

SENSITIVITY ANALYSIS OF THE  
DECISION MODELS

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DOCTOR OF PHILOSOPHY  
July 1967

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DECISION MODELS

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## PREFACE

This investigation is based upon management's need for operating ranges to provide managerial flexibility. These ranges must be provided in such a way that any selected value will yield an insignificant increase or decrease in the objective function (the total cost function or the total profit function).

This thesis consists of five chapters.

Chapter I, Introduction, is intended to present to the reader the need for undertaking a study of the Sensitivity Analysis of Decision Models. It explains why the Sensitivity Analysis of Decision Models is needed for management.

Chapter II, Review of the Literature, presents the results of the literature review. It summarizes the most of the work in this field which has been published as articles and technical notes.

Chapter III, Mathematical Models, develops the basis for the mathematical analysis required.

Chapter IV, Inventory Systems, derives a variety of inventory models. It also presents the sensitivity analysis of these models around their mathematical optimum for the purpose of establishing decision ranges.

Chapter V, Recommendations and Conclusions, summarizes the results achieved and recommendations for further studies.

Interest in this area of Sensitivity Analysis of Decision Models

began in the spring 1966 when the writer was a student of Dr. James E. Shamblin at Oklahoma State University. Interest in this area has continued to grow through my association with Dr. James E. Shamblin to whom I acknowledge my great indebtedness for his inspiration and encouragement.

The leadership of Professor Wilson J. Bentley, Head of the School of Industrial Engineering and Management, has been an inspiration. Gratitude and acknowledgement are extended to Dr. Paul E. Torgersen, who provided leadership in my graduate work. Dr. Leroy J. Folks and Professor Rollo E. Venn deserve credit for assisting in my doctoral program.

The Ph.D. program and the research resulting in this dissertation were supported by the Egyptian Iron and Steel Corporation. Indebtedness is acknowledged to the Corporation, particularly to Dr. Samir Taher.

A debt of gratitude is acknowledged to the staff of Cultural and Educational Bureau, UAR, Washington, D. C., for their contribution in facilitating the completion of my Ph.D. program.

Special thanks are due to each of the following for programming, typing, and photographing this thesis: Mrs. Mary Carney, Mrs. Louise Murphy, Mrs. Billie Sue Blevins, and Mr. Jack Shanklin.

Finally, I would like to express appreciation to my wife, Wegdan Charara; sons, Ayman and Louay; and daughter, Hanan, whose understanding, encouragement, and sacrifice were instrumental in the completion of my doctoral program and the preparation of this dissertation. God bless all of them and enable me to compensate their hardships and sacrifices.

## TABLE OF CONTENTS

Chapter	Page
I. INTRODUCTION . . . . .	1
II. REVIEW OF LITERATURE . . . . .	4
III. MATHEMATICAL MODELS . . . . .	11
Mathematical Techniques . . . . .	12
Models of One Standard Curve . . . . .	21
Models as a Summation of Two Different Standard Curves . . . . .	32
Models as a Summation of Three Different Standard Curves . . . . .	64
Summary of the Sensitivity Analysis of the Mathematical Models . . . . .	72
IV. INVENTORY MODELS . . . . .	78
Introduction . . . . .	78
Building the Total Cost Inventory Function . . . . .	83
Optimality and Dimensional Analysis . . . . .	93
Uniform Demand - Purchasing or Infinite Replenishment Models . . . . .	99
Power Demand - Purchasing or Infinite Replenishment Models . . . . .	121
Uniform Demand - Manufacturing or Finite Replenishment Models . . . . .	144
Factors Introducing Errors. . . . .	155
V. SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS . . . . .	157
BIBLIOGRAPHY . . . . .	161
APPENDIX A - COST OR MINIMIZATION FUNCTIONS . . . . .	163
APPENDIX B - PROFIT OR MAXIMIZATION FUNCTIONS . . . . .	192

LIST OF TABLES

Table	Page
I. Summary of the Sensitivity Analysis of Mathematical Models One Standard Curve Models . . . . .	73
II. Summary of the Sensitivity Analysis of Mathematical Models Summation of Two Different Standard Curves (Minimiz. ) Models. . . . .	74
III. Summary of the Sensitivity Analysis of Mathematical Models Summation of Two Different Standard Curves (Maximiz. ) Models. . . . .	76
IV. Summary of the Sensitivity Analysis of Mathematical Models Summation of Three Different Standard Curves Model . . . . .	77
V. Sensitivity Analysis of Minimization or Cost Functions Parabola Versus Straight Line Model . . . . .	172
VI. Sensitivity Analysis of Minimization or Cost Functions Straight Line Versus Rectangular Hyperbola Model. . . . .	173
VII. Sensitivity Analysis of Minimization or Cost Functions Parabola Versus Straight Line Model . . . . .	174
VIII. Sensitivity Analysis of Minimization or Cost Functions Parabola Versus Parabola Model. . . . .	175
IX. Sensitivity Analysis of Minimization or Cost Functions Parabola Versus Rectangular Hyperbola Model . . . . .	176
X. Sensitivity Analysis of Minimization or Cost Functions Cubical Parabola Versus Straight Line Model . . . . .	177
XI. Sensitivity Analysis of Minimization or Cost Functions Cubical Parabola Versus Parabola Model. . . . .	178
XII. Sensitivity Analysis of Minimization or Cost Functions Cubical Parabola Versus Rectangular Hyperbola Model . . . . .	179
XIII. Sensitivity Analysis of Minimization or Cost Functions Straight Line Versus Reciprocal of Quadratic Function Model. . . . .	180

Table	Page
XIV. Limits of Minimization or Cost Functions' Decision Ranges for 10% Increase in the Total Variable Cost Elements Above its Optimum . . . . .	181
XV. Sensitivity Analysis of Maximization or Profit Functions Parabola Versus Straight Line Model. . . . .	200
XVI. Sensitivity Analysis of Maximization or Profit Functions Straight Line Versus Parabola Model. . . . .	201
XVII. Sensitivity Analysis of Maximization or Profit Functions Parabola Versus Parabola Model . . . . .	202
XVIII. Sensitivity Analysis of Maximization or Profit Functions Straight Line Versus Cubical Parabola Model. . . . .	203
XIX. Sensitivity Analysis of Maximization or Profit Functions Parabola Versus Cubical Parabola Model . . . . .	204
XX. Limits of Maximization or Profit Functions' Decision Ranges for 10% Decrease in the Total Variable Profit Elements Under its Optimum . . . . .	205

## LIST OF FIGURES

Figure	Page
1. Minimization Curves . . . . .	17
2. Maximization Curves . . . . .	18
3. Replenishment Patterns . . . . .	84
4. Demand Patterns . . . . .	87
5. Uniform Replenishment and Demand Model . . . . .	90
6. Power Replenishment ( $n_1 < 1.0$ ) and Demand ( $n_2 < 1.0$ ) Model. .	90
7. Power Replenishment ( $n_1 > 1.0$ ) and Demand ( $n_2 > 1.0$ ) Model. .	91
8. Power Replenishment ( $n_1 < 1.0$ ) and Demand ( $n_2 > 1.0$ ) Model. .	91
9. Power Replenishment ( $n_1 > 1.0$ ) and Demand ( $n_2 < 1.0$ ) Model. .	91
10. Annual Analysis . . . . .	93
11. Periodic Analysis . . . . .	95
12. Purchasing, Uniform Demand, and Holding and Shortage Costs Elements Model. . . . .	100
13. Purchasing, Uniform Demand, and Holding and Procurement Costs Elements Model. . . . .	107
14. Purchasing, Uniform Demand, and Shortage and Procurement Cost Elements Model . . . . .	111
15. Uniform Demand, Purchasing, and Holding, Shortage, and Procurement Cost Elements Model . . . . .	114
16. Purchasing, Power Demand, Holding and Shortage Costs Model . . . . .	122
17. Power Demand, Purchasing, and Holding Shortage and Procurement Cost Elements Model . . . . .	136
18. Uniform Demand - Manufacturing or Finite Replenishment Model . . . . .	144



Figure	Page
19. Cost (Minimization) Functions Straight Line vs Parabola Model . . . . .	183
20. Cost (Minimization) Function Straight Line vs R. Hyperbola Model. . . . .	184
21. Cost (Minimization) Function-Parabola vs Straight Line Model . . . . .	185
22. Cost (Minimization) Function-Parabola vs Parabola Model. . . . .	186
23. Cost (Minimization) Function-Parabola vs Rectangular Hyperbola . . . . .	187
24. Cost (Minimization) Function-Cubical Parabola vs Straight Line. . . . .	188
25. Cost (Minimization) Function-Cubical Parabola vs Parabola Model . . . . .	189
26. Cost (Minimization) Function-Cubical Parabola vs Rectangular Hyperbola. . . . .	190
27. Cost (Minimization) Function-Straight Line vs Reciprocal of Q. Function. . . . .	191
28. Profit (Maximization) Function-Parabola vs Straight Line Model . . . . .	206
29. Profit (Maximization) Function-Straight Line vs Parabola Model . . . . .	207
30. Profit (Maximization) Function-Parabola vs Parabola Model . . . . .	208
31. Profit (Maximization) Function-Straight Line vs Cubical Parabola Model . . . . .	209
32. Profit (Maximization) Function-Parabola vs Cubical Parabola Model . . . . .	210

## CHAPTER I

### INTRODUCTION

When Operations Research techniques are applied to industrial situations, the most common procedure is to determine the optimum trade-off relationship between increasing and decreasing cost factors. To produce the least cost and the most profit, current Operations Research techniques have been carefully developed to determine the point at which the objective function is maximized or minimized.

The exact optimum point is not required in actual practice. From management's point of view, it is not even desirable due to the difficulties involved in its exact calculation. Instead, the vital information is the range or region of the operating variables which will result in an approximately minimum cost and maximum profit. Sometimes, the problem of determining this range is solved by asking from what range variables may be selected to avoid a significant increase in the cost for a system or a significant decrease in the profit. A knowledge of this range then provides management with a flexibility sufficient to perform its function of managing according to the dynamic needs of the situation and the environment.

The procedures which are used to establish these management ranges are broadly classified as sensitivity analysis. For a number of reasons, an understanding of these ranges and their effects as studied by sensitivity analysis is often of major importance to management. Under

changes of parameters, for instance, the stability of the optimum solution may frequently become critical. Using the exact optimum solution point, a slight variation of a parameter in one direction may result in a large, unfavorable difference in the objective function ( total cost function or total profit function). At the same time, a large variation of the parameters in a different direction may result in a small difference. In industrial situations, where there are certain inherent variabilities in the estimates of the parameters not taken account of in the model, it may sometimes be desirable to move away from the optimum solution in order to achieve a solution less likely to require essential modification.

The values of the coefficients of the objective function may be controllable to some extent. If this is the case, it is essential for management to know the effects which would result from changing these values. Even if these coefficients are not controllable, the estimates for their value may be only approximate. This makes it important to know for what ranges of their values the solution is still desirable. If it is determined that the optimum solution is extremely sensitive to their values, it may become necessary to obtain better estimates.

In a like manner, a model that involves a fixed or a first cost which does not vary with the decision variables may be insensitive as a whole. This might be in spite of the fact that variable parts in it are sensitive within a predetermined increase or decrease in the objective function above its optimum. This is yet another factor which serves to establish the validity and importance of sensitivity analysis.

In view of this importance, this research project proposes to develop several general mathematical models and investigate their

sensitivity. With these models as a guide, the project will continue to investigate the application of sensitivity analysis toward general trade-off relationships which might be encountered in the wide spectrum of problems facing the analysts who are currently engaged in applying quantitative methods to management.

Specifically, it is proposed that this research develops models for a variety of total cost curve or surface equations and total profit curve or surface equations in general mathematical forms.

It is further proposed that the project classify types of problems which may be treated by those generalized equations and fully develop this method of analysis for some of the classical Operations Research techniques now in use.

## CHAPTER II

### REVIEW OF LITERATURE

The initial reference to the use of an economic lot range to find the purchase quantity in situations where a quantity discount was offered apparently occurred in a journal article by Munn, Mary and Sarah Hayward (12) early in 1951. A similar approach to selecting buying quantity was published in 1957 by Norman, Lang E. (10).

Since that time, Solomon, M. J. (15) has published a significant and useful study on economic lot ranges. In a journal monograph of 1959, Solomon differentiated the total cost function to obtain the economic lot size and the optimum total cost as:

$$Q_o = \sqrt{\frac{2YS}{CI}}$$

and

$$T_o = YC + \frac{YS}{Q_o} + \frac{Q_o C}{2} I + \frac{S}{2} I.$$

Manipulating these functions, Solomon obtained:

$$\frac{(P-1)^2}{2P} = \lambda \left[ 1 + \frac{YC + S/2}{CIQ_o} \right].$$

Let:

- $T_o$  = the optimum total cost for the year
- $Q_o$  = the optimum lot size
- $Y$  = the number of units sold in one year
- $\delta$  = the cost per lot (set-up and paper-work)
- $Q$  = the lot size

$I$  = the cost of carrying inventory on an annual basis

$\lambda$  = the yearly cost increases of the optimum cost equal to some predetermined proportion

$P$  = the proportion of  $Q_o$  that corresponds to a penalty proportion of  $\lambda$ .

Using a numerical example, Solomon allowed  $\lambda = 0.01$  and finally derived values for the lower lot size  $Q_L$  and the upper lot size  $Q_u$ . He determined that:

$$Q_L = 0.72 Q_o$$

and

$$Q_u = 1.40 Q_o.$$

The author tabulated tables for lot size versus total cost, for  $P$  versus  $\frac{(P-1)^2}{2P}$ , and for different monthly useages for one product using the routine and expediting ranges in which the last ranges correspond to a penalty of  $k$  and in which  $k < 1$ . The last tables was computed for relatively homogeneous groups of products with approximately the same fixed and variable cost factors. If, however, a product involves an operation with a temporary bottleneck, the load on the bottleneck facility can be reduced by using the lower part of the range. If the situation is critical, then the critical range (expediting range) can be used. Thus, the expediting range can be resorted to in emergency situations. Such a table has the advantage of enabling a clerk to make scheduling and inventory decisions that are consistent with scheduling criteria on a roughly optimum basis. The proposed approach is a way of dealing with one dimension economy of manufacture for scheduling purposes.

The approach proposed by Solomon is a form of sensitivity analysis which should prove useful wherever the objective is to minimize or maximize the quadratic function  $\frac{(P-1)^2}{2P}$  subject to other considerations which are more intangible.

Another significant contribution to sensitivity analysis concerning the range of variables was made by Disney, R. L. (3) in a journal article published in 1962. In presenting a model of an inventory system for his research, Disney provided several assumptions. He assumed that demand is known and constant, lead time is known and constant, that running out of goods is prohibitively expensive, and that the yearly cost to hold and the cost to place are not inclusive. Disney neglected the fixed costs in his study because they do not affect the quantity to purchase nor the time of purchase.

Disney expressed the variable cost factors in the inventory system as a function of the exact data needed as:

$$100 \frac{C-C_o}{C_o} = \frac{(r-1)^2}{2r} .$$

When,

$$r = \frac{N}{N_o} = \frac{(\bar{N}/N) (\bar{C}_2/C_2)}{\bar{C}_1/C_1} .$$

Let:

$C$  = the cost of the system at any time

$C_o$  = the cost of the system when all needed information is known exactly

$r$  = the ratio of the quantity to purchase at any time to the quantity to purchase when all needed information is known exactly

$N$  = the demand for a year

$C_1$  = the yearly cost to hold

$C_2$  = the cost to place one order.

(Par values indicate estimated values; values without par indicate exact values).

The author tabulated the results of changing the ratios  $\frac{\bar{N}}{N}$ ,  $\frac{\bar{C}_2}{C_2}$ , and  $\frac{\bar{C}_1}{C_1}$  and then gave the corresponding effect on the total variable cost.

Disney concluded that the quantity to purchase can take a value between sixty-five per cent and one-hundred and fifty-five per cent of the exact value when the yield on the total variable cost is less than ten per cent above the optimum value.

A somewhat similar study was published by Withycombe, Richard (19) a year later in 1963. In his work, Withycombe defined  $q$  as the multiplier of the cost of ordering and carrying inventory in dollar units, and  $p$  as a multiplier of the economic order quantity (EOQ). This author then defined the relationship between these multipliers as:

$$q = \frac{1 + p^2}{2p} .$$

Giving a numerical example as an application to the above formula and table, the author tabulated  $q$  versus  $p$ . Withycombe concluded that a considerable latitude can be tolerated in making derivations from the calculated EOQ, and that the EOQ, as a general policy, may be rounded to within plus or minus ten per cent in order to form convenient lots. If other factors such as price concessions or standard packaging indicate that it may be desirable to deviate by more than ten per cent from the EOQ, the cost of such an action should be evaluated.

A later technical note by Rutenberg, H. Y. (14) in 1964 stressed the study of the ranges of set-up cost. Here, the variable is the set-up cost while the other parameters in the total cost equation are assumed to be constants. Rutenberg derived the optimum values as:

$$Q_o = \sqrt{\frac{2YS_o}{CI}}$$

and

$$T_o = \sqrt{2YCIS_o} .$$

Let:



$Q_o$  = the economic order quantity (EOQ) in units

$Y$  = the annual usage in units/year

$S_o$  = the true set-up cost in dollar/units

$C$  = the inventory cost at standard in dollar/units

$I$  = the inventory carrying cost in dollar/dollar of the inventory year

$T_o$  = the optimum total variable cost in dollars.

If the set-up cost is taken from the calculation as  $\bar{S}$  instead of as the exact value of  $S$ , then the above optimum values equations yield to become:

$$Q(\bar{S}) = \sqrt{\frac{2Y\bar{S}}{CI}}$$

and

$$T(\bar{S}) = \sqrt{2YCIS}.$$

The author defined  $P$  as the per cent of increase in the total annual cost when using  $\bar{S}$  instead of  $S$ . He then derived the expression:

$$P = \left( \sqrt{\frac{S}{4\bar{S}}} + \sqrt{\frac{\bar{S}}{4S}} - 1 \right) 100.$$

Manipulating this expression, the author substituted:

$$R = \frac{1}{2} \left( \frac{S}{\bar{S}} + \frac{\bar{S}}{S} \right)$$

and came to a quadratic function:

$$S^2 - 2RSS + \bar{S}^2 = 0.$$

The solutions of this quadratic equation are the lower limit of the set-up cost ( $S_L$ ) and the upper limit of the set-up cost ( $S_U$ ):

$$S_L = \bar{S} \left( R - \sqrt{R^2 - 1} \right)$$

and

$$S_U = \bar{S} \left( R + \sqrt{R^2 - 1} \right).$$

As a direct application to his study, Rutenberg gave a significant and interesting example demonstrating that even with a set-up cost

ranging from one to over one-thousand dollars, the use of the estimated value  $\bar{S}$  rather than the exact value  $S$  had very little effect on the total annual cost of ordering and carrying inventory. The benefit from Rutenberg's study is that it gives a wide range for set-up cost while, at the same time, the net increase in the inventory total variable cost is very little.

Important recent work has extended inventory theory beyond its strictly classical bounds. With the presentation of many new models by Dr. Naddor, Eliezer (11), there has been a new basis showing that sensitivity analysis can be based upon tabulated data.

In summarizing the related articles, notes, and citations which make up the bulk of the general literature prior and relevant to the present research, we find that all previous authors who have dealt with ordering and set-up cost and the carrying or holding cost have, regardless of their purpose, considered only the simple inventory model of the two cost elements. This simple inventory model can be represented mathematically as:

$$Y = AX + \frac{B}{X} .$$

The most recent work by Dr. Naddor does, indeed, include more extensive inventory models and their sensitivity analyses. But, again, these sensitivity analyses are based on the tabulated data only.

The purpose of this research project is to provide a variety of mathematical models and to include their mathematical sensitivity expressions. It later seeks the application of these models and their sensitivity expressions to decision models in the Operations Research field. Some of these applications might be inventory models, replacement models, production models, or queuing models. The scope of this

dissertation is limited to the continuous functions, leaving discrete functions to further investigation.

## CHAPTER III

### MATHEMATICAL MODELS

This chapter is devoted to the development of mathematical models of one independent variable  $X$ . The investigation is for the purpose of obtaining the optimum value  $X_0$  and the corresponding optimum value  $Y_0$ . Knowing the mathematical optimum  $(X_0, Y_0)$ , from what range may the independent variable be selected in a way that yields an insignificant change in the dependent variable  $Y_w$  above the optimum value  $Y_0$ ? The determination of the model range according to a predetermined setting standard allowance is made through the derived sensitivity formula. The setting standard allowance assumed in this study is a 10 per cent increase in the total variable elements over its optimum value (minimization function), and a 10 per cent decrease in the total variable elements, over its optimum value (maximization function).

The first part of this chapter describes the mathematical techniques using an illustrated example. The second part investigates the mathematical models graphed as one standard curve such as the straight line, circle, ellipse, hyperbola, rectangular hyperbola, Gaussian, and catenary curve. The third part analyzes the mathematical models graphed as a sum of two-standard curves such as a straight line versus a parabola, a straight line versus rectangular hyperbola, and others. The first curve

in these mathematical models is assumed to be an increasing function while the second curve is assumed to be a decreasing function. This assumption is necessary to provide the conditions for the existence of optimality for the model. The fourth part investigates the mathematical models graphed as the sum of three different standard curves such as a straight line versus a parabola and a rectangular hyperbola, a straight line versus a straight line and a rectangular hyperbola, and a parabola versus a straight line and a rectangular hyperbola. The first curve(s) is (are) assumed to be an increasing function(s) and the other(s) is (are) assumed to be decreasing function(s). This assumption is again essential in providing the optimality in the model. The last part summarizes the mathematical models discussed and their sensitivity formulas.

#### Mathematical Techniques

As was mentioned before, the study is concerned with obtaining the optimum point for these models as well as their sensitivity expressions. For the predetermined setting standard allowance increase or decrease above or under the optimum value  $Y_0$ , the study is devoted to establishing a range from which the variable  $X$  is selected. For the time being, we will call this range the decision range (defined in Chapter IV).

The mathematical techniques used are differentiation and transformation. Differentiation provides the optimality, and both differentiation and transformation are used for obtaining the general sensitivity expression. The model can be expressed mathematically as

$$Y = F(X).$$

By differentiating the function  $Y$  with respect to  $X$ , the derivative of the function is expressed as

$$\frac{dY}{dX} = F'(X).$$

At this point,  $F'(X)$  should be equated to zero to obtain the optimum value  $X_0$ . Thus

$$F'(X_0) = 0$$

and can be solved to find  $X_0$ . By substituting  $X_0$  in the original function  $Y = F(X)$  the corresponding optimum value  $Y_0$  can be obtained. This can be expressed mathematically as

$$Y_0 = F(X_0).$$

Therefore, the mathematical optimum point of this function is  $(X_0, Y_0)$ . The sensitivity expression of this mathematical model can be found by assuming an error  $w$  incurred in the calculating or estimating of the optimum variable  $X_0$  in such a way that

$$X_w = wX_0$$

where  $w$  is zero or a positive value by definition.

The effect of this error  $w$  on the optimum value  $Y_0$  is an increase in the case of a minimization curve or a decrease in the case of a maximization curve. The measurement of the sensitivity can be done through these ratios

$$\frac{Y_w}{Y_0} = \frac{\text{the error value } Y_w}{\text{the exact optimum value } Y_0}$$

or

$$\frac{Y_w - Y_o}{Y_o} = \frac{\text{the difference between the error value}}{\text{the exact optimum}}$$

$$\frac{Y_w \text{ and the exact optimum } Y_o}{\text{value } Y_o}$$

The result from the first expression is the same as the second, with 1.0 being the only difference between them. Thus, for the purpose of this dissertation we will deal with the first expression.

Both ratios define the sensitivity expression. The sensitivity expression is known as a general one if it is only a function of error  $w$ , otherwise the sensitivity expression is a special one. As is explained later in the illustrated example, the transformation technique is used to eliminate the constant term in the original function, and, subsequently, it is sometimes helpful to provide a/the general sensitivity expression.

#### Illustrated Example

The following cited example is the sum of two independent curves: an increasing parabola curve  $Y_1 = AX^2$  and a decreasing straight line curve with a positive intercept on the Y-axis  $Y_2 = C - BX$ . Thus, the sum of these two curves is

$$Y = Y_1 + Y_2 = AX^2 + C - BX.$$

The optimum point  $(X_o, Y_o)$  of this total function can be obtained by differentiating  $Y$  with respect to  $X$ , equating the result to zero, solving for  $X_o$ , and substituting  $X_o$  in the  $Y$  function to obtain  $Y_o$  as follows

$$\frac{dY}{dX} = 2AX + 0 - B$$

$$0 = 2AX_0 - B$$

$$X_0 = \frac{B}{2A}$$

$$Y_0 = AX_0^2 - BX_0 + C$$

$$= A \frac{B^2}{4A^2} - B \frac{B}{2A} + C$$

$$= -\frac{B^2}{4A} + C$$

Therefore the optimum point of the function  $(X_0, Y_0)$  is

$$\left(\frac{B}{2A}, -\frac{B^2}{4A} + C\right).$$

The sensitivity expression can be obtained by assuming that an error  $w$  has been associated with  $X_0$  in such a way that

$$X_w = wX_0$$

and

$$w = \text{zero or a positive value.}$$

Thus, the corresponding value  $Y_w$  which results from considering

$X_w = wX_0$  instead of the optimum value  $X_0$  is

$$Y_w = w^2 \frac{B^2}{4A} - w \frac{B^2}{2A} + C$$

and the sensitivity expression is

$$\frac{Y_w}{Y_0} = \frac{\frac{B^2}{4A} (w^2 - 2w) + C}{-\frac{B^2}{4A} + C}.$$



At this point the expression is not a general one. To obtain a general expression, transform the original function  $Y$  to another new function  $Y'$  such that the origin  $(0, 0)$  is shifted to another new origin  $(0, -C)$ . Thus,

$$X_w = X_w' = wX_o = wX_o'$$

$$w = w' = \text{zero or a positive value}$$

$$Y_w' = Y_w - C = \frac{B}{4A} (w^2 - 2w)$$

$$Y_o' = Y_o - C = -\frac{B}{4A}$$

and

$$\frac{Y_w'}{|Y_o'|} = \frac{Y_w - C}{|Y_o - C|} = w^2 - 2w.$$

By setting a standard 10 per cent increase over the transformed optimum  $Y_o'$  (the minimum value of the variable parts in the function) or the 10 per cent decrease under the transformed optimum  $Y_o'$  (the maximum of the variable parts of the function) four possibilities arise as in Figures 1 and 2:

- a) The original function is a minimization and the optimum of the transformed function  $Y_o'$  is a negative value. Thus

$$\frac{Y_w'}{|Y_o'|} = \frac{-0.90}{1.00} = -0.90$$

- b) The original function is a minimization and the transformed optimum  $Y_o$  is a positive value. Thus

$$\frac{Y_w'}{|Y_o'|} = \frac{1.10}{1.00} = 1.10$$

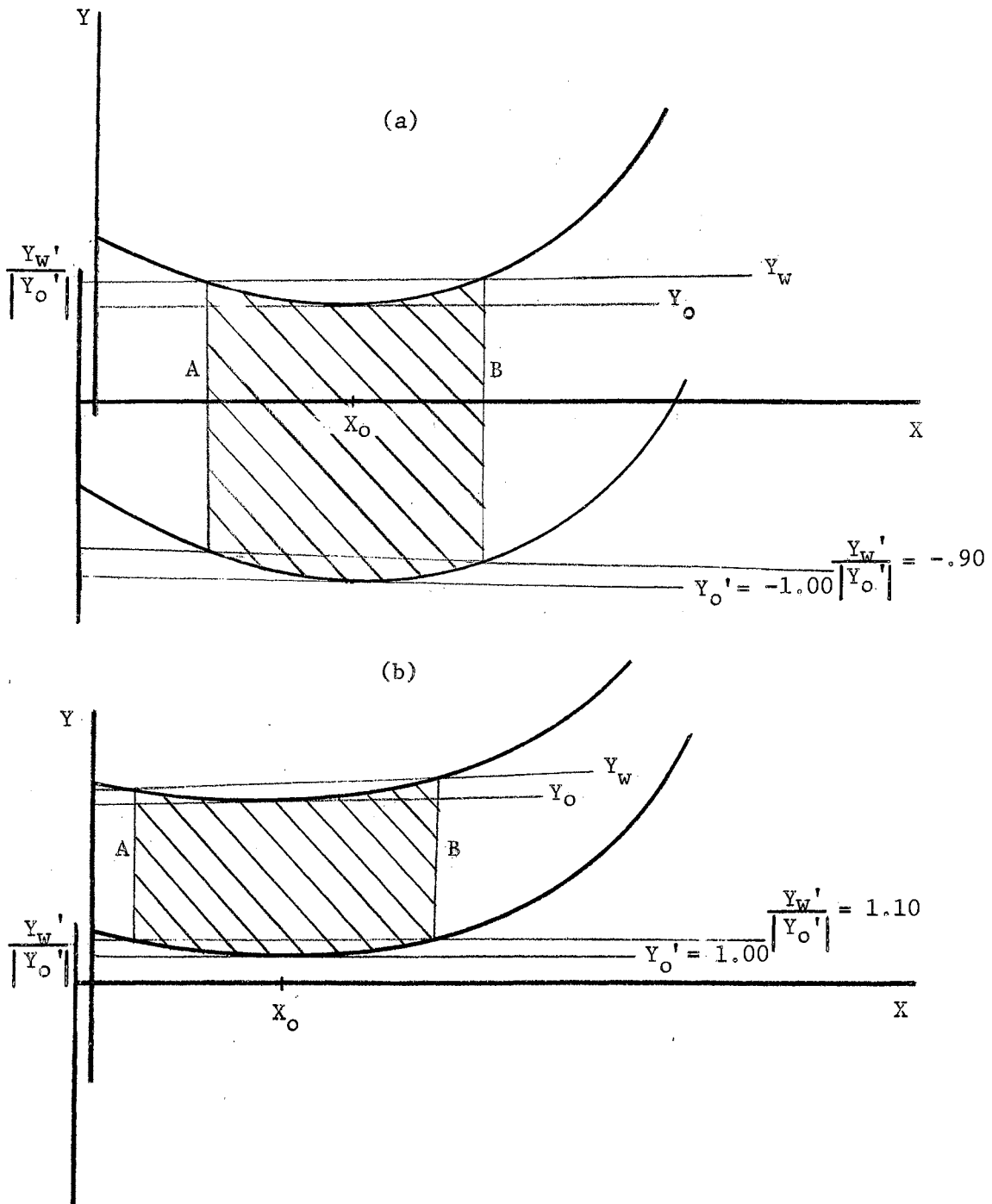


Figure (1). Minimization Curves

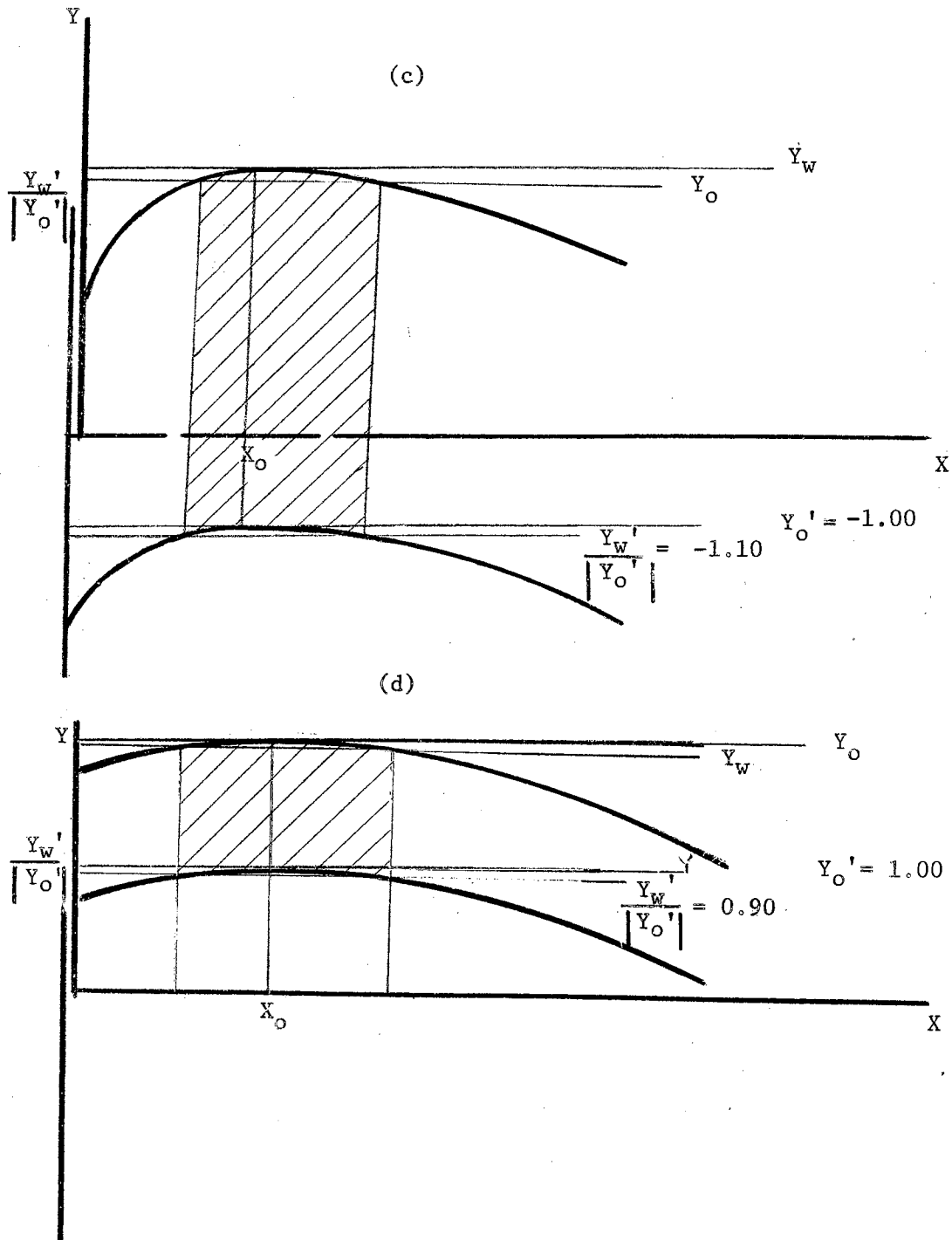


Figure (2). Maximization Curves

- c) The original function is a maximization and the transformed optimum value  $Y_o'$  is a negative value. Thus

$$\frac{Y_w'}{|Y_o'|} = \frac{-1.10}{1.00} = -1.10$$

- d) The original function is a maximization and the transformed optimum value  $Y_o'$  is a positive value. Thus

$$\frac{Y_w'}{|Y_o'|} = \frac{0.90}{1.00} = 0.90$$

The using of an absolute value of a transformed optimum value  $|Y_o'|$  is for the purpose to be consistent with the directions of the axes of the original function  $Y$ .

In our example the curve is a minimization and  $Y_o$  is negative. Thus, the sensitivity expression yields to

$$w^2 - 2w + .90 = 0.$$

The roots of this quadratic equation show the lower decision range  $w_1$  and the upper decision range  $w_u$  to be

$$\begin{aligned} w_1 &= \frac{2 - \sqrt{4 - 4 \times .90}}{2} \\ &= 1.0 - \sqrt{.10} = - .68377 \end{aligned}$$

and

$$\begin{aligned} w_u &= \frac{2 + \sqrt{4 - 4 \times .90}}{2} \\ &= 1.0 + \sqrt{.10} = 1.31622. \end{aligned}$$

The decision range is

$$0.68377 \leq w \leq 1.31622$$

and

$$0.68377X_o \leq X_w \leq 1.31622X_o.$$

Numerical example: For the above model, assume  $A = 1.00$ ,  $B = 1.00$ , and  $C = 10.0$ . Thus by substitution

$$X_o = X_o' = \frac{B}{2A} = 2.00$$

$$Y_o = -\frac{B^2}{4A} + C = 9.75$$

$$Y_o' = -\frac{B^2}{4A} = -.25$$

$$\frac{Y_w'}{Y_o'} = -.90 = w^2 - 2w$$

$$w_l = 0.68377$$

$$w_u = 1.31622$$

$$x_l = 0.68377 (2.00) = 1.36754$$

$$x_u = 1.31622 (2.00) = 2.63244$$

and the decision range

$$1.36754 \leq X_w \leq 2.63244.$$

The effect of an allowing a 10 per cent increase in the total variable parts of the function on the resulting net per cent increase is

$$\frac{Y_w'}{Y_o'} = \frac{Y_w - C}{Y_o - C}$$

By substituting we have

$$.90 = \frac{Y_w - 10.0}{Y_o - 10.0}$$

$$Y_w = .90Y_o + 1.0$$

$$\frac{Y_w}{Y_o} = .90 + \frac{1.0}{Y_o}$$

$$= \frac{.90 \times 9.75 + 1.0}{9.75}$$

$$= \frac{9.775}{9.75} = 1.0025$$

The 10 per cent increase in total variable items will yield only 0.25 per cent in total function which in practice is negligible.

### Models of One Standard Curve

#### Straight Line Model

The straight line model can be represented mathematically as

$$Y = MX + B$$

where

M = the slope of the line and a positive amount

B = the intercept of the line on the Y-axis and a positive amount.

Follow the same procedures as in the illustrated example pp. 14 - 21.

Thus

$$\frac{dY}{dX} = M + 0$$

therefore,

$$0 = M.$$

This result contradicts the assumption that  $M > 0$ . Therefore, one can conclude no optimum for the straight line model and consequently no sensitivity analysis around the optimum.

### Circle Model

The mathematical circle model can be represented as

$$Y = \pm \sqrt{R^2 - (X - H)^2} + K$$

The positive sign indicates the upper semi-circle, while the negative sign indicates the lower semi-circle. The following discussion is limited to the lower semi-circle, the minimization curve, where

(H, K) = the coordinates of the center of the circle

and

R = the radius of the circle.

Follow the same procedures given in the illustrated example pp.

14 - 21. Thus

$$\frac{dY}{dX} = -\frac{1}{2} \left[ R^2 - (X-H)^2 \right]^{-\frac{1}{2}} \left[ -2(X-H) \right]$$

$$0 = -\frac{1}{2} \left[ R^2 - (X_0-H)^2 \right]^{-\frac{1}{2}} \left[ -2(X_0-H) \right]$$

$$X_0 = H$$

and

$$Y_0 = -\sqrt{R^2} + K = -R + K$$

Transform the origin of the function (0, 0) to another new one (0, -K).

Thus,

$$Y_0' = Y_0 - K = -R$$

Assume an error  $w$  has been associated with  $X_0$  in such a way that

$$X_w = wX_0$$

and

$w =$  zero or a positive value.

The corresponding value due to this error  $w$  is

$$Y_w = -\sqrt{R^2 - H^2(w^2 - 1)} .$$

The sensitivity expression is

$$\frac{Y_w}{|Y_0|} = \sqrt{1 - \left(\frac{H}{R}\right)^2 (w^2 - 1)} .$$

This sensitivity expression is a function of the error  $w$ , the parameter of the circle  $R$ , and the abscissa of the origin of the circle. Thus, it is a special expression and the sensitivity analysis varies from one model to another. However, there is a general expression for the family of circle models when  $R = H$  and is presented as

$$\frac{Y_w}{|Y_0|} = \sqrt{1 - (w^2 - 1)} .$$

Numerical example: Let

$$R = H$$

The sensitivity expression for the lower semi-circle model is

$$\frac{Y_w}{|Y_0|} = -.90 = \sqrt{1 - (w - 1)^2} .$$

By squaring both sides, one can obtain



$$w^2 - 2w + 0.81 = 0.$$

The upper limit of the decision range  $w_u$  and the lower limit of the decision range  $w_l$  are

$$w_u = \frac{2 + \sqrt{4.00 - 4 \times 0.81}}{2} = 1.4357$$

and

$$w_l = \frac{2 - \sqrt{4.00 - 4 \times 0.81}}{2} = .5643$$

Thus, the decision range is

$$.5643 \leq w \leq 1.4357$$

The decision range, due to the 10 per cent increase in the total variable items for the family circle models when  $R = H$ , results in the 43 per cent above and under the optimum value  $X_0$ .

### Ellipse Model

The circle is a special form of an ellipse. Thus, results similar to those obtained from the circle model can be obtained for the ellipse model. The only difference is that the semi-major axis is replaced instead of the radius if the major axis is parallel to the x-axis.

### Parabola Model

The general mathematical expression for the parabola model is

$$Y = \frac{1}{4P} (X - H)^2 + K$$

where

$4p =$  the length of the latus rectum

and

$(H, K) =$  the coordinates of the vertex of the parabola.

Follow the same procedures as in the illustrated example pp.

14 - 21. Thus

$$\frac{dY}{dX} = \frac{2}{4P} (X - H) + 0$$

$$0 = \frac{1}{2P} (X_o - H)$$

$$X_o = H$$

and

$$Y_o = K .$$

Assume that an error  $w$  has been incurred in calculating or estimating the optimum value  $X_o$  in such a manner that

$$X_w = wX_o$$

and

$w =$  zero or a positive value

The corresponding value  $Y_w$  due to this error  $w$  is

$$Y_w = \frac{H^2}{4P} (w - 1)^2 + K$$

and the sensitivity expression for the total functions

$$\left| \frac{Y_w}{Y_o} \right| = \frac{H^2}{4P K} (w - 1)^2 + 1.00$$

The transformation here is not valid due to having yielded  $Y_o = 0$ .

Therefore, the sensitivity expression is not a general one and varies from one function to another.

Numerical example: The following two examples show that the sensitivity expression varies from one parabola model to another due to the different parameters.

Example 1.

Given for the parabola model:

$$H = 100 \quad K = 100 \quad \text{and} \quad P = 5.00.$$

Thus,

$$X_o = 100$$

$$Y_o = 100$$

and

$$\frac{H^2}{4PK} = \frac{100 \times 100}{4 \times 5 \times 100} = 5.00.$$

For a 10 per cent standard allowance increase in total variable elements above its optimum, the sensitivity expression is

$$\frac{Y_w}{|Y_o|} = 1.10 = 5(w - 1)^2$$

Solving this quadratic function the lower and upper limits are obtained as

$$w^2 - 2w + .88 = 0$$

$$w_1 = \frac{2 - \sqrt{4 - 3.52}}{2} = .65375$$

and

$$w_u = \frac{2 + \sqrt{4 - 3.52}}{2} = 1.34625.$$

Therefore, the decision range is

$$0.65375X_o \leq X_w \leq 1.34625X_o$$

$$65.375 \leq X_w \leq 134.625 .$$

Example 2.

