NON-RELATIVISTIC QUARK MODEL

FOR MESONS

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PREFACE

In this thesis, the non-relativistic quark model has been applied to mesons. A mass formula is developed and fitted into the experimentally confirmed mesons. The mass formula is found to be accurate in predicting the masses with errors of the orders of a few per cent. A complete table of mesons has been prepared with the help of this mass formula.

I would like to thank Dr. M. A. Samuel for his suggestion of the problem and his patient guidance during the course of this work. I would like to thank the Department of Physics, Oklahoma State University, for the financial help in the form of a teaching assistantship and Miss Bernadette O'Farrell for typing the manuscript.

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CHAPTER I

INTRODUCTION

A. Historical Background

The type of model for the strongly interacting "elementary particles" or hadrons to be discussed has a long history, beginning with the model discussed by Fermi and Yang (1) in which the pion is considered as a bound state of the nucleon-antinucleon system. These bound state models have never been considered fully respectable, perhaps not even today. Indeed, it is not really possible to meet all the objections to such models. It was realized by Fermi and Yang that, given the nucleons, it was unnecessary to consider π meson to be an independent particle, since a state having all the quantum numbers of the pion could be built up from nucleons and antinucleons. For a theory of the observed "elementary particles" in terms of a more primary object, it is clear that this should be chosen to be a fermion, the simplest possibility being that of spin $\frac{1}{2}$ for reasons of economy. Bosons can then be constructed from bound states of the particle and its antiparticle; some primary fermion object is necessary in order to allow the construction of states corresponding to the observed fermionic hadrons. At least two primary objects are needed, with differing charge values, in order to allow the possibility of constructing states of different charge values Q, for given baryon no. B. If the interactions between the primary objects are assumed charge-independent, then all the states formed from these objects can be classified into I-spin multiplets.

After the discovery of mesons and baryons with non-zero strangeness, it was pointed out by Sakata (2) that the model of Fermi and Yang could readily be extended to take into account the additional additive quantum no. of strangeness S (or of hypercharge Y defined by Y = S + B), simply by adding the Λ hyperon to the set of primary objects, giving rise to the primary triplet of Sakatans, (p,n,Λ)

Now, the charge independence long known for non-strange hadrons corresponds to the hypothesis that their interaction energy is invariant with respect to any unitary transformation between the states of the nucleon doublet (P,N) i.e. that the interactions are invariant with respect to the SU(2) group of isospin, whose properties are exactly parallel to those for the SU(2) group well known in connection with the Pauli spin theory. We shall represent these basis isospin states by the column matrix ξ, with $\xi_1 = p$ and $\xi_2 = n$, which have the same isospin transformations as P and N but need not be identical to them. Like P and N, they form a two-dimensional covariant isospinor

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{1}$$

which, under the transformations U of the SU(2) group, transforms as

$$\xi \rightarrow \xi' = U\xi \tag{2}$$

in which U is a 2 x 2 unitary matrix satisfying det U = 1. Any isospin rotation can be completely characterized by its effect on ξ as described by (2). The doublet (ξ_1, ξ_2) with isospin $I = \frac{1}{2}$ forms the basis for the fundamental representation of the isospin group SU(2)

We also now define contravariant spinors

$$\eta = (\eta^1, \eta^2) \tag{3}$$

which under the U transformations, transform in such a way that $\eta\xi = \eta^a \xi_a$ is invariant; (summation of repeated indices is understood throughout). η describes the transformation properties of the doublet of antiparticles $\bar{p}$ and $\bar{n}$. Higher isospin multiplets can be constructed by forming direct products of the spinors ξ or η or both. If we consider a system composed of a particle and an antiparticle, we obtain four states that can be written

$$M^i_k = \eta^i \xi_k \tag{4}$$

Then tensor M^i_k has mixed properties under isospin transformations; i.e. it does not correspond to an irreducible representation of SU(2). However, by judiciously taking linear combinations of the above states we can construct two sets of orthonormal states such that, under the action of SU(2), the states within each set transform among each other and as such, form the basis of an irreducible representation, i.e. a multiplet. Evidently one of these sets consists of the invariant or isoscalar $\eta^i \xi_i$ the remaining states form a triplet. The two sets in question are

$$\frac{1}{\sqrt{2}}(\eta^1 \xi_1 + \eta^2 \xi_2) = \frac{1}{\sqrt{2}}(\bar{p}p + \bar{n}n), \text{ singlet } I=0 \tag{5-a}$$

$$\left.\begin{aligned} \eta^1 \xi_2 &= \bar{p}n \\ \eta^2 \xi_1 &= p\bar{n} \\ \frac{1}{\sqrt{2}}(\eta^1 \xi_1 - \eta^2 \xi_2) &= \frac{1}{\sqrt{2}}(\bar{p}p - \bar{n}n) \end{aligned}\right\} \text{triplet } I=1 \tag{5-b}$$

showing that the direct product of the two isospin doublets breaks down into an isospin singlet and an isospin triplet. We can write this symbolically as

$$2 \times \bar{2} = 1 + 3 \tag{6}$$

With n and p carrying zero strangeness we can represent the triplet of pions by the triplet (5-b). This fact can mean two things. Either the fundamental objects $p, n, \bar{p}, \bar{n}$, are mathematical objects; thus identification of the pion triplet with (5-b) means only that the pion has the same isospin transformation properties as the combinations given by Eq. (5-b), or the objects $p, n, \bar{p}, \bar{n}$, are physical particles, hence the pion must be regarded as the bound state of these particles.

Similarly the η meson can be represented in this model by the singlet. In this way we can construct all nonstrange hadrons from our building blocks p,n, and their antiparticles. The assumption of invariance of the mechanics of the system under isospin transformation ensures that these hadrons fall into isospin multiplets, each of which is characterized by the value of the isospin I. If the symmetry is perfect, each multiplet is degenerate in mass. Electromagnetic forces, which break isospin symmetry, cause small mass splittings within the multiplets. Once one member of a given multiplet is found, all the other members of the multiplet must also exist.

It is clear that with this procedure we will never be able to construct the strange particles. For that purpose we must have at least one more fundamental object with nonzero strangeness. This requirement leads to SU(3).

