. The simplex method would yield optimum values for $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3$ and Y_4 . These in turn would be used to determine the value of the objective function or profit at the optimum point.

TABLE II
INITIAL SIMPLEX TABLEAU FOR CHARNES, COOPER,
AND MILLER MODEL

	-C ₁	-C ₂	-C ₃	-C ₄	P ₁	P ₂	P ₃	P ₄	0	0	0	0	0	0	0	0	0	Solution
	X ₁	X ₂	X ₃	X ₄	Y ₁	Y ₂	Y ₃	Y ₄	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈		
0 S ₁					1				1									A
0 S ₂	-1				1	1				1								A
0 S ₃	-1	-1			1	1	1				1							A
0 S ₄	-1	-1	-1		1	1	1	1				1						A
0 S ₅	1				-1								1					B-A
0 S ₆	1	1			-1	-1								1				B-A
0 S ₇	1	1	1		-1	-1	-1								1			B-A
0 S ₈	1	1	1	1	-1	-1	-1	-1								1		B-A

In this particular sample problem more constraints could be added when deemed necessary. For example, as was mentioned above, sales could be constrained. This could be complicated as necessary in order to take all variables into consideration.

Suppose that, because of the desire to keep salesmen busy, a minimum number of sales must be made in each period--say C number of units per period. In addition, it would be realistic to consider a maximum number of sales per period, say D units per period. These constraints would then be written as,

$$Y_j \geq C \quad (3.10)$$

and

$$Y_j \leq D \quad (3.11)$$

for $j = 1, 2, \dots, n$.

Similar constraints may be imposed upon purchases which would further complicate the problem. The constraints on purchases could take two forms where one limited the number of units purchased and another the number of dollars worth of goods purchased. The constraints on sales may also be in terms of dollars. The structure of the linear programming model leaves a great amount of flexibility in choosing the constraints placed on the objective function with the only requirement being that the relationships approximate linearity. In addition, the simplex method can handle inequalities which are in either direction, equalities, or approximations. The procedure involves inserting the proper slack variable or variables into the inequalities, equalities and approximations in order to facilitate the use of the simplex method.²

Weingartner Model³

The second model to be discussed involves the allocation of a fixed number of dollars, a budget of some sort, to independent competing investment proposals. This problem has been called the Lorie-Savage problem.⁴ The essential features of the problem are that a choice is to be made among a large number of alternative combinations of investments. The main difference between this problem and problems common to linear programming is that each of the investments presumably must be

²For an example, see Elwood S. Buffa, Model for Production and Operations Management (New York, 1963), pp. 352-354.

³H. Martin Weingartner, Mathematical Programming and the Analysis of Capital Budgeting Problems (New Jersey, 1963).

⁴Ibid., p. 16.

on an all-or-nothing basis. That is to say each solution must be either one or zero, where one indicates taking on the entire project and zero indicates complete rejection of the proposal. As we shall see, linear programming techniques do not solve the problem of fractional projects entirely. They do, however, minimize the problem to minor proportions. The desirability of linear programming over other methods which could more rigorously handle indivisibility lies in the simplicity of computation and interpretation.

This particular model, as compared with the previous one presented, has greater implication in finance. It deals specifically with capital budgeting which has long been a topic of discussion in the area of finance. In addition, this problem is of greater scope since in the first problem we were maximizing net returns on sales while here we maximize the present value of net returns on all possible combinations of investment proposals. More specifically the sale of a product can be looked upon as one alternative investment proposal for use of scarce funds. This single alternative could very well be included in the model presented here.

The basis of the model consists of the following symbols used to denote all the relevant variables:

c_{tj} = cost of project j in period t

C_t = budget ceiling in period t

b_j = present value of all revenues and costs associated with project j

X_j = fraction of project j undertaken

Underlying this model are certain assumptions similar to those made previously concerning the Charnes, Cooper and Miller Model. First

of all, it is assumed that the costs of each project can be determined for each period with a fair amount of certainty. In actual practice it may take a good deal of time to come up with such estimates. The same would hold true of the budget ceiling; however, it would seem to be a bit easier to estimate since most of the variables affecting the budget are under the control of management. The present value of all revenues and costs for each project are also assumed to be known with relative certainty.

These above assumptions are basic to the model presented here. Without such estimates no basis would exist. It is realized that each assumption is a difficult problem in itself. Each will have to be handled individually and are merely side issues in the discussion presented here.

Returning to the problem at hand, the object is to maximize the present value of revenues and costs of each project given the budget constraints. In mathematical notation using the symbols defined above the function to be maximized is,

$$\text{Maximize } Z = \sum_{j=1}^n b_j X_j \quad (3.12)$$

where n equals the number of projects being considered.