Given the parameters of the parabola model as

$$H = 100$$

$$K = 100$$

$$P = 10$$

Thus,

$$X_o = 100$$

$$Y_o = 100$$

and

$$\frac{H^2}{4PK} = \frac{100 \times 100}{4 \times 10 \times 100} = 2.50 .$$

For a 10 per cent standard allowance increase in the total variable elements above its optimum, the sensitivity expression is

$$\frac{Y_w}{|Y_o|} = 1.10 \quad = 2.5(w - 1)^2 .$$

Solving this quadratic function, the lower and upper limits of the decision range are obtained as

$$w^2 - 2w + .56 = 0$$

$$w_l = \frac{2 - \sqrt{4 - 2.24}}{2} = .3375$$

and

$$w_u = \frac{2 + \sqrt{4 - 2.24}}{2} = 1.6625.$$

Therefore the decision range is

$$0.3375X_o \leq X_w \leq 1.6625X_o$$

$$33.75 \leq X_w \leq 166.25 .$$

These two examples show that as the ratio  $\frac{H^2}{4PK}$  is smaller, the decision range is wider and vice versa.

#### Gaussian model

The general mathematical expression of the Gaussian curve is

$$Y = E^{-X^2} .$$

Follow the same procedures as in the illustrated example pp.14 - 21. Thus

$$\frac{dY}{dX} = E^{-X^2} (-2X)$$

$$0 = - \frac{E^{-X_o^2}}{2X_o}$$

$$0 = X_o$$

and

$$Y_o = E^{-0} = 1.00 .$$

Assumed that an error  $w$  has been associated with  $X_o$  in such a way that

$$X_w = wX_o$$

and

$w =$  zero or a positive value.

The corresponding value  $Y_w$  due to this error is

$$Y_w = E^{-w^2}$$

and the sensitivity expression is

$$\frac{Y_w}{|Y_o|} = E^{-w^2}$$

This sensitivity expression is a general one. For setting a 10 per cent standard allowance, the decision range can be obtained as

$$1.10 = E^{-w^2}$$

$$w^2 = .1005$$

$$w_1 = -0.317$$

and

$$w_u = 0.317$$

Thus, the decision range is

$$-0.317 \leq w \leq .317$$

and

$$-0.317X_o \leq X_w \leq .317X_o$$

### Catenary Model

The general expression of the catenary curve is

$$Y = A (E^{X/A} - E^{-X/A})$$

where

A = parameter of the catenary model or specifically the y-intercept.

Follow the same procedures given in the illustrated example pp.

14 - 21. Thus

$$\frac{dY}{dX} = \frac{A}{2} \left[ E^{X/A} \cdot \frac{1}{A} + E^{-X/A} \cdot \left(\frac{1}{A}\right) \right]$$

$$= \frac{1}{2} \left[ E^{X/A} - E^{-X/A} \right]$$

and

$$0 = \frac{1}{2} \left[ E^{X_0/A} - E^{-X_0/A} \right]$$

or

$$E^{X_0/A} = E^{-X_0/A}$$

This result implies that

$$X_0 = 0.00$$

By substitution, one can obtain

$$Y_0 = \frac{A}{2} E^0 + E^{-0} = A$$

Assume an error  $w$  occurs with  $X_0$  in such a way that

$$X_w = wX_0, w \geq 0$$

The resulting corresponding value  $Y_w$  is

$$Y_w = \frac{A}{2} \left[ E^{w/A} + E^{-w/A} \right]$$

and the sensitivity expression is

$$\frac{Y_w}{Y_o} = \frac{1}{2} \left[ E^{w/A} + E^{-w/A} \right].$$

The decision range for an allowance 10 per cent increase in  $Y_w$  over  $Y_o$  is

$$\frac{Y_w}{Y_o} = 1.10 = \frac{1}{2} \left[ E^{w/A} + E^{-w/A} \right].$$

Numerical example: If  $A = 1.00$  is given for the catenary model, the following results are obtained by substitution:

$$X_o = 0$$

$$Y_o = A = 1.00$$

and

$$\frac{Y_w}{Y_o} = 1.10 = \frac{1}{2} (E^w + E^{-w}).$$

To solve this equation, put  $E^w = Z$ . Thus,

$$2.20 = Z + \frac{1}{Z}$$

$$Z^2 - 2.20Z + 1.00 = 0.$$

The roots of this quadratic equation are

$$Z_1 = \frac{2.20 - \sqrt{4.84 - 4.00}}{2.00} = .6418$$

and

$$Z_u = \frac{2.20 + \sqrt{4.84 - 4.00}}{2.00} = 1.4582.$$



The lower and upper limits of the decision range are

$$w_l = -.376$$

and

$$w_u = 0.376 .$$

Therefore, the decision range is

$$-0.376 \leq w \leq .376.$$

### Models as a Summation of Two Different Standard Curves

A condition for providing the trade-off relationship in each of the following models is to have one curve as an increasing function while the other curve is a decreasing function. Throughout this discussion the first curve is referred to as an increasing function while the second curve is referred to as a decreasing one.

### Straight Line Versus Ellipse Model

The equation of an increasing straight line function is

$$Y_1 = MX .$$

The equation of a lower semi-half of the ellipse is

$$Y_2 = -A^2 - (X-H)^2 + K .$$

Therefore, the equation of the sum of the two curves is

$$Y = Y_1 + Y_2 = MX - A^2 - (X-H)^2 + K$$

where

M = the slope of the line and a positive amount

$A$  = the semi-major axis of the ellipse

and

$(H, K)$  = the coordinates of the center of the ellipse.

Following the same procedures, it is found that

$$\frac{dY}{dX} = M - \frac{1}{2} \left[ A^2 - (X - H)^2 \right]^{-\frac{1}{2}} \left[ 2(X - H) \right]$$

and

$$0 = M - \frac{1}{2} \left[ A^2 - (X_0 - H)^2 \right]^{-\frac{1}{2}} \left[ 2(X_0 - H) \right]$$

Thus,

$$M^2 = \frac{(X_0 - H)^2}{A^2 - (X_0 - H)^2}$$

$$(X_0 - H)^2 (M^2 + 1) = M^2 A^2$$

and

$$X_0 = \frac{MA}{\sqrt{1 + M^2}} + H$$

The corresponding optimum value  $Y_0$  is

$$\begin{aligned} Y_0 &= M \left[ \frac{MA}{\sqrt{1 + M^2}} + H \right] - \sqrt{A^2 - \left( \frac{MA}{1 + M^2} + H - H \right)^2} \\ &\quad + K \\ &= M \frac{MA}{\sqrt{1 + M^2}} + MH - \frac{\sqrt{A^2(1 + M^2) - M^2 A^2}}{\sqrt{1 + M^2}} + K \\ &= \frac{M^2 A}{\sqrt{1 + M^2}} + MH - \frac{\sqrt{A^2}}{\sqrt{1 + M^2}} + K \end{aligned}$$

$$= \frac{A}{\sqrt{1+M^2}} \left[ M^2 - 1 \right] + MH + K .$$

Transforming the origin of the function (0, 0) to another new one (0, -K), the result is

$$Y_o' = Y_o - K = \frac{A}{\sqrt{1+M^2}} \left[ M^2 - 1 \right] + MH.$$

Assume an error  $w$  has been incurred in calculating or estimating  $X_o$  in such a way that

$$X_w = wX_o$$

and

$w =$  zero or a positive value.

Thus, the corresponding value  $Y_w'$  which results from inducing this error  $w$  is

$$Y_w' = Mw \left[ \frac{MA}{\sqrt{1+M^2}} + H \right] - \sqrt{A^2 - \frac{MAw}{1+M^2} + wH - H^2}$$

and the sensitivity expression is

$$\frac{Y_w'}{Y_o'} = \frac{Mw \left[ \frac{MA}{\sqrt{1+M^2}} + H \right] - \sqrt{A^2 - \left( \frac{MAw}{1+M^2} + wH - H^2 \right)}}{\left| \frac{A}{\sqrt{1+M^2}} \left[ M^2 - 1 \right] + MH \right|} .$$

This expression is not a general one because it is a function of the error  $w$  and the parameters of the function. Thus, one can expect the sensitivity analysis to vary from one model to another according to the values of the parameters.

### Straight Line Versus Circle Model

The circle is a special case of the ellipse. Thus, for the straight line versus the circle model the same results can simply be obtained by replacing the radius R instead of the semi-major axis if the major axis of the ellipse is parallel to the x-axis.

### Straight Line Versus Parabola Model

The straight line as an increasing function can be represented mathematically as

$$Y_1 = MS$$

and the half of the parabola as a decreasing function as

$$Y_2 = -A\sqrt{X} + C.$$

Thus, the sum of the two functions is

$$Y = Y_1 + Y_2 = MX - A\sqrt{X} + C$$

where

M = the slope of the straight line and a positive value

A =  $2\sqrt{P}$ , the parameter of the decreasing parabola

4P = the latus rectum of the parabola

C = the ordinate of the parabola vertex of the y-axis and a positive amount.

Following the same procedures as before - the following steps can be obtained

$$\frac{dY}{dX} = M - \frac{A}{2\sqrt{X}}$$

$$0 = M - \frac{A}{2\sqrt{X_0}}$$

$$M^2 X_0 = \frac{A^2}{4}$$

$$X_0 = \frac{A^2}{4M^2}$$

and

$$\begin{aligned} Y_0 &= M \frac{A^2}{4M^2} - \sqrt{A^2 \frac{A^2}{M^2}} + C \\ &= -\frac{A^2}{4M} + C. \end{aligned}$$

As the function involves a constant term, the transformation is used to eliminate this constant term. Thus, after transforming the origin (0, 0) of the function to another origin (0, -C), it is found that

$$Y_0' = Y_0 - C = -\frac{A^2}{4M}.$$

Assume an error  $w$  has been associated with  $X_0$  in such a way that

$$X_w = X_w' = wX_0 = w'X_0$$

and

$$w' = w = \text{zero or a positive value.}$$

The transformed corresponding value  $Y_w'$  is

$$Y_w' = \frac{A^2}{4M} (w - 2\sqrt{w}),$$

and, thus, the sensitivity expression is

$$\left| \frac{Y_w'}{Y_0'} \right| = w - 2\sqrt{w}.$$

This mentioned function is a minimization and the transformed optimum value  $Y_0'$  is negative thus for a setting 10 per cent increase

in  $Y_w'$  above its optimum one can obtain (Illustrated example pp. 14 - 21)

$$-0.9 = w - 2\sqrt{w}.$$

This expression is a general one since it is a function of the error  $w$ .

To obtain the roots of this expression let  $\sqrt{w} = Z$  thus by substitution one can obtain

$$Z^2 - 2Z + .90 = 0.$$

The roots of this quadratic function are

$$Z_u = \frac{2 + \sqrt{4 - 3.6}}{2} = 1.31622$$

and

$$Z_l = \frac{2 - \sqrt{4 - 3.6}}{2} = 0.68377.$$

The corresponding roots  $w_l$  and  $w_u$  are the lower and upper limits of the error for predetermined 10 per cent increase in  $Y_w'$  above its optimum as

$$w_u = Z_u^2 = (1.31622)^2 = 1.73245$$

and

$$w_l = Z_l^2 = (0.68377)^2 = 0.46756.$$

Therefore the decision range is

$$0.46756 \leq w \leq 1.73245$$

or

$$0.46756X_o \leq X_w \leq 1.73245X_o.$$

$$0.46756 \frac{A^2}{4M^2} \leq x_w \leq 1.73245 \frac{A^2}{4M^2} .$$

The curve is flatter in the upper range than in the lower range within the decision range. A complete evaluation and ranges are obtained by the aid of digital computer in Appendix A.

Numerical example: For the straight line versus parabola model

let

$$M = 0.50$$

$$A = 2.0$$

$$C = 10.0 .$$

By substitution, the following results can be obtained

$$x_o' = x_o = \frac{A^2}{4M^2} = \frac{2 \times 2}{4 \times (1/2)^2} = 4.0$$

$$y_o = -\frac{A^2}{4M} + C = -2 + 10 = 8.0$$

and

$$y_o' = -\frac{A^2}{4M} = -2.0 .$$

The decision range is

$$0.46756(x_o) \leq x_w \leq 1.73245(x_o)$$

$$0.46756(4.0) \leq x_w \leq 6.92980 .$$

At this stage the question can be asked, what is the net increase in the total function if the constant term is taken into consideration?

The answer can be obtained by manipulating the ratios as

$$\frac{Y'_w}{Y'_o} = .90$$

$$Y'_w = .90 \times (-2.0) = -1.80$$

$$Y_w = Y'_w + c = 8.20$$

and

$$\frac{Y_w}{Y_o} = \frac{8.20}{8.00} = 1.025.$$

This means that the 10 per cent increase in  $Y'_w$  over  $Y'_o$  will yield only a 2.5 per cent increase in the total function above its optimum. This slight increase in the total function provides a wide range for decision making process that is a range of about 53 per cent in the lower side and 73 per cent in the upper side of the optimum value  $X_o$ .

The evaluation of this kind of models is given in the Appendix A.

#### Straight Line Versus Rectangular Hyperbola Model

The sum of the two curves, a straight line as an increasing function and a rectangular hyperbola as a decreasing function, is

$$Y = MX + \frac{B}{X}$$

where

M = the slope of the straight line and a positive amount

B = the parameter of the rectangular hyperbola and a positive amount.



Following the same procedures as in the illustrated example, pp.14 - 21, the following steps can be obtained

$$\frac{dY}{dX} = M - \frac{B}{X^2}$$

$$0 = M - \frac{B}{X_0^2}$$

$$X_0 = \sqrt{\frac{B}{M}}$$

and

$$Y_0 = \sqrt{M^2 \frac{B}{M}} + \sqrt{B^2 \frac{M}{B}} = 2\sqrt{MB}$$

The assumption of an error  $w$  is provided to describe the sensitivity analysis of the curve. Assuming an error  $w$  has been occurred in such a way that

$$X_w = wX_0$$

and

$$w = \text{zero or a positive value}$$

the corresponding value  $Y_w$  due to this error  $w$  is

$$Y_w = 2\sqrt{MB} \left[ \frac{w^2 + 1}{2w} \right]$$

and the sensitivity expression is

$$\frac{Y_w}{|Y_0|} = \frac{w^2 + 1}{2w}$$

This is a minimization type of expression curve with  $Y_0$  positive.

Thus

$$\frac{Y_w}{|Y_0|} = 1.10 = \frac{w^2 + 1}{2w}$$

The solution of this quadratic function gives the upper and lower limits of the decision range as

$$w^2 - 2.2w + 1 = 0$$

$$w_u = \frac{2.2 + \sqrt{4.84 - 4.00}}{2}$$

and

$$w_l = \frac{2.2 - \sqrt{4.84 - 4.00}}{2}$$

The decision range will be

$$0.641743 \leq w \leq 1.558256$$

and

$$0.641743X_o \leq X_w \leq 1.558256X_o$$

$$0.641743\sqrt{\frac{B}{M}} \leq X_w \leq 1.558256\sqrt{\frac{B}{M}} .$$

The curve is flatter in the upper range than in the lower range within the decision range.

Numerical example: For the above model assume  $M = 0.50$ , and  $B = 8.00$ .

By substitution, the results obtained are

$$X_o = \sqrt{\frac{B}{M}} = \sqrt{\frac{8}{1/2}} = 4.0$$

$$Y_o = 2\sqrt{BM} = 2\sqrt{1/2 \times 8} = 8.0$$

$$0.641743(4.0) \leq X_w \leq 1.558256(4.0)$$

$$2.566972 \leq X_w \leq 6.233024.$$

The evaluation of this kind of models is evaluated on the digital computer (Appendix A).

### Parabola Versus Straight Line Model

The sum of an increasing parabola curve and a decreasing straight line is

$$Y = AX^2 + C - MX$$

where

$A = \frac{1}{4P}$ , the parameter of the parabola

$4P$  = the length of the latus rectum of the parabola

$C$  = the ordinate intercept of the straight line and a positive value

$M$  = the slope of the straight line and a positive value.

Differentiate with respect to  $X$ , equate the result to zero, solve for the optimum value  $X_0$ , and substitute with  $X_0$  in  $Y$  function to obtain  $Y_0$  as

$$\frac{dY}{dX} = 2AX + 0 - M$$

$$0 = 2AX_0 - M$$

$$X_0 = \frac{M}{2A}$$

$$Y_0 = A \frac{M^2}{4A^2} + C - M \frac{M}{2A}$$

$$= - \frac{M^2}{4A} + C.$$

The sensitivity expression can be obtained by assuming an error  $w$  has been associated with  $X_0$  provided that

$$X_w = wX_o$$

and

$w =$  zero or a positive value.

The corresponding value  $Y_w$  due to this error  $w$  is

$$Y_w = \frac{M^2}{4A} [w^2 - 2w] + C.$$

Transform the origin of the function (0, 0) to another new origin (0, -C). Thus the transformed value  $Y_w'$  is

$$Y_w' = \frac{M^2}{4A} [w^2 - 2w].$$

The sensitivity expression is

$$\frac{Y_w'}{|Y_o|} = w^2 - 2w.$$

The type of this expression curve is minimization and  $Y_o'$  is negative. Thus

$$\frac{Y_w'}{|Y_o|} = -0.90 = w^2 - 2w$$

$$w^2 - 2w + 0.90 = 0.$$

The solution of this quadratic function gives the upper and lower limits of the decision range as

$$w_u = \frac{2 + \sqrt{4 - 3.60}}{2} = 1.316228$$

and

$$w_1 = \frac{2 - \sqrt{4 - 3.60}}{2} = 0.683774.$$

The decision range will be

$$0.683774 \leq w \leq 1.316228$$

and

$$0.683774X_0 \leq X_w \leq 1.316228X_0$$

$$0.683774 \frac{M}{2A} \leq X_w \leq 1.316228 \frac{M}{2A} .$$

The flatness of the curve is quite equal in the upper and lower range of the decision range. The 32 per cent upper the optimum  $X_0$  and 32 per cent lower the optimum  $X_0$  is the result of an allowance 10 per cent increase in the total variable elements. If the constant term is concerned in the function this 10 per cent is going to be less and as the constant term is high the reduction in this 10 per cent will be more.

Numerical example: For the above model let

$$A = 0.10$$

$$M = 0.50$$

$$C = 6.00 .$$

By substitution, the following results can be obtained

$$X_0 = \frac{M}{2A} = \frac{10}{2 \times 2} = 2.5$$

$$\begin{aligned} Y_0 &= -\frac{M^2}{4A} + C \\ &= -\frac{10}{4 \times 4} + 6.00 = 5.375 \end{aligned}$$

$$Y_0' = -\frac{10}{4 \times 4} = -.625.$$

The decision range is

$$0.683774(2.5) \leq X_w \leq 1.316228(2.5)$$

$$1.709435 \leq X_w \leq 3.292570 .$$

The evaluation of this kind of models is evaluated on the digital computer (Appendix A).

### Parabola Versus Parabola Model

The sum of an increasing parabola and a decreasing parabola is

$$Y = AX^2 - B\sqrt{X} + C$$

where

$$A = \frac{1}{4P_1} , \text{ the parameter of the increasing parabola}$$

$4P_1$  = the latus rectum of the increasing parabola

$$B = 2\sqrt{P_2}, \text{ the parameter of the decreasing parabola}$$

$4P_2$  = the latus rectum of the decreasing parabola

C = the ordinate of the decreasing parabola vertex and a positive value.

The optimum point  $(X_0, Y_0)$  of this function can be obtained by differentiating the y-function with respect to X, equating the result to zero, solving the past equation to obtain  $X_0$ , and again substituting with the value  $X_0$  in the Y-function to obtain  $Y_0$  as follows

$$\frac{dY}{dX} = 2AX - \frac{B}{2\sqrt{X}}$$

$$0 = 2AX_0 - \frac{B}{2\sqrt{X_0}}$$

$$X_0 = \left[ \frac{B}{4A} \right]^{2/3}$$

$$Y_0 = \frac{1}{4} \sqrt[3]{\frac{A^3 B^4}{4A^4}} - \sqrt[3]{\frac{B^3 B}{4A}} + C.$$

This function contains a constant term C. Thus, to obtain a general sensitivity expression, transform the origin (0, 0) of the original function to another new origin (0, -C).

The optimum of the transformed function ( $X_0'$ ,  $Y_0'$ ) is ( $X_0, Y_0 - C$ ). At this point assume an error w has been associated with  $X_0$  or  $X_0'$  in such a way that

$$X_w = X_w' = wX_0 = wX_w'$$

and

$$w = \text{zero or a positive value.}$$

The corresponding  $Y_w'$  is calculated by substituting  $wX_0$  instead of  $X_0$  in the y-function. Thus

$$Y_w' = \left[ \frac{B^4}{4A} \right]^{1/3} \left[ \frac{w^2}{4} - (w)^{1/2} \right].$$

and

$$Y_0' = \left[ \frac{B^4}{4A} \right]^{1/3} \left[ -\frac{3}{4} \right]$$

The sensitivity expression of this kind of models is

$$\left| \frac{Y_w'}{Y_0'} \right| = \frac{w^2}{3} - \frac{4}{3} (w)^{1/2}$$

The type of this curve for the sensitivity expression is a minimization with negative optimum value  $Y_0'$ . The sensitivity expression will be

$$\left| \frac{Y_w'}{Y_o'} \right| = -0.90 = \frac{w^2}{3} - \frac{4}{3} (w)^{1/2}$$

or

$$w^2 - 4(w)^{1/2} + 2.70 = 0.$$

The two positive roots of this equation around its optimum are obtained from Appendix A. These two positive roots are the lower and the upper limits of the decision range as

$$w_l = 0.5733071 \text{ and } w_u = 1.4612770$$

or

$$X_l = \text{the lower range} = 0.5733071(X_o)$$

and

$$X_u = \text{the upper range} = 1.4612770(X_o).$$

The decision range corresponding to the predetermined 10 per cent increase in transformed value  $Y_w'$  above its optimum  $Y_o'$  is

$$0.5733071(X_o) \leq X_w \leq 1.4612770(X_o)$$

$$0.5733071 \left( \frac{B}{4A} \right)^{2/3} \leq X_w \leq 1.4612770 \left( \frac{B}{4A} \right)^{2/3}.$$

The curve is slightly flatter in the upper range than in the lower range.

If the model is a summation of an increasing parabola of a type  $Y_1 = B(X)^{1/2}$  and a decreasing parabola of a type  $Y_2 = AX^2$  the discussion will lead to a sensitivity expression



$$\left| \frac{Y_w'}{Y_o'} \right| = -\frac{w^2}{3} + \frac{4}{3} (w)^{1/2}$$

This type of curve is a maximization with a negative optimum value  $Y_o'$ . Therefore, for 10 per cent decrease in the Y value under its optimum, the sensitivity expression will be

$$0.90 = -\frac{w^2}{3} + \frac{4}{3} (w)^{1/2}$$

or

$$w^2 - 4(w)^{1/2} + 2.70 = 0$$

The positive two roots of this equation around its optimum are the lower and upper limits of the decision range and are obtained from Appendix B as

$$w_l = 0.5734145 \text{ and } w_u = 1.4612045$$

which is the same result obtained in the minimization (neglecting these slight differences) and this is true due to the fact that the maximizing of the negative of the function is the minimizing of the same function.

Numerical example: For the above model let  $A = .10$ ,  $B = 2.0$ , and  $C = 10.0$ .

By substitution, the following results are calculated as

$$X_o = \frac{B^2}{16A}^{1/3} = \frac{2 \times 2 \times 10 \times 10}{16}^{1/3}$$

$$= (25)^{1/3} = 2.924$$

$$Y_o = \frac{B^4}{4A}^{1/3} - \frac{3}{4} + C$$

$$= \frac{16 \times 10}{4}^{1/3} \times (-3/4) + 10.0$$

$$= -2.4825 + 10.0 = 7.5175$$

$$Y_0' = -2.4825$$

and the decision range is

$$0.57330710 \left( \frac{B}{4A} \right)^{1/3} \leq X_w \leq 1.461277 \left( \frac{B}{4A} \right)^{1/3}$$

$$0.57330710(2.924) \leq X_w \leq 1.461277(2.924).$$

### Parabola Versus Rectangular Hyperbola Model

The sum of an increasing parabola and a decreasing rectangular hyperbola is

$$Y = AX^2 + \frac{B}{X}$$

where

$$A = \frac{1}{4P} \text{ the parameter of the parabola}$$

4P = the latus rectum of the parabola

B = the parameter of the rectangular hyperbola.

Differentiate with respect to X, equate the result to zero, solve for the optimum value  $X_0$ , and again substitute in Y-function with  $X_0$  to obtain  $Y_0$  as

$$\frac{dY}{dX} = 2AX - \frac{B}{X^2}$$

$$0 = 2AX_0 - \frac{B}{X_0^2}$$

$$X_0 = \sqrt[3]{\frac{B}{2A}}$$

$$Y_0 = \sqrt[3]{A^3 \frac{B^2}{4A^2}} + \sqrt[3]{B^3 \frac{2A}{B}}$$

$$= \sqrt[3]{(AB^2)} \left( \sqrt[3]{\frac{1}{4}} + \sqrt[3]{2} \right)$$

Assume an error  $w$  has been associated with  $X_0$  in such a way that

$$X_w = wX_0$$

and

$w =$  zero or a positive value.

The corresponding value  $Y_w$  due to this error  $w$  is

$$Y_w = (AB^2) \frac{1}{3} \frac{w^2}{(4)^{1/3}} + \frac{(2)^{1/3}}{w}$$

and the sensitivity expression is

$$\begin{aligned} \frac{Y_w}{|Y_0|} &= \frac{.6295}{1.8894} w^2 + \frac{1.2599}{1.8894w} \\ &= \frac{1}{3} w^2 + \frac{2}{3w} \end{aligned}$$

The type of this sensitivity expression is a minimization with a positive optimum value  $Y_0$ . Thus the sensitivity expression for a 10 per cent increase in the  $Y$  value above its optimum  $Y_0$  will yield to

$$\frac{Y_w}{|Y_0|} = 1.10 = \frac{1}{3} w^2 + \frac{2}{3w}$$

or

$$w^3 - 3.3w + 2.0 = 0 .$$

The two positive roots of this kind of model around its optimum are obtained from Appendix A. These two positive roots are the lower and upper limits of the decision range as

$$w_1 = 0.7186767 \text{ and } w_u = 1.3473911$$

or

or

$$X_1 = 0.7186767X_o \text{ and } X_u = 1.3473911X_o$$

or

$$X_1 = 0.7186767 \left( \frac{B}{2A} \right)^{1/3} \text{ and } X_u = 1.3473911 \left( \frac{B}{2A} \right)^{1/3}$$

and the decision range is

$$0.7186767 \left( \frac{B}{2A} \right)^{1/3} \leq X_w \leq 1.3473911 \left( \frac{B}{2A} \right)^{1/3} .$$

The model is flatter in the upper range than in the lower range within the decision range. Complete evaluation of this kind of models is given in Appendix A.

Numerical example: For the above model let  $A = \frac{1}{8}$  and  $B = 16.00$ .

By substitution the following results are calculated as

$$X_o = \left( \frac{B}{2A} \right)^{1/3} = \left( \frac{16 \times 8}{2} \right)^{1/3} = 4.0$$

$$Y_o = 1.8894 (AB^2)^{1/3} = 6.000$$

$$= 1.8894 \left( \frac{16 \times 16}{8} \right)^{1/3}$$

and the decision range is

$$0.7186767X_o \leq X_w \leq 1.3473911X_o$$

$$2.8747068 \leq X_w \leq 5.3895644 .$$

#### Cubical Parabola Versus Straight Line Model

The equation of the sum of a cubical parabola as an increasing function and a straight line as a decreasing function is

$$Y = AX^3 - MX + C$$

where

A = the parameter of the cubical parabola

M = the slope of the straight line and a positive value

C = the intercept of the slope of straight line on the y-axis and a positive value

Follow the same procedures as in the illustrated example pp. 14 - 21.

$$\frac{dY}{dX} = 3AX^2 - M$$

$$0 = 3AX_0^2 - M$$

$$X_0 = \left(\frac{M}{3A}\right)^{1/2}$$

$$Y_0 = \left[ A^2 \cdot \frac{M^3}{(3A)^3} \right]^{1/2} - \left[ \frac{M^2 M}{3A} \right]^{1/2} + C$$

$$= \left(\frac{M^3}{3A}\right)^{1/2} \left(\frac{1}{3} - 1\right) + C.$$

Transform the origin of the function (0,0) to (0,-C) thus

$$Y'_0 = \left(\frac{M^3}{3A}\right)^{1/2} \left(-\frac{2}{3}\right).$$

Assume an error w has been associated with  $X_0$  in such a way that

$$X_w = wX_0$$

and

$w =$  zero or a positive value.

The corresponding transformed value  $Y_w'$  due to this error  $w$  is

$$Y_w' = \left( \frac{M^3}{3A} \right)^{1/2} \left( \frac{w^3}{3} - w \right)$$

and the sensitivity expression is

$$\frac{Y_w'}{|Y_o'|} = -\frac{3}{2}w + \frac{w^3}{2},$$

where

$|Y_o'|$  = the absolute optimum transformed value in order to be consistent with the original directions of the axes.

The type of this sensitivity expression is a minimization with a negative optimum value. Thus

$$\frac{Y_w'}{|Y_o'|} = -.90 = -\frac{3}{2}w + \frac{w^3}{2}$$

and

$$w^3 - 3w + 1.80 = 0.$$

The two positive roots of this function around its optimum are the lower and the upper ranges of the decision range and are given in Appendix A as

$$w_l = 0.7293004 \text{ and } w_u = 1.2479858$$

or

$$X_l = 0.7293004X_o \text{ and } X_u = 1.2479858X_o$$

and the range is

$$0.7293004 \left( \frac{M}{3A} \right)^{1/2} \leq X_w \leq 1.2479858 \left( \frac{M}{3A} \right)^{1/2}$$

This kind of model is flatter in the lower range than in the upper range within the decision range. The complete evaluation of this kind of models is given in Appendix A.

Numerical example: For the above model let  $A = \frac{1}{16}$ ,  $M = \frac{1}{3}$  and  $C = 10$ .

By substitution, the following results can be obtained as

$$X_o = \left( \frac{M}{3A} \right)^{1/2} = \left( \frac{16}{3 \times 3} \right)^{1/2} = \frac{4}{3}$$

$$Y_o = \left( \frac{M^3}{3A} \right)^{1/2} \left( -\frac{2}{3} \right) + C$$

$$= -\frac{2}{3} \left( \frac{10}{3 \times 3 \times 3 \times 3} \right)^{1/2} + 10.0$$

$$= -\frac{2}{3} \times \frac{3.163}{9} + 10.0$$

$$= -0.2343 + 10.000 = 9.7667$$

$$Y_o^1 = Y_o - C = -0.2343$$

and the decision range is

$$0.7293004 X_o \leq X_w \leq 1.2479858 X_o$$

$$0.9724005 \leq X_w \leq 1.6639811$$

### Cubical Parabola Versus Parabola Model

The equation of the sum of a cubical parabola as an increasing function, and a parabola as a decreasing function is

$$Y = AX^3 + C - B\sqrt{X}$$

where

A = the parameter of the cubical parabola

C = the ordinate of the parabola vertex

B =  $2\sqrt{P}$ , the parameter of the parabola

4P = the latus rectum of the parabola.

Follow the same procedures as in the illustrated example pp. 14-21.

Thus

$$\frac{dY}{dX} = 3AX^2 - \frac{B}{2\sqrt{X}}$$

$$0 = 3AX_o^2 - \frac{B}{2\sqrt{X_o}}$$

$$X_o = \left(\frac{B}{6A}\right)^{2/5}$$

$$Y_o = \left(A^5 \frac{B^6}{6^6 X A^6}\right)^{1/5} + C - \left(\frac{B^5 B}{6A}\right)^{1/5}$$

$$= \left(\frac{B^6}{6A}\right)^{1/5} \left(\frac{1}{6} - 1\right) + C$$

$$= -\frac{5}{6} \left(\frac{B^6}{6A}\right)^{1/5} + C$$

Transform the origin (0,0) of the original function to another new origin (0, -C).



Thus,

$$Y_o' = Y_o - C = -\frac{5}{6} \left(\frac{B}{6A}\right)^{1/5}.$$

The two positive roots of this function around its optimum are the lower and the upper limits of the decision range for the assumption of 10 per cent increase in the transformed value  $Y_w'$  above its optimum  $Y_o'$ . These two roots are

$$w_l = 0.6252769 \text{ and } w_u = 1.3537807$$

or

$$X_u = 0.6252769X_o \text{ and } X_l = 1.3537807X_o$$

and the decision range is

$$0.6252769X_o \leq X_w \leq 1.3537807X_o$$

or

$$0.6252769 \left(\frac{B}{6A}\right)^{2/5} \leq X_w \leq 1.3537807 \left(\frac{B}{6A}\right)^{2/5}.$$

$$Y_w' = \left(\frac{B}{6A}\right)^{1/5} \left(\frac{w^3}{6} - w\right)$$

and the sensitivity expression is

$$\frac{Y_w'}{|Y_o'|} = -\frac{6w}{5} + \frac{w^3}{5}$$

where

$|Y_o'|$  = the absolute value of  $Y_o'$  to be consistent with the original directions of the axes.

The type of this sensitivity expression is a minimization and  $Y_o'$  is a negative value. Thus

$$\left| \frac{Y'}{Y_0} \right| = -0.90 = -\frac{6}{5}w + \frac{w^3}{5}$$

or

$$w^3 - 6w + 4.50 = 0.$$

The curve is flatter in the lower range than in the upper range.

The evaluation of this kind of model is evaluated on the digital computer (Appendix A).

Numerical example: Let for the above model  $A = .083$ ,  $B = 2.83$ , and  $C = 10.00$ .

By substitution, the following results can be obtained

$$X_0 = \left[ \frac{B^2}{6 \times 6A^2} \right]^{1/5} = \left[ \frac{(2.83)^2}{36 \times (.083)^2} \right]^{1/5} = 2/3$$

$$Y_0 = \left[ \frac{(2.83)^6}{6 \times .083} \right]^{1/5} \left[ -\frac{5}{6} \right] + 10.0$$

$$Y_0' = Y_0 - C = -3 \frac{1}{3}$$

and the decision range is

$$0.6252769 \left( \frac{2}{3} \right) \leq X_w \leq 1.3537807 \left( \frac{2}{3} \right)$$

$$0.4168506 \leq X_w \leq 0.9025204 .$$

#### Cubical Parabola Versus Rectangular Hyperbola Model

The sum of an increasing cubical parabola and a decreasing rectangular hyperbola is

$$Y = AX^3 + \frac{B}{X}$$

where

A = the parameter of the cubical parabola

B = the parameter of the rectangular hyperbola.

Follow the same procedures as pp. 14-21 in the illustrated example.

Thus

$$\frac{dY}{dX} = 3AX^2 - \frac{B}{X^2}$$

$$0 = 3AX_0^2 - \frac{B}{X_0^2}$$

$$X_0 = \left( \frac{B}{3A} \right)^{1/4}$$

$$Y_0 = \left( \frac{A^4 B^3}{(3A)^3} \right)^{1/4} + \left( \frac{B^4 3A}{B} \right)^{1/4}$$

$$= (AB^3)^{1/4} \left[ \sqrt[4]{\frac{1}{27}} + \sqrt[4]{3} \right]$$

$$= 1.741 (AB^3)^{1/4}$$

Assume an error has been associated with  $X_0$  in such a way that

$$X_w = wX_0$$

and

w = zero or a positive value.

Therefore, the corresponding value  $Y_w$  due to this error w is

$$Y_o = (AB^3)^{1/4} \left[ 0.425w^3 + \frac{1.316}{w} \right]$$

And the sensitivity expression is

$$\begin{aligned} \frac{Y_w}{Y_o} &= \frac{.425w^3}{1.741} + \frac{1.316}{1.741w} \\ &= \frac{1w^3}{4} + \frac{3}{4w} . \end{aligned}$$

The type of this sensitivity expression is a minimization with a positive optimum value  $Y_o$ . Thus,

$$1.10 = \frac{w^3}{4} + \frac{3}{4w}$$

and

$$w^4 - 4.4w + 3.0 = 0 .$$

The two positive roots of this equation around its optimum are the lower and upper ranges of the decision range for 10 per cent increase in the value  $Y_w$  above its optimum  $Y_o$ . The roots are obtained by the aid of the digital computer (Appendix A) as

$$w_l = 0.7561323 \text{ and } w_u = 1.2664232$$

or

$$X_l = 0.7561323X_o \text{ and } X_u = 1.2664232X_o$$

or

$$X_l = 0.7561323 \left( \frac{B}{3A} \right)^{1/4} \quad \text{and} \quad X_u = 1.2664232 \left( \frac{B}{3A} \right)^{1/4} .$$

And the decision range is

$$0.7561323 \left( \frac{B}{3A} \right)^{1/4} \leq X_w \leq 1.2664232 \left( \frac{B}{3A} \right)^{1/4} .$$

The curve is slightly flatter in the upper range than in the lower range within the decision range. A complete evaluation for the sensitivity expression is evaluated by the aid of a digital computer for  $w = .05$  to  $w = 3.95$  with an increment of  $w = 0.5$  (Appendix A).

Numerical example: For the above model let  $A = \frac{1}{12}$  and  $B = 4.0$ .

By substitution, these following results can be obtained as

$$X_o = \left( \frac{B}{3A} \right)^{1/4} = \left( \frac{4 \times 12}{3} \right)^{1/4} = 2.0$$

$$Y_o = 1.741 \left( \frac{4 \times 4 \times 4}{12} \right)^{1/4} \\ = \frac{2}{1.315} = 1.521$$

and the decision range will be

$$0.7561323 X_o \leq X_w \leq 1.2664232 X_o$$

$$1.5122646 \leq X_w \leq 2.5328464 .$$

#### Straight Line Versus Reciprocal of the Quadratic Function

The equation of the sum of an increasing straight line and a decreasing reciprocal of a quadratic function is

$$Y = MX + \frac{B}{X^2}$$

where

M = the slope of the straight line and a positive value

B = the parameter of the reciprocal of the quadratic function.

Follow the same procedures as in the illustrated example pp. 14-21.

Thus

$$\frac{dY}{dX} = M - \frac{2B}{X^3}$$

$$0 = M - \frac{2B}{X_o^3}$$

$$X_o = \left(\frac{2B}{M}\right)^{1/3}$$

$$Y_o = \left(\frac{M^3 2B}{M}\right)^{1/3} + \left(B^3 \frac{M^2}{4B^2}\right)^{1/3}$$

$$= \left(2BM^2\right)^{1/3} \left(1 + \frac{1}{2}\right)$$

$$= 1.50 \left(2BM^2\right)^{1/3}$$

Assume an error  $w$  has been associated with  $X_o$  in such a way that

$$X_w = wX_o$$

and

$$w = \text{zero or a positive value} .$$

The corresponding value  $Y_w$  due to this error is

$$Y_w = \left(2BM^2\right)^{1/3} \left(w + \frac{1}{2w^2}\right)$$

and the sensitivity expression is

$$\left|\frac{Y_w}{Y_o}\right| = \frac{2}{3}w + \frac{1}{3w^2} .$$

The type of this sensitivity expression is a minimization with a positive optimum value  $Y_o$ . Thus, according to the illustrated example, the sensitivity expression will be

$$\left|\frac{Y_w}{Y_o}\right| = 1.10 = \frac{2}{3}w + \frac{1}{3w^2}$$

or

$$2w^3 - 3.3w^2 + 1.0 = 0 .$$

The two positive roots of this equation around its optimum are the lower and the upper ranges of the decision range for 10 per cent increase in the value  $Y_w$  above its optimum  $Y_o$ . These two roots are obtained by the aid of the digital computer (Appendix A). The roots are

$$w_l = 0.742112 \text{ and } w_u = 1.3919315$$

or

$$X_l = 0.742112X_o \text{ and } X_u = 1.3919315X_o$$

or

$$X_l = 0.742112 \left( \frac{2B}{M} \right)^{1/3} \text{ and } X_u = 1.3919315 \left( \frac{2B}{M} \right)^{1/3} .$$

The decision range is

$$0.742112 \left( \frac{2B}{M} \right)^{1/3} \leq X_w \leq 1.3919315 \left( \frac{2B}{M} \right)^{1/3} .$$

The curve of this type of model is flatter in the upper range than in the lower range within the decision range. A complete evaluation for this kind of model is given in Appendix A.

Numerical example: For the above model let  $A = 1.0$  and  $B = 4.00$ .

By substitution, the following results can be obtained as

$$X_o = \left( \frac{2B}{M} \right)^{1/3} = \left( \frac{2 \times 4.0}{1} \right)^{1/3} = 2.0$$

$$Y_o = 1.50 \left( 2 \times 4.0 \times 1.0 \times 1.0 \right)^{1/3} = 3.0$$

and the decision range will be

$$0.742112X_o \leq X_w \leq 1.3919315X_o$$

$$1.4842224 \leq X_w \leq 2.7838630 .$$



### Models as a Summation of Three Different Standard Curves

The condition for providing the optimality (trade off relationship) in each of these models can be attained by having one increasing curve while the others are decreasing ones, or by having one decreasing curve while the others are increasing ones. The discussion here is limited to the three unknown curves occurring frequently in Operations Research field. These curves are straight line, rectangular hyperbola and parabola. The same analogy can be applied to any kind of curve.

#### Straight Line and Parabola Versus Rectangular Hyperbola

The straight line and parabola are increasing functions, and the rectangular hyperbola is a decreasing function. The sum of these three curves is

$$Y = MX + AX^2 + \frac{B}{X}$$

where

M = the slope of the straight line and a positive value

A = the parameter of the parabola

B = the parameter of the rectangular hyperbola.

Follow the same procedures given in the illustrated example pp. 15-19.

Thus

$$\frac{dY}{dX} = M + 2AX - \frac{B}{X^2}$$

$$0 = M + 2AX_0 - \frac{B}{X_0^2}$$

$$X_0^3 + \frac{M}{2A} X_0^2 - \frac{B}{2A} = 0$$

This is the cubic function and can be solved as follows.