B. The Quark Model

The hypothesis that this unitary symmetry for the interactions should be extended to SU(3) symmetry for the three-dimensional space of the Sakaton S = (p,n,Λ) was made by the Sakata school (3) by Yamaguchi (4) and Wess (5). Since the Λ state is observed to have mass about 176 MeV greater than that for the (n,p) states, this SU(3) symmetry cannot be satisfied to such accuracy as is observed for the SU(2) symmetry of isospin; there must exist interactions of nuclear strength which break this SU(3) symmetry. A particularly appealing model was the vecton model of Fujii, (6) discussed also by Kobzarev and Okun (7) and by Gell-Mann (8) in which the interaction arises from the coupling of a neutral vector field (the vector V_μ) with the baryon current

$$J_\mu^\beta = \left\{ \bar{p}\Gamma_\mu p + \bar{n}\Gamma_\mu n + \bar{\Lambda}\Gamma_\mu \Lambda \right\} \tag{7}$$

In this model, the vecton appears as a gauge field for baryon number and the invariance of the interaction $\lambda J_\mu V_\mu$ with respect to the SU(3) transformation appear as a consequence of baryon conservation.

In the SU(3) scheme, the states are labelled by the suffix α, thus u_α with $\alpha = 1, 2, 3$, the 3 - axis being associated with hypercharge. So, the only difference between SU(2) and SU(3) is that in SU(3) our basic state is a three-component spinor

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} p \\ n \\ \lambda \end{pmatrix} \tag{8}$$

Under the transformations of SU(3) this spinor transforms as

$$\xi \rightarrow \xi' = U\xi \qquad \text{(9a)}$$

where U is a 3 x 3 unitary matrix with det U = 1. The contravariant spinor describing the antiparticles are given by

$$\eta = \left(\eta^1\ \eta^2\ \eta^3\right) \equiv \left(\bar{p}\ \bar{n}\ \bar{\lambda}\right) \qquad \text{(9b)}$$

It transforms such that $\eta\xi$ is invariant. The triplets (p,n, λ) and ($\bar{p}$,$\bar{n}$, $\bar{\lambda}$) form the bases for the two fundamental representation of SU(3) These are denoted by $\{3\}$ and $\{\bar{3}\}$ respectively. The particles p,n, λ are called quarks and the antiparticles $\bar{p}$,$\bar{n}$,$\bar{\lambda}$ antiquarks, the names used by Gell-Mann (9). The consequences of quark model has been vigorously investigated by Zweig (10). The p and n quarks form an isodoublet (I = ½) of strangeness S = 0. ; The λ quark is an isoscalar (I = 0) to which we assign strangeness S =-1. An octet state can be formed from triplet quarks only from baryon no. B = 3nb where n is an integer and b is the quark baryon no. Hence it is necessary to assume a fractional value for b and the simplest possibility is b = 1/3, so that the observed baryon states are then composite states consisting of three quarks. Hence the hypercharge Y, defined by

$$Y = S + B \qquad \text{(9c)}$$

is + 1/3 for p and n, and -2/3 for λ . The Gell-Mann-Nishijima relation

$$Q = I_z + \tfrac{1}{2} Y \qquad \text{(9d)}$$

in which Q is the charge, then gives for the charges e_q of the quarks p,n, λ the fractional values 2/3 e,-1/3 e, -1/3e, respectively. Here e is the charge of the proton. We have collected the quantum numbers of the quarks in Table I :

TABLE I

QUANTUM NUMBERS OF THE QUARKS

	B	I	I_z	Y	S	e_q/e
p	1/3	1/2	1/2	1/3	0	2/3
n	1/3	1/2	-1/2	-1/2	0	-1/3
λ	1/3	0	0	-2/3	-1	-1/3

For the antiquarks the quantum numbers I_z, S, B, Y, and e_q are the opposites of those of the corresponding quarks. We can represent the basic triplets of SU(3) graphically as in Figure 1:

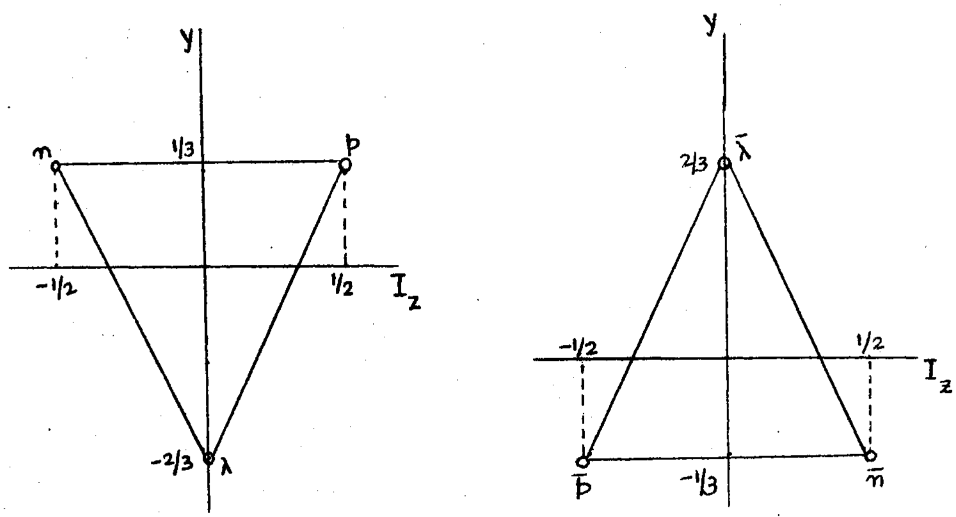

Figure 1. The Triplets of Quarks and Antiquarks

With these quantum nos. we conclude

(1) the quarks cannot decay completely into the observed particle states, since this would violate baryon conservation and charge conservation, both conservation laws being known to hold to an exceedingly

high accuracy. (11)

(2) The quark states can decay weakly into each other, following the rules known for weak interaction process. For example if q_3 is the heaviest quark then the weak decay processes

$$q_3 \rightarrow q_{1,2} + \pi \tag{10a}$$

$$\rightarrow q_2 + \gamma \tag{10b}$$

$$\rightarrow q_1 + e^- + \bar{\nu} \tag{10c}$$

are possible, at rates which depend on the mass differences. According as q_2 is heavier (or lighter) than q_1, then the beta decay processes

$$q_2 \rightarrow q_1 + e^- + \bar{\nu} \tag{11}$$

can occur, provided the mass difference is greater than m_e. In all cases, however the lightest quark state is necessarily stable; there are no decay processes consistent with the conservation laws.

Now, each hadron is supposed to be bound state of quarks or anti-quarks or both due to some strongly attractive force whose nature is unknown. SU(3) invariance means that the three quarks making up the triplet representation of SU(3) have the same mass and that the forces between them do not change under SU(3) transformation. This fact ensures the existence of SU(3) multiplets consisting of nq m$\bar{q}$ states (n,m = 0,1,2,....). With perfect symmetry the states within each multiplet are degenerate in mass. If the symmetry is broken, the degeneracy is lifted. Hence from the quark picture we arrive in a natural way at the classification of mesons, baryons and their resonances into certain SU(3) multiplets. In the simplest scheme, in which mesons are $q\bar{q}$ states and baryons qqq states, only singlets, octets and decuplets are allowed. Experimental verification of this ordering of hadrons into SU(3) multiplets has been one of the most striking discoveries in particle physics in recent years. The observed multiplets are only approximately degenerate, thus showing that SU(3) is only an approximate symmetry.