In maximizing the present values of the alternative projects the obvious solution would be to take on all projects which have a positive present value or which provide acceptable returns. In most instances in the real world, however, the financial manager is constrained from doing so by his budgeted limitations. These constraints must, therefore, be taken into consideration by the linear programming model.

If the cost of a particular project, j , in period t in terms of that period's dollars is c_{tj} and the budget ceiling for period t is C_t , the set of constraints can be summarized by the inequality,

$$\sum_{j=1}^n c_{tj}X_j \leq C_t \quad (3.13)$$

for $t = 1, 2, \dots, m$ where m is the number of periods under consideration. Additional constraints must also be met. These are the non-zero constraints placed on the fractions of projects undertaken which make it impossible to have negative projects. Also, it is desirable in most instances to require that not more than one of any project be undertaken. These considerations lead to the expression

$$0 \leq X_j \leq 1 \quad (3.14)$$

where $j = 1, 2, \dots, n$.

The upper limit of one on X_j prevents the allocation of budgeted funds to multiples of the best project. If for some reason, however, it is desired or plausible to take on the same project more than once, the above constraints may be altered. For example if project X_1 could be taken on three times, project X_2 , two times, and the rest only once, the constraints could be rewritten as:

$$0 \leq X_1 \leq 3 \quad (3.15)$$

$$0 \leq X_2 \leq 2 \quad (3.16)$$

$$0 \leq X_j \leq 1 \quad (3.17)$$

where $j = 3, 4, \dots, n$.

Other constraints may be added such as requiring the present value of all revenues and costs associated with each project to be above some

minimum value M . These would be written as:

$$b_j \geq M \quad (3.18)$$

where $j = 1, 2, \dots, n$.

From these examples of added constraints the flexibility of the linear programming model is well illustrated. Looking at the basic problem again, if two periods are considered, or $t = 2$, with nine projects being involved, or $j = 9$, then equations 3.12, 3.13 and 3.14 can be rewritten to form the linear programming model given by,

$$\text{maximize } Z = b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 + b_6x_6 + b_7x_7 + b_8x_8 + b_9x_9$$

constrained by:

$$c_{11}X_1 + c_{12}X_2 + c_{13}X_3 + c_{14}X_4 + c_{15}X_5 + c_{16}X_6 + c_{17}X_7 + c_{18}X_8 + c_{19}X_9 + S_1 = C_1$$

$$c_{21}X_1 + c_{22}X_2 + c_{23}X_3 + c_{24}X_4 + c_{25}X_5 + c_{26}X_6 + c_{27}X_7 + c_{28}X_8 + c_{29}X_9 + S_2 = C_2$$

$$X_1 + q_1 = 1 \quad X_4 + q_4 = 1 \quad X_7 + q_7 = 1$$

$$X_2 + q_2 = 1 \quad X_5 + q_5 = 1 \quad X_8 + q_8 = 1$$

$$X_3 + q_3 = 1 \quad X_6 + q_6 = 1 \quad X_9 + q_9 = 1$$

As required by the linear programming model the slack variables for each constraint are added to give equalities. The X_j 's, S_1 , S_2 , and q_j 's are all assumed positive and need not be written explicitly since the simplex method takes this into account. The slack variables S_1 and S_2 represent the present values of the unspent portions of the budgeted amounts in periods one and two, respectively. The slack variables q_1, q_2, \dots, q_9 correspond to the fractional parts of projects one through nine not undertaken, respectively.

The initial simplex matrix for the model using two period and nine projects is given in Table III. The simplex solution would yield optimum values for X_1, X_2, \dots, X_9 and these in turn could be used to

determine the total present value of the optimum projects.

TABLE III
INITIAL SIMPLEX TABLEAU FOR WEINGARTNER MODEL

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	0	0	0	0	0	0	0	0	0	0	Solution	
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	s_1	s_2	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	
$0 S_1$	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	1										C_1	
$0 S_2$	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}		1										C_2
$0 q_1$	1											1									1
$0 q_2$		1											1								1
$0 q_3$			1											1							1
$0 q_4$				1											1						1
$0 q_5$					1											1					1
$0 q_6$						1											1				1
$0 q_7$							1											1			1
$0 q_8$								1											1		1
$0 q_9$									1											1	1

It was mentioned above that the problem of coming up with fractional projects in the solution to a linear programming problem of this type would not be serious enough to warrant using more sophisticated techniques (such as integer programming). In the example given by Table III using two periods and nine projects, the optimal solution would yield no more than two fractional projects. This result is due to a fundamental property of the model and method of solution which restricts the number of fractional projects to the number of budgeted

periods.⁵ In actual practice the amount of funds to be used on fractional projects could be allocated to the projects which provide the greatest net present value in accordance with their respective fractional feasibilities. In other words, if it were possible to undertake a fraction of one of the remaining projects, then this would be done for the project yielding the highest return; or alternatively, the budget constraints could be increased slightly to allow the attainment of integral projects.