Let

$$A_1 = \text{coefficient of } X_o^3 = 1.00$$

$$A_2 = \text{coefficient of } X_o^2 = \frac{M}{2A}$$

$$A_3 = \text{coefficient of } X_o = -\frac{B}{2A} .$$

The above function can be rewritten as

$$A_1 X_o^3 + \frac{M}{2A} X_o^2 + A_3 = 0. \quad (1)$$

Let

$$X_o = Z_o - \frac{A_2}{3} .$$

Substitute with  $Z_o$  in the equation (1), and obtain

$$Z_o^3 + \left( A_1 - \frac{A_2^2}{3} \right) Z_o + \left( A_3 + \frac{A_1 A_2}{3} + \frac{2}{27} A_2^3 \right) = 0.$$

Let

$$-G = \left( A_1 - \frac{A_2^2}{3} \right)$$

and

$$R = - \left( A_3 + \frac{A_1 A_2}{3} + \frac{2}{27} A_2^3 \right) .$$

Thus,

$$Z_o^3 - GZ - R = 0 .$$

There are three cases for solving the cubic functions as follows

1) When  $27R^2 > 4G^3$ :

In this case, the solution contains one real root and two complex roots. As the interest here is for the real root, therefore the real root will be

a. If G and R are both positive, thus find  $\phi$  in such a way that

$$\cosh \phi = \left(\frac{3}{G}\right)^{3/2} \cdot \left(\frac{R}{2}\right)$$

Then, the real root is given by the equation

$$Z_0 = \frac{2}{(3)^{1/2}} G^{1/2} \cosh \left(\frac{\phi}{3}\right)$$

and consequently by substitution

$$\begin{aligned} X_0 &= \frac{2}{\sqrt{3}} G^{1/2} \cosh \left(\frac{\phi}{3}\right) - \frac{A_2}{3} \\ &= \frac{2}{\sqrt{3}} \sqrt{\frac{A_2^2}{3} - A_1} \cosh \left(\frac{\phi}{3}\right) - \frac{A_2}{3} \\ &= \frac{2}{\sqrt{3}} \sqrt{\frac{M^2}{12A^2} - 1.0} \cosh \left(\frac{\phi}{3}\right) - \frac{M}{6A} \end{aligned}$$

b. If G is a negative and R is a positive, find  $\phi$  in such a way that

$$\sinh \phi = \left(\frac{3}{-G}\right)^{3/2} \cdot \left(\frac{R}{2}\right)$$

Then, the real root is given by the equation

$$Z_0 = \frac{2}{\sqrt{3}} (-G)^{-1/2} \sinh \left(\frac{\phi}{3}\right)$$

and

$$\begin{aligned}
 X_0 &= \frac{2}{\sqrt{3}} \sqrt{A_1 - \frac{A_2^2}{3}} \operatorname{Sinh} \left( \frac{\phi}{3} \right) - \frac{A_2}{3} \\
 &= \frac{2}{\sqrt{3}} \sqrt{1.0 - \frac{M^2}{12A^2}} \operatorname{Sinh} \left( \frac{\phi}{3} \right) - \frac{M}{6A}
 \end{aligned}$$

2) When  $27R^2 < 4G^3$

In this case the cubic equation has three real roots as follows. Find  $\phi$  in such a way that

$$\cos \frac{\phi}{3} = \left( \frac{3}{G} \right)^{3/2} \cdot \left( \frac{R}{2} \right).$$

Then, the three roots are

$$Z_{10} = \frac{2}{(3)^{1/2}} G^{1/2} \cos \left( \frac{\phi}{3} \right)$$

$$Z_{20} = \frac{2}{(3)^{1/2}} G^{1/2} \cos \left( \frac{\pi - \phi}{3} \right)$$

and

$$Z_{30} = \frac{2}{(3)^{1/2}} G^{1/2} \cos \left( \frac{\pi + \phi}{3} \right)$$

and

$$X_{10} = \frac{2}{(3)^{1/2}} \left( \frac{A_2^2}{3} - A_1 \right)^{1/2} \cos \left( \frac{\phi}{3} \right) - \frac{M}{6A}$$

$$X_{20} = \frac{2}{(3)^{1/2}} \left( \frac{A_2^2}{3} - A_1 \right)^{1/2} \cos \left( \frac{\pi + \phi}{3} \right) - \frac{M}{6A}$$

and

$$X_{30} = \frac{2}{(3)^{1/2}} \left( \frac{A_2^2}{3} - A_1 \right)^{1/2} \cos \left( \frac{\pi + \phi}{3} \right) - \frac{M}{6A}$$

3) When  $27R^2 = 4G^3$ , the cubic equation has three real roots, two or them are equal as follows

$$Z_{10} = Z_{20} = \frac{-3R}{2P} \quad (\text{equal roots})$$

$$Z_{30} = \frac{3R}{2P} \quad (\text{single root}) .$$

The corresponding roots are

$$X_{10} = X_{20} = \frac{-3R}{2G} - \frac{M}{6A}$$

$$X_{30} = \frac{3R}{2G} - \frac{M}{6A} .$$

Let us assume that the equation yields one real root. Thus

$$X_o = \frac{2}{(3)^{1/2}} \left( \frac{M^2}{12A^2} - 1.00 \right)^{1/2} \cosh \left( \frac{\phi}{3} \right) - \frac{M}{6A} .$$

Substitute with  $X_o$  in Y function to obtain  $Y_o$

$$Y_o = MX_o + AX_o^2 + \frac{B}{X_o} .$$

Assume an error  $w$  has been associated with  $X_o$  in such a way that

$$X_w = wX_o$$

and

$w =$  zero or a positive value.

Thus, the corresponding value  $Y_w$  is

$$Y_w = \left( MX_o \right)_w + \left( AX_o^2 \right)_w + \left( \frac{B}{X_o} \right) \frac{1}{w}$$

and the sensitivity expression is

$$\begin{aligned} \frac{Y_w}{|Y_o|} &= \left( \frac{MX_o}{|Y_o|} \right) w + \left( \frac{AX_o^2}{|Y_o|} \right) w^2 + \left( \frac{B}{|Y_o X_o|} \right) \frac{1}{w} \\ &= A_o w + B_o w^2 + C_o \frac{1}{w} \end{aligned}$$

where

$|Y_o|$  = the absolute value of the optimum value  $Y_o$  to be consistent with the original directions of the axes.

The interpretation of this sensitivity expression is equal to another transformed curve as the sum of the straight line with a positive slope =  $A_o$  and the parabola with a parameter  $B_o$ , and the rectangular hyperbola with a parameter  $C_o$ . The values of  $A_o$ ,  $B_o$ , and  $C_o$  are

$A_o$  = the optimum part of the straight line  $MX_o$  over the absolute optimum of the total variable parts of the function  $|Y_o'|$ .

$B_o$  = the optimum part of the parabola  $AX_o^2$  over the absolute optimum of the total variable parts of the function  $|Y_o'|$ .

$C_o$  = the optimum part of the rectangular hyperbola  $\frac{B}{X_o}$  over the absolute optimum of the total variable parts of the function  $|Y_o'|$ .

The expression is general or in other words the expression is a function of the error  $w$  (the parameters and  $X_o$  are constants). Sensitivity expressions can be easily derived for each case around their optimum real roots. From this discussion one can notice that there are multiple real

optimum solutions.

### Straight Line Versus Parabola and Rectangular Hyperbola Model

In this model the straight line is an increasing, and the parabola and the rectangular hyperbola are decreasing functions. The sum of these three curves is

$$Y = MX - A(X)^{1/2} + C + \frac{B}{X}$$

where

M = the slope of the straight line and a positive value

A =  $2\sqrt{P}$  the parameter of the decreasing parabola

4P = the length of the latus rectum of the parabola

C = the ordinate of the vertex on the parabola on the y-axis

B = the parameter of the rectangular hyperbola.

Follow the same procedures as in the illustrated example pp. 14-21.

Thus,

$$\frac{dY}{dX} = M - \frac{A}{2}(X)^{-1/2} - \frac{B}{X^2}$$

$$0 = M - \frac{A}{2}(X_0)^{-1/2} - \frac{B}{X_0^2}$$

$$MX_0^2 - \frac{A}{2}(X_0)^{3/2} - B = 0$$

$$M^2 X_0^4 - 2MBX_0^2 + B^2 - \frac{A^2}{4} X_0^3 = 0$$

$$M^2 X_0^4 - \frac{A^2}{4} X_0^3 - 2MBX_0^2 + B^2 = 0.$$

As the coefficients of the fourth power equation are in letters instead of figures, the solution is not available. For situations which have this kind of equation, the sensitivity expression for the variable parts is as follows

$$\left| \frac{Y_w'}{Y_0'} \right| = A_0 w - B_0 w + C_0 \frac{1}{w}$$

where  $Y_0'$ ,  $A_0$ ,  $B_0$ , and  $C_0$  are defined in the previous model.

#### Parabola Versus Straight Line and Rectangular Hyperbola

This model is a summation of a parabola as an increasing function and a straight line and a rectangular hyperbola as decreasing functions.

Thus, the summation function is

$$Y = AX^2 - MX + C + \frac{B}{X}$$

where

A = the parameter of an increasing parabola

M = the slope of the straight line and a positive value

C = the intercept of the straight line of the y-axis and a positive value

B = the parameter of the rectangular hyperbola.

Follow the same procedures given in the illustrated example pp. 14-

21. Thus the sensitivity expression is

$$\left| \frac{Y_w'}{Y_0'} \right| = A_0 w^2 - B_0 w + C_0 \frac{1}{w}$$



where  $Y_0$ ,  $A_0$ ,  $B_0$ , and  $C_0$  are defined in the previous model.

Summary of the Sensitivity Analysis of the Mathematical Models

The following four tables summarize the optima, sensitivity expressions, and decision ranges of the discussed mathematical models in Chapter 3.

Table 1 deals with a variety of one standard curve models, Table 2 and Table 3 summarize a variety of a summation of two different standard curve models. However, Table 2 deals with the minimization functions and Table 3 treats the maximization functions. These two different standard curves are evaluated in detail by the aid of a digital computer in Appendices A and B. It is worth to mention that ranges and the optima of the X values of the models No. 3, 5, 6, 8, and 9 in Table 3 (minimization) are the same as the range and the optima of the X values of the models No. 1, 2, 3, 4, and 5 in Table IV (maximization). The optima of the transformed Y values (optimum of the variable parts of the function only) in Table 3 are the negative optima of the transformed Y values in Table 4. These results reconcile the fact that the minimizing of the objective function is the same as the maximizing of the negative of the same objective function. The final Table No. 4 summarizes the discussions of a summation of three different standard curves (straight line, parabola, and rectangular hyperbola). The models in Tables 2, 3, and 4 are discussed under the assumption that the first curve(s) is (are) an increasing function(s) while the other(s) is (are) a decreasing one(s).

TABLE I  
SUMMARY OF THE SENSITIVITY ANALYSIS OF MATHEMATICAL MODELS  
ONE STANDARD CURVE MODELS

No.	Type of the Model	Mathematical Equation	Optimum Values ( $X_o, Y_o$ )	Sensitivity Expression $\frac{Y_w}{Y_o}$	Decision Range for 10% Increase	
					Lower	Upper
1.	Circle <sup>*1</sup>			$1 - \left(\frac{H}{R}\right)^2 (w-1)^2$		
2.	Ellipse <sup>*2</sup>			$1 - \left(\frac{H}{A}\right)^2 (w-1)^2$		
3.	Parabola			$1 + \frac{H^2}{4PK} (w-1)^2$		
4.	Gaussian <sup>*3</sup>	$Y = E^{-X^2}$	(0, a)	$E^{-w^2}$	-0.317	0.317
5.	Catenary			$\frac{1}{2} E^{w/A} - E^{-w/A}$		

\*<sub>1</sub> General sensitivity expression only for the family of circle models  $H = R$

\*<sub>2</sub> General sensitivity expression only for the family of ellipse models  $H = A$

\*<sub>3</sub> General sensitivity expression

TABLE II  
 SUMMARY OF THE SENSITIVITY ANALYSIS OF MATHEMATICAL MODELS  
 SUMMATION OF TWO DIFFERENT STANDARD CURVES (MINIMIZ.\*) MODELS

No.	Type of the Model	Optimum Values $X_0, Y_0$	Sensitivity Expression $\frac{Y'}{Y} \frac{W}{W_0}$	Decision Range for 10% Increase	
				Lower	Upper
1.	Straight line vs. ellipse $Y =$	Look to the model	Look to the model		
2.	Straight line vs. circle $Y =$	Look to the model	Look to the model		
3.	Straight line vs. parabola $Y = MX - A(X)^{1/2} + C$	$\left( \frac{A^2}{4M^2}, C - \frac{A^2}{4M} \right)$	$w - 2\sqrt{w}$	0.467555	1.732452
4.	Straight line vs. R. hyperbola $Y = MX + \frac{B}{X} + C$	$\left( \sqrt{\frac{B}{M}}, C - 2\sqrt{MB} \right)$	$\frac{w^2 + 1}{2w}$	0.641743	1.558256
5.	Parabola vs. straight line $Y = AX^2 - MX + C$	$\left( \frac{M}{2A}, C - \frac{M^2}{4A} \right)$	$w^2 - 2w$	0.683774	1.316228

\* Appendix A

TABLE II (CONTINUED)

No.	Type of the Model	Optimum Values $X_o, Y_o$	Sensitivity Expression $\frac{Y'}{Y_o} \frac{w}{w'}$	Decision Range for 10% Increase	
				Lower	Upper
6.	Parabola vs. parabola $Y = AX^2 - B X + C$	$\left( \sqrt[3]{\frac{B^2}{16A^2}}, C - 3/4 \sqrt[3]{\frac{B^4}{4A}} \right)$	$\frac{1}{3} (w^2 - 4\sqrt{w})$	0.573307	1.461277
7.	Parabola vs. R. hyperbola $Y = AX^2 + \frac{B}{X} + C$	$\left( \sqrt[3]{\frac{B}{4A}}, C + 1.889 \sqrt[3]{AB^2} \right)$	$\frac{1}{3} (w^2 + \frac{2}{w})$	0.718677	1.347391
8.	C. parabola vs. st. line $Y = AX^3 - MX + C$	$\left( \sqrt{\frac{M}{2A}}, C - \frac{2}{3} \sqrt{\frac{M^3}{3A}} \right)$	$\frac{1}{2} (w^3 - 3w)$	0.729300	1.247986
9.	C. parabola vs. parabola $Y = AX^3 - B X + C$	$\left( \sqrt[5]{\frac{B^2}{36A^2}}, C - .5/6 \sqrt[5]{\frac{B^6}{6A}} \right)$	$\frac{1}{5} (w^3 - 6\sqrt{w})$	0.625277	1.353781
10.	C. parabola vs. R. hyperbola $Y = AX^3 + \frac{B}{X} + C$	$\left( \sqrt[4]{\frac{B}{3A}}, C + 1.741 \sqrt[4]{AB^3} \right)$	$\frac{1}{4} (w^3 + \frac{3}{w})$	0.756132	1.26642
11.	Straight line vs. $\frac{B}{X^2}$ $Y = MX + \frac{B}{X^2} + C$	$\left( \sqrt[3]{\frac{2B}{M}}, C + 1.500 \sqrt[3]{2BM^2} \right)$	$\frac{1}{3} (2w + \frac{1}{w^2})$	0.74211	1.391932

TABLE III  
SUMMARY OF THE SENSITIVITY ANALYSIS OF MATHEMATICAL MODELS  
SUMMATION OF TWO DIFFERENT STANDARD CURVES (MAXIMIZ.\*) MODELS

No.	Type of the Model	Optimum Values $X_o, Y_o$	Sensitivity Expression $\frac{Y'}{Y_o} \cdot \frac{w}{w}$	Decision Range for 10% Decrease	
				Lower	Upper
1.	Parabola vs. straight line $Y=A(X)^{1/2}-MX+C$	$\left( \frac{A^2}{4M^2}, C + \frac{A^2}{4M} \right)$	$2\sqrt{w} - w$	0.467547	1.732422
2.	Straight line vs. parabola $Y=MX-AC^2+C$	$\left( \frac{M}{2A}, C + \frac{M^2}{4A} \right)$	$2w - w^2$	0.683816	1.316189
3.	Parabola vs. parabola $Y=B(X)^{1/2}-AX^2+C$	$\left( \sqrt[3]{\frac{B^2}{16A^2}}, C + 3/4 \sqrt[3]{\frac{B^4}{4A}} \right)$	$\frac{1}{3} (4\sqrt{w} - w^2)$	0.573415	1.461205
4.	Straight line vs. C. parabola $Y=MX-AX^3+C$	$\left( \sqrt{\frac{M}{3A}}, C + \sqrt{\frac{M^3}{3A}} \right)$	$\frac{1}{2} (3w - w^3)$	0.729380	1.248141
5.	Parabola vs. C. parabola $Y=B(X)^{1/2}-AX^3+C$	$\left( \sqrt[5]{\frac{B^2}{36A^2}}, C + 5/6 \sqrt[5]{\frac{B^6}{6A}} \right)$	$\frac{1}{5} (6\sqrt{w} - w^3)$	0.625325	1.353622

\* Appendix B

TABLE IV  
 SUMMARY OF THE SENSITIVITY ANALYSIS OF MATHEMATICAL MODELS  
 SUMMATION OF THREE DIFFERENT STANDARD CURVES MODEL

No.	Type of the Model	Optimum Values $X_o, Y_o$	Sensitivity Expression $\frac{Y}{Y_o} \frac{w}{Y_o}$	Decision Range for 10% Increase	
				Lower	Upper
1.	Straight line and parabola vs. R. hyperbola $Y=MX+AX^2 + \frac{B}{X}$	Look to the model	$\frac{1}{ Y_o } \left( MX_o w + AX_o^2 w^2 + \frac{B}{X_o w} \right)$		
2.	Straight line vs. parabola and R. hyperbola $Y=MX-AX^2 + \frac{B}{X}$	Look to the model	$\frac{1}{ Y_o } \left( MX_o w - AX_o^2 w^2 + \frac{B}{X_o w} \right)$		
3.	Parabola vs. st. line and R. hyperbola $Y=AX^2-MX + \frac{B}{X}$	Look to the model	$\frac{1}{ Y_o } \left( AX_o^2 w^2 - MX_o w + \frac{B}{X_o w} \right)$		

## CHAPTER IV

### INVENTORY MODELS

#### Introduction

There are many different definitions for inventory systems according to the interesting points of view. Some people consider inventory systems as systems of keeping records of the amounts of commodities in stock, therefore making the problem one of where and when entries should be made. Other people look at inventory systems from the financial point of view, therefore making their problem one concerning the turnover and financing investments tied up in stocks. Other groups consider inventory systems as systems to deal with what items to stock, when to stock, how many to stock; therefore their problem is the labor stability, utilization of equipment and facilities, and customer relations. Because of this wide range of the inventory systems' definitions, it is necessary to define the inventory systems that will be dealt with in this dissertation. The following two definitions are well suited to the purpose of this research project.

Ackoff's Definition (2): "By an inventory process Operations Research has come to mean a process involving one or both of the following discussions: (a) how many (or much) to order (That is to produce or purchase) and (b) when to order. These discussions involve the balancing of inventory carrying costs against one or more of the following: order or run set-up costs, shortage or delay costs and cost associated with the level of production or purchasing."

Definition: An inventory is the system in which the decision maker is looking for the optimization of the decision variables normally called controllable variables in the light of inventory independent cost elements whose cost coefficients are mutually exclusive. These cost elements may be any two or the three of the following:

- (1) The cost of carrying or holding inventories.
- (2) The cost of running out of units or incurring shortages.
- (3) The cost of replenishing the inventory system.

#### The Nature of the Costs

This part is devoted for elaborating the last definition for the purpose of classifying the different independent cost elements in the inventory system. The nature of the first cost element is the cost incurred in carrying or holding inventories and is equal to one or more of the following:

- a. The cost of renting or providing warehouse space
- b. The cost incurred in the obsolescence risk
- c. The cost of warehouses' overheads such as the handling of equipment
- d. The cost of the tied capital in stored units
- e. The cost of insurance and taxes.

The second cost element is the cost incurred in suffering shortage of units or running out of units. This may be one or more of the following:

- a. The cost incurred in the loss of sales
- b. The cost incurred in the loss of good will
- c. The cost incurred in the shut-off of the related operations or the decrease in their productivities
- d. The cost due to the facility being idle for the next set-up manufacturing
- e. The cost due to special administrative efforts.

The third cost element is the cost incurred in the procurement and replenishment functions. This may be one or more of the following:

- a. The cost of ordering the lot size
- b. The cost of the paper work and the customs
- c. The cost of the commission



- d. The cost of receiving and inspecting the lot size
- e. The cost associated with the payment of invoices.

Later it is referred to as the holding cost element or HC; the shortage cost element as SC, and the procurement and replenishment cost element as PC. Some models involve the three cost elements while others involve only two of these three cost elements. These cost elements are independent and their cost factors are mutually exclusive.

#### Types of the Inventory Models

The inventory models can be classified on the basis of the number and type of cost elements as:

1. Models involving holding and shortage cost elements.
2. Models involving holding and procurement cost elements.
3. Models involving shortage and procurement cost elements.
4. Models involving holding, shortage, and procurement cost elements.

The models with two cost elements like 1, 2, and 3 are apparent in practice when the third absent cost element is not under the control of the decision maker. The model 1 appears in practice when the procurement cost is not charged on the inventory system or the replenishing of the items occurs over prescribed constant periods. The model 2 appears in practice when the shortage cost factor is extremely high and the processes cannot afford its occurrence. The model 3 appears in practice when there are no spaces provided for the holding units or the processes cannot afford to tie the capital in stored units for a period of time. There is another characteristic which distinguishes inventory models from the point of their replenishment periods as:

1. Purchasing models which have the replenishment periods are approximately equal to zero. Some times they are called instantaneous replenishment period models.
2. Manufacturing models which have the replenishment periods are greater than zero.

### Inventory Problem

The problem is to make an optimum decision for the inventory system. These decisions are always made in the light of cost elements, but they are rarely made in terms of costs or profits. They are usually made in time and quantity as:

1. When to replenish in time units
2. How much or many to replenish in quantity units.

The time and quantity are the decision variables and are subject to the control of the decision maker, usually called controllable variables.

Although finding the variables that give the optimum total cost is the main purpose of the inventory problem, the research maker has to provide the amount it will cost. Also the main purpose of this research is not only to provide the optimum variables and the optimum objective function, but also to establish the rules for the decision range for selecting the controllable variables in such a way that the net increase or decrease in the total objective function is insignificant or negligible.

### Seeking for Trade-Off Solution

The objective function has an optimum solution if and only if the

rates of the increasing cost elements are equal to the rates of the decreasing cost elements. This is the essence of the differentiation as a mathematical technique to obtain the optimality. In industrial or business situations, it is common to say the objective function has been traded-off instead of saying the objective function has an optimum solution. Thus, the traded off solutions for the decision models in inventory systems are obtained through using the differentiation for the continuous functions. However, if the objective function is a function of more than one decision variable, the partial derivative is used provided the limits exist. The objective function with one decision variable can be represented by a curve in two dimensions in space while the one with more than one decision variable can be represented on a surface whose dimensions in space are equal to the number of variables plus one.

#### Seeking for Trade-Off Range

The decision maker is frequently required to set a policy that will be flexible enough to meet the unpredictable situations in the future and to meet the dynamic needs of periodic change. As a rule of thumb, the total variable parts in the objective function is within plus or minus 10 per cent over or under its optimum value. The plus is correspondent to the minimizing of the objective function and the minus is correspondent to the maximizing of the objective function. The solution for this predetermined allowance in the objective function yields a range or a region for selecting the controllable variable or variables. This range or a region is called the decision range or the decision region. Looking back to the derived mathematical models in Chapter III, these ranges are obtained from the derived sensitivity expressions, which are the measurements of the sensitivity analyses of

these models around their optimum solutions ( their trade-off solutions).

At this point, it is necessary to define the decision range or region.

Decision range or region

Definition: the decision range or region is that where the decision maker can select a variable(s) in such a way that the yield in the objective function due to this selection is insignificant above or under its mathematical optimum.

### Building the Total Cost Inventory Function

The following discussions are for the purpose to build the total cost equation of an inventory system, by analyzing the type of the replenishment, the type of the demand, and the replenishment-demand interactions.

### Replenishment Patterns

Definition: The replenishment period is the length of time during which the replenishment lot size  $Q_0$  is being added to the inventory level as shown in the figure 3.

To discuss the situations of these five cases, let

$T_r$  = the replenishment period-time units

$R$  = the average replenishment during the replenishment period  $T_r$ -units

$Q_0$  = the optimum lot size to be replenished during the replenishment period  $T_r$ -units

$n_1$  = the power index of the replenishment curve.

Thus, five cases arise in Figure 3 as

- a. The replenishment period is insignificant that is  $T_r \cong 0$ .

Therefore

$$R = \frac{Q_0}{0} = \infty$$

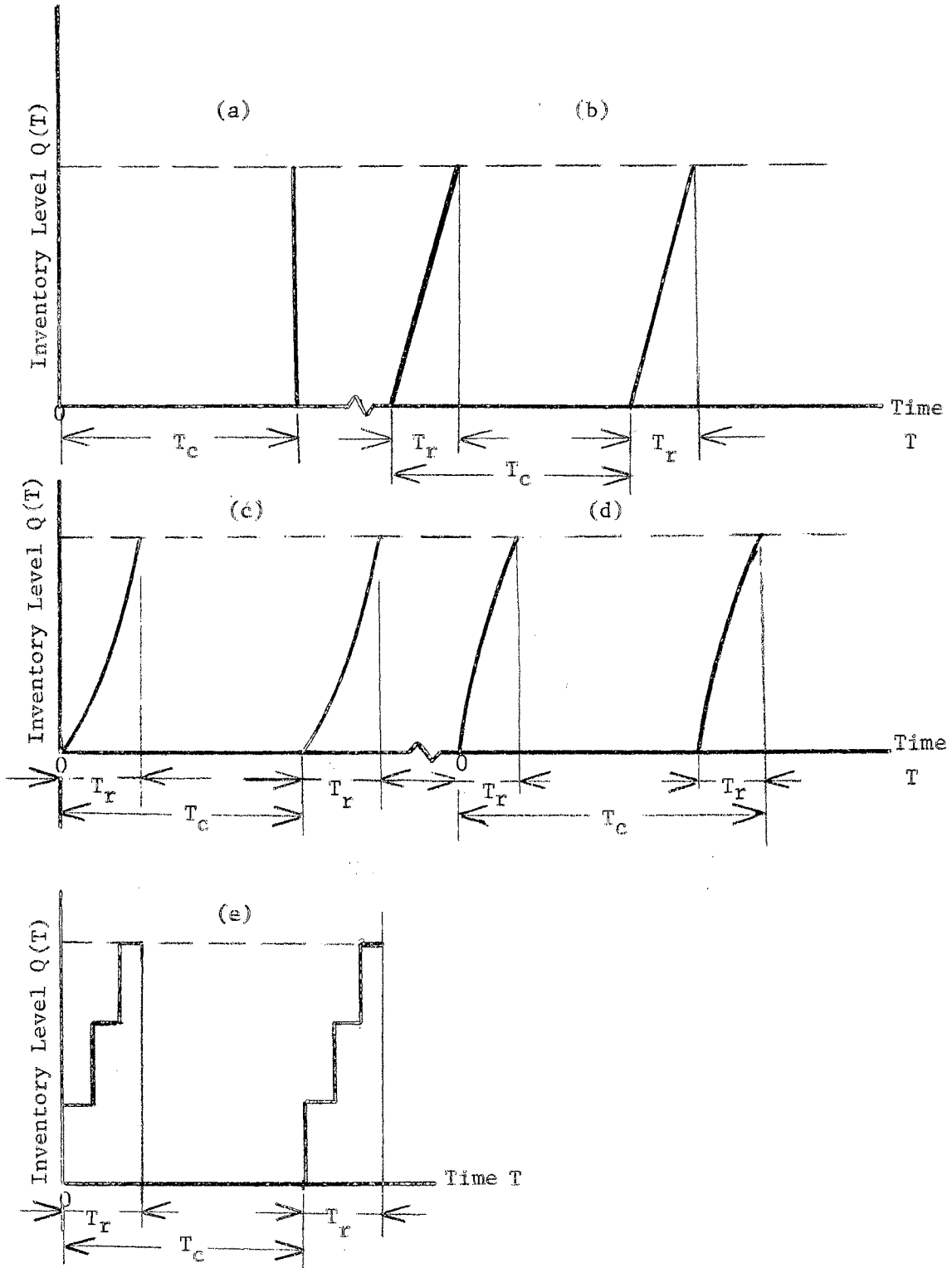


Figure (3). Replenishment Patterns

which implies that the average replenishment units during the replenishment period  $T_r$  are infinite, and the power index  $n_1$  is infinite.

- b. The power index of the replenishment curve is  $n_1 = 1.0$ , which implies a uniform replenishment during the replenishment period  $T_r$ .

Thus,

$$R = \frac{Q_0}{T_r} = \text{constant}$$

- c. The power index of the replenishment curve is  $n_1 > 1.0$ . This means that the rate of replenishment varies from time to time within the replenishment period  $T_r$  and the larger portion of the lot size occurs toward the end of the replenishment period.
- d. The power index of the replenishment curve is  $n_1 < 1.0$ . This means that the rate of replenishment varies from time to time within the replenishment period  $T_r$  and the larger portion of the lot size occurs towards the beginning of the replenishment period.
- e. It is a batch replenishment or discontinuous replenishment. Thus the general equation of the quantity replenished within the replenishment period  $T_r$  is (for a to d)

$$Q(T) = Q_0 \sqrt[n_1]{\frac{T}{T_r}}$$

where

$Q(T)$  = the quantity in the inventory at any time  $T$ .

$Q_0$  = the quantity in the inventory at the beginning of the replenishment period  $T_r$ .

$T_r$  = the replenishment period.

$$Q(T) = Q_0 \left[ \frac{T}{T_r} \right]^{\frac{1}{n_1}} .$$

### Demand Pattern

**Definition:** The demand period is the length of time during which the level of inventory came to a point where a new replenishment is starting to be added to the inventory level as in Figure 4.

To discuss the situations for these five cases, let

$T_d$  = the demand period

$D$  = the average demand during the demand period  $T_d$

$n_2$  = the power index of the demand curve.

Thus,

- a. The demand size  $X$  is drawn once at the end of the period.

This means that  $n_2$  is infinite and is called infinite demand pattern.

- b. The demand size  $X$  is drawn uniformly during the demand period

$T_d$ . This means that the average demand during the demand period  $T_d$  is constant as

$$D = \frac{X}{T_d} = \text{constant} .$$

- c. The demand size  $X$  is drawn through a curve with power index

$n_2 > 1.0$ . This means that the rate of demand varies from time to time within the demand period  $T_d$  and the larger portion of the demand size  $X$  is drawn towards the beginning of the demand period.

- d. The demand size  $X$  is drawn through a curve with power index

$n_2 < 1.0$ . This means that the rate of demand varies from time to time within the demand period  $T_d$  and the larger portion of the demand size  $X$  is drawn towards the end of the demand period.

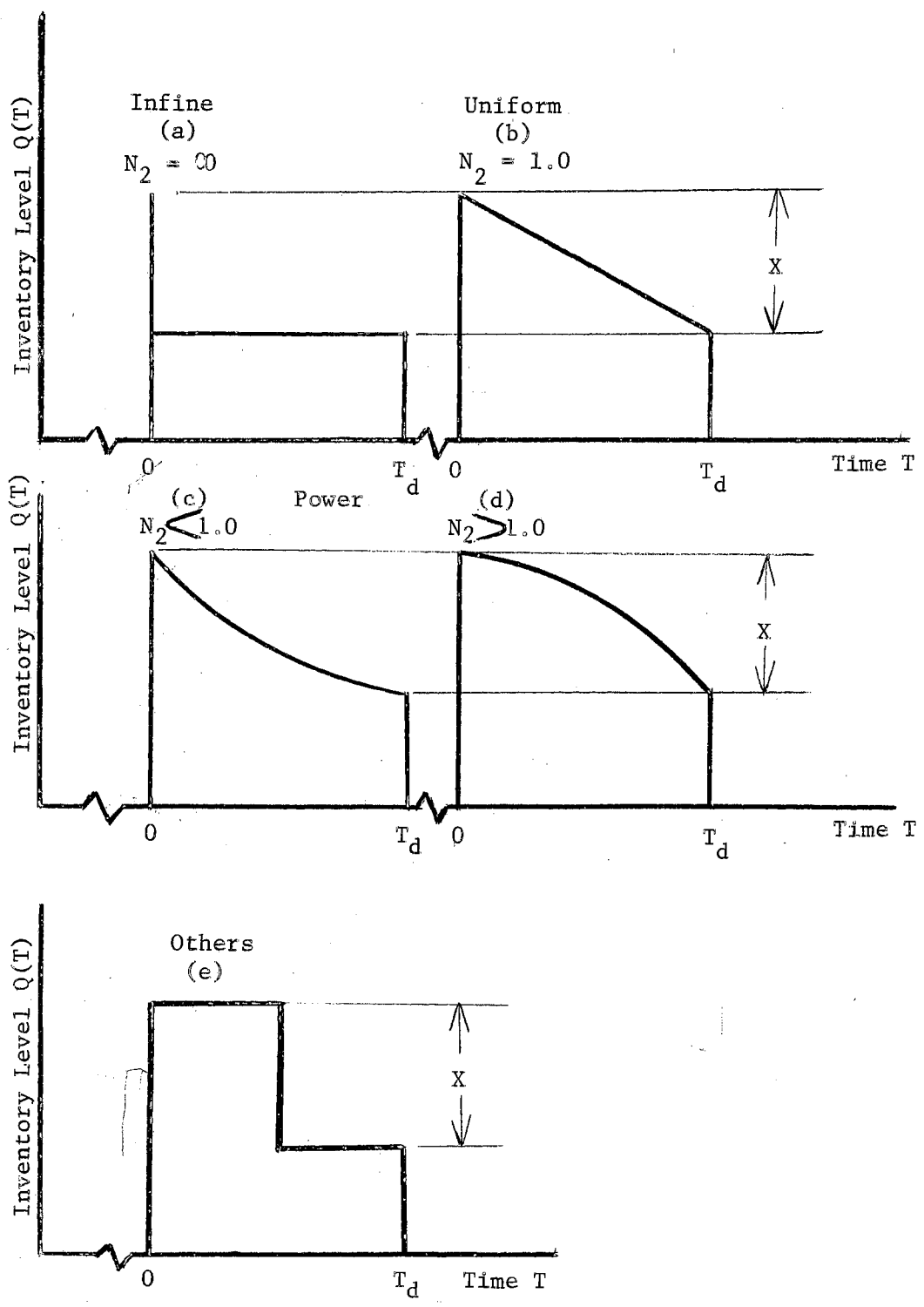


Figure (4). Demand Patterns



e. The demand is a discrete demand or batch demand pattern.

The general equation of the quantity in the inventory at time T within the demand period  $T_d$  is (for a to d)

$$Q(T) = S - X \sqrt[n_2]{\frac{T}{T_d}} .$$

where

$Q(T)$  = the quantity in the inventory at time T

$S$  = the level of the inventory at the beginning of the demand period  $T_d$

$X$  = the size of the demand to be withdrawn during the demand period  $T_d$

$T_d$  = the demand period and is equal to  $T_c$

For (a)

$$\begin{aligned} Q(T) &= S - X \left[ \frac{T_d}{T_d} \right]^{\frac{1}{n_2 = \infty}} \\ &= S - X . \end{aligned}$$

For (b)

$$\begin{aligned} Q(T) &= S - X \left[ \frac{T}{T_d} \right]^{\frac{1}{n_2 = 1.0}} \\ &= S - X \frac{T}{T_d} \\ &= S - DT . \end{aligned}$$

For (c) and (d)

$$Q(T) = S - X \left[ \frac{T}{T_d} \right]^{\frac{1}{n_2}} .$$

### Replenishment and Demand Interaction

In this kind of models, one can generate different combinations of the replenishment demand interactions. For example

- a. The power index of the replenishment curve  $n_1$  is equal to the power index of the demand curve  $n_2$  and each equals 1.0.
- b. The power index of the replenishment curve  $n_1$  is equal to the power index of the demand curve  $n_2$  and each is less than 1.0.
- c. The power index of the replenishment curve  $n_1$  is equal to the power index of the demand curve  $n_2$  and each is greater than 1.0.
- d. The power index of the replenishment curve  $n_1$  is less than 1.0, while that of the demand curve  $n_2$  is greater than 1.0.
- e. The power index of the replenishment curve  $n_1$  is greater than 1.0, while that of the demand curve  $n_2$  is less than 1.0.
- f. Or any other combination.

The following figures illustrate these cases.

In the figure 5,  $n_1 = n_2 = 1.0$ . This model has the following relations. The rates of the replenishment and the demand are constants. Thus the rate of accumulation is also constant.

$$R = \frac{Q_0}{T_r} \quad (\text{through the replenishment period})$$

$$D = \frac{X}{T_d} \quad (\text{through the demand period without replenishment})$$

And the rate of accumulation is

$$R - D = \frac{Q_0}{T_r} - \frac{X}{T_d} \quad (\text{through the replenishment period})$$

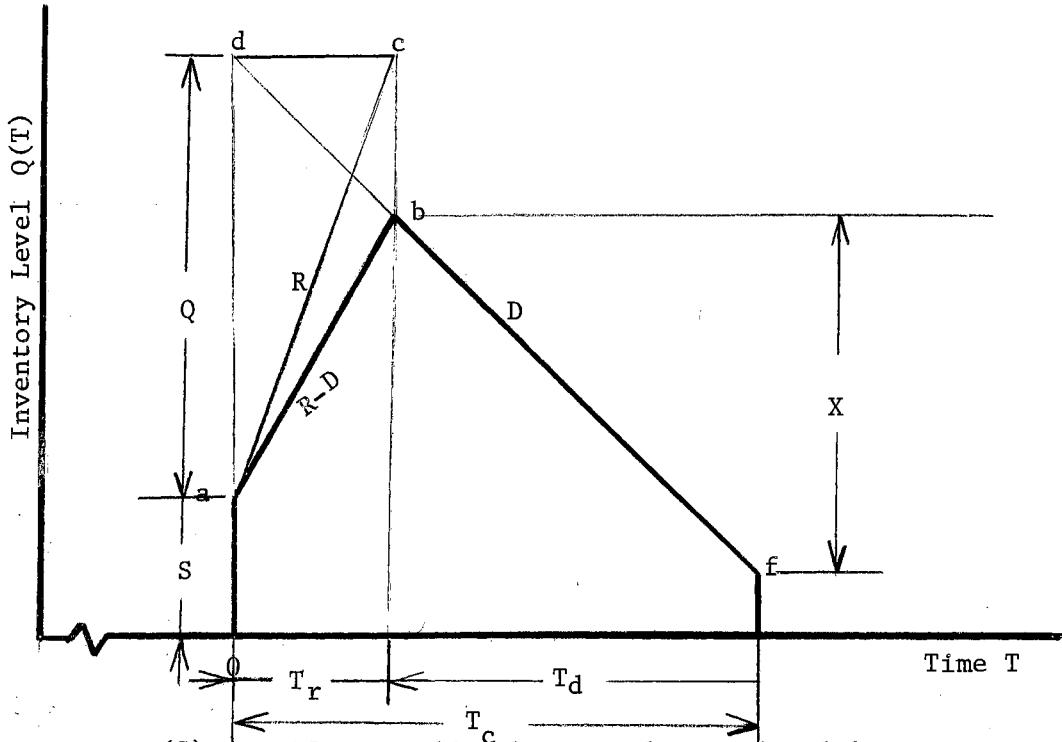


Figure (5). Uniform Replenishment and Demand Model

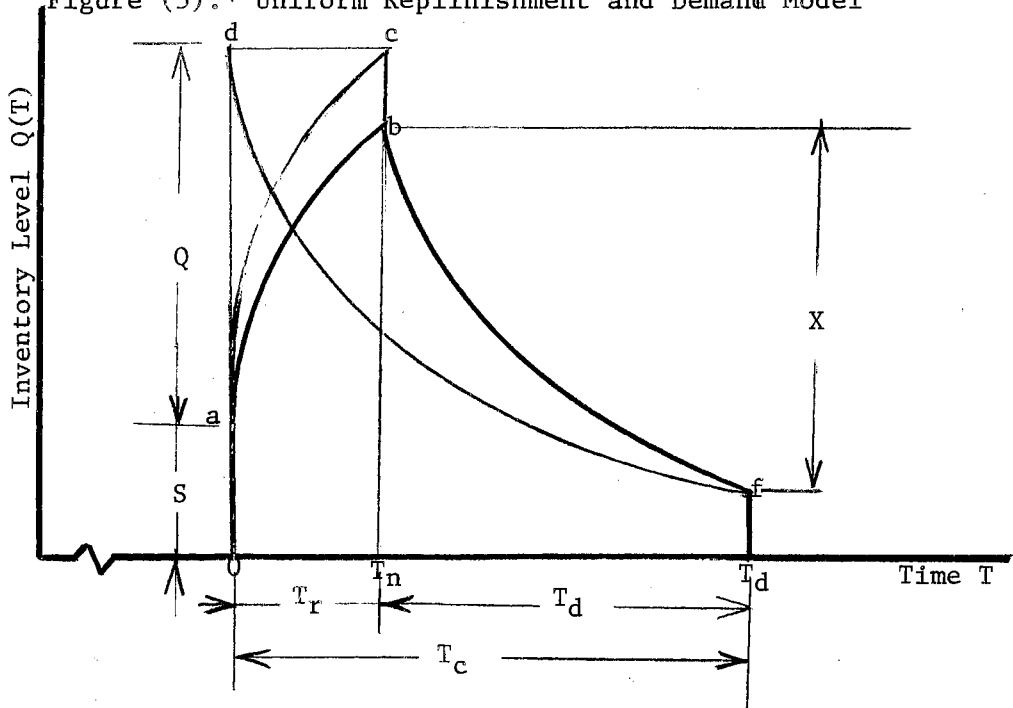


Figure (6). Power Replenishment ( $n_1 < 1.0$ ) and Demand ( $n_2 < 1.0$ ) Model

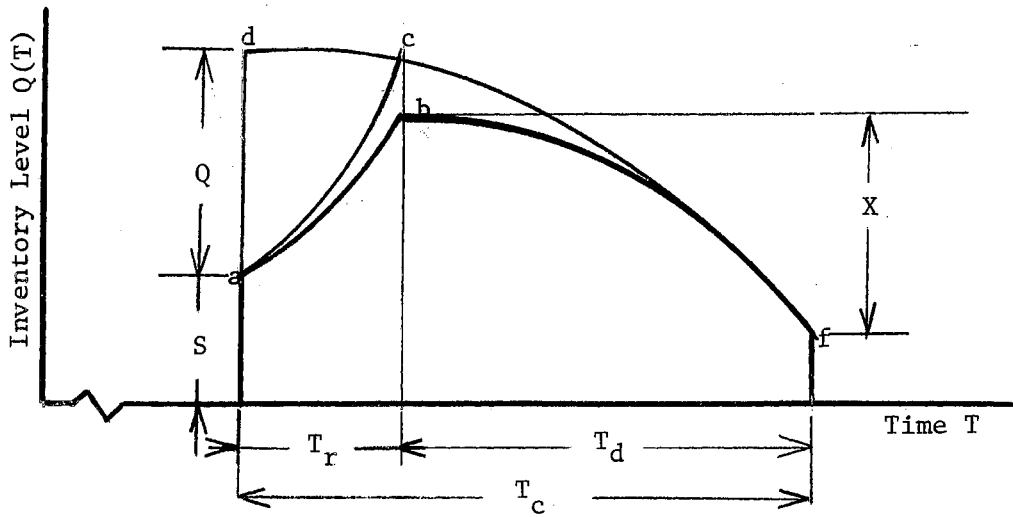


Figure (7). Power Replenishment ( $n_1 > 1.0$ ) and Demand ( $n_2 > 1.0$ ) Model

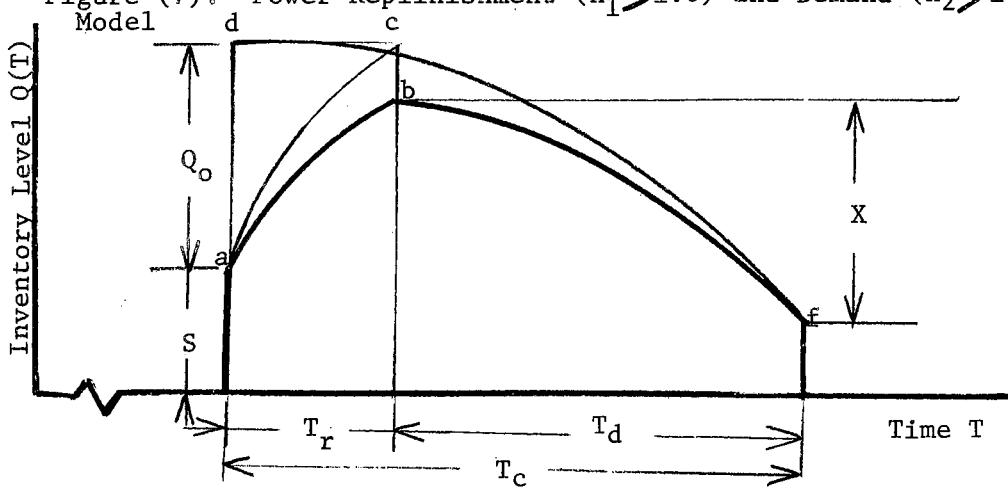


Figure (8). Power Replenishment ( $n_1 < 1.0$ ) and Demand ( $n_2 > 1.0$ ) Model

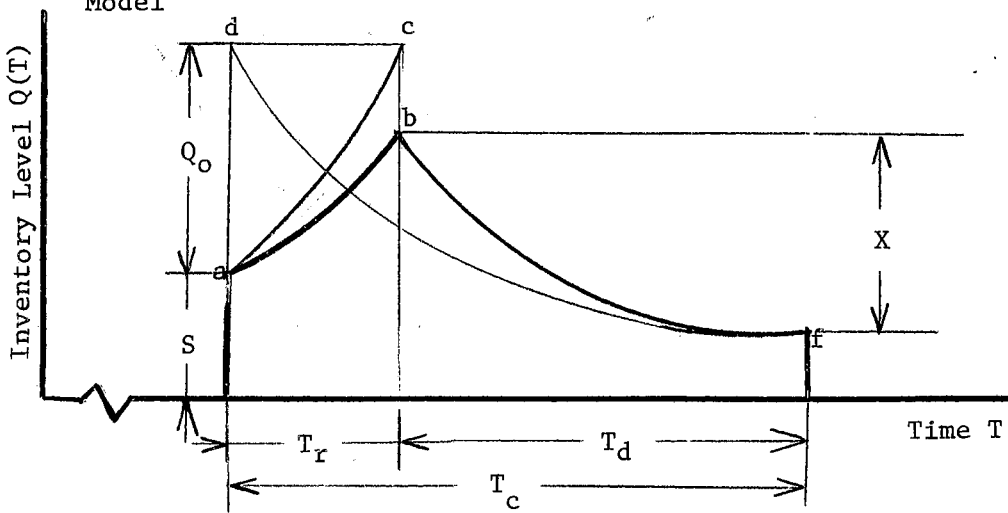


Figure (9). Power Replenishment ( $n_1 > 1.0$ ) and Demand ( $n_2 < 1.0$ ) Model

And the quantity at any time is

$$Q(T) = S + (R - D)T_r, \quad 0 \leq T \leq T_r.$$

And

$$\begin{aligned} Q(T) &= S + (R - D)T_r - \left[ S + (R - D)T_r \right] \left[ 1 - D(T - T_r) \right] \\ &= \left[ S + (R - D)T_r \right] \left[ 1 - 1 + D(T - T_r) \right] \\ &= \left[ S + (R - D)T_r \right] D(T - T_r), \quad T_r \leq T \leq T_d. \end{aligned}$$

The following four figures explain the last mentioned four cases.