The major problem about the quark hypothesis is the fact that no quark particle has yet been observed in nature. It must certainly be possible to produce $q\bar{q}$ pairs in high-energy nuclear collisions, although it is not easy to give a reliable estimate of the production cross-section to be expected. It's necessary to conclude that they must be very massive particles, so that their production rate in cosmic rays would be correspondingly low and their accumulated intensity in terrestrial matter sufficiently low that they would be sufficiently difficult to detect.

A number of accelerator experiments have been carried out to search for quark production in 30 GeV proton-nucleus collisions. Blum (12) searched for particles of charge e/3 or 2e/3 by examining particle tracks with subnormal bubble density in a hydrogen chamber exposes to a particle beam from the CERN accelerator. They concluded that if $M_q \leqslant 4$GeV then the quark production cross-section is not greater than $10^{-32}cm^2$ in nucleon - nucleon collisons at 27.5 GeV/c. Leipuner (13) made a counter search sensitive to particles of charge e/3 and concluded that, if $M \lesssim 2$ GeV, the production cross-section is not greater than $10^{-32}cm^2$ for 28 GeV protons. The most extensive accelerator search has been that recently reported by Lederman (14) which was sensitive to particles of charge $\geq$ 2e/3 and which could be interpreted more quantitatively as a result of their prior investigations of the effectiveness of the high momentum components of the nucleons within complex nuclei for the production of antiprotons. Estimating the quark pair production cross-section for the process

$$p + N \rightarrow p + N + q + \bar{q} \tag{12}$$

from the known cross-section for the corresponding proton - antiproton pair production process, with corrections for the phase space and with a factor $(M_p/M_q)^2$ to represent the charge in the intermediate propagator in this process, the observed upper limit cross section of 3 x 10^{-36} cm^2 sr^{-1} $(GeV/c)^{-1}$ corresponds to a lower limit of 4.5 GeV for the quark mass.

Cosmic ray experiments allow the possibility of exploring higher mass values. A recent experiment by Bowen (15) was sensitive to the low charge values $\pm$ e/3. The interpretation of their observations depend both on the production cross section assumed and on the quark interaction cross-section; for example, if the production cross-section is assumed to be 10^{-30} cm^2 for all energies above the threshold and σ_{qN} to be 15 mb, then the observations are consistent only with $M_q \geq$ 3 GeV.

McCusker and Cairns (16) claimed to have observed fractionally charged quarks in cloud-chamber photographs of the cores of very energetic cosmic ray showers while Chu (17) claimed to have observed a fractionally changed quark in a bubble-chamber photograph of energetic cosmic ray tracks. However, both of these experiments have alternative explanation which do not require fractionally changed quarks so that many physicists are not ready to accept the experiments of Cairns and McCusker and Chu until additional experimental work is performed to check these findings (18). Most physicists are now very sceptical about these claims.

More complicated triplet schemes have been put forward, with the purpose of allowing integral values of B and Q for the triplet states. We shall not discuss these more elaborate triplet models in detail, because there is a great deal of flexibility in their use and in their comparison with the properties of the observed particle states. The simple quark model of Gell-Mann and Zweig provides a very much less flexible framework for the interpretation of "elementary particle" properties and it is of particular interest to follow the development of this model until such time as it may prove inadequate to account for the observed phenomena.

CHAPTER II

QUARK MODEL FOR MESONS

A. Higher Multiplets in the Quark Model

We can obtain higher representations of SU(3) by forming direct products of the basic spinors ξ and η. Consider the states for a $q\bar{q}$ pair:

$$M^i_K = \eta^i \xi_K \tag{1}$$

There are nine of them that have mixed properties under SU(3) transformation. The combination

$$\frac{1}{\sqrt{3}} \eta^i \xi_i = \frac{1}{\sqrt{3}} \left(\bar{p}p + \bar{n}n + \bar{\lambda}\lambda \right) \tag{2}$$

is invariant under any U transformation and as such, forms the basis for a one-dimensional representation. This is a unitary singlet. The remaining eight states transform among each other and span the bases for an eight-dimensional representation. We call it an octet and so,

$$\{3\} \times \{\bar{3}\} = \{1\} + \{8\} \tag{3}$$

The two central states of the octet those with $I_z = 0$ are linear combinations of $p\bar{p}$, $n\bar{n}$, $\lambda\bar{\lambda}$. One of them forms an isotriplet with $\bar{p}n$ and $\bar{n}p$ and is

$$x = \frac{1}{\sqrt{2}} \left(\bar{p}p - \bar{n}n \right) \tag{4}$$

The remaining state y is an isosinglet and is given by

$$y = \frac{1}{\sqrt{6}} \left(\bar{p}p + \bar{n}n - 2\bar{\lambda}\lambda \right) \tag{5}$$

TABLE II

QUANTUM NOS. OF $q\bar{q}$ PAIR

	B	I	I_z	Y	S	e_q/e
$p\bar{p}$	0	1,0	1	0	0	0
$p\bar{n}$	0	1,0	1	0	0	1
$p\bar{\lambda}$	0	½	½	1	1	1
$n\bar{n}$	0	1,0	0	0	0	0
$n\bar{p}$	0	1,0	-1	0	0	-1
$n\bar{\lambda}$	0	½	-½	1	1	0
$\lambda\bar{\lambda}$	0	0	0	0	0	0
$\lambda\bar{n}$	0	½	½	-1	-1	0
$\lambda\bar{p}$	0	½	-½	-1	-1	-1/3

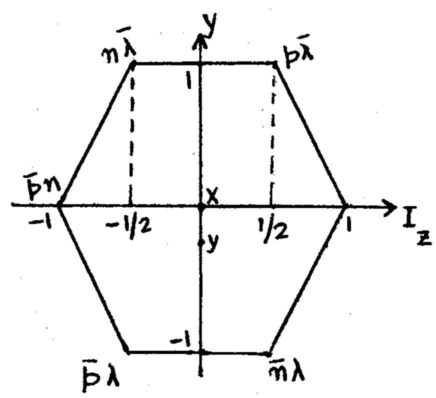

Figure 2. Octet of $q\bar{q}$ States

The basic states for two quark triplets are (19)

$$\xi_i \, \xi_k \quad (i, k = 1, 2, 3) \tag{6}$$

These nine states have mixed SU(3) transformation properties. We have six symmetric states:

$$\left.\begin{array}{ccc} pp & nn & \lambda\lambda \\ \frac{1}{\sqrt{2}}(pn + np) & \frac{1}{\sqrt{2}}(p\lambda + \lambda p) & \frac{1}{\sqrt{2}}(n\lambda + \lambda n) \end{array}\right\}\{6\} \tag{7}$$

and three anti symmetric states

$$\left.\begin{array}{l}\frac{1}{\sqrt{2}}(pn - np)\\ \frac{1}{\sqrt{2}}(p\lambda - \lambda p)\\ \frac{1}{\sqrt{2}}(n\lambda - \lambda n)\end{array}\right\}\{\bar{3}\} \quad \therefore \{3\}\times\{\bar{3}\} = \{\bar{3}\}+\{6\} \tag{8}$$

B. Pseudoscalar and Vector Meson States

In this model, the meson states are considered to be bound states of a $q\bar{q}$ pair, due to some strongly attractive interaction between them. This interaction could arise from the exchange of vector mesons between them, for example; A particular attractive possibility is provided by the vector model of Fujii (20).