This example is indeed of some value to the financial manager. It allows the determination of the optimum allocation of a fixed number of dollars to competing investment proposals. The main advantage of the linear programming approach to such problems lies in its computational efficiencies. If a problem of this type were solved by some other method such as that proposed by Lorie and Savage,⁶ it would be no doubt a long and laborous task of trial and error.

Ijiri, Levy, and Lyon Model⁷

The third model discussed here is related closely to finance in that it deals with budgeting and financial planning. The approach taken in the problem is from an accounting viewpoint; that is, the basic model is synthesized from a balance sheet. This balance sheet is shown in

⁵For a proof of this, see Weingartner, p. 35-38.

⁶Weingartner, Chapter 2.

⁷Y. Ijiri, P. F. Levy, and R. C. Lyon, "A Linear Programming Model for Budgeting and Financial Planning." Journal of Accounting Research, I (1963), pp. 198-212.

Table IV with the initial values of each account.

TABLE IV
BEGINNING BALANCE SHEET

C: Cash	\$ 7,260,000	P: Accounts Payable	\$ 3,592,000
B: Securities	12,000,000	D: Dividends and Taxes	
R: Accounts		Payable	2,922,000
Receivable	6,999,000	L: Loans Payable	4,400,000
G: Finished Goods	4,032,000	E: Stockholders'	
M: Raw Materials	1,499,000	Account	<u>46,876,000</u>
F: Fixed Assets	<u>26,000,000</u>		
	<u>\$57,790,000</u>		<u>\$57,790,000</u>

The notation used in the model refers to Table IV in that for any one transaction of X_{ij} number of dollars the i subscript refers to the account debited and j to the account credited. Specifically if $X_{CB} = \$5,000,000$, then the cash account is debited with \$5,000,000 and the securities account credited with the same amount. A profit and loss account is not used but instead each transaction is directly debited or credited to the stockholders' account. In addition, the initial value of each account is indicated by K with the appropriate subscript. For example, $K_B = \$12,000,000$.

The period covered in the problem is one month and the object is to maximize net additions to the stockholders' account. This can be stated as,

$$\text{maximize } Z = X_{CE} + X_{RE} + X_{GE} - X_{EF} - X_{EP} - X_{ED} \quad (3.19)$$

where X_{CE} represents income from interest, X_{RE} and X_{GE} represent contribution to profit from production and selling, X_{EF} represents depreciation, X_{EP} represents accounts paid, and X_{ED} represents dividends paid. These variables will become clearer as the constraints are explained.

There are nineteen constraints imposed upon the transactions involved. As shall be seen these are not as easily stated as the constraints given in the two previous problems. As each constraint is made certain obvious assumptions are made. To begin with it is assumed that the firm's sale of securities cannot exceed the beginning balance in the account--in other words, the firm cannot sell securities it has purchased during the period. This constraint is written as,

$$X_{CB} \leq K_B \quad (3.20)$$

Also purchases of securities are limited to the beginning cash balance as stated by,

$$X_{BC} \leq K_C \quad (3.21)$$

The maximum collection of receivables during the month is limited by the beginning receivables balance as stated by,

$$X_{CR} \leq K_R \quad (3.22)$$

Production capacity limits the amount of raw materials to be converted to finished goods which is written as

$$X_{GM} \leq 1,300,000 \quad (3.23)$$

where \$1,300,000 is the value of raw materials at standard cost. If conversion is not limited by 3.23, then it becomes limited by,

$$X_{GM} \leq K_M \quad (3.24)$$

Market conditions limit sales in this period to 2,000,000 units or \$4,200,000 at standard costs which is written as,

$$X_{RG} \leq 4,200,000 \quad (3.25)$$

If sales are not limited by market conditions, then they are limited by,

$$X_{RG} \leq K_G \quad (3.26)$$

Repayment of loans is limited by the outstanding loan balance at the beginning of the period as given by,

$$X_{LC} \leq K_L \quad (3.27)$$

The payment of accounts payable is limited to the sum of the beginning balance of accounts payable plus expenses of the current period expressed as,

$$X_{PC} \leq K_P + X_{EP} \quad (3.28)$$

Company policy requires a minimum cash balance of \$4,000,000 at the end of each period. This gives the constraint,

$$K_C + X_{CB} + X_{CR} + X_{CL} + X_{CE} - X_{BC} - X_{PC} - X_{DC} - X_{LC} \geq 4,000,000 \quad (3.29)$$