For these last four figures, the rates of replenishment and demand are varied with the time. Thus, the rate of accumulation also varied with time. Therefore the quantity in the inventory at any time is

$$\begin{aligned} Q(T) &= S + Q_0 \sqrt{\frac{T}{T_r}} \\ &\quad - \left[ S + Q_0 \sqrt{\frac{T}{T_r}} \right] \left[ \sqrt{\frac{T}{T_d}} \right] \\ &= \left[ S + Q_0 \sqrt{\frac{T}{T_r}} \right] \left[ 1 - \sqrt{\frac{T}{T_d}} \right] \end{aligned}$$

and

$$Q(T) = \left[ S + Q_0 \left( 1 - \sqrt{\frac{T}{T_d}} \right) \right] \left[ \sqrt{\frac{T - T_r}{T_d}} \right],$$

$$T_r \leq T \leq T_d.$$

In these models, if the replenishment is instantaneous that is

$T_r \approx 0$ , then the quantity in the inventory at zero time will be  $X + Q_0$  and is represented by the point d. Therefore the demand curve will follow the curve df instead of abf. If there is no demand during the replenishing period  $T_r$  the replenishment curve follows the curve ac instead of ab. If these situations exist, the above equations will change.

### Optimality and Dimensional Parameters

#### Annual Analysis

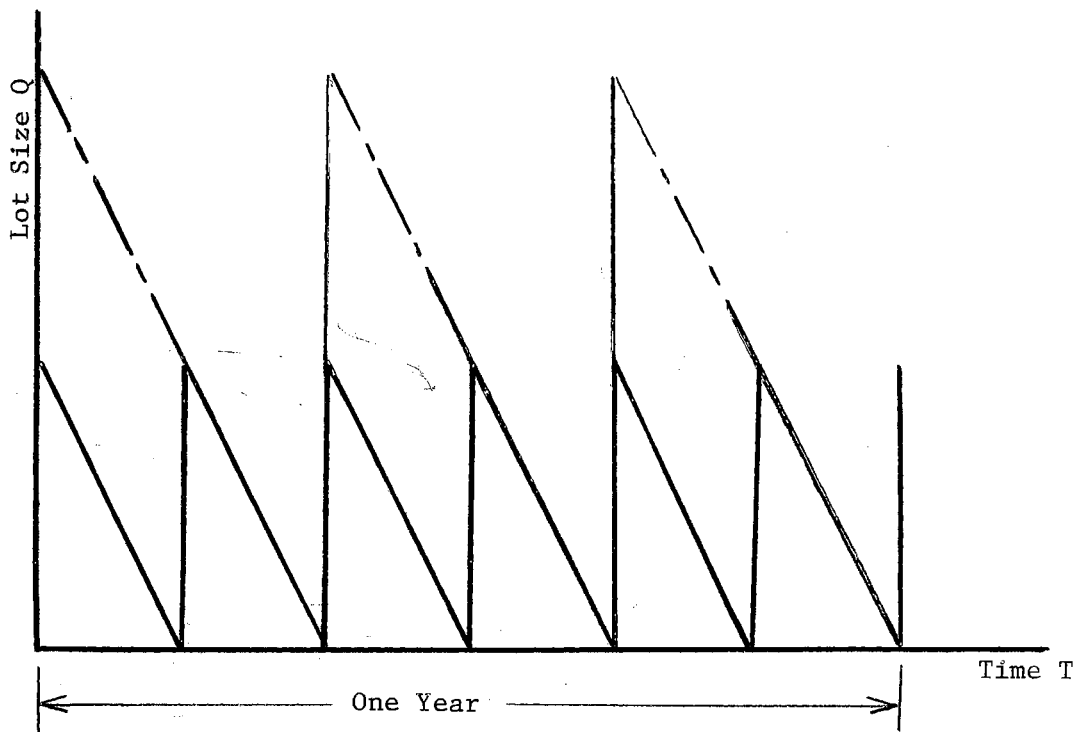


Figure (10). Annual Analysis

Let

$D$  = the demand - units per year

$C_h$  = the holding cost factor - dollars per item per year

$C_p$  = the procurement cost factor - dollars per order

$C_s$  = the shortage cost factor - dollars per unit per year

HC = the holding cost - dollars per year

PC = the procurement cost - dollars per year

and

SC = the shortage cost - dollars per year.

Thus,

HC = (average holding amount)(holding cost factor)

$$= \frac{Q}{2} \cdot C_h$$

= (Quantity units)(dollars per quantity unit per year)

= dollars per year.

PC = (number of replenished units per year)(procurement cost factor)

$$= \left( \frac{\text{total demand per year}}{\text{order lot size}} \right) \cdot C_p$$

$$= \frac{D}{Q} \cdot C_p$$

=  $\left( \frac{\text{quantity units per year}}{\text{quantity units per order}} \right) \cdot \text{dollars per order}$

= dollars per year.

Therefore,

$$TC = HC + PC$$

$$= \frac{Q}{2} \cdot C_h + \frac{D}{Q} \cdot C_p \quad \text{dollars per year.}$$

Differentiate with respect to  $Q$ , equate the result to zero, solve for the optimum lot size  $Q_o$ , and again substitute with  $Q_o$  in TC equation to obtain the optimum total cost  $TC_o$ .

$$\frac{dTC}{dQ} = \frac{C_h}{2} - \frac{D}{Q^2} C_p$$

$$0 = \frac{C_h}{2} - \frac{D}{Q_o^2} C_p$$

$$Q_o = \sqrt{\frac{2DC_p}{C_h}}$$

$$\begin{aligned} TC_o \text{ per year} &= \frac{C_h^2}{4C_h} 2DC_p + \frac{D^2 C_p^2 C_h}{2DC_p} \\ &= \sqrt{2DC_h C_p} \end{aligned}$$

### Periodic Analysis

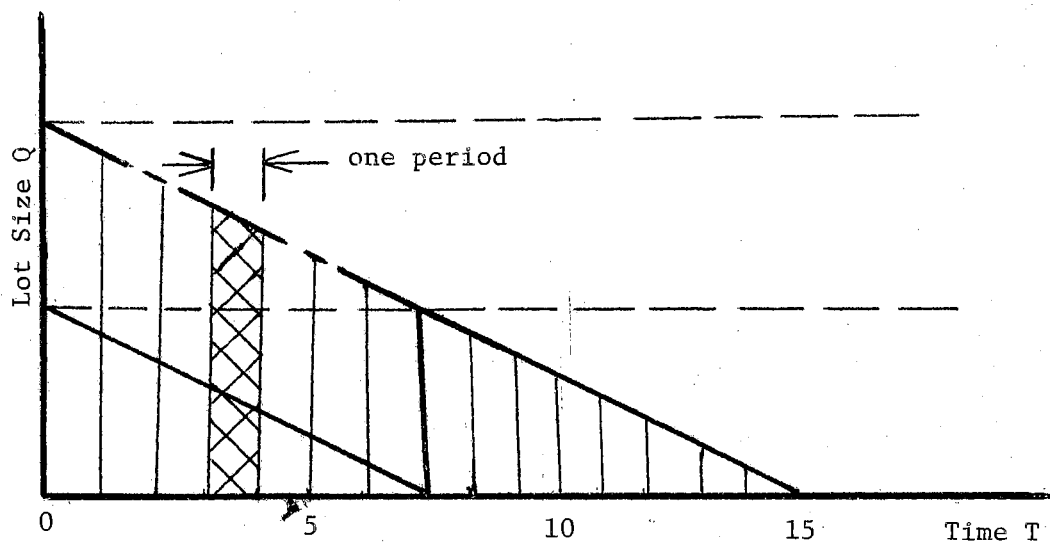


Figure (11). Periodic Analysis



Let

$D'$  = the rate of demand - units per period

$C_h'$  = the holding cost factor - dollars per unit per unit of time

$K$  = the number of periods per year.

Thus,

HC = (the average holding amount)(holding cost factor)

$$= \frac{Q}{2} C_h'$$

= (quantity units)(dollars per unit per period)

= dollars per period

PC = (the procurement cost factor)(sharing of one period)

$$= C_p \cdot \frac{1}{\text{number of periods in a cycle}}$$

$$= C_p \cdot \frac{1}{\left(\frac{Q}{D'}\right)}$$

$$= \frac{D'}{Q} \cdot C_p$$

Thus, the total cost per period is

$$\text{TC per period} = \frac{Q}{2} C_h' + \frac{D'}{Q} \cdot C_p$$

Thus,

$$Q_o = \sqrt{\frac{2D'C'_p}{C'_h}}$$

and

$$\text{TC}_o \text{ per period} = \sqrt{2D'C'_h C'_p}$$

$$\begin{aligned}
 Q_o &= \sqrt{\frac{2D' C_p'}{C_h'}} \\
 &= \sqrt{\frac{2C_p' (D'K)}{C_h' \times K}} \\
 &= \sqrt{\frac{2C_p' D}{C_h'}}
 \end{aligned}$$

and

$$TC_o \text{ per period} = \sqrt{2D' C_h' C_p'}$$

But

$$TC_o \text{ per year} = \text{number of periods per year} \times TC_o \text{ per period}$$

$$\begin{aligned}
 &= K \sqrt{2D' C_h' C_p'} \\
 &= \sqrt{2(D'K) (C_h' K) C_p'} \\
 &= \sqrt{2DC_h' C_p'}
 \end{aligned}$$

The same results are obtained as in Annual Analysis. Therefore, one can conclude the optimum lot size.  $Q_o$  does not affect by the different dimensional parameters however the optimum total cost per period has to be multiplied by the number of periods to obtain the optimum total cost per year.

### Cyclic Analysis

The time of the cycle is not known because this is the variable that the researcher is looking to optimize. Thus let us use the same cost parameters as in periodic analysis. Thus

$$\text{TC per cycle} = \frac{Q}{2} \times T_c \times C_h' + C_p$$

As  $PC$  is not a constant it is also a function of the number of cycles in the planned period (assume the planned period = one year). Thus

$$\text{TC per year} = \text{TC per cycle} \times \text{cycles per year}$$

$$\begin{aligned} \text{TC per year} &= \left[ \frac{Q}{2} \times T_c \times C_h' + C_p \right] \frac{D}{Q} \\ &= \frac{QDT_c}{2Q} C_h' + \frac{D}{Q} C_p \end{aligned}$$

But

$C_h' T_c$  is the holding cost factor per cycle ( $C_h''$ ) in dollars per unit per cycle

and

$C_h' T_c \frac{D}{Q}$  is the holding cost factor per year ( $C_h$ ) in dollars per unit per year.

Therefore

$$\text{TC per year} = \frac{Q}{2} C_h + \frac{D}{Q} C_p$$

which is the same function as in Annual Analysis before and the optimum values are the same as in Annual Analysis.

Uniform Demand, Purchasing or Infinite Replenishment Models

Models Involving Holding and Shortage Costs

This model has the characteristics of instantaneous replenishment period, uniform demand or usage, and prescribed cycle periods and/or the constant procurement cost for the whole planned period. The holding and shortage cost elements are subject to the control of the decision maker, however, the procurement cost is not under the control of the decision maker. The replica of this kind of model is the supplying of retain warehouses at constant intervals of time without previous requests. Another important assumption for all the models in Chapter IV is the mutual exclusiveness and independence of the cost parameters and the cost elements respectively.

Let

$Q$  = the inventory amount to be replenished every prescribed cycle period - units

$S$  = the maximum inventory quantity level during the cycle - units

$T_h$  = the holding time units per every cycle - time units

$T_s$  = the shortage time units per every cycle - time units

$T_c$  = the cycle time - time units

$HC$  = the holding cost per year - dollars per year

$SC$  = the shortage cost per year - dollars per year

$C_p$  = the procurement cost factor - dollars per order

$D$  = the demand or the usage - units per year

$C_h$  = the holding cost factor - dollars per unit quantity per year

$C_s$  = the shortage cost factor - dollars per unit quantity per year

Thus, the cost elements are

$$\begin{aligned} HC &= (\text{average holding amount})(\text{holding cost factor}) \\ &= (\text{quantity units})(\text{dollars per unit quantity per year}) \\ &= \text{dollars per year.} \end{aligned}$$

and

$$\begin{aligned} SC &= (\text{average shortage amounts})(\text{shortage cost factor}) \\ &= (\text{quantity units})(\text{dollars per unit quantity per year}) \\ &= \text{dollars per year.} \end{aligned}$$

This model is illustrated in the following diagram, Fig. 4-11

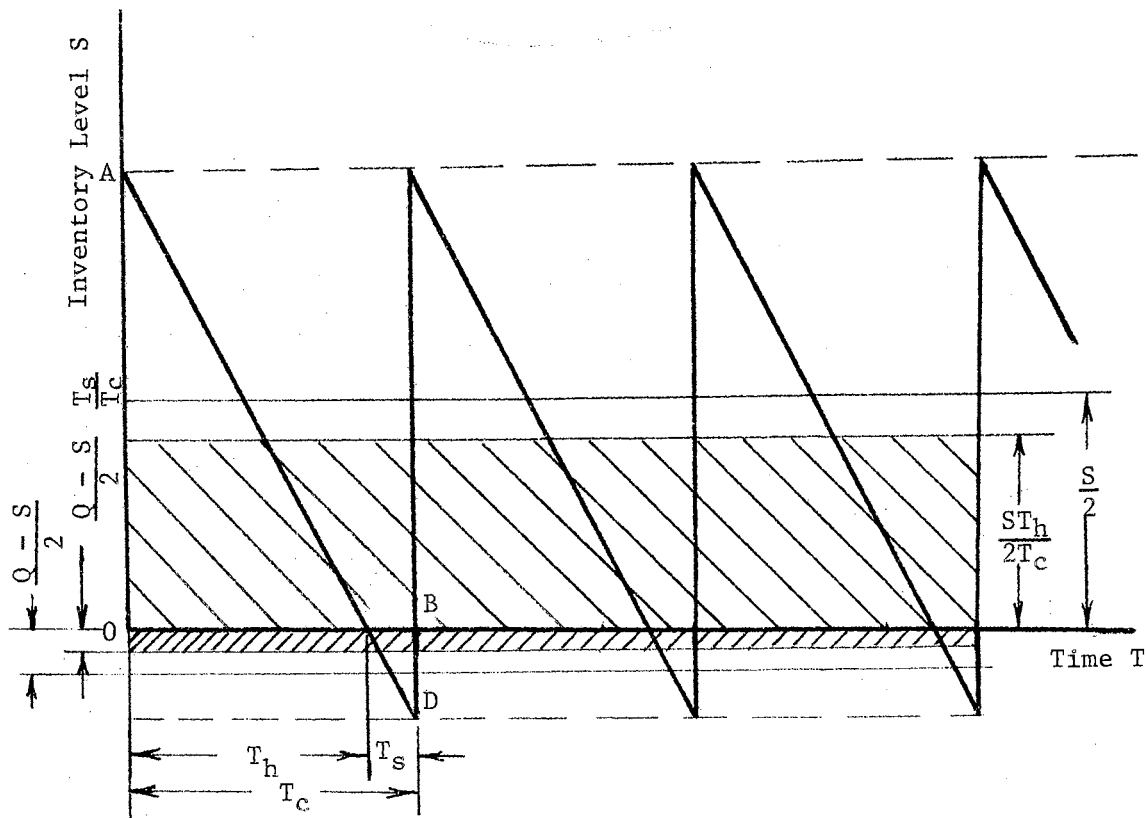


Figure (12). Purchasing, Uniform Demand, and Holding and Shortage Costs Elements Model

Since the optimality cannot be attained except the shortage amount is positive, then, by definition the shortage amount is always a positive amount in the preceding models in this chapter. The holding units occur during the holding time  $T_h$  while the shortage units occur during the shortage time  $T_s$ . Dimensionless multipliers  $\frac{T_h}{T_c}$  and  $\frac{T_s}{T_c}$  are multiplied by  $\frac{S}{2}$  and  $\frac{Q-S}{2}$  respectively in order to obtain the average of the holding amount and the average of the shortage amount. The shortage units are negative in the diagram, however,

$$TC = HC + SC$$

$$\begin{aligned} &= \frac{T_h}{T_c} \cdot \left( \frac{S}{2} \right) \cdot C_h + \frac{T_s}{T_c} \cdot \frac{Q-S}{2} \cdot C_s \\ &= \frac{S}{2} \frac{T_h}{T_c} \cdot C_h + \frac{Q-S}{2} \frac{T_s}{T_c} \cdot C_s. \end{aligned}$$

At this point, the values of these multipliers  $\frac{T_h}{T_c}$  and  $\frac{T_s}{T_c}$  in terms of the interesting variables  $S$  and  $Q$  can be obtained from the similarity of two triangles  $ABC$  and  $ADE$  (Figure 12).

$$\frac{T_h}{T_c} = \frac{S}{Q}$$

and

$$\frac{T_s}{T_c} = \frac{Q-S}{Q}.$$

Substitute these two values in the total cost function. Thus,

$$TC = \frac{s^2}{2Q} C_h + \frac{(Q-s)^2}{2Q} C_s .$$

If the procurement cost is not subject to the control of the decision maker, the model does not involve the procurement cost element. Expanding the TC before one can obtain,

$$\begin{aligned} TC(S) &= \frac{C_h}{2Q} \cdot s^2 + \frac{QC_s}{2} - C_s \cdot s + \frac{C_s}{2Q} \cdot s^2 \\ &= \frac{(C_h + C_s)}{2Q} \cdot s^2 + \frac{QC_s}{2} - C_s \cdot s . \end{aligned}$$

Follow the same procedure as in the illustrated example pp. 14-21 as

$$\frac{dTC}{ds} = \frac{2s}{2Q} (C_h + C_s) + 0 - C_s$$

$$0 = \frac{s_0}{Q} (C_h + C_s)$$

$$s_0 = Q \frac{C_s}{C_h + C_s}$$

$$\begin{aligned} TC_0 &= \frac{C_h + C_s}{2Q} Q^2 \frac{C_s^2}{(C_h + C_s)^2} + Q \frac{C_s}{2} \\ &\quad - C_s Q \frac{C_s}{C_h + C_s} \\ &= Q \frac{C_s^2}{2(C_h + C_s)} + Q \frac{C_s}{2} - Q \frac{C_s^2}{C_h + C_s} \\ &= Q \frac{C_s^2}{2(C_h + C_s)} + Q \frac{C_s}{2} \end{aligned}$$

$$\begin{aligned}
&= \frac{QC_s}{2} \left[ -\frac{C_s}{C_h + C_s} + 1 \right] \\
&= \frac{QC_s}{2} \left[ \frac{-C_s + C_h + C_s}{C_h + C_s} \right] \\
&= \frac{Q}{2} \frac{C_h C_s}{C_h + C_s} .
\end{aligned}$$

Assume an error  $w$  has been associated with  $S_o$  in such a way that

$$S_w = w S_o$$

and

$w =$  zero or a positive value.

Thus, the corresponding value  $Y_w$ , due to this error  $w$ , will be

$$\begin{aligned}
TC_w &= \frac{C_h + C_s}{2Q} Q^2 \frac{C_s^2}{(C_h + C_s)^2} w^2 + \frac{QC_s}{2} \\
&\quad - C_s Q \frac{C_s}{C_h + C_s} w \\
&= \frac{QC_s^2}{2(C_h + C_s)} \left[ + w^2 - 2w \right] + Q \frac{C_s}{2} \\
&= \frac{QC_s}{2} \left[ 1 + \frac{(w^2 - 2w) C_s}{C_h + C_s} \right] \\
&= \frac{QC_s}{2} \left[ \frac{C_h + C_s + w^2 C_s - 2w C_s}{C_h + C_s} \right] \\
&= \frac{QC_s}{2(C_h + C_s)} \left[ C_h + C_s + w^2 C_s - 2w C_s \right]
\end{aligned}$$



$$= \frac{QC_h C_s}{2(C_h + C_s)} \left[ \frac{(C_h + C_s + w^2 C_s - 2wC_s)}{C_h} \right].$$

Therefore, the sensitivity expression is

$$\frac{TC_w}{TC_o} = \frac{1}{C_h} \left[ (C_h + C_s) + C_s (w^2 - 2w) \right].$$

The reader has to notice that if  $w = 1.00$ , this expression yields 1.00, however, the expression is not a general one.

If one looks to the total cost function  $TC(S)$  he can say that it is similar to the mathematical model - parabola versus straight line model. This is not true because  $Q$  and  $S$  are not independent variables. They are related in a linear manner although  $Q$  is predetermined. That is why the sensitivity expression is not a general one and varies from one model to another according to different parameters.

Example: A grocery store estimates that the shortage cost factor for a certain item is 25 dollars per item per year; the holding cost factor, 5 dollars per item per year; and the demand, 1200 units per year. If the replenishment is done at the first day of every month and instantaneously, what is the range for the level of inventory at the beginning of each month for an allowance of a 10 per cent increase in the total cost above its optimum?

By substitution

$$Q = \frac{1200}{12} = 100 \text{ units.}$$

$$S_o = Q \frac{C_s}{C_h + C_s}$$

$$= 100 \frac{25}{.05 + 25} = 83 \text{ units.}$$

$$TC_o = \frac{Q}{2} \frac{C_h C_s}{C_h + C_s}$$

$$= \frac{100}{2} \frac{5 \times 25}{5 + 25} = 208.33 \text{ dollars/year.}$$

$$\frac{TC_w}{TC_o} = 1.10 \frac{(C_h + C_s + w^2 C_s - 2wC_s)}{C_h}$$

$$1.10 = \frac{.05 + 25 + 25w^2 - 50w}{.05}$$

$$1.10 = 6 + 5w^2 - 10w$$

$$w^2 - 2w + .98 = 0.$$

The upper and lower limits of the decision range are the solutions of this quadratic function as

$$w_{u,l} = \frac{2 - \sqrt{4 - 3.92}}{2} = 1.141, .859$$

These results imply that the model is very sensitive, only 14 per cent in lower range and upper range for not more than 10 per cent increase in the total cost above its optimum.

Example: For another item in the same grocery store the estimations are:  $C_h = 20$  dollars per item per year,  $C_s = 25$  dollars per item per year, and the other estimates are the same. What is the inventory level range at the beginning of each month for allowing

an increase of 10 per cent in the total cost function above its optimum?

By substitution

$$S_o = 100 \frac{20}{45} = 44 \text{ units.}$$

$$TC_o = \frac{100}{2} \frac{20 \times 25}{45} = 555.56 \text{ dollars/year}$$

$$\frac{TC_w}{TC_o} = \frac{20 + 25 + 25w^2 - 50w}{20} = 1.10$$

$$2.25 + 1.25w^2 - 2.50w = 1.10$$

$$w^2 - 2w + 0.92 = 0$$

The lower and upper limits of the decision range are

$$w_1 = \frac{2 - \sqrt{4 - 3.68}}{2} = 0.767$$

and

$$w_u = \frac{2 + \sqrt{4 - 3.68}}{2} = 1.233.$$

This model is less sensitive than the preceding example and its range is wider.

• These two examples emphasize that as the ratio of  $\frac{C_h}{C_s}$  is very high the model is insensitive and as the ratio is small the model is sensitive.

#### Model Involving Holding and Procurement Costs

For this model the shortage units are not allowed because the

shortage cost factor is extremely high. The replica of this model appears in procuring essential spare-parts for the continuous production and assembly lines and in procuring the essential medicines for the hospitals. The model can be represented in the following diagram Figure 13.

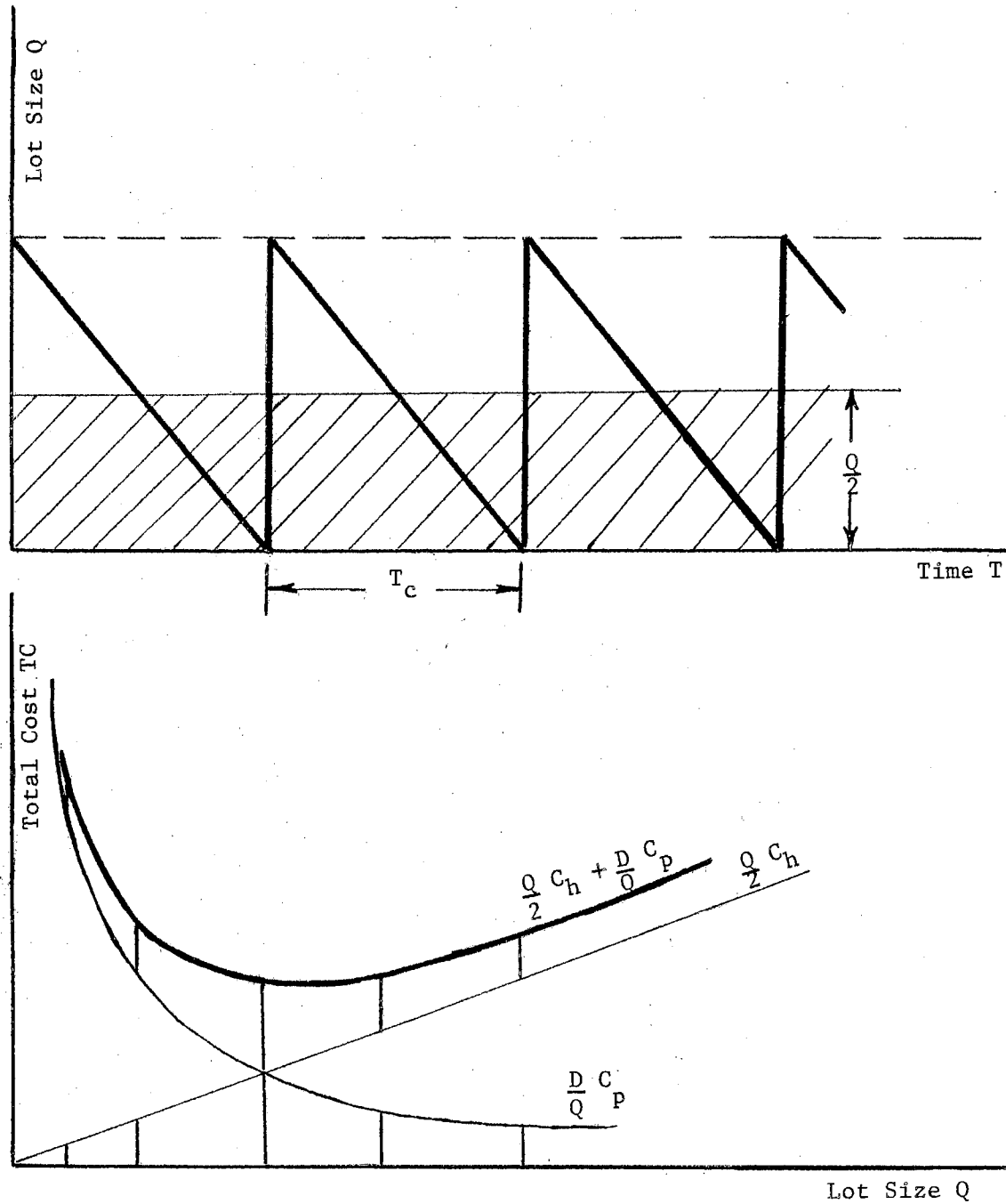


Figure (13). Purchasing, Uniform Demand, and Holding and Procurement Cost Elements Model

Using the same notations as in the annual analysis, the total cost function per year is

$$\begin{aligned} TC &= HC + RC \\ &= \frac{Q}{2} \cdot C_h + \frac{D}{Q} \cdot C_p \end{aligned}$$

The R. H. S. of the total cost equation is a summation of an increasing straight line and a rectangular hyperbola with the variable  $Q$ . Thus applying the results of the similar mathematical model in Chapter III as

$$Q_o = \sqrt{\frac{B}{M}}$$

and

$$TC_o = \sqrt{4MB}$$

Where

$$M = \frac{C_h}{2}$$

$$B = DC_p$$

By substitution, these results can be obtained

$$Q_o = \sqrt{\frac{2DC_p}{C_h}}$$

$$TVC_o = \sqrt{2DC_h C_p}$$

For an error  $w \geq 0$  associates with  $Q_o$ , the sensitivity expression

will be

$$\frac{TVC_w}{|TVC_o|} = \frac{w^2 + 1}{2w}$$

where

$|TVC_o|$  = the absolute value of the optimum total variable cost to be consistent with the directions of the axes.

And the decision range is

$$0.6837736 \frac{2DC}{C_h} P \leq Q_w \leq 1.3162279 \frac{2DC}{C_h} P .$$

Example: An assembly department estimates its demand of an essential spare part by 1,000 units per year; the procurement cost factor, 100 dollars per order; and the holding cost factor, 5 dollars per unit per year. What are the economical lot size, the economical total cost, and the decision range for the assumed predetermined allowance of a 10 per cent increase in the total variable cost above its optimum?

As the spare part is essential, the shortage units are not allowed to occur. Thus

$$Q_o = \sqrt{\frac{2DC}{C_h} P} = \sqrt{\frac{2 \times 1,000 \times 100}{5}} = 200 \text{ units}$$

$$\begin{aligned} \text{TVC}_o &= \sqrt{2DC_p C_h} = \sqrt{2 \times 100 \times 5 \times 100} \\ &= 1000 \text{ dollars/year} \end{aligned}$$

and the decision range is

$$0.6837736 (200) < \underline{Q_w} < 1.3162279 (200)$$

$$137 < \underline{Q_w} < 263 .$$

#### Models Involving Shortage and Procurement Costs

For this model, the holding units are not allowed due to space consideration. The replica of this model appears in the agencies or the offices taking the customers' orders and supplying them later. The delaying of these items will lose a portion of the total demand in the next period or losing the good will. The trade off relation is between the shortage cost and the procurement cost elements. The illustration of this model is as in the following diagram Figure 14.

Here, again, the assumption of the shortage units to be positive is essential for the trade off relationship. Thus, the total cost equation is

$$TC = \frac{Q}{2} C_s + \frac{D}{Q} C_p .$$

The R. H. S. of this total cost equation is a summation of an increasing straight line and a rectangular hyperbola. Thus, as before

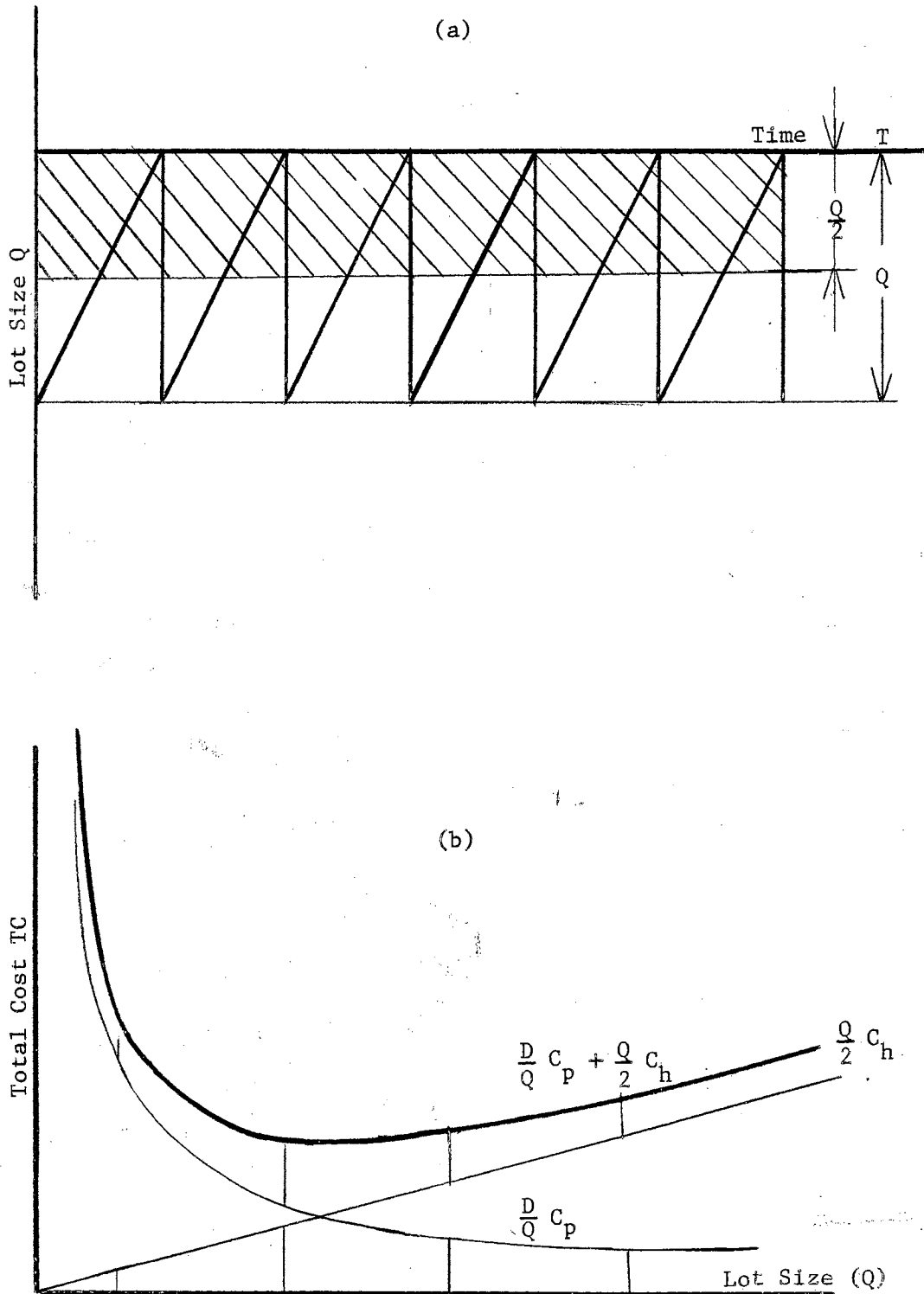


Figure (14). Purchasing, Uniform Demand, and Shortage and Procurement Cost Elements Model



$$Q_o = \sqrt{\frac{2DC_p}{C_s}}$$

$$TVC_o = \sqrt{2DC_s C_p}$$

For an error  $w \geq 0$  associates with  $Q_o$ , the sensitivity expression is

$$\frac{TVC_w}{|TVC_o|} = \frac{w^2 + 1}{2w}$$

where

$|TVC_o|$  = the absolute value of the optimum total variable cost to be consistent with the original direction of the axes.

And the decision range is

$$0.6837736 \frac{2DC_p}{C_s} < Q_w < 1.3162279 \frac{2DC_p}{C_s}$$

Example: An agency estimates that the demand from a certain item is 500 items per year; the procurement cost factor, 10 dollars per lot size; and the shortage cost factor, 5 dollars per item per year. What is the optimum policy? What is the decision range if 10 per cent increase is allowed in the total variable cost above its optimum?

By substitution, the following results are obtained

$$Q_o = \sqrt{\frac{2DC_p}{C_s}} = \sqrt{\frac{2 \times 500 \times 10}{5}} = 45 \text{ units.}$$

$$TVC_o = \sqrt{2DC_p C_s} = \sqrt{2 \times 500 \times 5 \times 10} = 223.61 \text{ dollars per year}$$

and the decision range is

$$0.6837736 \text{ (45)} \leq Q_w \leq 1.3162279 \text{ (45)}$$

$$31 \leq Q_w \leq 59 .$$

### Models Involving Holding, Shortage, and Procurement Costs

This model appears widely in business and industrial fields and is represented in the following diagram Figure (15).