This model allows only states which belong to $\{1\}$ or $\{8\}$ representations. The formation of meson states belonging to the $\{27\}$ representation requires the consideration of more complicated excitations, such as the structure $\bar{q}\bar{q}qq$ and we interpret the absence of evidence for the existence of $\{27\}$ states to the higher excitation energies needed for these more complicated structures. For mesons, a particle and it's antiparticle are always in the same SU(3) multiplet. Now since quark and antiquark have opposite intrinsic parity, the parity P of the $q\bar{q}$ state is given by

$$P = (-)^{L+1} \tag{9}$$

and charge conjugation quantum numbers C for the neutral states is

$$C = (-)^{L+S} \tag{10}$$

where S is the total intrinsic spin, which is 0 or 1 according to whether the quark spins are parallel or antiparallel. This implies that $J^{PC}=0^{--}$, $(odd)^{-+}$, $(even)^{+-}$ are excluded in quark model. The lowest $q\bar{q}$ states are the L = 0 states. Depending on S, there are two sets of nine S states having the following quantum nos.

$$\begin{array}{llll} \text{(a)} & S = 0, & P = -1, & C = +1 \\ \text{(b)} & S = 1, & P = -1, & C = -1 \end{array} \qquad (11)$$

each of which falls into an SU(3) singlet and an SU(3) octet. Sets (a) and (b) may be identified with the two nonets of observed pseudoscalar and vector mesons respectively.

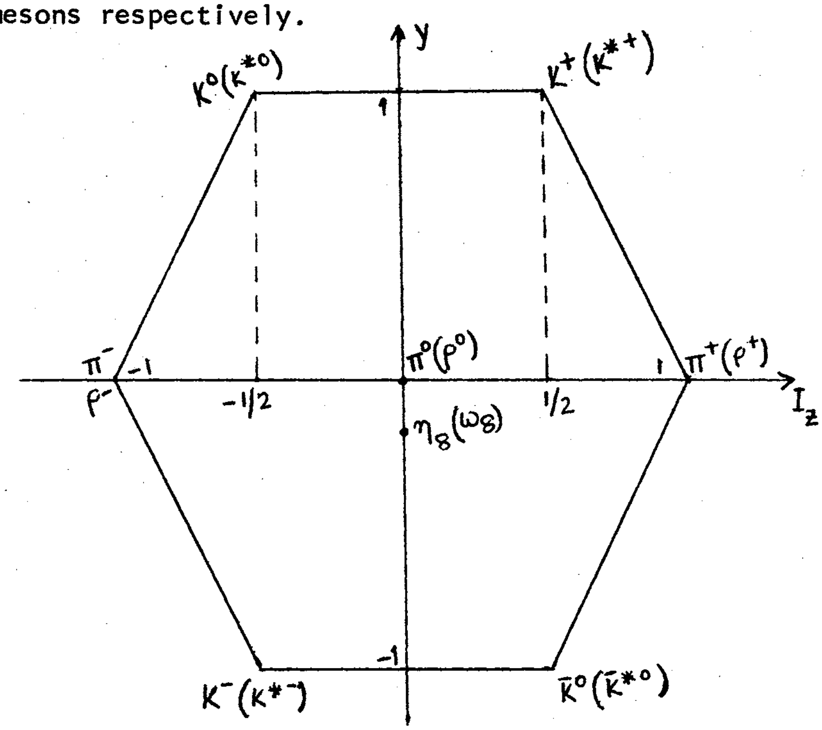

Figure 3. Octet of Pseudoscalar and Vector Mesons

The wavefunctions of the substates for these L = 0 unitary multiplets may be written

$$\psi(\{\alpha\}, S; Y, I, I_3) = \phi(\{\alpha\}, S; \underline{r})\, \chi_S\, g(\{\alpha\}; Y, I, I_3) \qquad (12)$$

where

$\phi \longrightarrow$ radial wavefunction

$\chi \longrightarrow$ spin wavefunction

$g \longrightarrow$ unitary - spin wavefunction (13)

Octet States:

$$I = 1 \begin{cases} g(\rho^+) = g(\pi^+) = \bar{q}_2 q_1 \\ g(\rho^0) = g(\pi^0) = (\bar{q}_1 q_2 - \bar{q}_2 q_1)/\sqrt{2} \\ g(\rho^-) = g(\pi^-) = \bar{q}_1 q_2 \end{cases} \qquad (14)$$

$$I = \frac{1}{2} \begin{cases} g(K^{*+}) = g(K^{+}) = \bar{q}_3 q_1 \ ; \ g(\bar{K}^{*-}) = g(K^{-}) = \bar{q}_1 q_3 \\ g(K^{*0}) = g(K^{0}) = \bar{q}_3 q_2 \ ; \ g(\bar{K}^{*0}) = g(\bar{K}^{0}) = \bar{q}_2 q_3 \end{cases}$$

$$I = 0: \quad g(\phi_8) = g(\eta_8) = (\bar{q}_1 q_1 + \bar{q}_2 q_2 - 2\bar{q}_3 q_3)/\sqrt{6} \tag{15}$$

Singlet State:

$$g(\omega) = g\ (X) = (\bar{q}_1 q_1 + \bar{q}_2 q_2 + \bar{q}_3 q_3)/\sqrt{3} \tag{16}$$

The interaction energy in these states must be very large. The masses of the observed particles are quite low, relative to $q\bar{q}$ total mass $2M_q$, so that the $q\bar{q}$ binding energy must be very large.