The firm is assumed to require an ending balance of finished goods inventory at least as great as \$3,750,000 which is stated as,

$$K_G + X_{GM} + X_{GE} - X_{RG} \geq 3,570,000 \quad (3.30)$$

The ending balance of raw materials must be great enough to cover the next period's production. The value of this production in raw materials is \$1,200,000 and the constraint is written as,

$$K_M + X_{MP} - X_{GN} \geq 1,200,000 \quad (3.31)$$

In addition to the inequalities as stated above, there are certain equality constraints which must be met. First, interest earned is equal to the end-of-period security balance multiplied by the periodic rate of 0.229 percent so that,

$$X_{CE} = .00229(K_B + X_{BC} - X_{CB}) \quad (3.32)$$

The contribution of a unit sale in terms of the corresponding deduction from finished goods is calculated to be 3.76⁸ times each unit so that

⁸This was derived by using a selling price of \$9.996+ and a standard cost of production of \$2.10/unit. Each unit sold gives rise to a contribution of \$7.896+ which is 3.76•\$2.10.

$$X_{RE} = 3.76X_{RG} \quad (3.33)$$

In addition, standard costs of finished goods is divided into material costs of \$1.00/unit and direct labor and overhead costs of \$1.10/unit. This gives the equality,

$$X_{GE} = 1.1X_{GM} \quad (3.34)$$

Equations 3.33 and 3.34 are necessary since the profit and loss statement has been bypassed.

Monthly depreciation charges are .833 percent of the net fixed assets owned at the beginning of the period which is stated by,

$$X_{EF} = .00833K_F \quad (3.35)$$

The costs to be incurred during the period include fixed operating costs of \$2,675,000, variable conversion costs of X_{GE} , interest penalty for not taking discounts on accounts payable (3.09 percent per period), and interest on loans payable at the end of the period at a rate of .291 percent on the remaining balance. These can be summarized in the equation,

$$X_{EP} = 2,675,000 + X_{GE} + .0309(K_P + X_{EP} - X_{PC}) + .0029(K_L + X_{CL} - X_{LC}) \quad (3.36)$$

The payables account is consolidated for all such costs.

Income tax is accrued at a rate of 52 percent of net profit. In addition, the firm declares a dividend of \$83,000 plus or minus five percent of the excess or shortage of profit after taxes of \$1,860,000. This relationship can be written as,

$$X_{ED} = .52(X_{CE} + X_{RE} + X_{GE} - X_{EF} - X_{EP}) + 83,000 + .05[.48(X_{CE} + X_{RE} + X_{GE} - X_{EF} - X_{EP}) - 1,860,000]$$

Simplified the above equation is

$$X_{ED} = .544X_{CE} + .544X_{RE} + .544X_{GE} - .544X_{EF} - .544X_{EP} - 10,000 \quad (3.37)$$

And as a final constraint all outstanding income taxes payable and dividends declared or accrued must be paid so that

$$K_D + X_{ED} - X_{DC} = 0 \quad (3.38)$$

Summarizing and putting in the necessary slack variables, we have the following linear programming model:

$$\text{maximize } Z = X_{CE} + X_{RE} + X_{GE} - X_{EF} - X_{EP} - X_{ED}$$

constrained by:

$$(3.20) \quad X_{CB} + S_1 = K_B$$

$$(3.21) \quad X_{BC} + S_2 = K_C$$

$$(3.22) \quad X_{CR} + S_3 = K_R$$

$$(3.23) \quad X_{GM} + S_4 = 1,300,000$$

$$(3.24) \quad X_{GM} + S_5 = K_M$$

$$(3.25) \quad X_{RG} + S_6 = 4,200,000$$

$$(3.26) \quad X_{RG} + S_7 = K_G$$

$$(3.27) \quad X_{LC} + S_8 = K_L$$

$$(3.28) \quad X_{PC} - X_{EP} + S_9 = K_P$$

$$(3.29) \quad X_{CB} + X_{CR} + X_{CL} + X_{CE} - X_{BC} - X_{PC} - X_{DC} - X_{CL} - S_{10} + A_{10} = 4,000,000 - K_C$$