The total cost function is

$$TC = HC + SC + RC$$

$$= \frac{S^2}{2Q} C_h + \frac{(Q - S)^2}{2Q} C_s + \frac{D}{Q} C_p$$

which is a function of S and Q. Thus, differentiate partially with respect to each Q and S, equate each of these partial derivatives to zero, solve for the optimum economic order  $Q_o$  and the optimum inventory level at the beginning period of the cycle  $S_o$ , and finally substitute with these two optimum values in TC equation to obtain the optimum total cost  $TC_o$ .

$$\left( \frac{\partial TC}{\partial S} \right) = \frac{2S}{2Q} C_h + \frac{2(Q - S)(-1)}{2Q} C_s$$

$$0 = S_o C_h - (Q - S_o) C_s$$

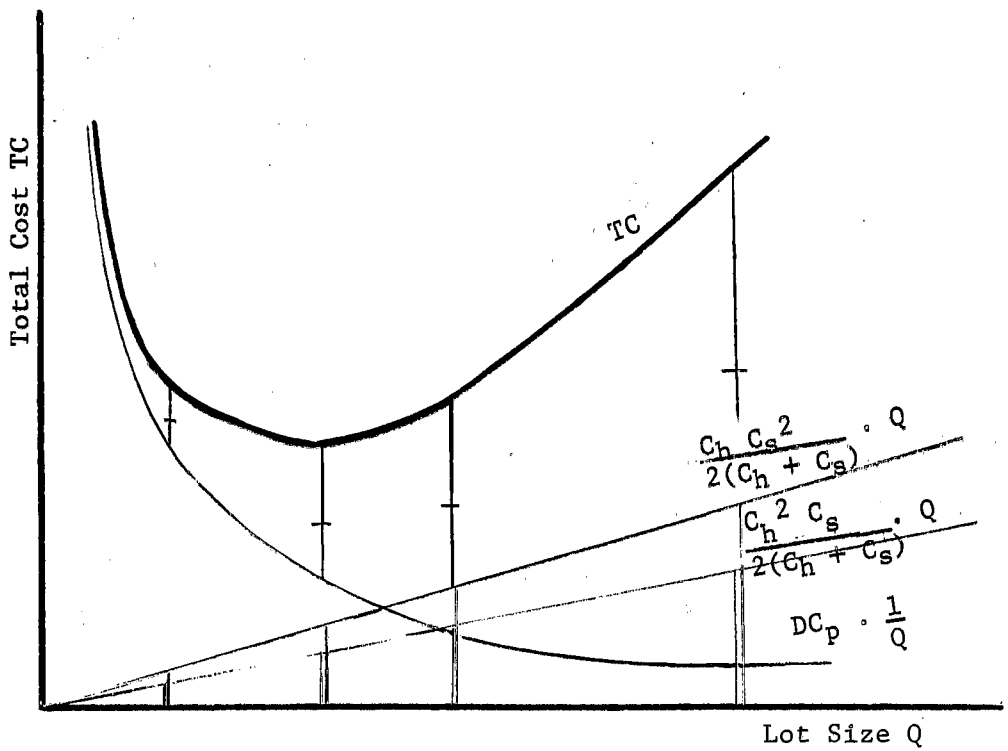
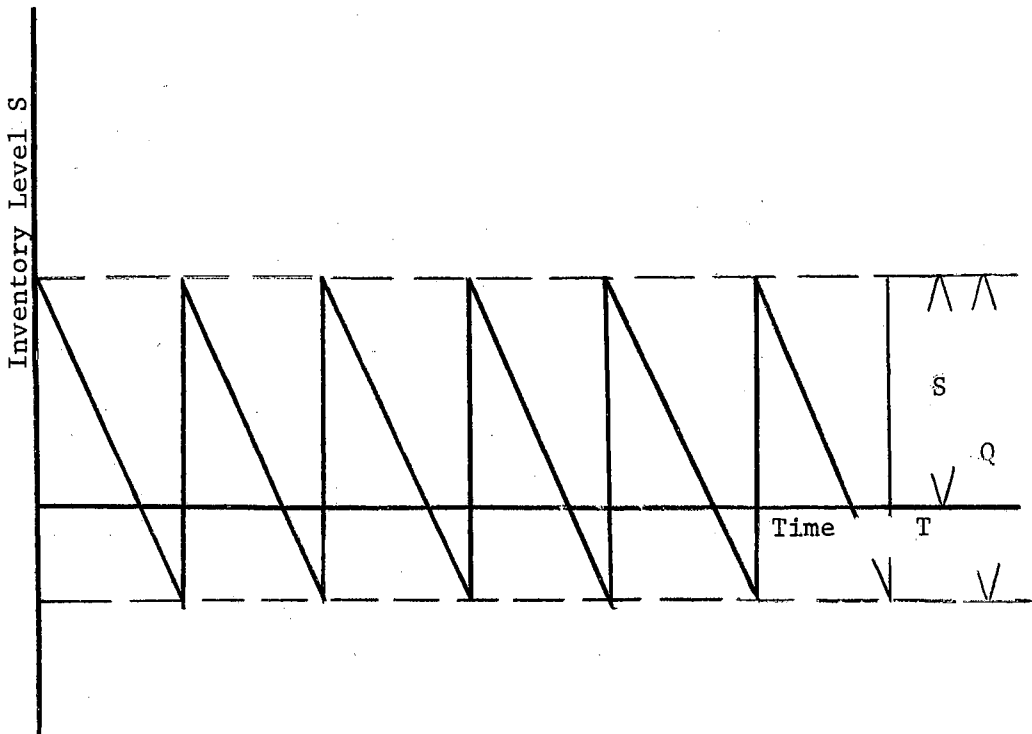


Figure (15). Uniform Demand, Purchasing, and Holding, Shortage, and Procurement Cost Elements Model

$$S_o = Q \frac{C_s}{C_h + C_s} .$$

$$\begin{aligned} \left( \frac{\partial TC}{\partial Q} \right) &= - \frac{S^2}{2Q^2} C_h + \frac{2Q \times 2(Q-S) - (Q-S)^2 \times 2}{4Q^2} C_s - \frac{D}{Q^2} C_p \\ &= - \frac{1}{Q^2} \left[ \frac{S^2}{2} C_h + DC_p \right] + \frac{4Q^2 - 4QS - 2Q^2 + 4QS - 2S^2}{4Q^2} C_s \\ &= - \frac{1}{Q^2} \left[ \frac{S^2}{2} C_h + DC_p \right] + \frac{C_s}{2} - \frac{S^2}{2Q^2} C_s \\ &= - \frac{1}{Q^2} \left[ \frac{S^2}{2} (C_h + C_s) + DC_p \right] + \frac{C_s}{2} \\ 0 &= \frac{-1}{Q_o^2} \left[ S^2 (C_h + C_s) + 2DC_p \right] + C_s . \end{aligned}$$

Substitute with the value of  $S_o$  in the last equation. Thus,

$$\begin{aligned} 0 &= - \frac{1}{Q_o^2} \left[ Q_o^2 \left( \frac{C_s}{C_h + C_s} \right)^2 (C_h + C_s) + 2DC_p \right] + C_s \\ \frac{1}{Q_o^2} 2DC_p &= - \frac{C_s^2}{C_h + C_s} + C_s \\ &= \frac{-C_s^2 + C_h C_s + C_s^2}{C_h + C_s} \\ Q_o &= \sqrt{\frac{2DC_p (C_h + C_s)}{C_h + C_s}} . \end{aligned}$$

$$= \sqrt{\frac{2DC_p}{C_h}} \cdot \sqrt{\frac{C_h + C_s}{C_s}}$$

$$Q_o = \sqrt{\frac{2DC_p}{C_s}} \cdot \sqrt{\frac{C_h + C_s}{C_h}}$$

and

$$TC_o = \frac{Q_o^2 C_s^2 C_h}{2Q_o (C_h + C_s)^2} + \frac{Q_o^2}{2Q_o} \left[ 1 - \frac{S_o}{Q_o} \right]^2 C_s + \frac{DC_p}{Q_o}$$

$$= \frac{Q_o}{2} \frac{C_h C_s^2}{(C_h + C_s)^2} + \frac{Q_o}{2} \left[ 1 - \frac{C_s}{C_h + C_s} \right]^2 C_s + \frac{DC_p}{Q_o}$$

$$= \sqrt{\frac{2DC_p (C_h + C_s)}{4C_h C_s}} \left[ \frac{C_h C_s^2 + C_h^2 C_s}{(C_h + C_s)^2} \right]$$

$$+ DC_p \sqrt{\frac{C_h C_s}{2DC_p (C_h + C_s)}}$$

$$= \sqrt{\frac{2DC_p (C_h + C_s)}{4C_h C_s}} \left[ \frac{C_h C_s (C_h + C_s)}{(C_h + C_s)^2} \right]$$

$$+ \sqrt{\frac{DC_p^2 C_h C_s}{2DC_p (C_h + C_s)}}$$

$$= \sqrt{\frac{DC_p (C_h + C_s) C_h^2 C_s^2}{2C_h C_s (C_h + C_s)^2}} + \sqrt{\frac{DC_p C_h C_s}{2(C_h + C_s)}}$$

$$= \sqrt{\frac{DC_p C_h C_s}{2(C_h + C_s)}} + \sqrt{\frac{DC_p C_h C_s}{2(C_h + C_s)}}$$

$$\begin{aligned}
 TC_o &= \sqrt{\frac{2DC_p C_h C_s}{(C_h + C_s)}} \\
 &= \sqrt{2DC_p C_h} \cdot \sqrt{\frac{C_s}{C_h + C_s}} \\
 TC_o &= \sqrt{2DC_p C_s} \cdot \sqrt{\frac{C_h}{C_h + C_s}}
 \end{aligned}$$

The last term in the 1st relation is due to allowing shortage units while the last term in the 2nd relation is due to allowing holding units for both the optimum values of lot size and total cost.

Assume an error  $w_1$  has been associated with the optimum lot size  $Q_o$  in such a way that

$$Q_{w1} = w_1 Q_o$$

and

$$w_1 = \text{zero or a positive value.}$$

Thus the error  $w_2$  associates with the optimum inventory level at the beginning of the cycle  $S_o$  can be obtained from the linear relation derived before as

$$S_o = (Q_o) \frac{C_s}{C_h + C_s}$$

$$S_{w2} = (Q_{w1}) \frac{C_s}{C_h + C_s}$$

$$w_2(S_o) = (w_1)(Q_o) \frac{C_s}{C_s + C_h} .$$

But, the error  $w_1$  in estimating the optimum lot size  $Q_o$  is

$$w_1 = \frac{Q_{w1}}{Q_o} = \sqrt{\frac{\frac{D'}{D} \frac{C_p'}{C_p} \frac{(C_h' + C_s')}{(C_h + C_s)}}{\frac{(C_h' C_s')}{(C_h + C_s)}}}$$

and the error  $w_2$  in estimating the optimum inventory level at the beginning period of the cycle is

$$w_2 = \frac{w_1 Q_o}{Q_o} \left[ \frac{\frac{C_s'}{C_s}}{\frac{(C_h' + C_s')}{(C_h + C_s)}} \right] .$$

If there are no errors in the estimates of  $C_s'$  and  $C_h'$ , that is the estimates of  $C_s'$  and  $C_h'$  are equal to the exact values  $C_s$  and  $C_h$  respectively, thus

$$w_2 = w_1 \sqrt{\frac{D'}{D} \frac{C_p'}{C_p}} .$$

On the other hand if there are errors in the estimates of  $C_s'$  and  $C_h'$ , then

$$w_2 = Kw_1$$

where

$$K = \text{constant} = \frac{\frac{C_s'}{C_h}}{\frac{C_h' + C_s'}{C_h + C_s}} .$$

and the values with the primes indicate the estimate values.

The value of the total cost due to the error  $w_1$  is obtained by substituting  $Q_w = w_1 Q_o$  in the TC function. Thus,

$$TC_w = \left( \frac{w}{2} + \frac{1}{2w} \right) (TC_o)$$

and the sensitivity expression is

$$\frac{TC_w}{|TC_o|} = \frac{w^2 + 1}{2w}$$

where

$|TC_o|$  = the absolute value of the optimum total cost to be consistent with the original directions of the axes.

Example: The procurement and inventory department in a company estimates the holding cost factor for an item by 4 dollars per unit per year; the shortage cost factor, 16 dollars per unit per year; setting cost factor, 40 dollars per lot size; and the demand, 10,000 units per year. What is the optimum policy and the decision range policy for 10 per cent increase in the total variable cost above its optimum?

By using the results of the above model, one can obtain

$$Q_o = \sqrt{\frac{2DC_p(C_h + C_s)}{C_h C_s}}$$

$$= \sqrt{\frac{2 \times 10,000 \times 40 \times (4 + 16)}{4 \times 16}} = 500 \text{ items}$$

$$S_o = Q_o \frac{C_s}{C_s + C_h}$$



$$= (500) \left( \frac{16}{20} \right) = 400 \text{ items}$$

$$\begin{aligned} \text{TVC}_o &= \sqrt{\frac{2DC_h C_s C_p}{(C_h + C_s)}} \\ &= \sqrt{\frac{2 \times 10,000 \times 4 \times 16 \times 40}{(4 + 16)}} \\ &= \sqrt{10,000 \times 16 \times 16} = 1600 \text{ dollars/year.} \end{aligned}$$

The decision range is

$$0.6837736Q_o \leq Q_{w1} \leq 1.3162279Q_o$$

$$342 \leq Q_{w1} \leq 658 \text{ units.}$$

Example: The PID in the preceding example can control the parameters  $C_h$ ,  $C_s$ , and  $C_p$ . What are the limits on the demand fluctuation for this policy?

$$w_1 = \frac{Q_{w1}}{Q_o} = \sqrt{\frac{D'}{D}}$$

Thus,

$$\left[ \frac{D'}{D} \right]_l = (.68377)^2 = .467856 .$$

And,

$$\left[ \frac{D'}{D} \right]_u = (1.31623)^2 = 1.7424$$

Thus,

$$0.46786 (10,000) \leq D' \leq 1.7424 (10,000)$$

$$4676 \leq D' \leq 17,424 \text{ units.}$$

Therefore, the demand can fluctuate between 4676 and 17424 units per year while the net increase in the total variable cost over its optimum is not more than 10 per cent. This quite wide range of demand makes the model insensitive to the change in the demand within the above limits.

#### Power Demand - Purchasing or Infinite Replenishment Models

##### Model Involving Holding and Shortage Costs

The same assumptions as in the similar model with uniform demand except that the rate of demand varies from time to time during the cycle. This model can be illustrated in the following diagram.

The amount in the inventory at any time T is

$$Q(T) = S - Q_0 \sqrt{\frac{T}{T_c}} \cdot n_2$$

And the total cost equation is

$$TC = HC + SC .$$

The total holding units per cycle can be obtained by integration from zero to  $T_h$  time units. The average amount carried in the inventory is obtained by dividing the result of the integration by the total cycle time  $T_c$  time units as

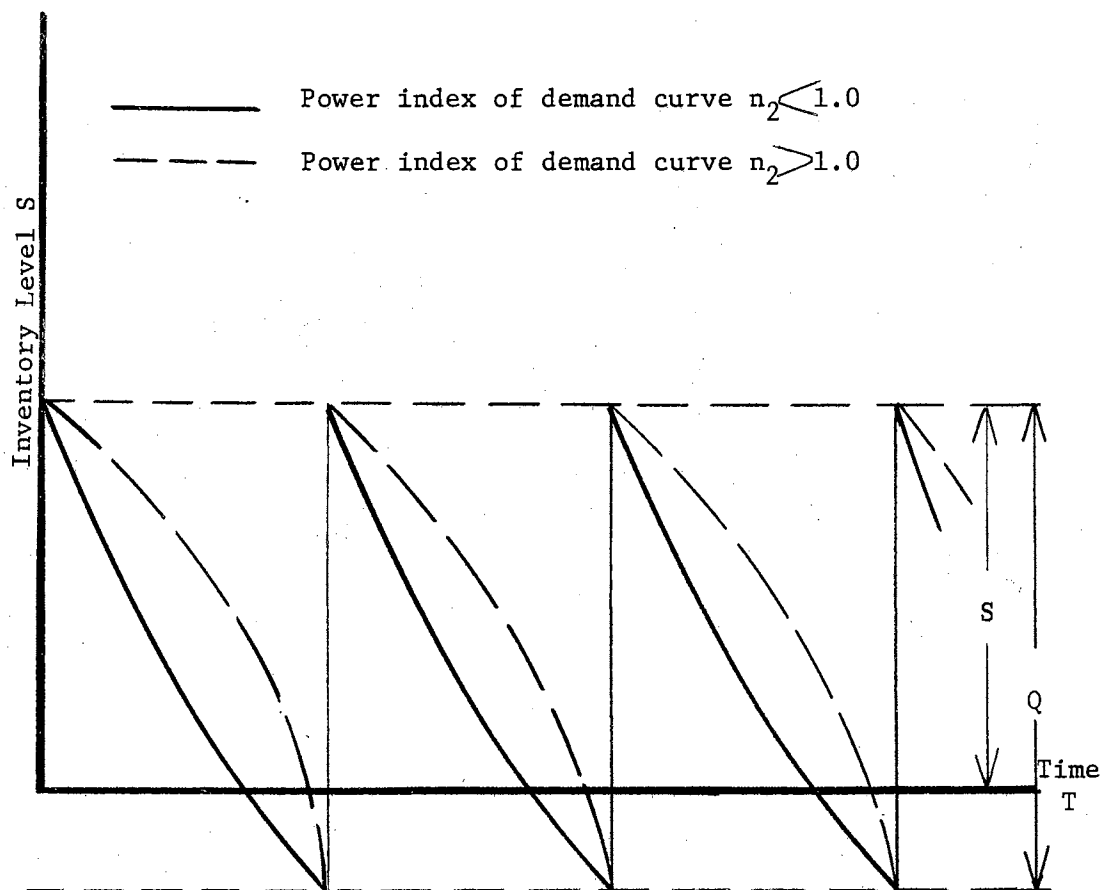


Figure (16). Purchasing, Power Demand, Holding and Shortage Costs Model

$$I_1(S) = \frac{1}{T_c} \int_0^{T_h} Q(T) dT .$$

Where

$I_1(S)$  = the average amount carrying in the inventory.

The shortage units per cycle (positive amounts) can also be obtained by integration from  $T_h$  to  $T_c$  time units. The average amount of shortage is obtained by dividing the result of integration by the total cycle time  $T_c$  time units as

$$-I_2(S) = \frac{1}{T_c} \int_{T_h}^{T_c} Q(T) dT .$$

Where

$I_2(S)$  = the positive amount of shortage in the inventory.

Let us clear these integrations first as follows,

$$\begin{aligned} I_1(S) &= \frac{1}{T_c} \int_0^{T_h} Q(T) dT \\ &= \frac{1}{T_c} \int_0^{T_h} \left[ S - Q_0 \left( \frac{T}{T_c} \right)^{\frac{1}{n_2}} \right] dT \\ &= \frac{1}{T_c} \left[ ST - Q_0 \left( \frac{1}{T_c} \right)^{\frac{1}{n_2}} \left( T^{\frac{1}{n_2} + 1} \right) \frac{n_2}{n_2 + 1} \right]_{0}^{T_h} \\ &= \frac{1}{T_c} \left[ ST_h - Q_0 \left( \frac{1}{T_c} \right)^{\frac{1}{n_2}} \cdot (T_h)^{\frac{1}{n_2} + 1} \frac{n_2}{n_2 + 1} \right] \end{aligned}$$

$$= \frac{1}{T_c} \left[ ST_h - Q_o T_h \left( \frac{T_h}{T_c} \right)^{\frac{1}{n_2} \frac{n_2}{n_2+1}} \right].$$

But from the diagram of the model Figure 4-15 the following relation can be obtained as

$$Q(T = T_h) = 0 = S - Q_o \left( \frac{T_h}{T_c} \right)^{\frac{1}{n_2}}$$

$$\frac{S}{Q_o} = \left( \frac{T_h}{T_c} \right)^{\frac{1}{n_2}}$$

$$\left( \frac{S}{Q_o} \right)^{n_2} = \frac{T_h}{T_c}.$$

Thus,

$$T_h = T_c \left( \frac{S}{Q_o} \right)^{n_2}.$$

Substituting with the value of  $T_h$  in  $I_1(S)$  as

$$\begin{aligned} I_1(S) &= \frac{1}{T_c} \left( ST_c \left( \frac{S}{Q_o} \right)^{n_2} - \frac{n_2}{n_2+1} Q_o T_c \left( \frac{S}{Q_o} \right)^{n_2} \right) \\ &\quad \left[ \frac{T_c \left( \frac{S}{Q_o} \right)^{n_2}}{T_c} \right]^{\frac{1}{n_2}} \\ &= S \left( \frac{S}{Q_o} \right)^{n_2} - \frac{n_2}{n_2+1} Q_o \frac{S}{Q_o} \left( \frac{S}{Q_o} \right)^{n_2} \\ &= S \left( \frac{S}{Q_o} \right)^{n_2} \left[ 1 - \frac{n_2}{n_2+1} \right] \end{aligned}$$

$$\begin{aligned}
 &= s \left( \frac{s}{Q_0} \right)^{n_2} \left( \frac{n_2 + 1 - n_2}{n_2 + 1} \right) \\
 &= s \left( \frac{s}{Q_0} \right)^{n_2} \left( \frac{1}{n_2 + 1} \right) .
 \end{aligned}$$

The average shortage amount is

$$\begin{aligned}
 -I_2(s) &= \frac{1}{T_c} \int_{T_h}^{T_c} Q(T) dT \\
 &= \frac{1}{T_c} \int_{T_h}^{T_c} \left[ s - Q_0 \left( \frac{T}{T_c} \right)^{n_2} \right] dT \\
 &= \frac{1}{T_c} \left[ sT - Q_0 \left( \frac{T}{T_c} \right)^{\frac{1}{n_2} + 1} \frac{n_2}{n_2 + 1} \right]_{T_h}^{T_c} \\
 &= \frac{1}{T_c} \left[ s(T_c - T_h) - \frac{n_2}{n_2 + 1} \left( \frac{T_c}{T_c} \right)^{\frac{1}{n_2} + 1} \right. \\
 &\quad \left. Q_0 \left( T_c^{\frac{1}{n_2} + 1} - T_h^{\frac{1}{n_2} + 1} \right) \right] .
 \end{aligned}$$

Using the relation derived before as

$$T_h = T_c \left( \frac{s}{Q_0} \right)^{\frac{1}{n_2}} .$$

Thus,

$$I_2(s) = \frac{1}{T_c} \left( s \left[ T_c - T_c \left( \frac{s}{Q_0} \right)^{\frac{1}{n_2}} \right] - \frac{n_2}{n_2 + 1} \left( \frac{T_c}{T_c} \right)^{\frac{1}{n_2} + 1} \right)$$

$$\begin{aligned}
& Q_o \left[ T_c \frac{1}{n_2} + 1 - T_c \frac{1}{n_2} + 1 \left( \frac{s}{Q_o} \right)^{\left( \frac{1}{n_2} + 1 \right) n_2} \right] \\
&= s - s \left( \frac{s}{Q_o} \right)^{n_2} - \frac{n_2}{n_2+1} Q_o \left[ 1 - \left( \frac{s}{Q_o} \right)^{n_2+1} \right] \\
&= s - s \left( \frac{s}{Q_o} \right)^{n_2} - \frac{n_2}{n_2+1} Q_o \left[ 1 - \frac{s}{Q_o} \left( \frac{s}{Q_o} \right)^{n_2} \right] \\
&= s - s \left( \frac{s}{Q_o} \right)^{n_2} - \frac{n_2}{n_2+1} \left[ Q_o - s \left( \frac{s}{Q_o} \right)^{n_2} \right] \\
&= s - s \left( \frac{s}{Q_o} \right)^{n_2} \left( 1 - \frac{n_2}{n_2+1} \right) - \frac{n_2}{n_2+1} Q_o \\
&= s - s \left( \frac{s}{Q_o} \right)^{n_2} \left( \frac{n_2+1-n_2}{n_2+1} \right) - \frac{n_2}{n_2+1} Q_o \\
&= s - \frac{s \left( \frac{s}{Q_o} \right)^{n_2} + n_2 Q_o}{n_2 + 1} .
\end{aligned}$$

Therefore, the total cost function is

$$TC = I_1(S) C_h + I_2(S) C_s .$$

Where

$C_h$  = the holding cost factor - dollars per unit per year.

$C_s$  = the shortage cost factor - dollars per unit per year.

Thus,

$$TC = S \left( \frac{S}{Q_o} \right)^{n_2} \frac{C_h}{n_2+1} - \left[ S - \frac{S \left( \frac{S}{Q_o} \right)^{n_2} + n_2 Q_o}{n_2 + 1} \right] C_s.$$

which is a function of the inventory level  $S$ .

If  $n_2 = 1.00$ , that is a model with uniform demand then the total cost function yields to

$$\begin{aligned} TC &= \frac{S^2}{2Q_o} C_h + \left[ S - \frac{\frac{S^2}{Q_o} + Q_o}{2} \right] C_s \\ &= \frac{S^2}{2Q_o} C_h - \left[ S - \frac{S^2}{2Q_o} + \frac{Q_o}{2} \right] C_s \\ &= \frac{S^2}{2Q_o} C_h - \left[ \frac{2SQ_o - S^2 + Q_o^2}{2Q_o} \right] C_s \\ &= \frac{S^2}{2Q_o} C_h + \frac{(Q_o - S)^2}{2Q_o} C_s. \end{aligned}$$

which is the same result derived before for the similar model with uniform demand.

Returning to our model with the power demand index  $n_2$ , the total cost function is

$$TC = S \left( \frac{S}{Q_o} \right)^{n_2} \frac{C_h}{n_2+1} + \left[ \frac{S \left( \frac{S}{Q_o} \right)^{n_2} + n_2 Q_o}{n_2 + 1} - S \right] C_s.$$

The trade-off solution and trade-off decision range are obtained through the differentiation of TC function and the derivation of the sensitivity expression respectively. Thus, differentiate with respect to  $S$ , equate the result to zero, solve for the optimum level of inventory  $S_o$ , and substitute with  $S_o$  in the TC function to obtain the optimum total cost  $TC_o$  as



$$\frac{dT_C}{dS} = (n_2+1) \left( \frac{S}{Q_0} \right)^{n_2} C_h \frac{1}{n_2+1} + (n_2+1) \left( \frac{S}{Q_0} \right)^{n_2} \cdot$$

$$\frac{C_s}{n_2+1} = C_s$$

$$0 = \left( \frac{S}{Q_0} \right)^{n_2} (C_h + C_s) - C_s$$

$$S_0 = Q_0 \sqrt[n_2]{\frac{C_s}{C_h + C_s}}$$

$$TC_0 = S \left( \frac{S}{Q_0} \right)^{n_2} \frac{C_h}{C_h+1} + S \left( \frac{S}{Q_0} \right)^{n_2} \frac{C_s}{n_2+1}$$

$$+ \frac{n_2 Q_0}{n_2+1} C_s - S C_s$$

$$= Q_0 \sqrt[n_2]{\frac{C_s}{C_h + C_s}} \cdot \frac{C_s}{C_h + C_s} (C_h + C_s) \frac{1}{n_2+1}$$

$$+ \frac{n_2}{n_2+1} Q_0 C_s - Q_0 \sqrt[n_2]{\frac{C_s}{C_h + C_s}} \cdot C_s$$

$$= Q_0 C_s \sqrt[n_2]{\frac{C_s}{C_h + C_s}} \left[ 1 - \frac{1}{n_2+1} \right] + \frac{n_2}{n_2+1} Q_0 C_s$$

$$= - \frac{n_2}{n_2+1} Q_0 C_s \sqrt[n_2]{\frac{C_s}{C_h + C_s}} + \frac{n_2}{n_2+1} Q_0 C_s$$

$$= \frac{n_2}{n_2+1} Q_0 C_s \left[ 1 - \sqrt[n_2]{\frac{C_s}{C_h + C_s}} \right]$$

Assume an error  $w$  has been associated with  $S_0$  in such a way that

$$S_w = wS_o = wQ_o \sqrt{\frac{C_s}{C_h + C_s}}^{n_2}$$

and

$w =$  zero or a positive value.

The corresponding value  $TC_w$  will be

$$\begin{aligned} TC_w &= Q_o \sqrt{\frac{C_s}{C_h + C_s}}^{n_2} \cdot \frac{C_s}{n_2 + 1} (w)^{n_2 + 1} \\ &\quad + \frac{n_2}{n_2 + 1} Q_o C_s - Q_o \sqrt{\frac{C_s}{C_h + C_s}}^{n_2} \cdot C_s \cdot w \\ &= Q_o C_s \sqrt{\frac{C_s}{C_h + C_s}}^{n_2} \left[ -w + \frac{w^{n_2 + 1}}{n_2 + 1} \right] \\ &\quad + \frac{n_2}{n_2 + 1} Q_o C_s \\ &= Q_o C_s \sqrt{\frac{C_s}{C_h + C_s}}^{n_2} \left[ \frac{w^{n_2 + 1} - wn_2 - w}{n_2 + 1} \right] \\ &\quad + \frac{n_2}{n_2 + 1} Q_o C_s \\ &= \frac{n_2}{n_2 + 1} Q_o C_s \left[ \frac{w^{n_2 + 1} - wn_2 - w}{n_2} \sqrt{\frac{C_s}{C_h + C_s}}^{n_2} + 1 \right] \\ &= \frac{n_2}{n_2 + 1} Q_o C_s \left[ \frac{(w^{n_2 + 1} - wn_2 - w) \sqrt{\frac{C_s}{C_h + C_s}}^{n_2} + n_2 \sqrt{\frac{C_s}{C_h + C_s}}}{n_2 \sqrt{\frac{C_s}{C_h + C_s}}^{n_2}} \right] \end{aligned}$$

The sensitivity expression of this model is

$$= \frac{(w^{n_2+1} - wn_2 - w) \sqrt[n_2]{C_s} + n_2 \sqrt[n_2]{C_h + C_s}}{n_2 \sqrt[n_2]{C_h + C_s} - \sqrt[n_2]{C_s}}$$

where

$TC_0$  = the absolute value of the optimum total cost.

If  $n_2 = 1.00$ , the case when the model has a uniform demand, the sensitivity expression yields to

$$\begin{aligned} \frac{TC_w}{TC_0} &= \frac{(w^2 - w - w)C_s + C_h + C_s}{C_h + C_s - C_s} \\ &= \frac{(C_h + C_s) + C_s(w^2 - 2w)}{C_h} \\ &= 1 + \frac{C_s}{C_h} + \frac{C_s}{C_h}(w^2 - 2w) \end{aligned}$$

which is the same result as in the purchasing model with uniform demand.

If either uniform demand or power demand model involves an addition PC as a constant element, then the total cost function is

$$TC = HC + SC + PC.$$

This total cost function is a function of  $S$  only due to the assumption  $PC = \frac{DC}{Q} = \text{constant}$  which consequently implies  $Q$  and  $T_c$  are constants.

Then

$$\begin{aligned} TC &= \frac{C_h + C_s}{n_2 + 1} \cdot \frac{S^{n_2+1}}{Q^{n_2}} + \frac{n_2}{n_2 + 1} C_s Q - C_s S + \frac{D}{Q} C_p \\ S &= Q \sqrt[n_2]{\frac{C_s}{C_h + C_s}} \end{aligned}$$

If  $n_2 = 1$  that is the model has a uniform demand, then

$$TC = \frac{C_h + C_s}{2} \frac{S^2}{Q} + \frac{1}{2} QC_s - C_s S + \frac{D}{Q} C_p$$

This model is similar to the parabola versus the straight line model in Chapter III. Thus

$$S_o = Q \frac{C_s}{C_h + C_s}$$

$$TC_o = -\frac{1}{2} Q \frac{C_s^2}{C_h + C_s} + \frac{1}{2} QC_s + \frac{D}{Q} C_p$$

$$TVC_o = -\frac{1}{2} Q \frac{C_s^2}{C_h + C_s}$$

and the decision range is

$$0.6837736 Q \frac{C_s}{C_s + C_h} < S < 1.3162278 Q \frac{C_s}{C_s + C_h}$$

If  $n_2 = 2.00$ , that is the cubical parabola versus the straight line model. Applying the results of the similar model Chapter III, thus

$$S_o = \frac{2QC_s}{C_h + C_s}$$

$$TC_o = -\frac{2}{3} Q \frac{C_s^3}{(C_h + C_s)} + \frac{D}{Q} C_p + \frac{2}{3} C_s Q$$

$$TVC_o = -\frac{2}{3} Q \frac{C_s^3}{(C_h + C_s)}$$

and the decision range is

$$0.7293004 \frac{2QC_s}{C_h + C_s} < S < 1.2479858 \frac{2QC_s}{C_h + C_s}$$

The total cost function is

$$TC = HC + PC.$$

Where

HC = (average amount held)(holding cost factor)

$$= [B + I_1(S)] C_h$$

$I_1(S)$  = the average amount in inventory above the buffering stock level B

PC = the procurement cost per year

$$= \frac{D}{Q}(C_p)$$

D = the demand-units per year

Q = the lot size units

and

B = the buffering stock units.

The value of  $I_1(S)$  can be obtained by calculating the area under the demand curve and divide this area by  $T_c$ . Using the integration

as

$$I_1(S) = \frac{1}{T_c} \int_0^{T_c} Q(T) dT$$

$$= \frac{1}{T_c} \int_0^{T_c} \left[ S - Q \sqrt{\frac{T}{T_c}} \right] dT$$

$$\begin{aligned}
 &= \frac{1}{T_c} \left[ ST_c - \frac{Q}{n_2 \sqrt{T_c}} - T_c^{\frac{1}{n_2+1}} \times \frac{n_2}{n_2+1} \right] \\
 &= S - Q \frac{n_2}{n_2+1} .
 \end{aligned}$$

But from the diagram of the model figure 4-16 the following relations can be obtained as

$$Q(T-T_c) = B = S - Q \left( \frac{T_c}{T} \right)^{\frac{1}{n_2}}$$

$$S = Q + B$$

thus,

$$I_1(S) = Q + B - Q \frac{n_2}{n_2+1} .$$

And

$$\begin{aligned}
 TC &= \left[ (Q + B) - Q \frac{n_2}{n_2+1} \right] C_h + \frac{D}{Q} C_p \\
 &= BC_h + Q \left( \frac{n_2+1-n_2}{n_2+1} \right) C_h + \frac{D}{Q} C_p \\
 &= BC_h + \frac{QC_h}{n_2+1} + \frac{D}{Q} C_p .
 \end{aligned}$$

Which is the equation of a straight line with slope  $M = \frac{C_h}{n_2+1}$  and the intercept on the y-axis is  $BC_h$ , and a rectangular hyperbola with parameter  $B = DC_p$ . By applying the results of the similar mathematical model, the following results are obtained by substitution

$$Q_o = \sqrt{\frac{2DC_p}{C_h}} \cdot \sqrt{\frac{n_2+1}{2}}$$

$$TC_o = BC_h + \sqrt{2DC_h C_p} \sqrt{\frac{1}{(n_2+1)}}$$

$$TVC_o = TC_o - BC_h = 2DC_h C_p \cdot \frac{1}{n_2+1}$$

and the sensitivity expression is

$$\frac{TVC_w}{|TVC_o|} = \frac{w+1}{2w}$$

where

$|TVC_o|$  = the absolute value of the total variable cost to be consistent with the directions of the original axes.

And the decision range is

$$0.6837736 \sqrt{\frac{2DC_p (n_2+1)}{C_h}} < Q_w < 1.3162279 \sqrt{\frac{2DC_p (n_2+1)}{C_h}}$$

The buffering stock units do not affect the location of the optimum lot size  $Q_o$  but it does affect the location of the optimum total cost, that is it shifts the curve of total cost up by the value  $BC_h$ .

Example: Solve the example given at the end of the similar purchasing model with uniform demand for  $n_2 = 3.00$  and  $B = 50$  units.

By substitution one can obtain

$$Q_o = \sqrt{\frac{2DC_p (n_2+1)}{C_h}}$$

$$= \sqrt{\frac{2 \times 1000 \times 100 \times 4}{5}} = 400 \text{ units}$$

$$TVC_o = \sqrt{\frac{2DC_h C_p}{n_2 + 1}}$$

$$= \sqrt{\frac{2 \times 1000 \times 5 \times 100}{4}} = 500 \text{ dollars/year}$$

$$TC_o = 500 + BC_h$$

$$= 500 + 50 \times 5 = 750 \text{ dollars/year.}$$

And the decision range for 10 per cent increase in the total variable cost above its optimum is

$$0.6837736Q_o \leq Q_w \leq 1.3162279Q_o$$

$$273 \leq Q_w \leq 525$$

To find the net increase in the total cost follow as

$$\frac{TVC_w}{TVC_o} = 1.10 = \frac{TC_w - BC_h}{TC_o - BC_h}$$

$$1.10TC_o - 275 = TC_w - 500$$

$$\frac{TC_w}{TC_o} = 1.10 - \frac{25}{TC_o}$$

$$= \frac{1.10 \times 750 - 25}{750}$$

$$= \frac{825 - 25}{750} = 1.067.$$



This last calculation shows the net increase in the total cost function (including the cost of the buffering stocks) is only 6.7 per cent corresponding to 10 per cent increase in the total variable cost above their optima.

The same results can be obtained for the model with shortage and procurement cost elements - power demand patterns. The only difference is to substitute  $C_s$  instead of  $C_h$ .

#### Model Involving Holding, Shortage and Procurement Costs

The same assumptions as in the similar model with uniform demand. This model is represented in the following diagram Figure (17).

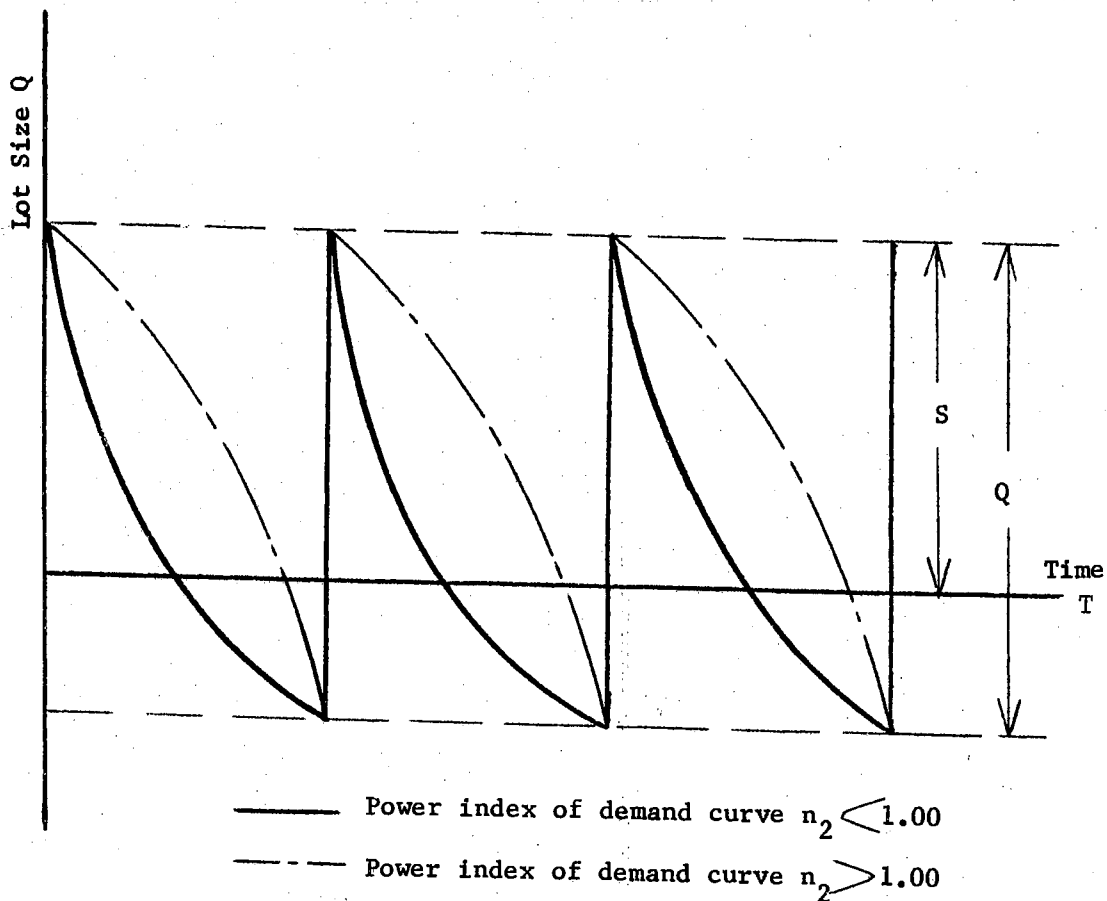


Figure (17). Power Demand, Purchasing, and Holding Shortage and Procurement Cost Elements Model

The average values of the holding and the shortage units derived in the previous model are used for this model. Thus the total cost function is

$$TC = HC + SC + RC$$

$$= \frac{S \left(\frac{S}{Q}\right)^{n_2}}{n_2 + 1} C_h + \frac{n_2 Q + S \left(\frac{S}{Q}\right)^{n_2}}{n_2 + 1} - S C_s + \frac{D}{Q} C_p$$

$$= \frac{S \left(\frac{S}{Q}\right)^{n_2}}{n_2 + 1} (C_h + C_s) + \frac{n_2}{n_2 + 1} Q C_s - S C_s + \frac{D}{Q} C_p$$

The total cost function is a function of two variables S and Q but these two variables are related in a linear relationship as it will be shown later.

Differentiate partially with respect to S and Q, equate each of the partial derivative to zero to obtain the optimum values  $S_0$  and  $Q_0$ , and again substitute with these two optimum values in TC equation to obtain the optimum total cost  $TC_0$  as

$$\left(\frac{\partial TC}{\partial S}\right) = \frac{n_2 + 1}{n_2 + 1} \frac{S^{n_2}}{Q^{n_2}} (C_h + C_s) - C_s$$

$$Q = \text{constant}$$

$$0 = \left(\frac{S}{Q}\right)^{n_2} (C_h + C_s) - C_s$$

$$S_0 = Q \sqrt[n_2]{\frac{C_s}{C_h + C_s}}$$

$$\left(\frac{\partial TC}{\partial Q}\right) = -\frac{n_2}{n_2 + 1} (C_h + C_s) \left(\frac{S}{Q}\right)^{n_2 + 1} \left(\frac{C_s}{C_h + C_s}\right)^{\frac{1}{n_2} (n_2 + 1)}$$

$$+ C_s \frac{n_2}{n_2+1} + \frac{C_p D}{Q^2}$$

$$0 = - \frac{n_2}{n_2+1} C_s \sqrt{\frac{C_s}{C_h+C_s}} + C_s \frac{n_2}{n_2+1} - \frac{C_p D}{Q^2}$$

$$Q_o = \sqrt{\frac{(n_2+1) D C_p}{n_2 C_s \left( 1 - \sqrt{\frac{C_s}{C_h+C_s}} \right)}}$$

$$TC_o = \left[ \frac{C_h+C_s}{n_2+1} \frac{Q_o}{Q_o} \right] \left[ \frac{C_s}{C_h+C_s} \right]^{\frac{1}{n_2}} (n_2+1)$$

$$- C_s Q_o \left[ \frac{C_s}{C_h+C_s} \right]^{\frac{1}{n_2}} + C_s Q_o \frac{n_2}{n_2+1} + \frac{C_p D}{Q_o}$$

$$= C_s Q_o \frac{1}{n_2+1} \sqrt{\frac{C_s}{C_h+C_s}} - C_s Q_o \sqrt{\frac{C_s}{C_h+C_s}}$$

$$+ C_s Q_o \frac{n_2}{n_2+1} + \frac{C_p D}{Q_o}$$

$$= Q_o C_s \sqrt{\frac{C_s}{C_h+C_s}} \left[ \frac{1}{n_2+1} - 1 \right] + C_s Q_o \frac{n_2}{n_2+1}$$

$$+ \frac{C_p D}{Q_o}$$

$$\begin{aligned}
&= Q_o C_s \sqrt{\frac{n_2}{C_h + C_s}} \left( \frac{-n_2}{n_2 + 1} \right) + C_s Q_o \frac{n_2}{n_2 + 1} + \\
&\quad \frac{C D}{Q_c} \\
&= Q_o C_s \frac{n_2}{n_2 + 1} \left[ 1 - \sqrt{\frac{n_2}{C_h + C_s}} \right] + \frac{D}{Q_o} C_p \\
&= \sqrt{\frac{(n_2 + 1) D C_p}{n_2 C_s} \frac{1 - \frac{n_2}{C_h + C_s}}{1 - \frac{n_2}{C_h + C_s}} \cdot C_s^2 \frac{n_2^2}{(n_2 + 1)^2}} \\
&\quad \left[ 1 - \frac{n_2}{C_h + C_s} \right] \\
&\quad + \sqrt{\frac{D^2 C_p^2 n_2 C_s \left[ 1 - \sqrt{\frac{n_2}{C_h + C_s}} \right]}{(n_2 + 1) D C_p}} \\
&= \sqrt{\frac{n_2}{n_2 + 1} D C_p C_s \frac{1}{1 - \frac{n_2}{C_h + C_s}} \left[ 1 - \sqrt{\frac{n_2}{C_h + C_s}} \right]^2} \\
&\quad + \sqrt{\frac{n_2}{n_2 + 1} D C_p C_s \left[ 1 - \sqrt{\frac{n_2}{C_h + C_s}} \right]} \\
&= \sqrt{\frac{4n_2}{n_2 + 1} D C_p C_s \left[ 1 - \sqrt{\frac{n_2}{C_h + C_s}} \right]}
\end{aligned}$$

Assume that errors  $w_1$  and  $w_2$  have been associated with  $Q_o$  and  $S_o$  respectively in such a way that

$$Q_{w_1} = w_1 Q_o$$

$$S_{w_2} = w_2 S_o$$

and

$w_1, w_2$  are zeros or positive values.