We shall generally use non-relativistic concepts. Morpurgo (21) pointed out this is not unreasonable. The range of the $q\bar{q}$ force is likely to be of the order $R \approx \hbar/m_v c$. So in the $\bar{q}$-q wavefunction typical quark momenta will be $\hbar/R \approx m_v c$ to be compared with the quark mass energy $M_q c^2 \gtrsim 5$ GeV. So the quark vels. in these states are therefore

$$V/C \sim \hbar/(RM_q c) \sim m_v/M_q \lesssim 1/5 \tag{17}$$

So non-relativistic concepts are quite appropriate

Vector Meson States:

With exact unitary symmetry, there will be two mass values for the vector mesons, m_8 for the octet states and m_1, for the singlet state. In general, these mass values will differ, since the $\bar{q}$-q potential U may be expected to depend on the unitary representation $\{\alpha\}$ to which the state belongs.

The vector mesons observed show appreciable mass splittings between the various isospin multiplets. For example $m(\rho) = 765$ MeV whereas $m(k^*) = 892$ MeV. The simplest hypothesis about these SU(3) breaking interactions is that the mass splittings are simply due to a mass difference between quark q_3 and the quarks q_1, q_2 with $m_1 = m_2 = m$ required by isospin conservation and

$$m_3 = m + \Delta \qquad (18)$$

whereas the mass ρ is given by m_8, this additional quark mass leads to

$$K^* = m_8 + \Delta \qquad (19)$$

So, to a first approximation

$$\Delta = K^* - \rho = 127 \text{ MeV} \qquad (20)$$

The expectation values of the mass for the states ϕ_8 and ω_1 are obtained using the unitary spin wavefunction, with the results

$$\phi_8 = m_8 + 4\ /3, \quad \omega_1 = m_1 + 2\ /3 \qquad (21)$$

With this symmetry - breaking term, the mass operator has a matrix element linking the ϕ_8 and ω_1 states, given by

$$(\phi_8/m/\omega_1) = (-2\sqrt{2}/3)\ I\Delta \qquad (22)$$

where I denotes the overlap integral between the radial wavefunctions appropriate to the octet and singlet potentials.

A case of special interest is that in which the $\bar{q}$-q potential does not depend on the quark labels, thus

$$(\bar{q}_\alpha q_\beta /U/\bar{q}_\gamma q_\delta) = U\,\delta_{\alpha\gamma}\,\delta_{\beta\delta} \qquad (23)$$

This property holds automatically for the potential resulting from the exchange of a vecton coupled with the baryon current. With this property, the potentials U ($\{8\}$) are U ($\{1\}$) are identical and we have

$$m_8 = m_1 \qquad (24)$$

The I = Y = 0 eigenstates of the energy are not the ϕ_8 and ω_1 states, but are given by the states $(\bar{q}_1 q_1 + \bar{q}_2 q_2)/\sqrt{2}$ and $\bar{q}_3 q_3$, corresponding to mass values m_8 and $m_8 + 2\Delta$ respectively. These states are naturally to be identified with the observed ω and ϕ states, so that

$$g(\omega) = (\bar{q}_1 q_1 + \bar{q}_2 q_2)/\sqrt{2} = \cos\theta_v\, g(\omega_1) + \sin\theta_v\, g(\phi_8)$$
$$g(\phi) = -\bar{q}_3 q_3 = \sin\theta_v\, g(\omega_1) + \cos\theta_v\, g(\phi_8) \tag{25}$$

where the mixing angle θ_v is given by $\cos\theta_v = \sqrt{2/3}$, $\sin\theta_v = \sqrt{1/3}$. So we have the mass predictions

$$\omega = \rho$$
$$\omega + \phi = 2k^* \tag{26}$$

and leads to the further estimate

$\Delta = (\phi - \omega)/2 = 118$ MeV, very close to the estimate obtained above from $(K^* - \rho)$.

More generally, we consider the I=Y=O states for the case $m_8 \neq m_1$ The mass operator has this form

$$\begin{pmatrix} m_8 + 4\Delta/3 & -(2\sqrt{2}/3)I\Delta \\ -(2\sqrt{2}/3)I\Delta & m_1 + 2\Delta/3 \end{pmatrix} \tag{27}$$

and has the eigenvalues ω and ϕ. Hence

$$\omega + \phi = m_1 + m_8 + 2\Delta$$
$$\omega\phi = (m_8 + 4\Delta/3)(m_1 + 2\Delta/3) - 8I^2\Delta^2/9 \tag{28}$$

With $\rho = m_8$ and $K^* = m_8 + \Delta$, We can eliminate m_1, m_8 and from these equations to give the inequality (22)

$$\left\{(\omega-\rho)(\phi-\rho) - \frac{4}{3}(K^*-\rho)(\omega+\phi-2K^*)\right\} = \frac{8}{9} *$$
$$(K^*-\rho)^2(1-I^2) \geqslant 0 \tag{29}$$

Assuming I = 1,

$$(\omega-\rho)(\phi-\rho) = \frac{4}{3}(K^*-\rho)(\phi+\omega-2K^*) \tag{30}$$

At this point, we shall go over to the conventional use of the $(\text{mass})^2$ operator for bosons. This appears rather appropriate since the boson mass appears only in the combination $(\text{mass})^2$ in the energy operator so that the mass splitting perturbations calculated are contributions directly to $(\text{mass})^2$. Insofar as perturbation theory is valid for the mass splitting effects, it should be equally valid to use perturbation theory for (mass) or $(\text{mass})^2$ and in fact, for the vector mesons it generally makes little difference whether (mass) or $(\text{mass})^2$ is used. However, there are very good reasons to prefer the use of the $(\text{mass})^2$ operator in the case of the pseudoscalor mesons and so for consistency, we shall use the $(\text{mass})^2$ operator for the vector mesons. We have

$$\rho^2 = m_8{}^2$$

$K^{*2} = m_8{}^2 + \delta$ and for the $(\text{mass})^2$ matrix

$$\begin{pmatrix} m_8{}^2 + 4\delta/3 & -(2\sqrt{2}/3)\delta \\ -(2\sqrt{2}/3)\delta & m_1{}^2 + 2\delta/3 \end{pmatrix} \tag{31}$$

where the correction δ is proportional to the quark mass difference (Δ) . With the first approximation $m_8 = m_1$

$$\begin{aligned} \delta &= K^{*2} - \rho^2 = 2.025 \times 10^5 \ (\text{MeV})^2 \\ 2\delta &= \phi^2 - \omega^2 = 4.27 \times 10^5 \ (\text{MeV})^2 \end{aligned} \tag{32}$$

in good agreement with each other, confirming that $m_1 \approx m_8$. Writing $\delta = 2m_8\Delta$, we have $\Delta \sim 135$ MeV. Allowing $m_8 \neq m_1$, we have Schwinger's relation

$$(\omega^2 - \rho^2)\ (\phi^2 - \rho^2) = \frac{4}{3}\ (K^* - \rho^2)\ (\omega^2 + \phi^2 - 2K^{*2}) \tag{33}$$

This requires $(\omega - \rho) = 25.0$ MeV, Somewhat larger than the present value of 19 MeV.