$$(3.30) \quad X_{GM} + X_{GE} - X_{RG} - S_{11} + A_{11} = 3,570,000 - K_G$$

$$(3.31) \quad X_{MP} - X_{GM} - S_{12} + A_{12} = 1,200,000 - K_M$$

$$(3.32) \quad X_{CE} - .00229X_{BC} + .00229X_{CB} + A_{13} = .00229K_B$$

$$(3.33) \quad X_{RE} - 3.76X_{RG} + A_{14} = 0$$

$$(3.34) \quad X_{GE} - 1.1X_{GM} + A_{15} = 0$$

$$(3.35) \quad X_{EF} + A_{16} = .00833K_F$$

$$(3.36) \quad .9691 X_{EP} - X_{GE} + .0309 X_{PC} - .00291 X_{CL} + .00291 X_{LC} + A_{17} = \\ 2,675,000 + .0309 K_P + .00291 K_L$$

$$(3.37) \quad X_{ED} - .544 X_{CE} - .544 X_{RE} - .544 X_{GE} + .544 X_{EF} + .544 X_{EP} + A_{18} = 10,000$$

$$(3.38) \quad X_{ED} - X_{DC} + A_{19} = -K_D$$

and $X_{CB}, X_{BC}, X_{CR}, X_{GM}, X_{RG}, X_{LC}, X_{PC}, X_{EP}, X_{CE}, X_{DC}, X_{CL}, X_{GE}, X_{MP},$
 $X_{GM}, X_{RE}, X_{EF}, X_{ED} \geq 0$

The artificial slack variables A_{10}, \dots, A_{19} are added in order to satisfy the simplex requirements. The simplex method will yield a solution of 19 values in terms of the 37 variables both transaction and slack. If, however, an arbitrarily large negative contribution⁹ is assigned to the artificial slack variables, then these variables will not be in the final solution. From the variables in the optimal solution the financial manager is able to construct the optimum transactions to be made during the period that will yield the greatest increase to the stockholders' account or net profit.

This problem exemplifies the versatility of linear programming. The constraints and the basic model from which they were made does, however, make many simplifying assumptions. These assumptions could be made in different ways. For example, the minimum cash balance in equation 3.29 could be changed or the firm could be allowed to borrow and repay a loan during the period instead of the requirement of 3.27. The model itself is, therefore, adaptable to an unlimited number of situations.

⁹The contribution of a variable is simply the coefficient of that variable in the objective function.

Comparing this model with the last two it seems to be much more complex. In this situation one of the primary attributes of linear programming is apparent. This attribute is the power to conceive complex relationships in an organized and simplified manner. If the financial manager were to attempt to arrive at the optimal transactions as proposed by this problem, he would most likely spend most of the month and probably more using trial and error processes.

CHAPTER IV

SOME ASPECTS OF THE DUAL PROBLEM

In the previous chapter linear programming as a financial tool was shown to be of value in determining the optimum allocation of a scarce resource. Its flexibility was demonstrated by the addition and rearrangement of constraining equations. Linear programming, however, offers additional power as an aid to the financial manager through its dual. The attention of this chapter is focused upon the explanation and demonstration of the dual problem and its aspects which lead to more information useful to the financial manager.

To each linear programming problem there corresponds another linear programming problem related to it which is called the dual. The dual is extracted from the same data used in the original or primal problem and it is solved in the same way using the simplex method. The solution of a dual problem can easily be converted to a solution of the primal problem. This property is useful in the case where a problem contains a large number of rows and a small number of columns. If the dual is solved instead of the primal, there would be less total computational effort involved. This, however, not the only use to be made of the dual. Its real value lies in the analytical power of its objective function as will be shown below. A great deal of insight into the nature of the problem as well as valuable information is gained by careful analysis of the characteristics of the objective function.

It might be well to begin the discussion by referring back to the linear programming problem presented in Chapter II. The necessary data for the formulation of the dual is presented in Table V which is essentially the same as Table I except that the slack variables have not been entered. Two additional variables have been introduced, A and B, which constitute the framework for the formulation of the dual problem.

TABLE V
DUAL VARIABLES FOR PRODUCTION PROBLEM

	X	Y		
A	10	20	\leq	30,000 Forming
B	15	5	\leq	30,000 Stamping
	\downarrow	\downarrow		
	.02	.03		

Instead of maximizing some function as in the primal problem, the objective of the dual is to minimize the linear function. This function is shown by equation 4.1 and is formed by multiplying the corresponding constraining capacities in each row then taking their sum.

The constraints shown in equation 4.2 and 4.3 are obtained in a similar manner by multiplying A and B by the corresponding numbers in each row and taking their sum for each column. These sums are then set to be greater than or equal to the corresponding numbers at the bottom as indicated in Table V. These numbers, .02 and .03, are the original coefficients of X and Y, respectively.

$$\text{Minimize } d = 30,000A + 30,000B \quad (4.1)$$

constrained by:

$$10A + 15B \geq .02 \quad (4.2)$$

$$20A + 5B \geq .03 \quad (4.3)$$

$$A, B \geq 0 \quad (4.4)$$

The reason for reversing the inequality is obvious for if some minimum were not placed on A and B, the values of A and B at the optimum would be zero. The non-negative constraints are given by equation 4.4 which require that the dual variables be greater than or equal to zero as were the variables in the primal problem. Equations 4.1 through 4.4 constitute the dual problem and from here it is only necessary to add the slack variables and solve using the simplex method.