By the same method as in the previous models the sensitivity expression is

$$\frac{TVC_w}{|TVC_o|} = \frac{w_1^2 + 1}{2w_2} = \frac{w_2^2 + 1}{2w_2}$$

where

$|TVC_o|$  = the absolute value of the optimum total variable cost to be consistent with the directions of the axes.

The decision ranges are

$$0.6837736Q_o \leq Q_{w_1} \leq 1.3162279Q_o$$

and

$$0.6837736S_o \leq S_{w_2} \leq 1.3162279S_o$$

$$0.6837736Q_o \frac{C_s}{C_s + C_h} \leq S_{w_2} \leq 1.3162279Q_o \frac{C_s}{C_s + C_h}$$

If  $n_2 = 1.00$ , that is the model has a uniform demand

$$S_o = Q_o \frac{C_s}{C_h + C_s}$$

$$Q_o = \sqrt{\frac{2DC_p}{C_s \left( \frac{C_h + C_s - C_s}{C_h + C_s} \right)}}$$

$$= \sqrt{\frac{2DC_p}{C_h}} \cdot \sqrt{\frac{C_h + C_s}{C_s}}$$

$$TC_o = \sqrt{2DC_p C_s \left( \frac{C_h + C_s - C_s}{C_h + C_s} \right)}$$

$$= \sqrt{2DC_h C_p} \sqrt{\frac{C_s}{C_s + C_h}}$$

The same results obtained in the previous purchasing model with uniform demand.

The relation between the two errors are

$$Q_{w_1} = w_1 Q_o$$

and

$$S_{w_2} = w_2 S_o$$

Thus,

$$\frac{Q_{w_1}}{S_{w_2}} = \frac{w_1 Q_o}{w_2 S_o} = \frac{w_1}{w_2} \sqrt{n_2 \frac{C_s}{C_h + C_s}}$$

Example: Solve the previous example at the end of the similar model with uniform demand for  $n_2 = 3$ .

By substitution in the results of the above model one can obtain

$$Q_o = \sqrt{\frac{(n_2+1)DC_p}{n_2 C_s \left(1 - \sqrt[3]{\frac{C_s}{C_h+C_s}}\right)}}$$

$$= \sqrt{\frac{4 \times 10,000 \times 40}{3 \times 16 \left(1 - \sqrt[3]{\frac{16}{20}}\right)}}$$

$$= \sqrt{\frac{100,000}{3 \times .072}} = \sqrt{462916}$$

$$= 686 \text{ units}$$

$$S_o = Q_o \sqrt[3]{\frac{C_s}{C_h+C_s}}$$

$$= 686 \sqrt[3]{\frac{16}{20}} = 638 \text{ units}$$

$$TC_o = \sqrt{\frac{4n_2}{n_2+1} DC_p C_s \left(1 - \sqrt[3]{\frac{C_s}{C_h+C_s}}\right)}$$

$$= \sqrt{\frac{4 \times 3}{4} (10,000) (40) (16) \left(1 - \sqrt[3]{\frac{16}{20}}\right)}$$

$$= 1162 \text{ dollars per year.}$$

And the decision range is

$$(0.6837736)686 \underset{\text{w}}{\leq} Q \underset{\text{w}}{\leq} 1.3162279(686)$$

$$469 \underset{\text{w}}{\leq} Q \underset{\text{w}}{\leq} 904$$

and

$$435 \underset{\text{w}}{\leq} S \underset{\text{w}}{\leq} 840 .$$



Uniform Demand - Manufacturing or  
Finite Replenishment Models

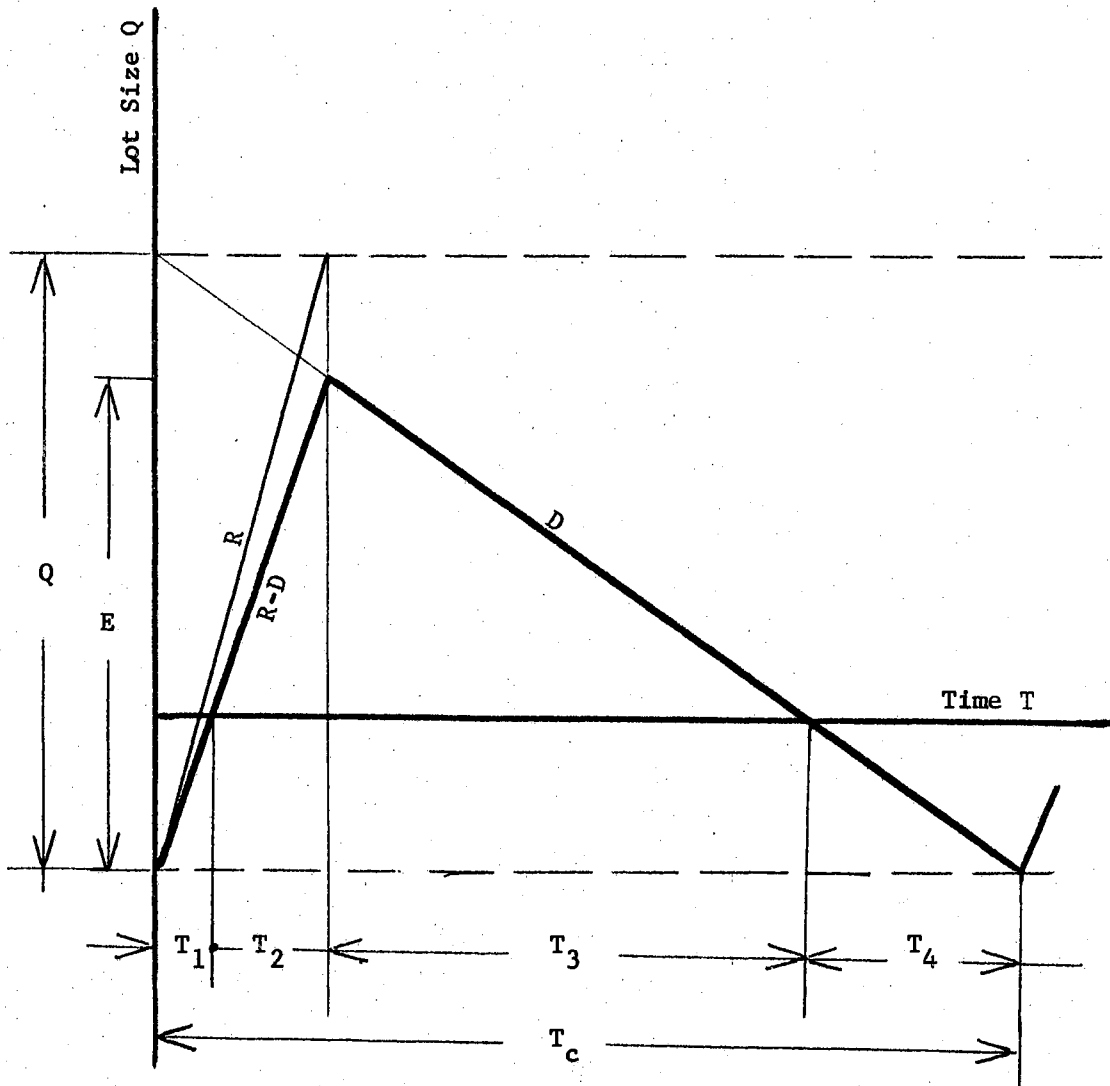


Figure (18). Uniform Demand - Manufacturing or Finite Replenishment Model

This model can be represented in the following diagram, Figure (18).

Let

$R$  = the rate of the manufacturing during the manufacturing periods  $T_1$  and  $T_2$

$D$  = the rate of the demand and it is less than  $R$

$R - D$  = the rate of the accumulation during the manufacturing periods  $T_1 + T_2$

$T_1$  = the manufacturing time periods while the shortage units are occurred that is the inventory level is less than or equal to zero units.

$T_2$  = the manufacturing time periods while the holding units are occurred that is the inventory level is greater than zero units

$T_1 + T_2$  = the manufacturing time periods

$T_3 + T_4$  = the remaining time periods of the cycle - when the replenishment rate is zero

$T_c$  = the time periods of one cycle

$Q$  = the lot size

$S$  = the maximum inventory level and it occurs at the end of the last manufacturing period

$E$  = the maximum amount replenished and it occurs at the end of the last manufacturing period

and

$E - S$  = the maximum positive shortage level and it occurs at the beginning of manufacturing period.

From the symmetry of the triangles in Figure 4-18 the following relations can be obtained

$$R = \frac{Q}{T_1 + T_2}$$

$$R - D = \frac{E}{T_1 + T_2} .$$

Thus

$$T_1 + T_2 = \frac{Q}{R} = \frac{E}{R-D} .$$

And

$$\begin{aligned} E &= Q \frac{R-D}{R} \\ &= Q \left( 1 - \frac{D}{R} \right) \end{aligned}$$

$$T_1 = \frac{E-S}{R-D}$$

$$T_2 = \frac{S}{R-D}$$

$$T_3 = \frac{S}{D}$$

and

$$T_4 = \frac{E-S}{D} .$$

#### Models Involving Holding and Shortage Costs

Assume that this model involves a procurement cost as a constant, that is  $\frac{DC}{Q}$  is constant which consequently implies that  $Q$  and  $T_c$  are constants. Thus, the total cost function is

$$TC = HC + SC + PC.$$

The holding cost  $HC$  can be obtained by using the preceding relations discussed at the beginning of the manufacturing models as

HC = (average holding amount)(holding cost factor)

$$= \frac{S}{2} \left( \frac{T_2 + T_3}{T_c} \right) (C_h)$$

$$= \frac{S}{2} (T_2 + T_3) \left( \frac{1}{T_c} \right) (C_h)$$

$$= \frac{S}{2} \left( \frac{S}{R-D} + \frac{S}{D} \right) \left( \frac{D}{Q} \right) (C_h)$$

$$= \frac{S^2}{2} \frac{1}{Q(1-D/R)} C_h$$

$$= \frac{S^2}{2E} C_h.$$

The shortage cost HC can also be obtained by using the preceding relations discussed at the beginning of the manufacturing models as

SC = (average shortage amount)(shortage cost factor)

$$= \frac{E-S}{2} \left( \frac{T_1 + T_4}{T_c} \right) (C_s)$$

$$= \frac{E-S}{2} (T_1 + T_4) \left( \frac{1}{T_c} \right) (C_s)$$

$$= \frac{E-S}{2} \left( \frac{E-S}{R-D} + \frac{E-S}{D} \right) \left( \frac{D}{Q} \right) (C_s)$$

$$= \frac{(E-S)^2}{2} \frac{1}{Q(1-D/R)} C_s$$

$$= \frac{(E-S)^2}{2E} C_s.$$

Therefore, the total cost equation is

$$TC = \frac{S^2}{2E} C_h + \frac{(E-S)^2}{2E} C_s + \frac{D}{Q} C_p.$$

According to the assumption, this TC function is a function of S only.

Thus, expand the function and inspect its right hand side and apply the results of the similar mathematical model in Chapter III as

$$TC = \frac{S^2}{2E} (C_h + C_s) + \frac{E}{2} C_s + \frac{D}{Q} C_p - S C_s.$$

This is similar to the parabola versus the straight line model. Thus,

$$S_o = E_o \frac{C_s}{(C_s + C_h)}$$

$$TC_o = -\frac{E_o}{2} \frac{C_s^2}{(C_h + C_s)} + \frac{E}{2} C_s + \frac{D}{Q} C_p$$

$$TVC_o = -\frac{E_o}{2} \frac{C_s^2}{(C_h + C_s)}.$$

For an error  $w \geq 0$  associates with  $S_o$ , the sensitivity expression is

$$\frac{TVC_w}{|TVC_o|} =$$

where

$|TVC_o|$  = the absolute value of the optimum total variable cost to be consistent with the directions of the original axes.

And the decision range is

$$0.6837736E_o \frac{C_s}{(C_s + C_h)} \leq S_w \leq 1.3162278E_o \frac{C_s}{(C_s + C_h)}.$$

If this model has a procurement cost element which is not under the control of the decision maker, that is Q and  $T_c$  are not constants, results

Let

$T_m$  = manufacturing periods in time units

$T_o$  = the remaining periods of the cycle in time units

and

$T_m + T_o$  = the cycle periods in time units.

Thus, the total cost equation can be obtained by using the preceding relations at the beginning of manufacturing models as

$$\begin{aligned} TC &= HC + PC \\ &= \frac{E}{2} C_h + \frac{D}{Q} C_p \\ &= \frac{E}{2} \frac{(1-D/R)}{(1-D/R)} C_h + \frac{D}{Q} C_p \\ &= \frac{Q}{2} \left(1 - \frac{D}{R}\right) C_h + \frac{D}{Q} C_p . \end{aligned}$$

This is similar to the mathematical model - the straight line versus the rectangular hyperbola in Chapter III. One can find that

$$Q_o = \sqrt{\frac{2DC_p}{C_h}} \sqrt{\frac{1}{(1-D/R)}}$$

$$TC_o = TVC_o = \sqrt{2DC_h C_p} \sqrt{(1 - D/R)}$$

for an error  $w$  0 associates with  $Q_{o1}$  the sensitivity expression is

$$\frac{TVC_w}{|TVC_o|} = \frac{w^2 + 1}{2w}$$

where

$|TVC_o|$  = the absolute value of the optimum total variable cost.

Assume an error has been associated with  $E_o$  in such a way that

$$E_{w_3} = w_3 E_o$$

where

$$w_3 = \text{zero or a positive value.}$$

The corresponding value in TC due to this error is  $TC_w$  as

$$\begin{aligned} TC_{w_3} &= w_3 E_o \frac{C_h C_s}{2(C_h + C_s)} \frac{1}{w_3 E_o} D(1-D/R)C_p \\ &= w_3 \frac{DC_p C_h C_s}{2(C_h + C_s)} (1-D/R) \\ &\quad + \frac{1}{w_3} \frac{DC_p C_h C_s}{2(C_h + C_s)} (1-D/R) \\ &= \frac{DC_p C_h C_s}{2(C_h + C_s)} (1-D/R) \left( w_3 + \frac{1}{w_3} \right) \\ &= \frac{2DC_p C_h C_s}{(C_h + C_s)} (1-D/R) \left( \frac{w_3}{2} + \frac{1}{2w_3} \right) \end{aligned}$$

The sensitivity expression is

$$\frac{TVC_{w_3}}{|TVC_o|} = \frac{w_3^2 + 1}{2w_3}$$

where

and the decision range is

$$0.6837736Q_o \leq Q_w \leq 1.3162279Q_o$$

$$0.6837736 \sqrt{\frac{2DC_p}{C_h(1-D/R)}} \leq Q_w \leq 1.3162279 \sqrt{\frac{2DC_p}{C_h(1-D/R)}}$$

#### Models Involving Shortage and Procurement Costs

Results similar to the preceding model can be obtained with the only difference  $C_s$  is to be substituted instead of  $C_h$ .

#### Models Involving Holding, Shortage, and Procurement Costs

The diagram of this model is similar to the Figure 4-18. The total cost equation is

$$TC = HC + SC + PC$$

$$= \frac{S^2}{2E} C_h + \frac{(E-S)^2}{2E} C_s + \frac{D}{Q} C_p$$

$$= \frac{S^2}{2E} C_h + \frac{(E-S)^2}{2E} C_s + \frac{D(1-D/R)}{E} C_p$$

$$= \frac{S^2}{2E} (C_h + C_s) + \frac{E}{2} C_s - SC_s - \frac{D(1-D/R)}{E} C_p$$

This TC equation is a function of S and E. Thus, differentiate partially the TC equation with respect to each S and E, equate each partial derivative to zero, solve for the optimum value  $S_o$  and  $E_o$ , and finally substitute with these two optimum values  $S_o$  and  $E_o$  in TC to obtain the optimum total cost  $TC_o$  as



$$\left[ \left( \frac{\partial TC}{\partial S} \right) \right] = -C_s + \frac{2S}{2E} (C_h + C_s) = 0$$

$E = \text{constant}$

$$S_o = E_o \frac{C_s}{C_s + C_h}$$

$$\left[ \left( \frac{\partial TC}{\partial E} \right) \right] = \frac{C_s}{2} - \frac{S^2}{2E^2} (C_h + C_s) - \frac{D(1-D/R)}{E} C_p = 0$$

$S = \text{constant}$

thus,

$$\frac{C_s}{2} = \frac{S_o^2}{2E_o^2} (C_h + C_s) + \frac{D}{E_o^2 (1-D/R)} C_p$$

Substitute with the value of  $S_o$  in the last equation. Thus

$$\frac{C_s}{2} = \frac{E^2}{2E^2} \frac{C_s^2}{(C_s + C_h)^2} (C_h + C_s) + \frac{D(1-D/R)}{E} C_p$$

$$\frac{2D(1-D/R)}{E_o^2} C_p = \frac{C_s^2 + C_h C_s - C_s^2}{(C_s + C_h)}$$

$$= \frac{C_h C_s}{C_s + C_h}$$

thus

$$E_o = \sqrt{\frac{2DC_p}{C_h}} \sqrt{\frac{C_h + C_s}{C_s}} \sqrt{(1-D/R)}$$

Thus,

$$\begin{aligned}
 Q_o &= \sqrt{\frac{2DC_p}{C_h}} \cdot \sqrt{\frac{C_h+C_s}{C_s}} \sqrt{\frac{1}{(1-D/R)}} \\
 TC_o &= \frac{C_h+C_s}{2E_o} E_o^2 \frac{C_s^2}{(C_h+C_s)^2} + \frac{C_s}{2} E_o \\
 &\quad - \frac{C_s^2}{(C_h+C_s)} E_o + \frac{D(1-D/R)}{E} C_p \\
 &= E_o \left[ \frac{C_s^2}{2(C_h+C_s)} + \frac{C_s}{2} - \frac{C_s^2}{(C_h+C_s)} \right] \\
 &\quad + \frac{1}{E_o} \left[ \frac{D}{(1-D/R)} C_p \right] \\
 &= E_o \left[ \frac{C_s(C_h+C_s) - C_s^2}{2(C_h+C_s)} \right] + \frac{1}{E_o} \left[ D(1-D/R)C_p \right] \\
 &= E_o \left[ \frac{C_h C_s}{2(C_h+C_s)} \right] + \frac{1}{E_o} \left[ D(1-D/R)C_p \right] \\
 &= \sqrt{\frac{2DC_p(C_h+C_s)(1-D/R)C_h^2 C_s^2}{(C_h C_s) 4(C_h+C_s)^2}} \\
 &\quad + \sqrt{D^2(1-D/R)^2 C_p^2 \frac{C_h C_s}{2DC_p(C_h+C_s)(1-D/R)}} \\
 &= \sqrt{2DC_h C_p} \sqrt{\frac{C_s}{C_h+C_s}} \sqrt{1-D/R} .
 \end{aligned}$$

$$\text{TVC} = \text{TC}$$

$|\text{TVC}_o|$  = the absolute value of the optimum total variable cost.

And the decision range is

$$0.6837736 \frac{2DC_p (C_h + C_s)(1-D/R)}{C_h C_s} \leq E_{w_3} \leq 1.3162279 \frac{2DC_p (C_h + C_s)(1-D/R)}{C_h C_s}$$

This model is a general model for all models with a uniform replenishment and a uniform demand patterns excluding the models involved holding and shortage costs as variable elements in the total cost function. It is easily to generate all other models from this general model as follows:

For the general model the optimum values are

$$Q_o = \sqrt{\frac{2DC_p}{C_h}} \cdot \sqrt{\frac{C_h + C_s}{C_s}} \cdot \sqrt{\frac{1}{(1-D/R)}}$$

or

$$Q_o = \sqrt{\frac{2DC_p}{C_s}} \cdot \sqrt{\frac{C_h + C_s}{C_h}} \cdot \sqrt{\frac{1}{(1-D/R)}}$$

And

$$\text{TC}_o = \sqrt{2DC_p C_h} \cdot \sqrt{\frac{C_s}{C_h + C_s}} \cdot \sqrt{(1-D/R)}$$

or

$$= \sqrt{2DC_p C_s} \sqrt{\frac{C_h}{C_h + C_s}} \sqrt{(1-D/R)} .$$

The last square roots in these two relations are due to the effect of manufacturing or finite replenishment models over the infinite replenishment or purchasing models. If  $R$  approaches  $\infty$ , the last square roots will equal 1.00 and the optimum values are for the purchasing models. If the model does not allow shortage units or holding units then each median square root will equal 1.00 in the first two relations or in the second two relations respectively. If the model has an infinite replenishment and it does not allow shortage units or holding units then the last square root is equal to 1.00 and each of the median square roots in the first two relations or the second two relations is equal to 1.00. By applying these different combinations one can easily come to the same results derived before.

#### Factors Introducing Errors

If one assumes that there are errors occurring in estimating or calculating the variables, he would desire to know the parameters which introduce these errors. Let us assume that there are errors  $w_1$ ,  $w_2$ , and  $w_3$  which have been associated with the optimum lot size  $Q_0$ , the optimum inventory level at the beginning of the cycle period  $S_0$ , and the maximum amount replenished during the cycle  $E_0$  respectively. Thus for the general model

$$w_1 = \frac{Q_{w1}}{Q_0} = \sqrt{\frac{D^*}{D} \frac{C_p^*}{C_p} \frac{C_h^*}{C_h} \frac{C_s^*}{C_s} \frac{C_h^* + C_s^*}{C_h + C_s} \frac{(1-D^*/R^*)}{(1-D/R)}}$$

$$w_2 = \frac{Sw_2}{S_o} = \sqrt{\frac{D'}{D} \frac{C_p'}{C_p} \frac{C_h}{C_h'} \frac{C_s'}{C_s} \frac{C_h+C_s}{C_h'+C_s'} \frac{(1-D/R)}{(1-D'/R')}}}$$

$$w_3 = \frac{Ew_3}{E_o} = \sqrt{\frac{D'}{D} \frac{C_p'}{C_p} \frac{C_h}{C_h'} \frac{C_s'}{C_s} \frac{C_h'+C_s'}{C_h+C_s} \frac{(1-D/R)}{(1-D'/R')}}}$$

Where the parameter with prime indicates the over estimated or the under estimated value while the parameter without prime indicates exact estimated value.

By inspecting these ratios under the square roots one can conclude that the parameters introduce these errors are the demand, the procurement cost factor, the holding cost factor, the shortage cost factor, the sum of the holding and shortage cost factors, and the ratio of the rate of the demand over the rate of manufacturing subtracted from 1.00. Fortunately these ratios are tending to balance each other if the estimator is consistent with his estimates that is the estimator is overestimating for all parameters or underestimating for all parameters. Sometimes if these ratios are well balanced the final calculation yields to the optimum lot size  $Q_o$  and the optimum total cost  $TC_o$  regardless of the errors in estimating the parameters inside the inventory system. That is to say there are errors inside the inventory system but due to the consistency in the estimating of the ratios of the parameters are well balanced and yields to an output optimum policy.

## CHAPTER V

### SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

In the summary, sensitivity analysis of decision models, represented in continuous mathematical equations, are obtained through the using of the mathematical techniques, differentiation, or partial derivative, and transformation. The differentiation or the partial derivative is used for providing the optimum solution if it exists. The transformation is used to eliminate the constant term or terms from the original equation due to the fact that the curvature of the curve or the shape of the surface is only affected by the variable parts in the equation. General sensitivity expression for any decision model can now be obtained by eliminating this constant term or terms by dividing and the transformed function by the absolute transformed optimum value  $Y_0'$  or  $TC_0'$  or  $TP_0'$ . The absolute value is used as a divisor to be consistent with the original directions of the axes.

Chapter III discussed groups of mathematical models or decision models. Each model in the first group can be represented as a one standard curve. In the second group each model can be represented as a summation of two different standard curves, the first curve is an increasing function while the second is a decreasing one.

In the last group, each model can be represented as a summation of three different standard curves; one of them is an increasing function and the others are decreasing ones and vice versa. The

assumption of one increasing function and the other(s) is (are) decreasing one(s) and vice versa is important for providing the optimality. The management is not only interested in the optimum values but they are also interested in the sensitivity of these models around these optimum values. Tables I, II, III, IV, are the summary of the mathematical models and their sensitivity expressions. They show that general sensitivity expressions are achieved; three for the one standard curve models, nine for summation of two standard curves models, five for summation of two standard curves models, and three for summation of three standard curves models. The third table is referred to maximization while the others are referred to minimization.

Chapter IV seeks the application of the Chapter III models in decision making process areas. Although the application is devoted to the inventory models but it can also be applied in other areas such as the replacement and the retirement models, the queuing models and so on.

One can itemize the decision models into the following conclusions:

- a. Most of the ordinary decision models have the characteristics that the upper ranges of the decision models are flatter than in the lower ranges.
- b. Quadratic functions' decision models have equal decision range limits from the optimum values.
- c. Decision models, containing one of their summation curves as a third degree equation or more have the characteristic that the lower ranges are flatter than in the upper ranges, and sometimes the sensitivity in the upper ranges is very high.

The flatness is reduced if the other curve(s) in the models is (are) rectangular hyperbolas or reciprocal of quadratic functions.

- d. Although the decision range is varied from one model to another, all of them are very satisfactory for the same pre-determined allowance increase or decrease above or under the optimum.
- e. Care and caution have to be exercised in handling the holding and shortage cost elements models in the inventory system. The sensitivity expression is a special one and very sensitive especially when the ratio of the holding cost factor over the shortage cost factor is high.
- f. Appendices A and B provide complete evaluations and decision ranges for  $\pm 10$  per cent above or under the optimum of the variable parts in the decision model.

The following recommendations are suggested for further studies and investigations:

- a. Sensitivity analysis for discrete decision models.
- b. Sensitivity analysis for more than one independent decision variable.
- c. Application of the sensitivity analysis upon the replacement and the retirement models.
- d. Application of the sensitivity analysis upon the queuing models.
- e. Application of the sensitivity analysis upon the linear programming models.

Finally, the sensitivity analysis emphasizes that there is a wide



range for making decision around the optimum value while the net yield in the objective function is insignificant. This fact fortunately meets the management's dynamic needs and liquidates the enormous problems facing the management in searching the exact parameters and consequently the mathematical optimum point as it was mentioned in Chapter I.

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APPENDIX A  
COMPLETE EVALUATION OF MINIMIZATION  
OR COST FUNCTIONS AND THE LIMITS  
OF DECISION RANGES FOR 10% INCREASE  
IN THE TOTAL VARIABLE COST ELEMENTS  
ABOVE ITS OPTIMUM VALUE

APPENDIX A  
COMPUTER PROGRAMS  
FOR  
MINIMIZATION OR COST FUNCTIONS

In Chapter III - models as a summation of two different standard curves - nine minimization or cost functions had been derived and it was shown that these functions after transformation (if needed) are functions of the error  $w$  which was associated with the optimum value of independent variable  $X_0$ . The transformation was used only when the function contained a constant term. The representations of a constant term in the decision models are the buffering stock cost in the inventory models, the first period maintenance or operating cost in the gradient maintenance or operating cost throughout the asset life for the retirement or replacement models, the unit variable cost of the linear relationship in the marginal or incremental production models and so on.

These nine minimization or cost functions are evaluated on the digital computer (Computer Program I) for an error  $w$  varying from 0.05 to 3.95 with an incremental error  $w = 0.05$ . At  $w = 1.0$  which is corresponding to the mathematical optimum point (minimum point)  $(X_0, Y_0)$ , the value of the transformed dependent variable (total variable cost elements) is the optimum one  $(Y_0'$  or  $TVC_0')$  its absolute optimum value  $(Y_0'$  or  $TVC_0')$  and this is equal to  $\pm 1.00$ . The value of  $+ 1.00$  refers to a positive optimum transformed value  $(Y_0')$  or a positive optimum of the total variable profit elements  $(TVP_0')$ , otherwise the value is  $- 1.00$ .

The limits of the minimization or the cost functions' decision ranges were obtained for a predetermined allowance of 10% increase in

the total variable elements of the cost function above its optimum ( $Y_0'$  or  $TVC_0'$ ) through the Computer Program II and by using a sub-routine program.

The complete evaluations of these cost functions are given in Tables V to XIII. The limits of cost functions' decision ranges are given in Table XIV.

The graphs of these minimization or cost functions are drawn in Figure 19 through 27.

I. COMPUTER PROGRAM FOR EVALUATING  
MINIMIZATION OR COST FUNCTIONS

Aboue1-Nour

```

C      EVALUATION OF 9 FORMULAE FROM .05 to .4. IN INCREMENTS OF .005
C      C-ARRAY IS LOWER STRADDLE OF ROOT, D-ARRAY IS UPPER STRADDLE
C      SLIM= STARTING VALUE FOR EVALUATION
C      ADD = INCREMENT FOR EVALUATION
C      NIT= No. EVALUATIONS TO BE MADE
C      NF= NO. FORMULAE
      DIMENSION C(2,10),D(2,10),W(99),Y(99)
      READ (1,101)SLIM,ADD,NIT,NF
      WRITE (3,101),SLIM,ADD,NIT,NF
101    FORMAT(2F10.5,2I5)
      DO 2 J=1,9
      READ(1,102) (C(I,J),D(I,J),I=1,2)
      WRITE (3,102) (C(I,J),D(I,J),I=1,2)
102    FORMAT(4F5.3)
      DO 3 J=1,NF
      IF(I-1)31,31,32
31     W(I)=0.05
      GO TO 33
32     W(I)=W(I-1)+ADD
33     WW=W(I)
      SRTW=SQRT(WW)
C      EVALUATE THE FUNCTIONS, PRINT OUTPUT, FIND ROOT
      GO TO(5,6,7,8,9,10,11,12,13),J
      5     Y(I)=WW -2.*SRTW
      GO TO 4
      6     Y(I)= ((WW*WW)+1)/(2.0* WW)
      GO TO 4
      7     Y(I) = WW *WW -2. * WW
      GO TO 4
      8     Y(I)=(WW * WW)/3.0-( 4.0/3.0)* SRTW
      GO TO 4
      9     Y(I)= (WW*WW)/3.0 + 2./(3*WW)
      GO TO 4
      10    Y(I)= .5* WW**3 - 1.5*WW
      GO TO 4
      11    Y(I)= -1.2*SRTW + (WW**3)/5.
      GO TO 4
      12    Y(I)= WW**3/4 + 3.0/(4.0*WW)
      GO TO 4
      13    Y(I)= (2.0/3.0)*WW + 1.0/(3.0*WW*WW)
      4     CONTINUE
      GO TO(14,15,16,17,18,19,20,21,22),J
C      WRITE HEADINGS AND OUTPUT
      14    WRITE(3,114)
114    FORMAT('1 SENSITIVITY ANALYSIS OF STRAIGHT LINE VS. PARABOLA MODEL
1'//20X,' FUNCTION... Y= W-2*SQRT(W)' ..10X,'ERROR W',10X'TOTAL
2VALUE')

```

```

23  WRITE(3,30)(W(K),Y(K),K=1,NIT)
30  FORMAT(5X,F10.5,12X,F12.6)
    GO TO 3
15  WRITE(3,115)
115 FORMAT('1 SENSITIVITY ANALYSIS OF STRAIGHT LINE VS. RECTANGULAR HY
1PERBOLA MODEL'//20X, 'FUNCTION... Y=(W*W+1)/2W' //10X, 'ERROR
2W',10X, ' TOTAL VALUE')
    GO TO 23
16  WRITE(3,116)
116 FORMAT('1 SENSITIVITY ANALYSIS OF PARABOLA VS. STRAIGHT LINE MODE
1L',//20X, 'FUNCTION ... Y=W*W - 2W' //10X, ' ERROR W',10X, 'TOTA
1L VALUE')
    GO TO 23
17  WRITE(3,117)
117 FORMAT('1 SENSITIVITY ANALYSIS OF PARABOLA VS. PARABOLA MODEL',//
120X, ' FUNCTION... Y =(W*W)/3-(4/3)*SQRT(W)'//10X 'ERROR W',10X 'TOTAL
2VALUE')
    GO TO 23
18  WRITE(3,118)
118 FORMAT('1 SENSITIVITY ANALYSIS OF PARABOLA VS. RECTANGULAR HYPERBO
1LA MODEL'//20X ' FUNCTION... Y = (W*W)/3 + 2/3W',//10X, ' ERROR W',10
2X, ' TOTAL VALUE')
    GO TO 23
19  WRITE(3,119)
119 FORMAT('1 SENSITIVITY ANALYSIS OF CUBICAL PARABOLA VS. STRAIGHT LI
1NE'// 20X' FUNCTION... Y= .5W**3 - 1.5W ' //10X'ERROR W',10X'TOT
2AL VALUE')
    GO TO 23
20  WRITE(3,120)
120 FORMAT('1 SENSITIVITY ANALYSIS OF CUBICAL PARABOLA VS. PARABOLA'//
120X, ' FUNCTION... Y= -1.2*SQRT(W)+ W**3/5.'// 10X, 'ERROR W',10X 'T
2OTAL VALUE')
    GO TO 23
21  WRITE(3,121)
121 FORMAT('1 SENSITIVITY ANALYSIS OF CUBICAL PARABOLA VS. RECTANGULAR
1 HYPER BOLA' // 20X, 'FUNCTION... Y+ W**3/4 + 3/4W' // 10X, 'ERROR W
2', 10X 'TOTAL VALUE')
    GO TO 23
22  WRITE(3,122)
122 FORMAT('1 STRAIGHT LINE VS. RECIPROCAL OF QUADRATIC FUNCTION'//20X
1, 'FUNCTION... 2/3 W + 1/3W*W'// 10X 'ERROR W',10X, 'TOTAL VALUE')
    GO TO 23
3   CONTINUE
    STOP

```



II. COMPUTER PROGRAM FOR OBTAINING LIMITS OF MINIMIZATION  
OR COST FUNCTIONS' DECISION RANGES FOR A PREDETERMINED  
ALLOWANCE 10% INCREASE ABOVE ITS OPTIMUM

Abouel-Nour

```

FUNCTION F(X,JFW)
GO TO (1,2,3,4,5,6,7,8,9),JFW
1 F=X-2.*SQRT(X)+1.10
GO TO 10
2 F=X*X-2.2*X+1.0
GO TO 10
3 F=X*X-2.*X+.9
GO TO 10
4 F=X*X-4.*SQRT(X)+2.7
GO TO 10
5 F=X**3 - 3.3*X +2
GO TO 10
6 F= X**3-3*X+1.8
GO TO 10
7 F=X**3-6.*SQRT(X) + 4.5
GO TO 10
8 F=X**4 - 4.4*X + 3.0
GO TO 10
9 F=2.*X**3 - 3.3*X*X + 1.0
10 RETURN
END

```

```

BEGIN COMPILATION
DIMENSION C(2,10),D(2,10)
DO 2 J=1,9
READ(1,102) (C(I,J),D(I,J),I=1,2)
2 WRITE (3,102) (C(I,J),D(I,J),I=1,2)
102 FORMAT(4F5.3)
DO 3 J=1.9
24 DO 26 I=1,2
A=C(I,J)
B=D(I,J)
N1=5
N2=50
E1=.0005
E2=.0005
JX=J
CALL RTFIND(A,B,X,Y,E1,E2,N1,N2,N,JX)
26 WRITE(3,25) X,Y,N,J
25 FORMAT(//' LIMIT =',E15.8,' Y=',E15.8,' N=',I10,'FORMULA NO.',
115)
3 CONTINUE
STOP
END

```

## BEGIN COMPILATION

```

SUBROUTINE RTFIND(A,B,X,Y,E1,E2,N1,N2,N,JFW)
JFW=JFW
N=0
F1=F(A,JFW)
F2=F(B,JFW)
IF(F1*F2)1,11,20
1 IF(F1)2,3,3
2 FM=F1
XM=A
FP=F2
XP=B
GO TO 4
3 FM=F2
XM=B
FP=F 1
XP=A
4 N=N+1
IF(N-N1)5,5,21
5 C=FP*(XP-XM)/(FP-FM)
X=XP-C
Y=F(X,JFW)
IF(ABS(C)-E1)6,6,8
6 IF(ABS(Y)-E2)7,7,8
7 RETURN
8 IF(Y)9,10,10
9 FM=Y
XM=X
10 FP=Y
XP=X
11 IF(F1)12,13,12
12 X=B
Y=F2
GO TO 7
13 X=A
Y=P1
GO TO 7
20 WRITE(3,300)
300 FORMAT(' NO STRADDLE ')
X1=A
X2=B
GO TO 24
21 IF(FM-Y)22,23,22
22 F2=FM
X2=XM
GO TO 24
23 F2=FP
X2=XP
24 N=0
F1=Y
X1=X
25 N=N+1
IF(N-N2)26,26,30

```

```

26  C=((X1-X2)/(P1-F2))*F1
    X=X1-C
    Y=F(X, JFW)
    IF (ABS(C)-E1) 27, 27, 28
27  IF (ABS(Y)-E2) 7, 7, 28
28  F2=F1
    X2=X1
    F1=Y
    X1=X
    GO TO 25
30  WRITE(3, 301)Y, X, N
301  FORMAT(' CONVERGENCE INCOMPLETE.  LAST F(X)= 'F10.6, 'AT X='F10.4, '
      1 FOR '15, ' ITERATIONS')
    GO TO 7
    END

```

LIMIT = 0.46755546E 00	Y = -0.49471855E-05	N =	1FORMULA NO.	1
LIMIT = 0.17324524E 01	Y = 0.35762787E-06	N =	1FORMULA NO.	1
LIMIT = 0.64174318E 00	Y = 0.0	N =	1FORMULA NO.	2
LIMIT = 0.15582561E 01	Y = -0.95367432E-06	N =	1FORMULA NO.	2
LIMIT = 0.68377364E 00	Y = -0.71525574E-06	N =	1FORMULA NO.	3
LIMIT = 0.13162279E 01	Y = -0.59604645E-06	N =	1FORMULA NO.	3
LIMIT = 0.57330710E 00	Y = -0.95367432E-06	N =	1FORMULA NO.	4
LIMIT = 0.14612770E 01	Y = -0.95367432E-06	N =	1FORMULA NO.	4
LIMIT = 0.71867669E 00	Y = -0.43773651E-03	N =	1FORMULA NO.	5
LIMIT = 0.13473911E 01	Y = -0.25177002E-03	N =	1FORMULA NO.	5
LIMIT = 0.72930038E 00	Y = -0.95367432E-06	N =	1FORMULA NO.	6
LIMIT = 0.12479858E 01	Y = -0.26035309E-03	N =	1FORMULA NO.	6
LIMIT = 0.62527692E 00	Y = -0.95367432E-06	N =	1FORMULA NO.	7
LIMIT = 0.13537807E 01	Y = -0.21934509E-04	N =	1FORMULA NO.	7
LIMIT = 0.75613230E 00	Y = -0.99182129E-04	N =	1FORMULA NO.	8
LIMIT = 0.12664232E 01	Y = 0.95367432E-06	N =	3FORMULA NO.	8
LIMIT = 0.74211115E 00	Y = 0.0	N =	5FORMULA NO.	9
LIMIT = 0.13919315E 01	Y = 0.0	N =	5FORMULA NO.	9

APPENDIX A - TABLES  
SENSITIVITY ANALYSIS  
FOR  
MINIMIZATION OR COST FUNCTIONS

TABLE V  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 PARABOLA VERSUS STRAIGHT LINE MODEL

$$\frac{Y_w'}{|Y_o'|} = -2\sqrt{w} + w$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	-0.397214	2.10	-0.798280
0.10	-0.532455	2.15	-0.782580
0.15	-0.624597	2.20	-0.766484
0.20	-0.694427	2.25	-0.750007
0.25	-0.750000	2.30	-0.733157
0.30	-0.795445	2.35	-0.715949
0.35	-0.833215	2.40	-0.698394
0.40	-0.864910	2.45	-0.680503
0.45	-0.891640	2.50	-0.662286
0.50	-0.914213	2.55	-0.643752
0.55	-0.933240	2.60	-0.624911
0.60	-0.949193	2.65	-0.605773
0.65	-0.972451	2.70	-0.586345
0.70	-0.973319	2.75	-0.566634
0.75	-0.982051	2.80	-0.546651
0.80	-0.988854	2.85	-0.526399
0.85	-0.993908	2.90	-0.505889
0.90	-0.997366	2.95	-0.485126
0.95	-0.99359	3.00	-0.464114
1.00	-1.00000	3.05	-0.442863
1.05	-0.999389	3.10	-0.421376
1.10	-0.997617	3.15	-0.399661
1.15	-0.994760	3.20	-0.377723
1.20	-0.990890	3.25	-0.355567
1.25	-0.986068	3.30	-0.333196
1.30	-0.980351	3.35	-0.310616
1.35	-0.973790	3.40	-0.287833
1.40	-0.966432	3.45	-0.264852
1.45	-0.958320	3.50	-0.241675
1.50	-0.949491	3.55	-0.218307
1.55	-0.949981	3.60	-0.194751
1.60	-0.929824	3.65	-0.171014
1.65	-0.919047	3.70	-0.147096
1.70	-0.907683	3.75	-0.123004
1.75	-0.895753	3.80	-0.098739
1.80	-0.883285	3.85	-0.074304
1.85	-0.870297	3.90	-0.049705
1.90	-0.856812	3.95	-0.024943
1.95	-0.842851		
2.00	-0.828431		
2.05	-0.813569		