CHAPTER III

A. Pseudoscalar Mesons

We now return to our discussion of pseudoscalar mesons. Their main properties with their decay modes and quantum numbers are shown below.

TABLE III

PSEUDOSCALAR MESONS

Particle	Mass (MeV)	J^P	I^G	Main Decay Mode	C	Y	σ	L
$\pi^\pm$	139.6	0^-	1^-	$\mu\nu$	+	0	0	0
π°	134.97	0^-	1^-	$\gamma\gamma$	+	0	0	0
η	548.8 $\pm$0.6	0^-	0^+	$\gamma\gamma$	+	0	0	0
X^o	957.7 $\pm$0.8	0^-	0^+	$\eta\pi\pi$	+	0	0	0
$K^\pm$	493.8	0^-	0^-	$\mu\nu$	+	$+^-$	0	0
$K^o,\bar{K}^o$	498.8	0^-	0^-	$\mu\nu$	+	+1,-1	0	0

Looking at the table we find that the mass values for pseudoscalar mesons appear widely separated. Of the I = Y = 0 states, the η meson at 549 MeV lies relatively close to the π triplet and the K doublets and is usually identified as the eight member of the pseudoscalar octet. The use of linear mass expressions gives rather poor agreement for pseudoscalar mesons. With the use of $(\text{mass})^2$ expressions, the Gell-Mann

Okubo mass formula (23)

$$\eta_8^2 = \left(4K^2 - \pi^2\right)/3 \tag{1}$$

gives good agreement to the experimental mass. We usually take X^o to be the ninth pseudoscalar meson. This is not strictly necessary. Another candidate is the E (1422). The η_8 is pure unitary octet and η_1' pure singlet. Since these states have the same quantum numbers I = Y = 0, they can mix in broken SU(3) when belonging to the same nonet and the observed particles η and X^o are coherent superpositions of them. Explicitly,

$$\eta = \psi_8 \cos\theta - \psi_1 \sin\theta \tag{2}$$

$$X^o = \psi_8 \sin\theta + \psi_1 \cos\theta \tag{3}$$

in which ψ_8 and ψ_1 denote the pure octet and singlet states respectively. Kokkedee has given the following relations

$$m_\eta^2 + m_{X^o}^2 = m_1^2 + m_8^2 + 2\delta \tag{4}$$

$$m_\eta^2 m_{X^o}^2 = m_1^2 m_8^2 + \frac{2}{3}\delta\left(2m_1^2 + m_8^2\right) + \frac{8}{9}\delta^2\left(1 - F^2\right) \tag{5}$$

$$\tan 2\theta_p = \frac{(4\sqrt{2}/3)F\delta}{m_8^2 - m_1^2 + \left(\frac{2}{3}\right)\delta} \tag{6}$$

$$m_\pi^2 = m_8^2 \tag{7}$$

This leads to

$$m_1 = 863 \text{ MeV} \qquad m_8 = 135 \text{ MeV} \tag{8}$$

$$F = 0.52 \qquad \theta_p \simeq -11^\circ \tag{9}$$

where F is the overlap integral F (0) between the space wave functions of η_8 and η_1'.

Gursey (24) has given an interesting argument for the use of $(\text{mass})^2$ for pseudoscalar mesons. This argument depends on the hypothesis that the pseudoscalar octet masses are all zero in the limit of exact unitary symmetry, when the symmetry - breaking interactions are turned off. In this situation, to obtain the mass, generated in first order by the introduction of the symmetry-breaking interaction, once calculates the energy of the meson state for a given linear momentum. The energy for momentum p then changes fromp to $E\ (p) = \sqrt{(m^2+p^2)} = p + m^2/2p + \ldots$ so that the first-order correction to the energy gives directly the value of m^2. This argument has been given support by explicit calculation based on a covariant model by Wick (25) and Cutkosky (26)

If we now consider E(1422) meson instead of X^o as the ninth member of the pseudoscalar nonet, then

$$m_1 = 1360 \text{ MeV} \qquad m_8 = 135 \text{ MeV} \tag{10}$$

$$F = 0.88 \qquad \theta_p = -6^\circ \tag{11}$$

The actual situation may be more complicated in the sense that, in principle, mixing can occur between the states η, X^o and E. Samuel (27) has examined the mixing of the pure octet member and two SU(3) singlets. His results are quoted below:

$$|E\rangle = 0.08\,|\eta_8\rangle + 0.43\,|\eta_0\rangle + 0.90\,|\eta_0'\rangle \tag{12}$$

$$|X^o\rangle = 0.08\,|\eta_8\rangle + 0.90\,|\eta_0\rangle - 0.43\,|\eta_0'\rangle \tag{13}$$

$$|\eta\rangle = 0.99\,|\eta_8\rangle - 0.11\,|\eta_0\rangle - 0.04\,|\eta_0'\rangle \tag{14}$$

B. Vector Mesons

The main properties of vector mesons with their decay modes and quantum numbers are shown below.

TABLE IV

VECTOR MESONS

Particle	Mass (MeV)	J^P	I^G	Main Decay Mode	C	Y
$\rho^{\pm}$	769 ± 3	1^-	1^+	2π	-	0
ρ^{0}	769 ± 3	1^-	1^+	2π	-	0
ω	783.7 ± 0.4	1^-	0^-	$\pi^+\pi^-\pi^0$	-	0
ϕ	1018.8 ± 0.5	1^-	0^-	K^+K^-	-	0
$K^{*\pm}$		1^-	$\frac{1}{2}$	$K\pi$	-	± 1
	891 ± 1					
$K^{*0}, \bar{K}^{*0}$		1^-	$\frac{1}{2}$		-	+1,-1

Kokkedee has given the following relations for the vector mesons

$$m_1 = 799 \text{ MeV} \qquad m_8 = 777 \text{ MeV} \tag{15}$$

$$\tan 2\theta_V = \frac{(4\sqrt{2}/3)\, F\Delta}{m_8 - m_1 + \frac{2}{3}\Delta} \tag{16}$$

where he has given the mass m_A of particle A within SU(3) multiplet $\{\alpha\}$ as

$$m_A = \langle \psi(A) | \sum_i m_{q_i} - U(\{\alpha\}) | \psi(A) \rangle \tag{17}$$

in which the sum runs over the quarks composing hadrons A and U denotes the $q\bar{q}$ potential. U does not depend on the quark labels and U ($\{\alpha\}$) may include all possible SU(3) - invariant contributions. The near equality of m_1 and m_8 is what we expect if within the 35 - plet, the dominant SU(6) - breaking forces are the spin-spin forces. In that case, for vector mesons U ($\{1\}$) = U ($\{8\}$) and

$$m_1 \simeq m_8 \tag{18}$$

So within the experimental uncertainty in the value of m_ρ, the vector meson nonet is consistent with $F \simeq 1$ and this leads to θ_V = arc tan $[(1/2)\sqrt{2}] = 35^o$.