In order to satisfy the requirements of the simplex method it is necessary to introduce what are called artificial slack variables into equations 4.2 and 4.3. This was done in some of the constraints for the Ijiri, Levy and Lyon Model given in Chapter III. First, the ordinary slack variables are subtracted from the left-hand side of the equation since the right-hand side of the equation is less than the left. Then the artificial slack variables are introduced, A_1 and A_2 , which are added to the left-hand side to provide a slack variable with a coefficient of "+ 1" required by the simplex method. These artificial slack variables are included as computational devices to aid in the solution and are naturally not wanted in the optimum solution. To insure that A_1 and A_2 will not end up as part of the solution, an arbitrarily large negative contribution is assigned to each and is denoted by $-M$. This $-M$ value is taken to be larger than any other number in the program and will thus insure the elimination of A_1 and A_2 .

Equations 4.2 and 4.3 are now rewritten as,

$$10A + 15B - S_1 + A_1 = .02 \quad (4.5)$$

$$20A + 5B - S_2 + A_2 = .03 \quad (4.6)$$

and the initial simplex tableau is given in Table VI. The problem is solved in exactly the same way as the primal problem with the only difference being that the coefficients of the objective function are now opposite in sign. The objective of minimization necessitates the use of the opposite signs.

TABLE VI
INITIAL SIMPLEX TABLEAU FOR DUAL OF PRODUCTION PROBLEM

		-30,000	-30,000	0	0	-M	-M	
		A	B	S ₁	S ₂	A ₁	A ₂	Solution
-M	A ₁	10	15	-1	0	1	0	.02
-M	A ₂	20	5	0	-1	0	1	.03

The solution of the dual at the optimum yields the values for A and B:

$$A = .0014$$

$$B = .0004$$

When these values are substituted into equation 4.1, the optimum value of the objective function becomes,

$$d = 30,000(.0014) + 30,000(.0004) = 42 + 12 = \$54.00 \quad (4.7)$$

The optimum value of the objective function of the dual thus obtained is exactly the same as the optimum value of the primal problem as given by equation 2.10 in Chapter II. This relationship between the

dual and the primal is guaranteed by the dual theorem of linear programming.¹ It allows the solution of the dual and the objective function of the primal at the same time. In addition, it is possible to find the optimum values of the primal variables from the final simplex tableau of the dual problem. This tableau is shown in Table VII and the optimal solution to the primal is given by the numbers in the bottom row under the columns headed by S_1 and S_2 . These numbers are part of the result in solving the problem by the simplex method and are easily interpreted. The optimum value of X is indicated by the number in the column headed S_1 and is taken as having the opposite sign or 1800. Similarly under the column headed S_2 the indicated value of Y is 600. These values are exactly the same as those yielded by the primal problem.

TABLE VII
FINAL SIMPLEX TABLEAU FOR DUAL OF PRODUCTION PROBLEM

		-30,000	-30,000	0	0	-M	-M	
		A	B	S_1	S_2	A_1	A_2	Solution
-30,000	B	0	1	-.08	.04	.08	-.04	.0004
-30,000	A	1	0	.02	-.06	-.02	.06	.0014
		0	0	-1800	-600	30,000	30,000	
						-M	-M	

¹A. Charnes, et al., p. 23.

More important than this relationship, however, is the role of the dual variables as evaluators. This role can be demonstrated by an analysis of the optimum values of A and B. The value of A in the optimal solution gives the marginal contributions to profit of an additional unit of time on the forming machine and the value of B at the optimum gives the marginal contribution for the stamping machine. If for some reason management wished to determine the results of adding 1000 seconds of time to the capacity of the forming machine, then the dual used as an evaluator would show the increase in profits. This investment could then be compared to others to determine which is best.

The actual determination of profits assuming a 1000 second increase in the forming machines capacity is accomplished by substituting 31,000 for 30,000 as the coefficient of A in equation 4.1. It then becomes:

$$d = 31,000A + 30,000B \quad (4.8)$$

Then, substituting the optimum values of A and B equation 4.8 becomes,

$$d = 31,000(.0014) + 30,000(.0004) = 43.40 + 12.00 = \$55.40 \quad (4.9)$$

This gives the new profit level of \$55.40 with an increase in forming capacity of 1000 seconds. Similarly, any other increment for either the forming or stamping time capacities will yield a new profit level by simple substitution into the objective function of the dual.