TABLE VI  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 STRAIGHT LINE VERSUS RECTANGULAR HYPERBOLA MODEL

$$\frac{Y_w'}{|Y_o^v|} = \frac{2}{w+1}$$

Error w	Value $\frac{Y_w'}{ Y_o^v }$	Error w	Value $\frac{Y_w'}{ Y_o^v }$
0.05	10.024999	2.10	1.288088
0.10	5.049997	2.15	1.307550
0.15	3.408332	2.20	1.327265
0.20	2.599998	2.25	1.347214
0.25	2.124999	2.30	1.367382
0.30	1.816667	2.35	1.387756
0.35	1.603572	2.40	1.408323
0.40	1.449999	2.45	1.429071
0.45	1.336111	2.50	1.449989
0.50	1.250000	2.55	1.471067
0.55	1.184091	2.60	1.492296
0.60	1.133333	2.65	1.513667
0.65	1.094231	2.70	1.535173
0.70	1.064285	2.75	1.556806
0.75	1.041666	2.80	1.578558
0.80	1.025000	2.85	1.600426
0.85	1.013235	2.90	1.622400
0.90	1.005555	2.95	1.644477
0.95	1.001316	3.00	1.666652
1.00	1.000000	3.05	1.688920
1.05	1.001190	3.10	1.711275
1.10	1.004544	3.15	1.733714
1.15	1.009781	3.20	1.756234
1.20	1.016665	3.25	1.778830
1.25	1.024999	3.30	1.801498
1.30	1.034614	3.35	1.824237
1.35	1.045368	3.40	1.847041
1.40	1.057140	3.45	1.869909
1.45	1.069825	3.50	1.892838
1.50	1.083330	3.55	1.915826
1.55	1.097577	3.60	1.938869
1.60	1.112496	3.65	1.961967
1.65	1.128026	3.70	1.985115
1.70	1.144114	3.75	2.008313
1.75	1.160709	3.80	2.031558
1.80	1.177773	3.85	2.054849
1.85	1.195265	3.90	2.078182
1.90	1.213152	3.95	2.101559
1.95	1.231404		
2.00	1.249993		
2.05	1.268895		

TABLE VII  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 PARABOLA VERSUS STRAIGHT LINE MODEL

$$\frac{Y_w'}{|Y_o'|} = w^2 - 2w$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	-0.097500	2.10	0.209961
0.10	-0.190000	2.15	0.322457
0.15	-0.277500	2.20	0.439953
0.20	-0.360000	2.25	0.562449
0.25	-0.437500	2.30	0.689945
0.30	-0.510000	2.35	0.822441
0.35	-0.577500	2.40	0.959937
0.40	-0.640000	2.45	1.102432
0.45	-0.697500	2.50	1.249928
0.50	-0.750000	2.55	1.402423
0.55	-0.797499	2.60	1.559918
0.60	-0.839999	2.65	1.722413
0.65	-0.877499	2.70	1.889908
0.70	-0.910000	2.75	2.062403
0.75	-0.937499	2.80	2.239897
0.80	-0.960000	2.85	2.422392
0.85	-0.977499	2.90	2.609886
0.90	-0.990000	2.95	2.802380
0.95	-0.997499	3.00	2.999874
1.00	-1.000000	3.05	3.202368
1.05	-0.997500	3.10	3.409861
1.10	-0.990001	3.15	3.622355
1.15	-0.977502	3.20	3.839848
1.20	-0.960002	3.25	4.062341
1.25	-0.937503	3.30	4.289834
1.30	-0.910004	3.35	4.522326
1.35	-0.877505	3.40	4.759819
1.40	-0.840006	3.45	5.002312
1.45	-0.797507	3.50	5.249804
1.50	-0.750009	3.55	5.502296
1.55	-0.697511	3.60	5.759789
1.60	-0.640013	3.65	6.022280
1.65	-0.577515	3.70	6.289772
1.70	-0.510016	3.75	6.562263
1.75	-0.437519	3.80	6.839755
1.80	-0.360022	3.85	7.122246
1.85	-0.277524	3.90	7.409738
1.90	-0.190027	3.95	7.702229
1.95	-0.097529		
2.00	-0.000032		
2.05	0.102464		

TABLE VIII  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 PARABOLA VERSUS PARABOLA MODEL

$$\frac{Y_w^*}{|Y_o^*|} = \frac{w^2}{3} - \frac{4}{3}\sqrt{w}$$

Error w	Value $\frac{Y_w^*}{ Y_o^* }$	Error w	Value $\frac{Y_w^*}{ Y_o^* }$
0.05	-0.297309	2.10	-0.462199
0.10	-0.418303	2.15	-0.414233
0.15	-0.508897	2.20	-0.364338
0.20	-0.582951	2.25	-0.312521
0.25	-0.645833	2.30	-0.258789
0.30	-0.700296	2.35	-0.203152
0.35	-0.747977	2.40	-0.145617
0.40	-0.789940	2.45	-0.086190
0.45	-0.826927	2.50	-0.024880
0.50	-0.859475	2.55	0.038307
0.55	-0.887993	2.60	0.103366
0.60	-0.912794	2.65	0.170290
0.65	-0.934133	2.70	0.239073
0.70	-0.952212	2.75	0.309712
0.75	-0.967200	2.80	0.382198
0.80	-0.979236	2.85	0.456532
0.85	-0.988438	2.90	0.532703
0.90	-0.994910	2.95	0.610710
0.95	-0.998738	3.00	0.690550
1.00	-0.999999	3.05	0.772215
1.05	-0.998758	3.10	0.855703
1.10	-0.995077	3.15	0.941011
1.15	-0.989006	3.20	1.028135
1.20	-0.980593	3.25	1.117070
1.25	-0.969878	3.30	1.207814
1.30	-0.956901	3.35	1.300364
1.35	-0.941695	3.40	1.394718
1.40	-0.924289	3.45	1.490870
1.45	-0.904715	3.50	1.588819
1.50	-0.882996	3.55	1.688562
1.55	-0.859156	3.60	1.790096
1.60	-0.833219	3.65	1.893418
1.65	-0.805202	3.70	1.998528
1.70	-0.775127	3.75	2.105419
1.75	-0.743008	3.80	2.214093
1.80	-0.708863	3.85	2.324547
1.85	-0.672706	3.90	2.436777
1.90	-0.634550	3.95	2.550781
1.95	-0.594409		
2.00	-0.552298		
2.05	-0.508224		



TABLE IX  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 PARABOLA VERSUS RECTANGULAR HYPERBOLA MODEL

$$\frac{Y_w'}{|Y_o'|} = \frac{w^2}{3} + \frac{2}{3w}$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	13.334167	2.10	1.787437
0.10	6.670002	2.15	1.850886
0.15	4.451946	2.20	1.916337
0.20	3.346667	2.25	1.983768
0.25	2.687501	2.30	2.053158
0.30	2.252223	2.35	2.124490
0.35	1.945597	2.40	2.197743
0.40	1.720000	2.45	2.272906
0.45	1.548982	2.50	2.349961
0.50	1.416668	2.55	2.428898
0.55	1.312955	2.60	2.509702
0.60	1.231111	2.65	2.592361
0.65	1.166474	2.70	2.676867
0.70	1.115715	2.75	2.763208
0.75	1.076389	2.80	2.851377
0.80	1.046666	2.85	2.941364
0.85	1.025146	2.90	3.033162
0.90	1.010740	2.95	3.126762
0.95	1.002587	3.00	3.222161
1.00	1.000000	3.05	3.319348
1.05	1.002419	3.10	3.418320
1.10	1.009393	3.15	3.519071
1.15	1.020542	3.20	3.621594
1.20	1.035554	3.25	3.725886
1.25	1.054164	3.30	3.831942
1.30	1.076151	3.35	3.939757
1.35	1.101323	3.40	4.049328
1.40	1.129519	3.45	4.160649
1.45	1.160598	3.50	4.273719
1.50	1.194438	3.55	4.388534
1.55	1.230933	3.60	4.505089
1.60	1.269991	3.65	4.623382
1.65	1.311530	3.70	4.743410
1.70	1.355479	3.75	4.865171
1.75	1.401773	3.80	4.988662
1.80	1.450356	3.85	5.113880
1.85	1.501179	3.90	5.240824
1.90	1.554193	3.95	5.369490
1.95	1.609363		
2.00	1.666647		
2.05	1.726015		

TABLE X  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 CUBICAL PARABOLA VERSUS STRAIGHT LINE MODEL

$$\frac{Y_w'}{|Y_o'|} = \frac{w^3}{2} - \frac{3}{2} w$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	-0.074937	2.10	1.480410
0.10	-0.149500	2.15	1.744087
0.15	-0.223312	2.20	2.023888
0.20	-0.296000	2.25	2.320189
0.25	-0.367187	2.30	2.633366
0.30	-0.436500	2.35	2.963791
0.35	-0.503562	2.40	3.311840
0.40	-0.568000	2.45	3.677888
0.45	-0.629437	2.50	4.062312
0.50	-0.687500	2.55	4.465481
0.55	-0.741812	2.60	4.887774
0.60	-0.792000	2.65	5.329575
0.65	-0.837687	2.70	5.791243
0.70	-0.878499	2.75	6.273159
0.75	-0.914061	2.80	6.775705
0.80	-0.944000	2.85	7.299248
0.85	-0.967937	2.90	7.844160
0.90	-0.985500	2.95	8.410831
0.95	-0.996312	3.00	8.999620
1.00	-0.999999	3.05	9.610909
1.05	-0.996188	3.10	10.245070
1.10	-0.984501	3.15	10.902485
1.15	-0.964564	3.20	11.583521
1.20	-0.936003	3.25	12.288551
1.25	-0.898442	3.30	13.017965
1.30	-0.851506	3.35	13.772114
1.35	-0.794821	3.40	14.551394
1.40	-0.728010	3.45	15.356171
1.45	-0.650701	3.50	16.186813
1.50	-0.562516	3.55	17.043732
1.55	-0.463082	3.60	17.927261
1.60	-0.352024	3.65	18.837784
1.65	-0.228966	3.70	19.775681
1.70	-0.093534	3.75	20.741333
1.75	0.054648	3.80	21.735107
1.80	0.215956	3.85	22.757385
1.85	0.390761	3.90	23.808517
1.90	0.579442	3.95	24.888916
1.95	0.782372		
2.00	0.999928		
2.05	1.232480		

TABLE XI  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 CUBICAL PARABOLA VERSUS PARABOLA MODEL

$$\frac{Y_w'}{|Y_o'}| = \frac{w^3}{5} - \frac{6}{5}\sqrt{w}$$

Error w	Value $\frac{Y_w'}{ Y_o'} $	Error w	Value $\frac{Y_w'}{ Y_o'} $
0.05	-0.268303	2.10	0.113196
0.10	-0.379273	2.15	0.228086
0.15	-0.464083	2.20	0.349665
0.20	-0.535056	2.25	0.478072
0.25	-0.596875	2.30	0.613452
0.30	-0.651867	2.35	0.755947
0.35	-0.701354	2.40	0.905700
0.40	-0.746146	2.45	1.062854
0.45	-0.786759	2.50	1.227554
0.50	-0.823528	2.55	1.399941
0.55	-0.856668	2.60	1.580163
0.60	-0.886316	2.65	1.768366
0.65	-0.912546	2.70	1.964690
0.70	-0.935391	2.75	2.169283
0.75	-0.954855	2.80	2.382292
0.80	-0.970912	2.85	2.603860
0.85	-0.983520	2.90	2.834131
0.90	-0.992620	2.95	3.073257
0.95	-0.998139	3.00	3.321380
1.00	-0.999999	3.05	3.578646
1.05	-0.998108	3.10	3.845202
1.10	-0.992370	3.15	4.121197
1.15	-0.982681	3.20	4.406775
1.20	-0.968934	3.25	4.702080
1.25	-0.951016	3.30	5.007269
1.30	-0.928813	3.35	5.322478
1.35	-0.902202	3.40	5.647861
1.40	-0.871063	3.45	5.983561
1.45	-0.835271	3.50	6.329729
1.50	-0.794700	3.55	6.686513
1.55	-0.749221	3.60	7.054058
1.60	-0.698703	3.65	7.432508
1.65	-0.643014	3.70	7.822021
1.70	-0.582022	3.75	8.222735
1.75	-0.515592	3.80	8.634804
1.80	-0.443588	3.85	9.058372
1.85	-0.365874	3.90	9.493589
1.90	-0.282310	3.95	9.940605
1.95	-0.192760		
2.00	-0.097088		
2.05	0.004851		

TABLE XII  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 CUBICAL PARABOLA VERSUS RECTANGULAR HYPERBOLA MODEL

$$\frac{Y_w'}{|Y_o'|} = \frac{w^3}{4} + \frac{3}{4w}$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	15.000031	2.10	2.672337
0.10	7.500252	2.15	2.833369
0.15	5.000845	2.20	3.002841
0.20	3.752002	2.25	3.180915
0.25	3.003907	2.30	3.367757
0.30	2.506751	2.35	3.563530
0.35	2.153577	2.40	3.768406
0.40	1.891001	2.45	3.982551
0.45	1.689448	2.50	4.206140
0.50	1.531250	2.55	4.439342
0.55	1.405231	2.60	4.682331
0.60	1.304000	2.65	4.935288
0.65	1.222503	2.70	5.198381
0.70	1.157179	2.75	5.471788
0.75	1.105469	2.80	5.755691
0.80	1.065500	2.85	6.050261
0.85	1.035884	2.90	6.355680
0.90	1.015583	2.95	6.672132
0.95	1.003818	3.00	6.999788
1.00	1.000000	3.05	7.338834
1.05	1.003691	3.10	7.689447
1.10	1.014566	3.15	8.051814
1.15	1.032391	3.20	8.426111
1.20	1.056997	3.25	8.812520
1.25	1.088277	3.30	9.211230
1.30	1.126168	3.35	9.622416
1.35	1.170642	3.40	10.046263
1.40	1.221705	3.45	10.482953
1.45	1.279387	3.50	10.932675
1.50	1.343738	3.55	11.395611
1.55	1.414825	3.60	11.871940
1.60	1.492733	3.65	12.361846
1.65	1.577557	3.70	12.865520
1.70	1.669403	3.75	13.383142
1.75	1.768388	3.80	13.914896
1.80	1.874637	3.85	14.460966
1.85	1.988277	3.90	15.021541
1.90	2.109448	3.95	15.596805
1.95	2.238292		
2.00	2.374954		
2.05	2.519583		

TABLE XIII  
 SENSITIVITY ANALYSIS OF MINIMIZATION OR COST FUNCTIONS  
 STRAIGHT LINE VERSUS RECIPROCAL OF QUADRATIC FUNCTION MODEL

$$\frac{Y_w'}{|Y_o'|} = \frac{2}{3} w + \frac{1}{3w^2}$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	133.366714	2.10	1.475574
0.10	33.400024	2.15	1.505432
0.15	14.914841	2.20	1.535524
0.20	8.466679	2.25	1.565830
0.25	5.500008	2.30	1.596332
0.30	3.903709	2.35	1.627011
0.35	2.954428	2.40	1.657855
0.40	2.350003	2.45	1.688850
0.45	1.946093	2.50	1.719984
0.50	1.666669	2.55	1.751245
0.55	1.468596	2.60	1.782625
0.60	1.325927	2.65	1.814116
0.65	1.222289	2.70	1.845706
0.70	1.146939	2.75	1.877392
0.75	1.092593	2.80	1.909164
0.80	1.054167	2.85	1.941018
0.85	1.028028	2.90	1.972948
0.90	1.011522	2.95	2.004949
0.95	1.002678	3.00	2.037015
1.00	1.000000	3.05	2.069144
1.05	1.002342	3.10	2.101330
1.10	1.008815	3.15	2.133571
1.15	1.018713	3.20	2.165862
1.20	1.031480	3.25	2.198200
1.25	1.046664	3.30	2.230584
1.30	1.063903	3.35	2.263010
1.35	1.082896	3.40	2.295476
1.40	1.103398	3.45	2.327979
1.45	1.125204	3.50	2.360518
1.50	1.148144	3.55	2.393089
1.55	1.172071	3.60	2.425693
1.60	1.196869	3.65	2.458325
1.65	1.222429	3.70	2.490987
1.70	1.248666	3.75	2.523674
1.75	1.275502	3.80	2.556387
1.80	1.302872	3.85	2.589125
1.85	1.330719	3.90	2.621884
1.90	1.358993	3.95	2.654666
1.95	1.387651		
2.00	1.416656		
2.05	1.445973		

TABLE XIV  
LIMITS OF MINIMIZATION OR COST FUNCTIONS' DECISION RANGES  
FOR  
10% INCREASE IN THE TOTAL VARIABLE COST ELEMENTS ABOVE ITS OPTIMUM ( $Y_0$ )

No.	Type of the Model	Optimum Values $X_0, Y_0$	Decision Range for 10% Increase	
			Lower	Upper
1.	Straight line vs. parabola $Y = MX - B(X)^{1/2} + C$	$\left( \frac{A^2}{4M^2}, C - \frac{A^2}{4M} \right)$	$0.467555X_0$	$1.732452X_0$
2.	Straight line vs. rectangular hyperbola $Y = MX + \frac{B}{X} + C$	$\left( \left(\frac{B}{M}\right)^{1/2}, C - 2(MB)^{1/2} \right)$	$0.641743X_0$	$1.558256X_0$
3.	Parabola vs. straight line $Y = AX^2 - MX + C$	$\left( \frac{M}{2A}, C - \frac{M^2}{4A} \right)$	$0.683774X_0$	$1.316228X_0$
4.	Parabola vs. parabola $Y = AX^2 - B(X)^{1/2} + C$	$\left( \left(\frac{B}{4A}\right)^{2/3}, C - \frac{3}{4}\left(\frac{B}{4A}\right)^{4/3} \right)$	$0.573307X_0$	$1.461277X_0$
5.	Parabola vs. Rect. hyperbola $Y = AX^2 + \frac{B}{X} + C$	$\left( \left(\frac{B}{2A}\right)^{1/3}, C - 1.889(AB^2)^{1/3} \right)$	$0.718677X_0$	$1.347391X_0$
6.	Cubical parabola vs. straight line $Y = AX^3 - MX + C$	$\left( \left(\frac{M}{2A}\right)^{1/2}, C - \frac{2}{3}\left(\frac{M}{3A}\right)^{3/2} \right)$	$0.729300X_0$	$1.247986X_0$

TABLE XIV (CONTINUED)

No.	Type of the Model	Optimum Values $X_0, Y_0$	Decision Range for 10% Increase	
			Lower	Upper
7.	Cubical parabola vs. parabola $Y = AX^3 - B(X)^{1/2} + C$	$\left( \left(\frac{B}{6A}\right)^{2/5}, C - \frac{5}{6} \left(\frac{B}{6A}\right)^{1/5} \right)$	$0.625277X_0$	$1.353781X_0$
8.	Cubical parabola vs. Rect. hyperbola $Y = AX^3 + \frac{B}{X} + C$	$\left( \left(\frac{B}{3A}\right)^{1/4}, C + 1.74 (AB^3)^{1/4} \right)$	$0.756132X_0$	$1.26642X_0$
9.	Straight line vs. $\frac{B}{X^2}$ $Y = MX + \frac{B}{X^2} + C$	$\left( \left(\frac{2B}{M}\right)^{1/3}, C + \frac{3}{2} (2BM^2)^{1/3} \right)$	$0.74211X_0$	$1.391932X_0$

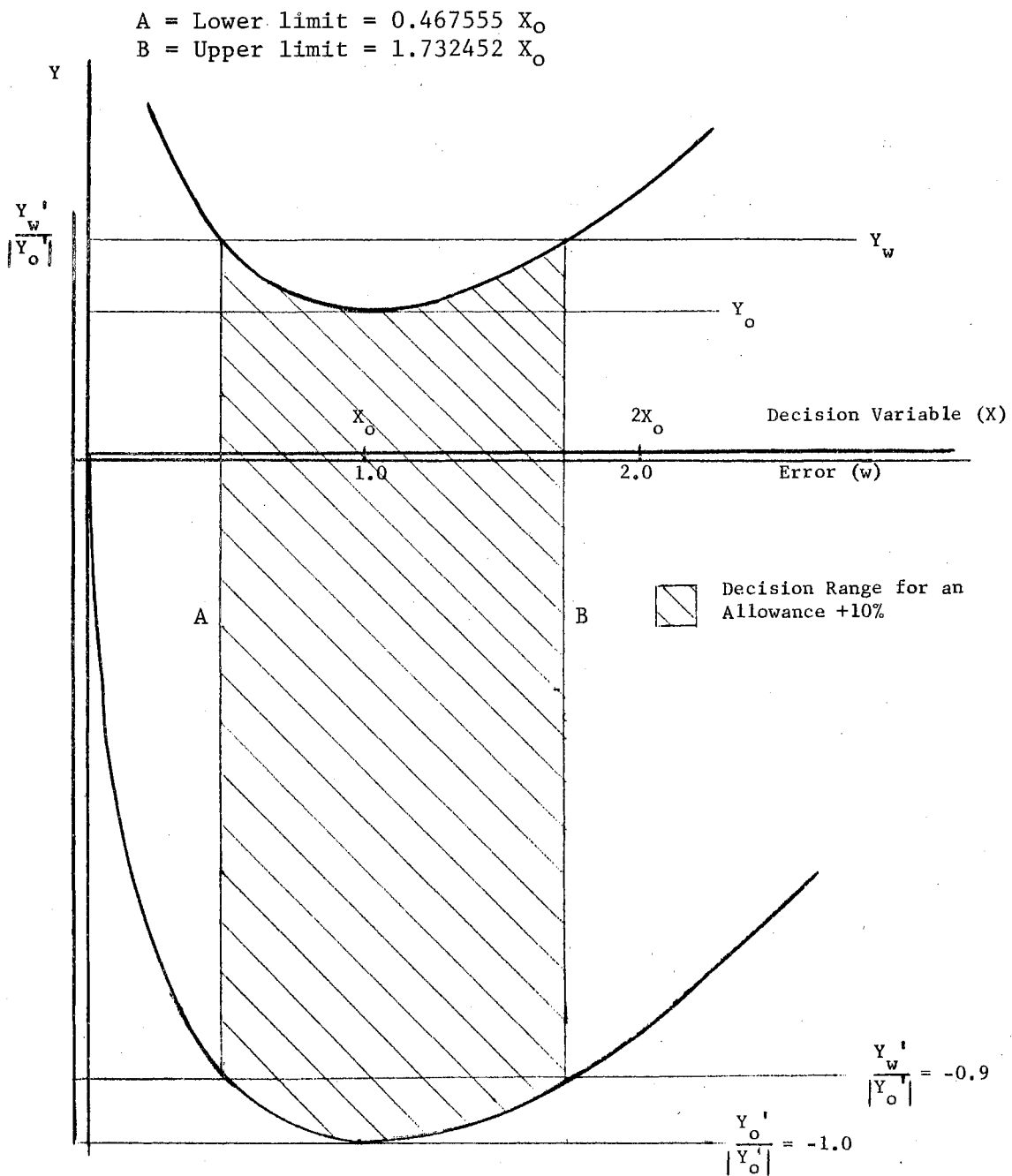


Figure (19). Cost (Minimization) Functions Straight Line vs Parabola Model



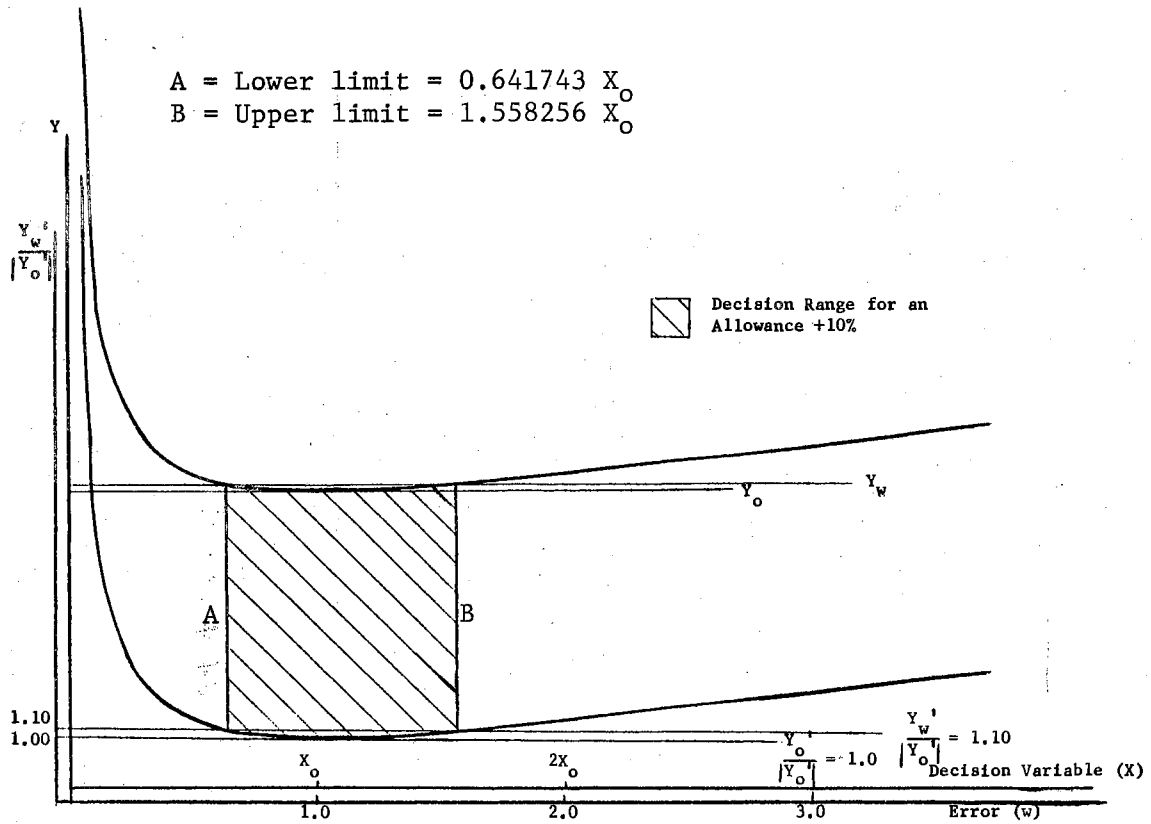


Figure (20). Cost (Minimization) Function Straight Line vs R. Hyperbola Model

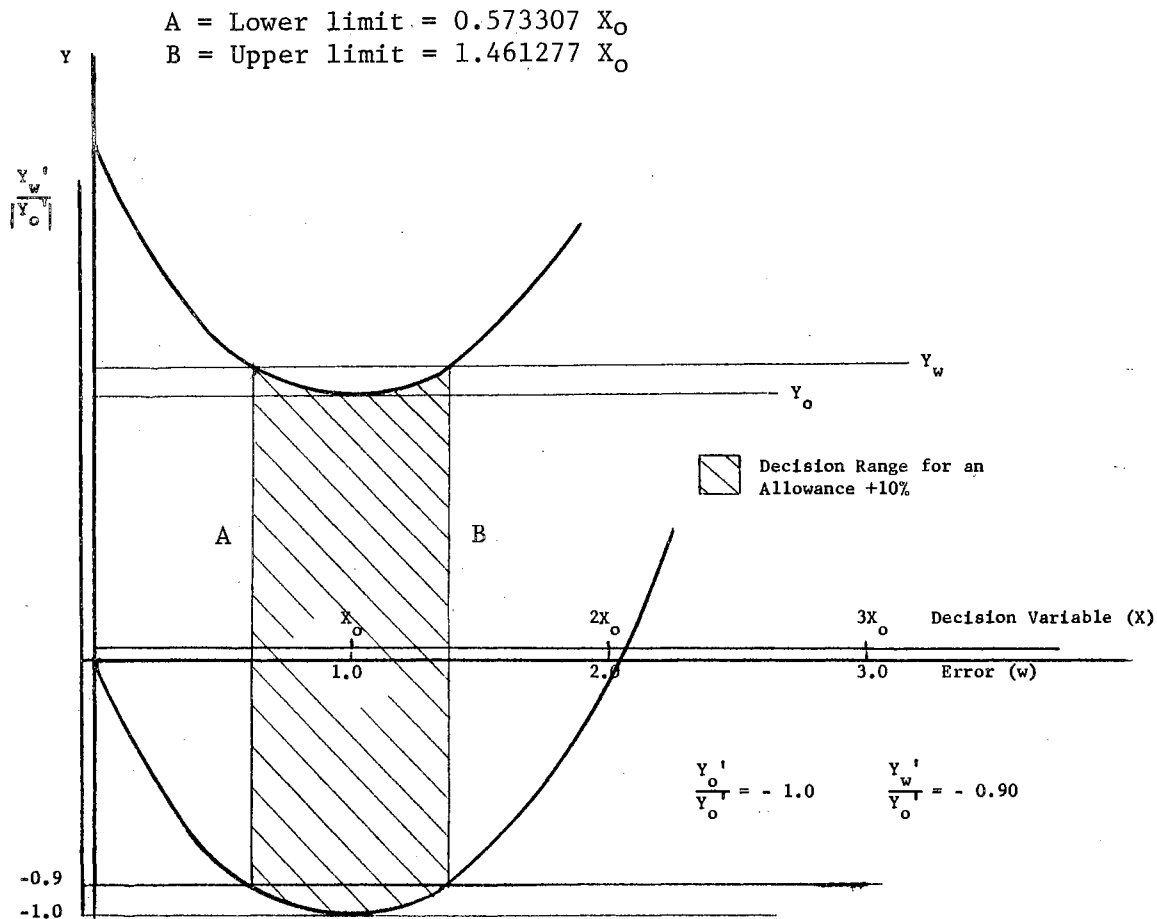


Figure (21). Cost (Minimization) Function-Parabola vs Straight Line Model

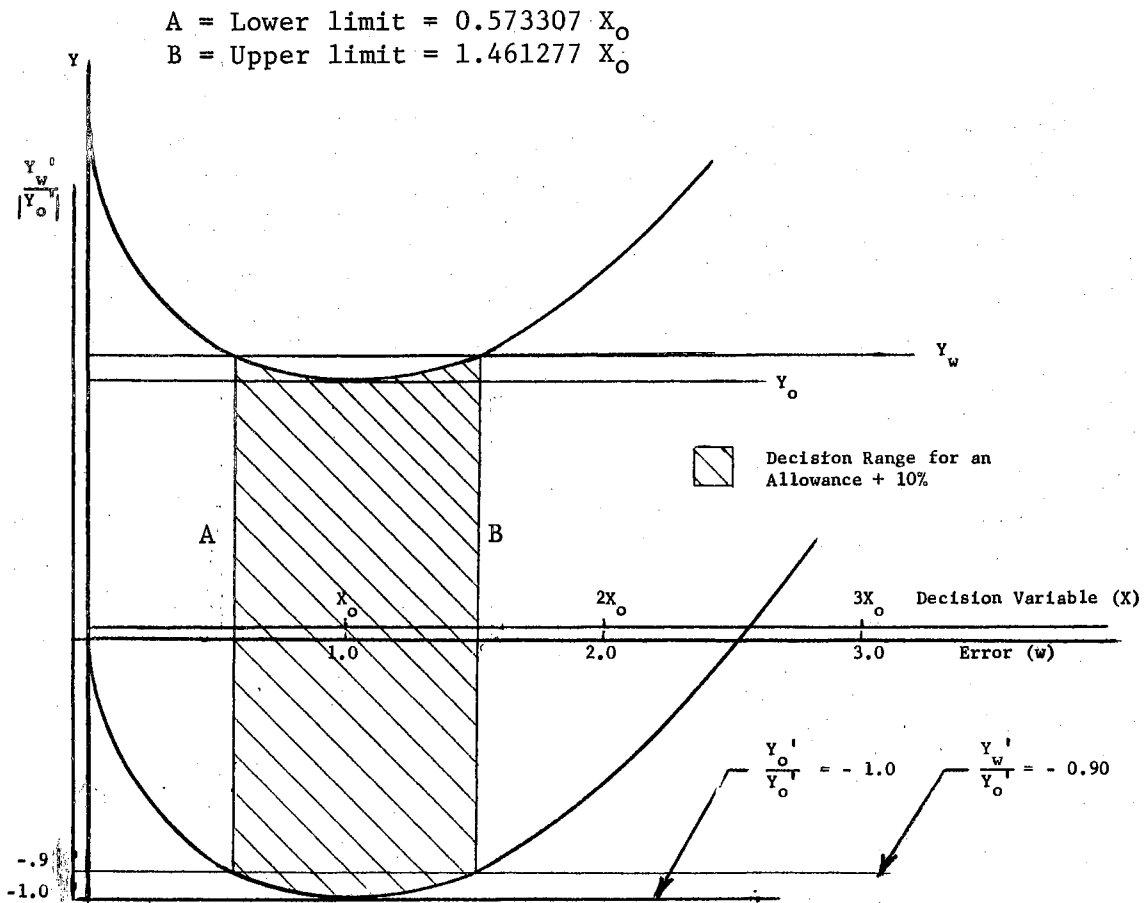


Figure (22). Cost (Minimization) Function-Parabola vs Parabola Model

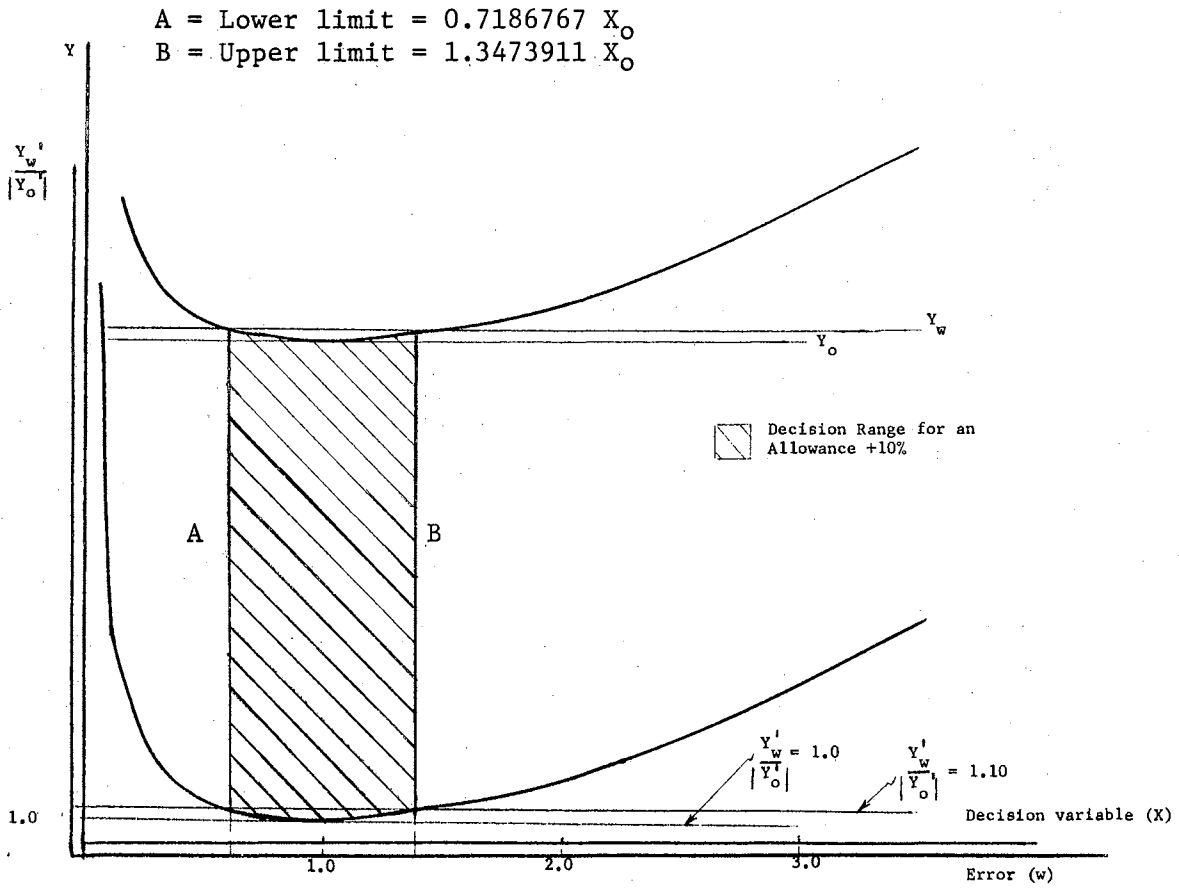


Figure (23). Cost (Minimization) Function-Parabola vs Rectangular Hyperbola



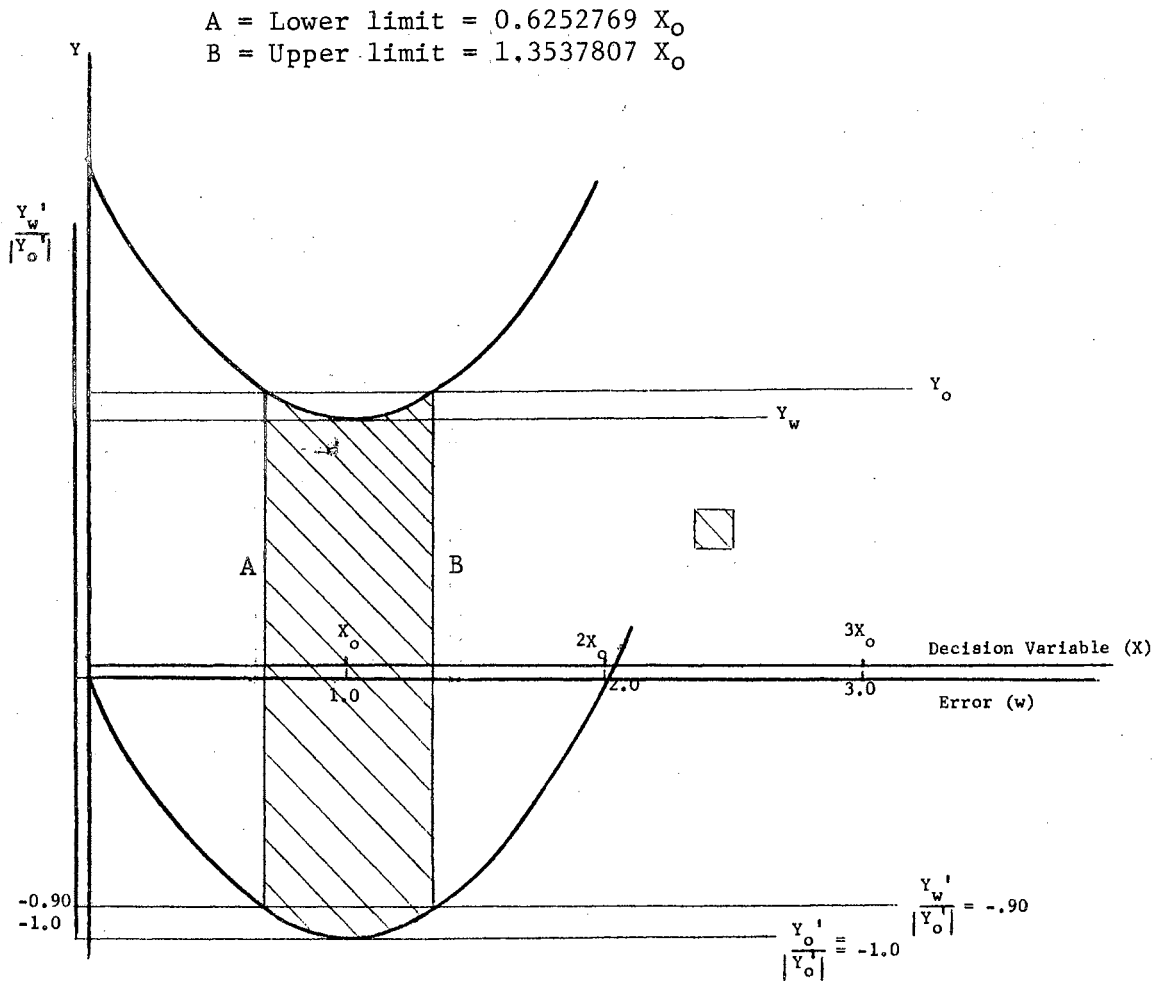


Figure (25). Cost (Minimization) Function - Cubical Parabola vs Parabola Model

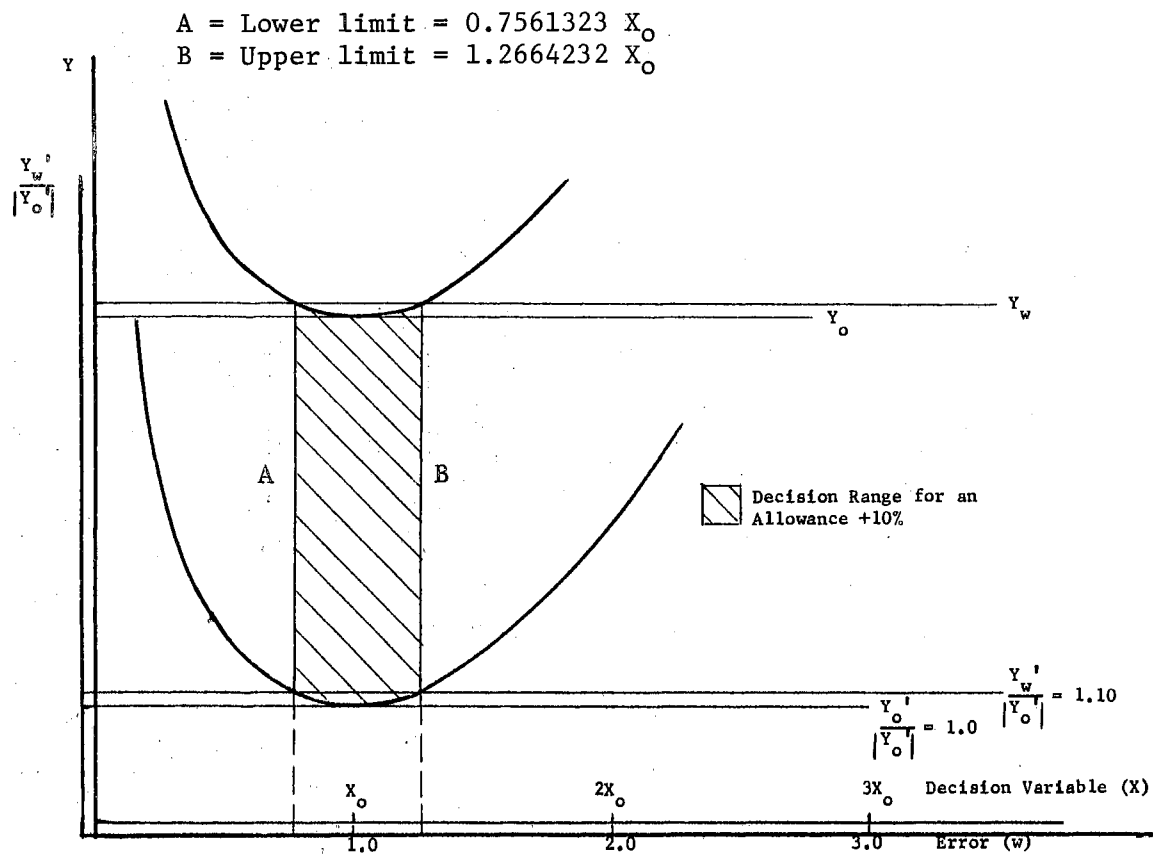


Figure (26). Cost (Minimization) Function - Cubical Parabola vs Rectangular Hyperbola

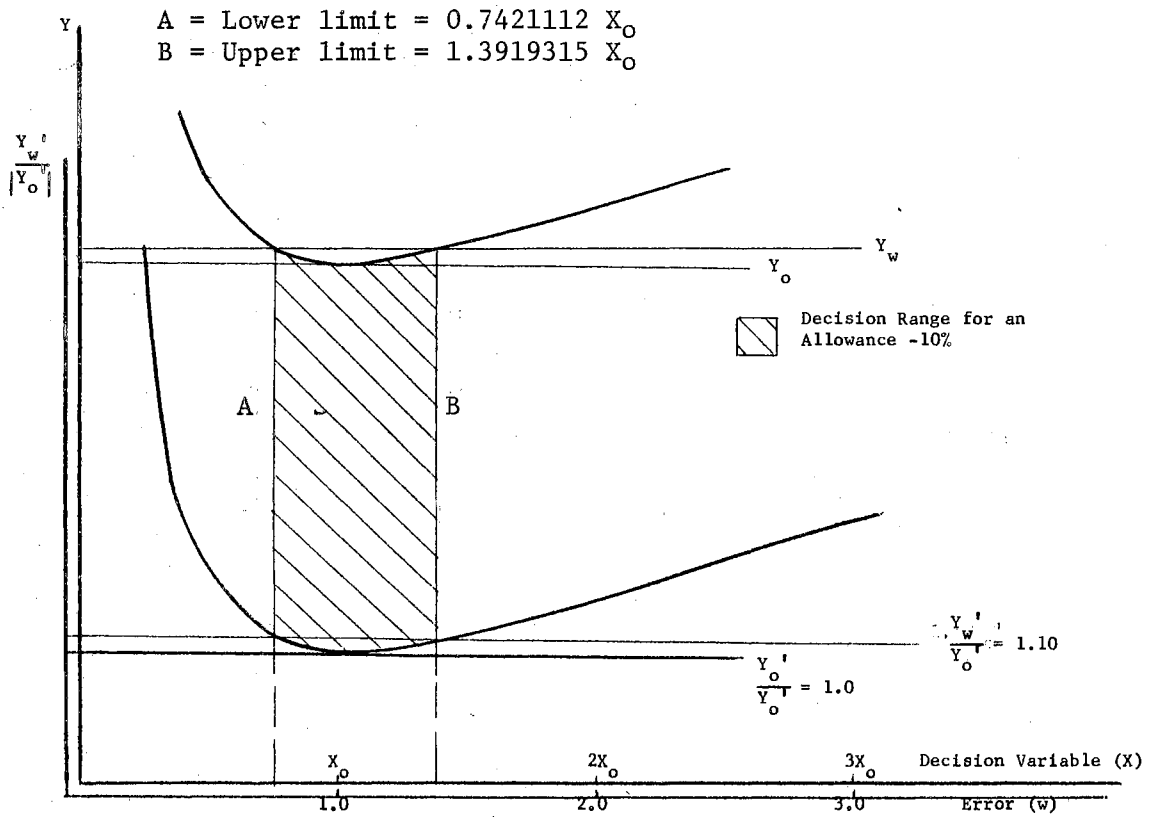


Figure (27). Cost (Minimization) Function - Straight Line vs Reciprocal of Q. Function



APPENDIX B  
COMPLETE EVALUATION OF MAXIMIZATION  
OR PROFIT FUNCTIONS AND THE LIMITS  
OF DECISION RANGES FOR 10% DECREASE  
IN THE TOTAL VARIABLE PROFIT ELEMENTS  
UNDER ITS OPTIMUM VALUE

APPENDIX B  
COMPUTER PROGRAMS  
FOR  
MAXIMIZATION OR PROFIT FUNCTIONS

In Chapter III - models as a summation of two different standard curves - minimization models can be transferred to maximization models if one can multiply the cost function by -1 provided that each standard curve in the cost function can be represented as an increasing or a decreasing function. Thus, the models contain rectangular hyperbola or the inverse of the quadratic function which are always decreasing functions are excluded from this idea. Therefore, there are five models out of the nine studied models that can be applied to this idea. The Computer Program III evaluated these five maximization functions for  $w = 0.05$  till  $w = 3.95$  with an increment  $w = 0.05$ . At  $w = 1.00$ , which is corresponding to the mathematical optimum point (maximum point)  $(X_0, Y_0)$ , the value of the transformed dependent variable (total variable profit elements) is the optimum one ( $Y_0'$  of  $TPC_0'$ ) over its absolute optimum value ( $Y_0'$  or  $TPC_0'$ ) and this is equal to  $\pm 1.00$ . The value of  $+1.00$  refers to a positive optimum transformed value ( $Y_0'$ ) or a positive optimum total variable profit element ( $TVP_0'$ ) otherwise the value is  $-1.00$ .