If we compare these results with those for the pseudoscalar mesons, the large difference between the values of m_8 for the two nonets points to the presence of strong SU(6) breaking, spin-dependent forces, at least within the 35-plet. We may have roughly the picture shown below:

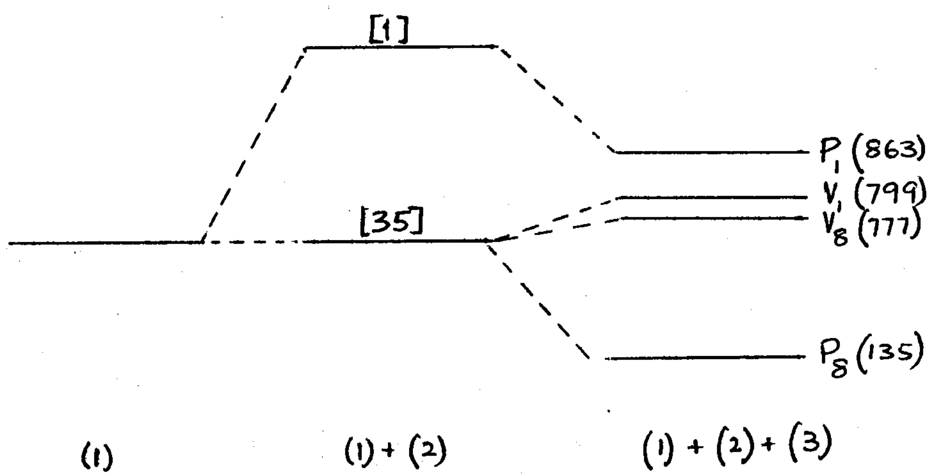

Figure 4. Mass Splittings among the 36 Mesonic States with L = 0 due to forces of Types (1), (2) and (3). The figure is not according to scale

C. Excited Mesonic States

In the last few years, an amazing number of mesonic and baryonic resonances has been established in the mass region from 1 to about 3 GeV. This number is steadily rising and, witness the skill of the experimentalists, will undoubtedly continue to do so for quite a while. It is logical within the framework of the quark model to try to interpret these higher resonance states as excitations of the $q\bar{q}$ systems. This spectroscopic aspect of the quark model has been vigorously investigated by Dalitz (28). Now in the quark model, excited meson states may be generated in two distinct ways (which can occur combined):

(i) more complicated quark - antiquark excitations, for example the configurations $\bar{q}\bar{q}qq$. The SU(6) and relativistic $\tilde{U}$ (12) schemes which have been discussed in the literature usually attribute higher resonances to these excitations.

(ii) non-zero orbital angular momentum for the quarks. These are the most natural to consider, within the framework of our model.

A $q\bar{q}$ system with orbital angular momentum $L \neq 0$ generates four sets of nonets of parity $(-1)^{L+1}$, namely three for $S = 1$ and $C = (-)^{L+1}$ and $J = L + 1, L, L - 1$ and one for $S = 0$ having $C = (-)^L$ and $J = L$ in which J is the total angular momentum. For $L = 0$, there are, of course, only two nonets. We denote these nonets by ${}^3L_{L+1}$, ${}^3L_L$, ${}^3L_{L-1}$ and ${}^1L_L$ respectively. Each of them consists of an SU(3) singlet and octet. The possible pattern for mass splittings among the $q\bar{q}$ states for general L is shown below:

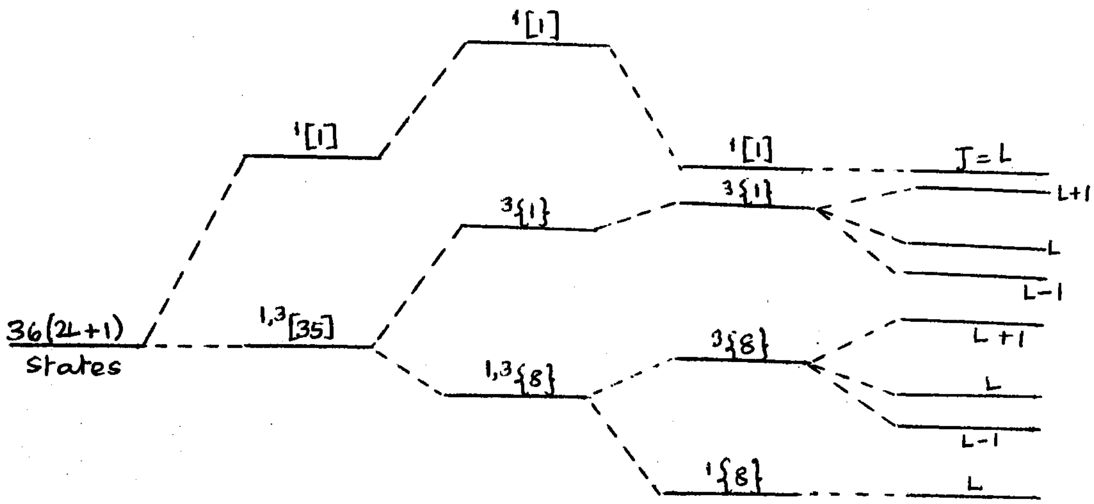

Figure 5. Possible Pattern for Mass Splittings Among the Quark-Antiquark States for General L

For L = 0 mesons, the observed pattern is consistent with the above scheme. Here ${}^1[1]$, ${}^3\{1\}$ and ${}^3\{8\}$ are close together in mass, whereas ${}^1\{8\}$ is pushed down considerably. Now, the first excited configurations will be those corresponding to L = 1. These four nonets will have the spin-parity values (2+), (1+), (0+) with C = 1 and (1+) with C = -1. The four nonets will be separated in mass by the spin-orbit coupling; in each nonet, there may be some difference between the m_1 and m_8 masses and there will be mixing between the I = Y = 0 states and mass splitting for the other states, introduced by the quark mass difference Δ . There will be no mixing between the Y = 0 states of the two (1+) nonets with C = $\pm$1, since charge-conjugation invariance holds for the strong-interactions. Mixing between these two nonets can occur for the Y = $\pm$1 states, in general, through the symmetry-breaking interactions; this mixing could arise only from symmetry-breaking potentials which couple S = 0 and S = 1 states.