The concept of the dual and its use as an evaluator can be extended to any linear programming problem. As an example, it might be worthwhile to look at the dual of one of the problems given in Chapter III.

The dual of the Charnes, Cooper and Miller model can be derived using the generalized data given in equations 3.4, 3.5 and 3.6. In addition, equations 3.8 and 3.9 help to visualize the forming of the

dual. These equations form an array of rows and columns; for example row one of equations 3.8 is $Y_1 \leq A$. Using the variables t_K and u_K and multiplying the coefficients of equations 3.8 by the u_K 's and the coefficients of equation 3.9 by the t_K 's the constraints would be obtained. For example, the first column would produce the equation,

$$-u_1 - u_2 - \dots - u_n + t_1 + t_2 + \dots + t_n \geq -C_1 \quad (4.10)$$

where C_1 is the coefficient of X_1 in the objective function. Proceeding in this way the following sets of equations are obtained to form the dual problem.

$$\text{minimize } d = \sum_{K=1}^n Au_K + \sum_{K=1}^n (B-A)t_K \quad (4.11)$$

constrained by,

$$- \sum_{i=K+1}^n u_i + \sum_{i=K}^n t_i \geq -C_K \quad K = 1, 2, \dots, n \quad (4.12)$$

$$\sum_{i=L}^n u_i - \sum_{i=L}^n t_i \geq P_L \quad n = 1, 2, \dots, n \quad (4.13)$$

$$t_i, u_i \geq 0 \quad i = 1, 2, \dots, n \quad (4.14)$$

If the optimum values of t_K and u_K are assumed to be t_K^* and u_K^* , then the objective function or equation 4.11 becomes,

$$d^* = \sum_{K=1}^n Au_K^* + \sum_{K=1}^n (B-A)t_K^* \quad (4.15)$$

By the dual theorem of linear programming d^* is equal to Z^* of equation 3.4 where the star indicates the optimum value of the objective function in both the dual and primal problems. This relationship is

represented as,

$$d^* = Z^* = \sum_{j=1}^n P_j Y_j^* - \sum_{j=1}^n C_j X_j^* \quad (4.16)$$

Theoretically then, any change in d^* reflects a change in Z^* and vice versa. Equation 4.15 can now be used to evaluate the constraining values of the primal problem, namely the asset of warehouse capacity. To illustrate this, assume that warehouse capacity, represented by B , is in terms of tons and that the firm is considering the addition of an extra ton of capacity. Total warehouse capacity then becomes $B+1$ which can now be substituted into equation 4.15 in place of B to get,

$$\sum_{K=1}^n Au_K^* + \sum_{K=1}^n (B+1-A)t_K^* \quad (4.17)$$

Rewriting equation 4.17 at a new optimum, it becomes,

$$d^{**} = \sum_{K=1}^n Au_K^* + \sum_{K=1}^n (B-A)t_K^* + \sum_{K=1}^n t_K^* \quad (4.18)$$

This new optimum value of the objective function is larger than d^* and this increment can be written as,

$$d^{**} = d^* + \sum_{K=1}^n t_K^* \quad (4.19)$$

Also, since the dual theorem implies that $d^{**} = Z^{**}$, it must be true that

$$Z^{**} = Z^* + \sum_{K=1}^n t_K^* \quad (4.20)$$

Equation 4.20 gives a new total profit level with the added ton of warehouse capacity.

From this and the previous example it is clear that the dual in linear programming gives to the financial manager additional guidance

in analyzing such problems. The analysis and interpretations of the dual evaluators as given here is only illustrative of the type of extensions which can be made using such procedures. The real value of the dual lies in the fact that it allows an evaluation of changes in the program without actually going through the process of solving the problem each time a change is made.

CHAPTER V

VALUE OF LINEAR PROGRAMMING IN FINANCE

The three examples of linear programming as applied to problems in finance are the extent of development of the tool in this area. Further development will undoubtedly come as has already been indicated in the June issue of the Journal of Financial and Quantitative Analysis.¹ In this article Van Horne adds a new twist to the problem of allocating a fixed amount of dollars among competing investment proposals by considering constraints imposed by the terms of the bond indenture or loan agreement. By including these the financial manager is able to evaluate the effects of certain protective covenants of the indenture or agreement. More specifically, the dual variables allow the determination of opportunity costs of these covenants; and armed with this knowledge the financial manager is in a much better position when negotiating the bond indenture or loan agreement.