The limits of the maximization or profit functions' decision ranges were obtained for a predetermined allowance of 10% decrease in the total variable profit elements of the profit or maximization function under its optimum ( $Y_0'$  or  $TVP_0'$ ) through the Computer Program IV and by using a subrouting program.

The complete evaluation of these profit functions is given in Tables XV to XIX. The limits of the profit functions' decision ranges

are given in Table XX. The graphs of these maximization or profit functions are drawn in Figures 28 through 32.

III. COMPUTER PROGRAM FOR EVALUATING  
MAXIMIZATION OR PROFIT FUNCTIONS

Abouel-Nour

```

C      C-ARRAY IS LOWER STRADDLE OF ROOT, D-ARRAY IS UPPER STRADDLE
C      SLIM= STARTING VALUE FOR EVALUATION
C      ADD = INCREMENT FOR EVALUATION
C      NIT= NO. EVALUATIONS TO BE MADE
C      NF= NO. FORMULAE
      DIMENSION C (2,10),D(2,10),W(99),Y(99)
      READ (1,101)SLIM, ADD, NIT, NF
      WRITE(3,101)SLIM, ADD, NIT, NF
101    FORMAT(2F10.5,215)
      DO 3 J=1,NF
      DO 4 I=1,NIT
      IF(I-1)31,31,32
      31  W(I)=0.05
      GO TO 33
      32  W(I)=W(I-1)+ADD
      33  WW=W(I)
      SRTW=SQRT(WW)
C      EVALUATE THE FUNCTIONS, PRINT OUTPUT, FIND ROOT
      GO TO(5,6,7,8,9,10,11,12,13),J
      5  Y(I)=-WW+ 2.*SRTW
      GO TO 4
      6  Y(I) = -WW *WW + 2. * WW
      GO TO 4
      7  Y(I)=(-WW*WW)/3.0+( 4.0/3.0)*SRTW
      GO TO 4
      8  Y(I)=-.5* WW**3 + 1.5*WW
      GO TO 4
      9  Y(I)= =1.2*SRTW-(WW**3)/5.
      10 GO TO 4
      11 GO TO 4
      12 GO TO 4
      13 GO TO 4
      4  CONTINUE
      GO TO(14,15,16,17,18,19,20,21,22),J
C      WRITE HEADINGS AND OUTPUT
      14  WRITE(3,114)
114    FORMAT('1 MAXIMIZATION OR PROFIT FUNCTIONS',////'PARABOLA VERSUS
1 STRAIGHT LINE  Y= -W+2*SQRT(W)')
115    FORMAT('1 STRAIGHT LINE VERSUS PARABOLA      Y=-W*W+2W')
116    FORMAT('1 PARABOLA VERSUS PARABOLA      Y=-W*W/3+4/3*SQRT(W)')
117    FORMAT('1 STRAIGHT LINE VERSUS CUBICAL PARABOLA  Y=-(W**3)/2+1
1 .5W')
118    FORMAT('1 PARABOLA VERSUS CUBICAL PARABOLA  Y=1.2*SQRT(W)-W**3
1/5')
      23  WRITE (3,30)(W(K),Y(K),K=1,NIT)
      30  FORMAT(5X,F10.5,12X,F12.6)
      GO TO 3

```

```

15 WRITE(3,115)
   GO TO 23
16 WRITE(3,116)
   GO TO 23
17 WRITE(3,117)
   GO TO 23
18 WRITE(3,118)
   GO TO 23
19 GO TO 23
20 GO TO 23
21 GO TO 23
22 GO TO 23
  3 CONTINUE
    STOP
    END

```

IV. COMPUTER PROGRAM FOR OBTAINING LIMITS OF  
MAXIMIZATION OR PROFIT FUNCTIONS' DECISION  
RANGES FOR A PREDETERMINED ALLOWANCE 10%  
DECREASE UNDER ITS OPTIMUM

Abouel-Nour

```

BEGIN COMPILATION
  DIMENSION C(2,10),D(2,10)
  DO 2 J=1,5
    READ(1,102) (C(I,J),D(I,J),I=1,2)
  2  WRITE (3,102) (C(I,J),D(I,J),I=1,2)
102  FORMAT(4F5.3)
    DO 3 J=1,5
  24  DO 26 I=1,2
      A=C(I,J)
      B=D(I,J)
      N1=5
      N2=50
      E1=.0005
      E2=.0005
      JX=J
      CALL RTFIND(A,B,X,Y,E1,E2,N1,N2,N,JX)
  26  WRITE(3,25) X,Y,N,J
  25  FORMAT(// 'LIMIT =',E15.8, ' Y=',E15.8, ' N=',110, 'FORMULA NO.',
115)
  3  CONTINUE
    STOP
    END

```

```

BEGIN COMPILATION
  FUNCTION F(X,JFW)
  GO TO (1,2,3,4,5),JFW
  1  F=X-2.*SQRT(X)+.9
    GO TO 10
  2  F=X*X-2.*X+.9
    GO TO 10

```

```

2   F=X*X-2.*X+.9
    GO TO 10
3   F=(X*X-4.*SQRT(X))/3.+9
    GO TO 10
4   F=(X**3-3.*X)/2.+9
    GO TO 10
5   F=(X**3-6.*SQRT(X))/5.+9
10  RETURN
    END

```

## BEGIN COMPILATION

```

SUBROUTINE RTFIND(A,B,X,Y,E1,E2,N1,N2,N,JFW)
JFW=JFW
N=0
F1=F(A,JFW)
F2=F(B,JFW)
IF(F1*F2)1,11,20
1  IF(F1)2,3,3
2  FM=F1
   XM=A
   FP=F2
   XP=B
   GO TO 4
3  FM=F2
   XM=B
   FP=F1
   XP=A
4  N=N+1
   IF(N-N1)5,5,21
5  C=FP*(XP-XM)/(FP-FM)
   X=XP-C
   Y=F(X,JFW)
   IF(ABS(C)-E1)6,6,8
6  IF(ABS(Y)-E2)7,7,8
7  RETURN
8  IF(Y)9,10,10
9  FM=Y
   XM=X
   GO TO 4
10 FP=Y
   XP=X
   GO TO 4
11 IF(F1)12,13,12
12 X=B
   Y=F2
   GO TO 7
13 X=A
   Y=F1
   GO TO 7
20 WRITE(3,300)
300 FORMAT(' NO STRADDLE ')
    X1=A
    X2=B
    GO TO 24

```

```

21  IF (FM-Y)22,23,22
22  F2=FM
    X2=XM
    GO TO 24
23  F2=FP
    X2=XP
24  N=0
    F1=Y
    X1=X
25  N=N+1
    IF (N-N2)26,26,30
26  C=((X1-X2)/(f1-F2))*F1
    X=X1-C
    Y=f(X,JFW)
    IF (ABS(C)-E1)27,27,28
27  IF (ABS(Y)-E2)7,7,28
28  F2=F1
    X2=X1
    F1=Y
    X1=X
    GO TO 25
30  WRITE(3,301)Y,X,N
301  FORMAT(' CONVERGENCE INCOMPLETE. LAST F(X)= 'F10.6,' AT X='F10.4,'
1 FOR '15,' ITERATIONS')
    GO TO 7
    END

```

LIMIT = 0.46754456E 00	Y = 0.35762787E-06	N =	1FORMULA NO.	1
LIMIT = 0.17324514E 01	Y =-0.59604645E-06	N =	1FORMULA NO.	1
LIMIT = 0.68381584E 00	Y =-0.26941299E-04	N =	1FORMULA NO.	2
LIMIT = 0.13161860E 01	Y =-0.27298927E-04	N =	1FORMULA NO.	2
LIMIT = 0.57341456E 00	Y =-0.53346157E-04	N =	1FORMULA NO.	3
LIMIT = 0.14612637E 01	Y =-0.56624413E-05	N =	1FORMULA NO.	3
LIMIT = 0.72938055E 00	Y =-0.56385994E-04	N =	1FORMULA NO.	4
LIMIT = 0.12481413E 01	Y = 0.35762787E-06	N =	3FORMULA NO.	4
LIMIT = 0.62532508E 00	Y =-0.25570393E-04	N =	1FORMULA NO.	5
LIMIT = 0.13536224E 01	Y =-0.95963478E-04	N =	1FORMULA NO.	5

APPENDIX B - TABLES  
SENSITIVITY ANALYSIS  
FOR  
MAXIMIZATION OR PROFIT FUNCTIONS



TABLE XV  
 SENSITIVITY ANALYSIS OF MAXIMIZATION OR PROFIT FUNCTIONS  
 PARABOLA VERSUS STRAIGHT LINE MODEL

$$\frac{Y_w'}{|Y_o'|} = 2\sqrt{w} - w$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	0.397214	2.10	0.798280
0.10	0.532455	2.15	0.782580
0.15	0.624597	2.20	0.766484
0.20	0.694427	2.25	0.750007
0.25	0.750000	2.30	0.733157
0.30	0.795445	2.35	0.715949
0.35	0.833215	2.40	0.698394
0.40	0.864910	2.45	0.680503
0.45	0.891640	2.50	0.662286
0.50	0.914213	2.55	0.643752
0.55	0.933240	2.60	0.624911
0.60	0.949193	2.65	0.605773
0.65	0.962451	2.70	0.586345
0.70	0.973319	2.75	0.566634
0.75	0.982051	2.80	0.546651
0.80	0.988854	2.85	0.526399
0.85	0.993908	2.90	0.505889
0.90	0.997366	2.95	0.485126
0.95	0.999359	3.00	0.464114
1.00	1.000000	3.05	0.442863
1.05	0.999389	3.10	0.421376
1.10	0.997617	3.15	0.399661
1.15	0.994760	3.20	0.377723
1.20	0.990890	3.25	0.355567
1.25	0.986068	3.30	0.333196
1.30	0.980351	3.35	0.310616
1.35	0.973790	3.40	0.287833
1.40	0.966432	3.45	0.264852
1.45	0.958320	3.50	0.241675
1.50	0.949491	3.55	0.218307
1.55	0.939981	3.60	0.194751
1.60	0.929824	3.65	0.171014
1.65	0.919047	3.70	0.147096
1.70	0.907683	3.75	0.123004
1.75	0.895753	3.80	0.098739
1.80	0.883285	3.85	0.074304
1.85	0.870297	3.90	0.049705
1.90	0.856812	3.95	0.024943
1.95	0.842851		
2.00	0.828431		
2.05	0.813569		

TABLE XVI  
 SENSITIVITY ANALYSIS OF MAXIMIZATION OR PROFIT FUNCTIONS  
 STRAIGHT LINE VERSUS PARABOLA MODEL

$$\frac{Y_w'}{|Y_o'} = 2w - w^2$$

Error w	Value $\frac{Y_w'}{ Y_o'}$	Error w	Value $\frac{Y_w'}{ Y_o'}$
0.05	0.097500	2.10	-0.209961
0.10	0.190000	2.15	-0.322457
0.15	0.277500	2.20	-0.439953
0.20	0.360000	2.25	-0.562449
0.25	0.437500	2.30	-0.689945
0.30	0.510000	2.35	-0.822441
0.35	0.577500	2.40	-0.959937
0.40	0.640000	2.45	-1.102432
0.45	0.697500	2.50	-1.249928
0.50	0.750000	2.55	-1.402423
0.55	0.797499	2.60	-1.559918
0.60	0.839999	2.65	-1.722413
0.65	0.877499	2.70	-1.889908
0.70	0.910000	2.75	-2.062403
0.75	0.937499	2.80	-2.239897
0.80	0.960000	2.85	-2.422392
0.85	0.977499	2.90	-2.609886
0.90	0.990000	2.95	-2.802380
0.95	0.997499	3.00	-2.999874
1.00	1.000000	3.05	-3.202368
1.05	0.997500	3.10	-3.409861
1.10	0.990001	3.15	-3.622355
1.15	0.977502	3.20	-3.839848
1.20	0.960002	3.25	-4.062341
1.25	0.937503	3.30	-4.289834
1.30	0.910004	3.35	-4.522326
1.35	0.877505	3.40	-4.759819
1.40	0.840006	3.45	-5.002312
1.45	0.797507	3.50	-5.249804
1.50	0.750009	3.55	-5.502296
1.55	0.697511	3.60	-5.759789
1.60	0.640013	3.65	-6.022280
1.65	0.577515	3.70	-6.289772
1.70	0.510016	3.75	-6.562263
1.75	0.437519	3.80	-6.839755
1.80	0.360022	3.85	-7.122246
1.85	0.277524	3.90	-7.409738
1.90	0.190027	3.95	-7.702229
1.95	0.097529		
2.00	0.000032		
2.05	-0.102464		

TABLE XVII  
 SENSITIVITY ANALYSIS OF MAXIMIZATION OR PROFIT FUNCTIONS  
 PARABOLA VERSUS PARABOLA MODEL

$$\frac{Y_w'}{|Y_o'|} = \frac{4}{3} w - \frac{w^2}{3}$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	0.297309	2.10	0.462199
0.10	0.418303	2.15	0.414233
0.15	0.508897	2.20	0.364338
0.20	0.582951	2.25	0.312521
0.25	0.645833	2.30	0.258789
0.30	0.700296	2.35	0.203152
0.35	0.747977	2.40	0.145617
0.40	0.789940	2.45	0.086190
0.45	0.826927	2.50	0.024880
0.50	0.859475	2.55	-0.038307
0.55	0.887993	2.60	-0.103366
0.60	0.912794	2.65	-0.170290
0.65	0.934133	2.70	-0.239073
0.70	0.952212	2.75	-0.309712
0.75	0.967200	2.80	-0.382198
0.80	0.979236	2.85	-0.456432
0.85	0.988438	2.90	-0.532703
0.90	0.994910	2.95	-0.610710
0.95	0.998738	3.00	-0.690550
1.00	0.999999	3.05	-0.772215
1.05	0.998758	3.10	-0.855703
1.10	0.995077	3.15	-0.941011
1.15	0.989006	3.20	-1.028135
1.20	0.980593	3.25	-1.117070
1.25	0.969878	3.30	-1.207814
1.30	0.956901	3.35	-1.300364
1.35	0.941695	3.40	-1.394718
1.40	0.924289	3.45	-1.490870
1.45	0.904715	3.50	-1.588819
1.50	0.882996	3.55	-1.688562
1.55	0.859156	3.60	-1.790096
1.60	0.833219	3.65	-1.893418
1.65	0.805202	3.70	-1.998528
1.70	0.775127	3.75	-2.105419
1.75	0.743008	3.80	-2.214093
1.80	0.708863	3.85	-2.324547
1.85	0.672706	3.90	-2.436777
1.90	0.634550	3.95	-2.550781
1.95	0.594409		
2.00	0.552298		
2.05	0.508224		

TABLE XVIII  
 SENSITIVITY ANALYSIS OF MAXIMIZATION OR PROFIT FUNCTIONS  
 STRAIGHT LINE VERSUS CUBICAL PARABOLA MODEL

$$\frac{Y_w^*}{|Y_o^*|} = \frac{3}{2} w - \frac{w^3}{2}$$

Error w	Value $\frac{Y_w^*}{ Y_o^* }$	Error w	Value $\frac{Y_w^*}{ Y_o^* }$
0.05	0.074937	2.10	-1.480410
0.10	0.149500	2.15	-1.744087
0.15	0.223312	2.20	-2.023888
0.20	0.296000	2.25	-2.320189
0.25	0.367187	2.30	-2.633366
0.30	0.436500	2.35	-2.963791
0.35	0.503562	2.40	-3.311840
0.40	0.568000	2.45	-3.677888
0.45	0.629437	2.50	-4.062312
0.50	0.687500	2.55	-4.465481
0.55	0.741812	2.60	-4.887774
0.60	0.792000	2.65	-5.329575
0.65	0.837687	2.70	-5.791243
0.70	0.878499	2.75	-6.273159
0.75	0.914061	2.80	-6.775705
0.80	0.944000	2.85	-7.299248
0.85	0.967937	2.90	-7.844160
0.90	0.985500	2.95	-8.410831
0.95	0.996312	3.00	-8.999620
1.00	0.999999	3.05	-9.610909
1.05	0.996188	3.10	-10.245070
1.10	0.984501	3.15	-10.902485
1.15	0.964564	3.20	-11.583521
1.20	0.936003	3.25	-12.288551
1.25	0.898442	3.30	-13.017965
1.30	0.851506	3.35	-13.772114
1.35	0.794821	3.40	-14.551394
1.40	0.728010	3.45	-15.356171
1.45	0.650701	3.50	-16.186813
1.50	0.562516	3.55	-17.043732
1.55	0.463082	3.60	-17.927261
1.60	0.352024	3.65	-18.837784
1.65	0.228966	3.70	-19.775681
1.70	0.093534	3.75	-20.741333
1.75	-0.054648	3.80	-21.735107
1.80	-0.215956	3.85	-22.757385
1.85	-0.390761	3.90	-23.808517
1.90	-0.579442	3.95	-24.888916
1.95	-0.782372		
2.00	-0.999928		
2.05	-1.232480		

TABLE XIX  
 SENSITIVITY ANALYSIS OF MAXIMIZATION OR PROFIT FUNCTIONS  
 PARABOLA VERSUS CUBICAL PARABOLA MODEL

$$\frac{Y_w'}{|Y_o'|} = \frac{6}{5}\sqrt{w} - \frac{w^3}{5}$$

Error w	Value $\frac{Y_w'}{ Y_o' }$	Error w	Value $\frac{Y_w'}{ Y_o' }$
0.05	0.268303	2.10	-0.113196
0.10	0.379273	2.15	-0.228086
0.15	0.464083	2.20	-0.349665
0.20	0.535056	2.25	-0.478072
0.25	0.596875	2.30	-0.613452
0.30	0.651867	2.35	-0.755947
0.35	0.701354	2.40	-0.905700
0.40	0.746146	2.45	-1.062854
0.45	0.786759	2.50	-1.227554
0.50	0.823528	2.55	-1.399941
0.55	0.856668	2.60	-1.580163
0.60	0.886316	2.65	-1.768366
0.65	0.912546	2.70	-1.964690
0.70	0.935391	2.75	-2.169283
0.75	0.954855	2.80	-2.382292
0.80	0.970912	2.85	-2.603860
0.85	0.983520	2.90	-2.834131
0.90	0.992620	2.95	-3.073257
0.95	0.998139	3.00	-3.321380
1.00	0.999999	3.05	-3.578646
1.05	0.998108	3.10	-3.845202
1.10	0.992370	3.15	-4.121197
1.15	0.982681	3.20	-4.406775
1.20	0.968934	3.25	-4.702080
1.25	0.951016	3.30	-5.007269
1.30	0.928813	3.35	-5.322478
1.35	0.902202	3.40	-5.647861
1.40	0.871063	3.45	-5.983561
1.45	0.835271	3.50	-6.329729
1.50	0.794700	3.55	-6.686513
1.55	0.749221	3.60	-7.054058
1.60	0.698703	3.65	-7.432508
1.65	0.643014	3.70	-7.822021
1.70	0.582022	3.75	-8.222735
1.75	0.515592	3.80	-8.634804
1.80	0.443588	3.85	-9.058372
1.85	0.365874	3.90	-9.493589
1.90	0.282310	3.95	-9.940605
1.95	0.192760		
2.00	0.097088		
2.05	-0.004851		

TABLE XX  
 LIMITS OF MAXIMIZATION OR PROFIT FUNCTIONS' DECISION RANGES  
 FOR  
 10% DECREASE IN THE TOTAL VARIABLE PROFIT ELEMENTS UNDER ITS OPTIMUM ( $Y_0$ )

No.	Type of the Model	Optimum Values $X_0, Y_0$	Decision Range for 10% Decrease	
			Lower	Upper
1.	Parabola vs. straight line $Y = B(X)^{1/2} - MX + C$	$\left( \frac{A^2}{4M^2}, C + \frac{A^2}{4M} \right)$	$0.467547X_0$	$1.732422X_0$
2.	Straight line vs. parabola $Y = MX - AX^2 + C$	$\left( \frac{M}{2A}, C + \frac{M^2}{4A} \right)$	$0.683816X_0$	$1.316189X_0$
3.	Parabola vs. parabola $Y = B(X)^{1/2} - AX^2 + C$	$\left( \left(\frac{B}{4A}\right)^{2/3}, C + \frac{3}{4} \left(\frac{B}{4A}\right)^{1/3} \right)$	$0.573415X_0$	$1.461205X_0$
4.	Straight line vs. cubical parabola $Y = MX - AX^3 + C$	$\left( \left(\frac{M}{3A}\right)^{1/2}, C + \frac{2}{3} \left(\frac{M}{3A}\right)^{3/2} \right)$	$0.729380X_0$	$1.248141X_0$
5.	Parabola vs. cubical parabola $Y = B(X)^{1/2} - AX^3 + C$	$\left( \left(\frac{B}{6A}\right)^{2/5}, C + \frac{5}{6} \left(\frac{B}{6A}\right)^{1/5} \right)$	$0.625325X_0$	$1.353622X_0$

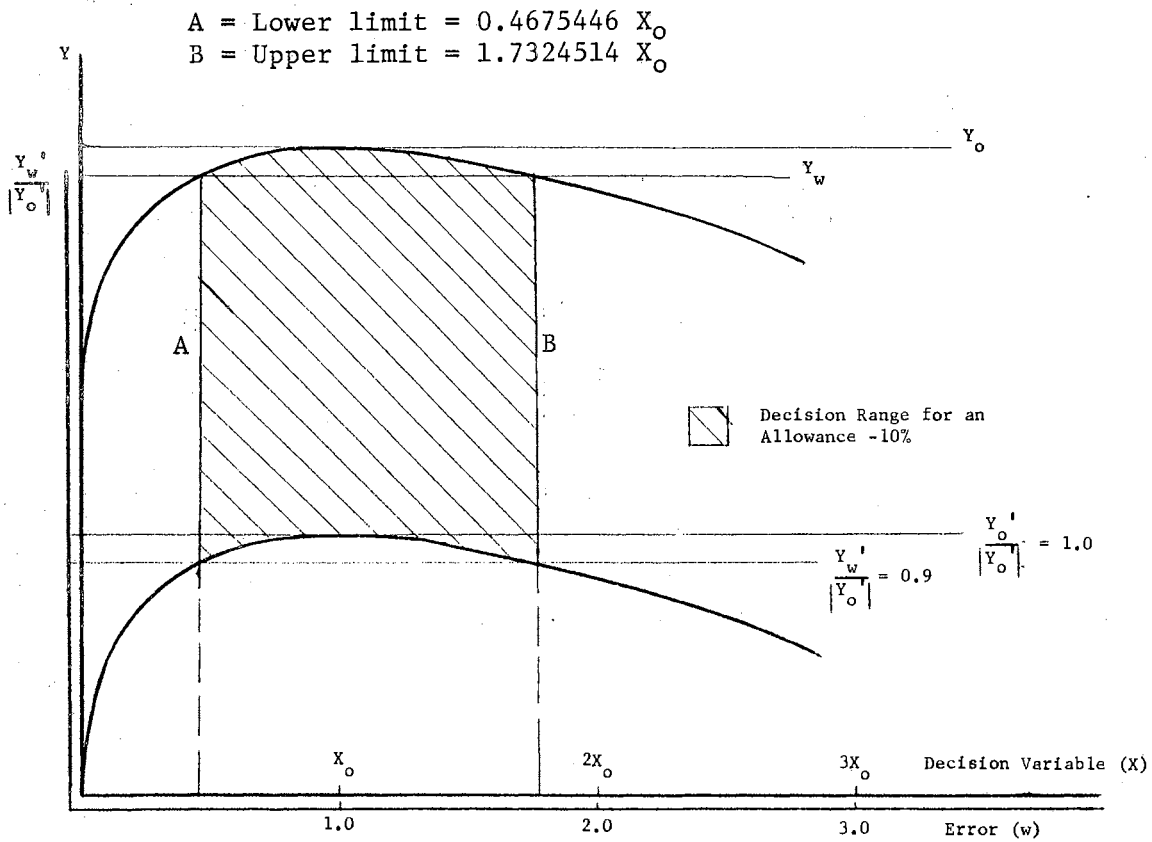


Figure (28). Profit (Maximization) Function - Parabola vs Straight Line Model

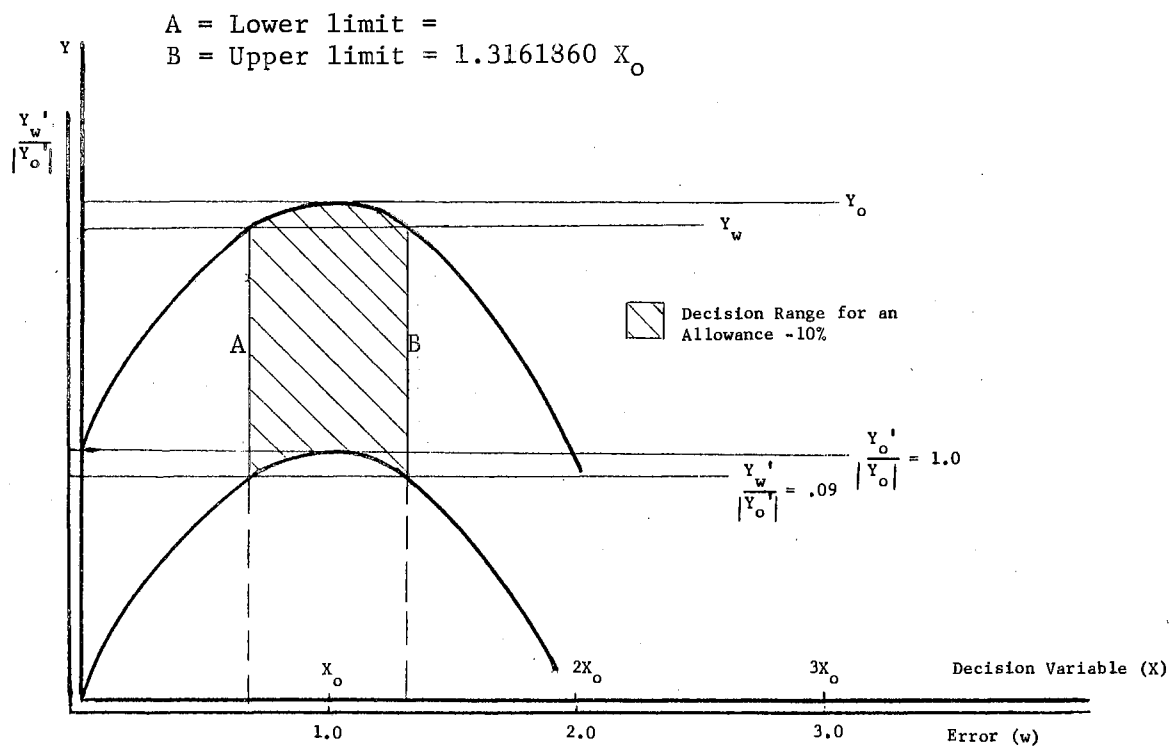


Figure (29). Profit (Maximization) Function - Straight Line vs Parabola Model



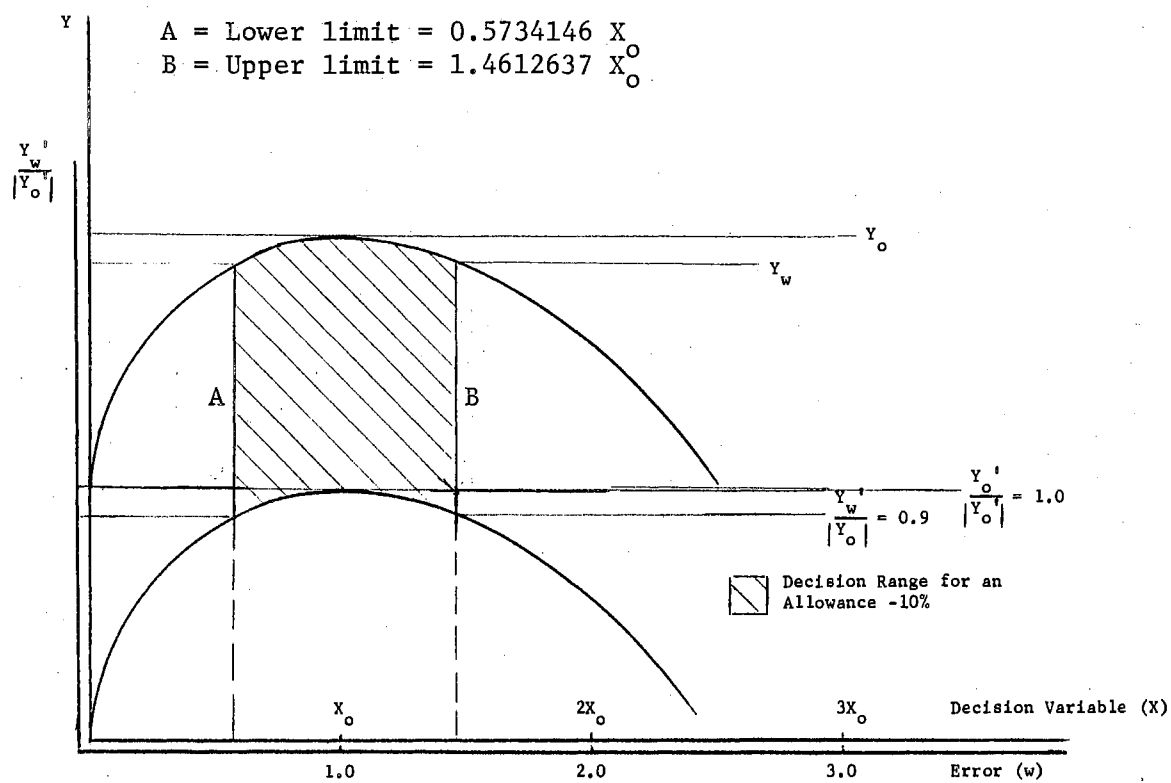


Figure (30). Profit (Maximization) Function - Parabola vs Parabola Model

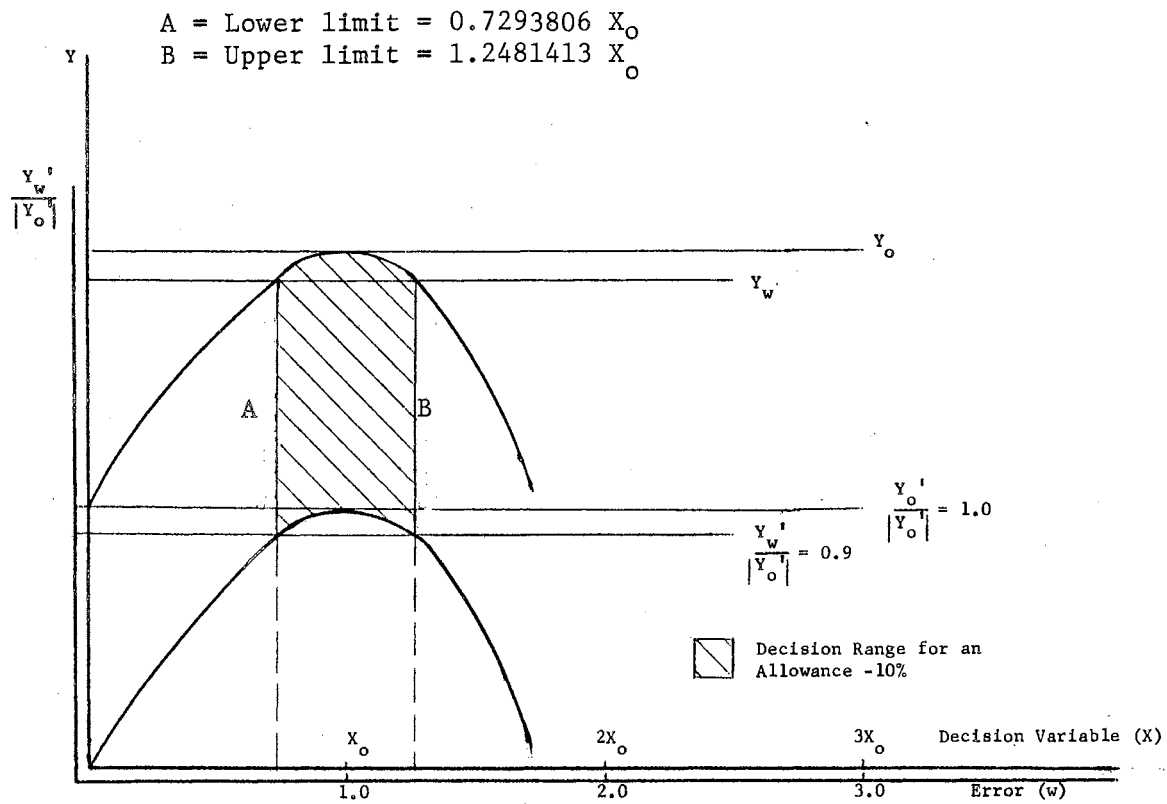


Figure (31). Profit (Maximization) Function.- Straight Line vs Cubical Parabola Model

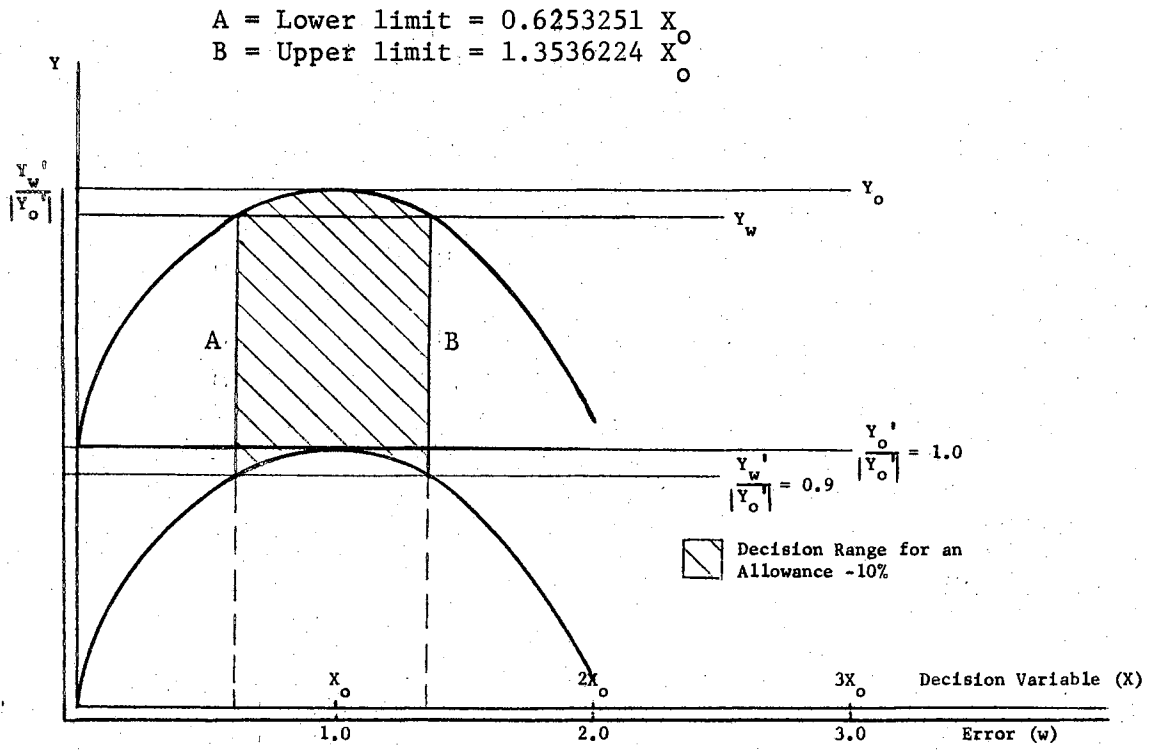


Figure (32). Profit (Maximization) Function -Parabola vs Cubical Parabola Model

VITA

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Doctor of Philosophy

Thesis: SENSITIVITY ANALYSIS OF THE DECISION MODELS

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Professional Experience: Industrial Engineer expert in the Vocational Training and Productivity Department; Ministry of industry from July 1956 to July 1960. During these four years, contributed with International Labor Office Mission - UN - for studying and executing industrial projects in different Egyptian industrial firms for the purpose of raising the firms' productivities. Head of Material Control Department - Egyptian Iron and Steel Company from July 1960 to September 1964. Part time as an instructor in Petroleum College in Sues UAR. Assistant Professor in Iowa University, Iowa, January 1966.

Professional Memberships: Member in Egyptian Production Engineering Society; member in American Industrial Engineering Society; full member in Operations Research Society of America.

Conferences Attended and Participation:

A. In UAR Organized by Ministry of Industry and International Labor Office Un.

<u>Year</u>	<u>Hours</u>	<u>Course Work</u>
1957	100	Ind. Human Relations
1957	100	Work Study
1959	48	Ind. Quality Control
1959	40	Ind. Production Planning and Control
1960	60	Raising Productivity in Industrial Firms

B. In USA, Iowa State University.

<u>Year</u>	<u>Days or Weeks</u>	<u>Course Work</u>
1967	(5 weeks)	Capital Budgeting & Eng. Economy
1967	(4 days)	Utility and Interest Rates
1967	(2 days)	Industrial Quality Control