From all indications linear programming seems to be an invaluable tool to the financial manager. Possible areas for future development of linear programming in finance are very difficult to forecast since this tool is so specialized. It would seem fair to predict that the main emphasis in the future will be toward the development and extension

¹James Van Horne, "A Linear Programming Approach to Evaluating Restrictions Under Bond Indenture or Loan Agreement," Journal of Financial and Quantitative Analysis, June, 1966, pp. 68-83.

of the investment allocation model such as the Weingartner and Van Horne models. This model could possibly begin with a larger model which would first allocate all available financial resources of the company to each specific use such as production, investment, dividends, etc. Then sub-models could be constructed to handle each of the specific uses of funds such as the investment model. By interconnecting each model a comprehensive structure is obtained that would determine the financial plans of the entire company.

Obviously before linear programming is developed to such an advanced stage as this, many problems would have to be overcome. These problems are now seriously limiting the accuracy of the predictions given by linear programming and the next section is devoted to a short discussion of these disadvantages.

Summary of Disadvantages and Advantages

Two major types of problems are encountered in applying linear programming to a specific situation. First, linear programming, as well as most other mathematical models, deems necessary a considerable amount of oversimplification in order to fit the real world to the model. As was mentioned in an earlier chapter, the major difficulty with linear programming in this area is the linearity requirements of the relationships involved. Key factors are often completely disregarded, and as a result the simplified version of the real world does not adequately predict future outcomes. This disadvantage will only be overcome by exploring possible alterations of the existing state of mathematical programming into other areas such as non-linear programming.

A second difficulty encountered in using linear programming is the necessity of quantification of the variables involved. In many instances these variables are seemingly qualitative in nature and can be expressed in quantitative terms only by using subjective judgments. This problem was discussed in Chapter I in connection with uncertainty and risk. The degree to which uncertainty is involved will determine the seriousness of the problem. If uncertainty can be resolved in some way and refinements in forecasts and predictions are made, then the results of the linear programming model will come closer to reality. Thus, the disadvantage will be overcome not by changing the model but by changing or creating ways of measuring and forecasting the factors involved in the problem. In addition, the lack of necessary data has previously hampered the accurate measurements and forecasts often encountered in a business situation. With the increased use of computers more data than ever thought possible will be made available to the decision maker.

Certain advantages inherent in the linear programming model also exist. First, linear programming requires that the objective be clearly and concisely stated and that all key factors be identified. Consequently, relationships between variables become readily apparent and much insight into the actual problem is gained by going through such an analysis. This procedure forces the attention of the financial decision maker upon all details while maintaining an organized structure.

Secondly, and most important, linear programming provides a method whereby the financial decision maker can consider many alternatives within the framework of a single model. In solving the linear programming problem virtually all alternatives are considered, and the optimum

choice is made from among them. To accomplish this task with conventional methods would be in most instances an impossibility.

Adaptability and flexibility also provide the financial manager with many choices as to how the model can be built around his specific problem. This advantage was demonstrated throughout the paper by the simple addition or alteration of constraining equations.

Lastly, the effects of any one variable may be exemplified. The dual aspects were shown to give this added advantage in Chapter VI where a marginal analysis may be made on the effects of changing the value of the variable involved.

Conclusion

The conclusion reached here is that linear programming used in financial decision making is indeed an improvement over past methods. The areas of finance to which it is applicable at this time are limited to financial budgeting and analysis of investment proposals. Further development of applications will no doubt be coming in the future as soon as new techniques are developed. As of now linear programming has prompted research in other areas such as dynamic programming, non-linear programming, and integer programming all of which come under the heading of mathematical programming. These new techniques all hold great promise of becoming applicable to problems of a financial nature.

Before closing it must be emphasized that linear programming is not meant to be a substitute for good judgment on the part of the financial decision maker but only a supplement. The results are not used as judgments in themselves, but rather as guides to good judgment.

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Scope and Method of Study: This report consists of a description of linear programming and its uses in financial management. The theme exemplified throughout the report is that linear programming is indeed a viable approach to problem solving in finance. Three possible applications are presented with an analysis of each. One application concerns costing of funds, a second concerns capital budgeting and the third, budgeting and financial planning. In addition the dual aspects of linear programming and their relevance to financial problems are analyzed in a separate chapter.

Findings and Conclusions: The conclusion reached in this report is that linear programming used as a tool in financial decision making is an improvement over past methods. The main attribute of value to the financial manager is that all alternatives of a given situation can be considered within the framework of this single model, linear programming. Also of measurable advantage is the adaptability and flexibility of the technique. Other areas of research along this line include investigation into possible applications of dynamic, integer, and non-linear programming.

ADVISER'S APPROVAL