Now, of the L = 1 states, the nonet ${}^3P_2$ with JPC = 2 ++ is well established. The I = Y = 0 members are the well known f meson of mass 1260 $\pm$ 20 MeV and width 100 MeV and the f' meson of mass 1514 $\pm$ 20 MeV and width 85 MeV., recently discovered by Barnes (29). For the f meson, the decay mode $f \rightarrow \pi\pi$ is dominant; for the f' meson, the decay mode $f' \rightarrow K\bar{K}$ is dominant. Both states therefore have C = +1. The I = 1, Y = 0 state is the A2 meson, of mass 1300 $\pm$ 10 MeV and width 85$\pm$10 MeV, known from its decay modes $A2 \rightarrow \rho\pi$ and $K\bar{K}$; the $\rho\pi$ mode requires G =-1 for the A2 meson, which corresponds again to C = +1. The Y = +1 (-1) state is the K** meson of mass 1420 $\pm$ 10 MeV and width 100 $\pm$ 20 MeV, established from the work of Haque et al.(30) and Hardy et al., (31) whose dominant decay mode is $K^{**} \rightarrow K\pi$.

Kokkedee has given the following relations for the $^3P_2$ nonet:

$$\delta = 3 \times 10^5 \ (\text{MeV})^2; \quad \theta \simeq 28^\circ \quad F \simeq 1 \tag{19}$$

$$m_8 = 1315 \ \text{MeV} \qquad m_1 = 1230 \ \text{MeV} \tag{20}$$

Where θ is the mixing angle for the I = Y = 0 states and F the overlap integral of their space wave-functions. The value of δ is in resonable agreement with those found for the L = 0 states. The qualitative features of the partial widths observed for the decay processes of these mesons is also in accord with the nonet structure. We show below the octet pattern of 2^+ meson.

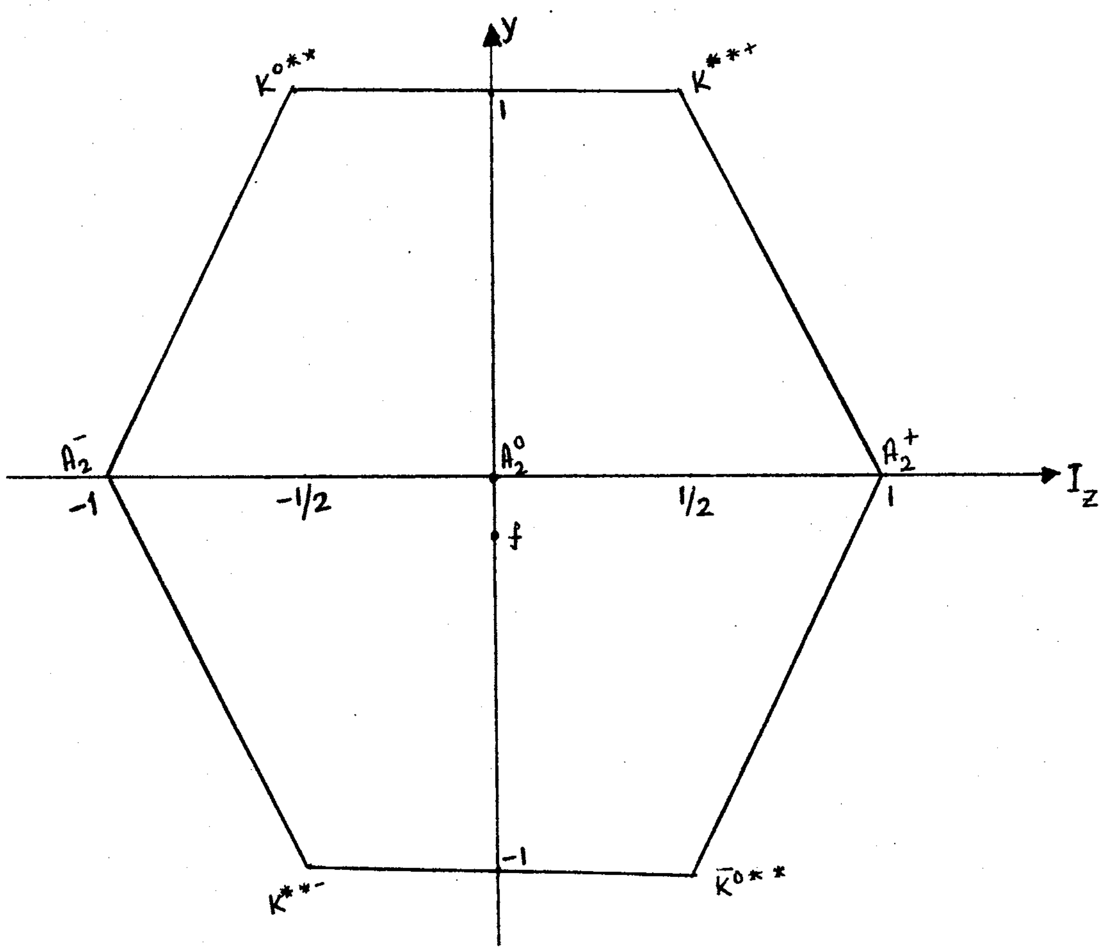

Figure 6. Octet Pattern for 2^+ Mesonic Nonet

The evidence concerning (1+) states is on a less secure fitting. The D meson at 1285 MeV with width 40 MeV, established recently by Miller et al. (32) and by d'Andlau et al. (33), from the decay modes $D \rightarrow K\bar{K}\pi$, has I = Y = 0 and is consistent with spin parity (1+) or (2-). On the basis of our model, the (1+) assignment would be favoured since the (2-) states require L = 2 and would be expected to lie in a much higher mass region. The properties of this decay mode also indicate G = +1; with I = 0 we then have C = +1 for the D meson. The A1 meson at 1070 $\pm$ 13 MeV and width 125 $\pm$ 25 MeV, has been established for the decay mode $A1 \rightarrow \rho\pi$ whose characteristics strongly favour the spin-parity assignment (34) (1+) and which has I = 1, Y = 0. The $\pi\rho$ decay mode requires G = -1 for the A1 meson and hence C=+1. The K* - meson, of mass 1230 $\pm$ 10 MeV and width 60 $\pm$ 10 MeV, and with the decay mode $K^* \rightarrow K\pi\pi$ has been reported by Armenteros et al. (35) to have I = ½ and decay characteristics strongly suggestive of spin-parity (1+). It is rather difficult to fit the above mass values into a nonet picture. The identification of the E meson already discussed as the D' meson is barely compatible with the Schwinger inequality and would require the overlap integral between the $A_1^{(+)}$ and $A_8^{(+)}$ states to be essentially zero i.e. that these states should not mix. For the C=-1 (1+) nonet, we have to date,two condidates. The B meson, with mass 1235 MeV and width 125 $\pm$ 30 MeV has been identified from its decay to $\pi + \omega$ and therefore has I = 1, G = +1 whence C =- 1. The K* meson, with mass 1320 MeV also has the J^P assignment 1 + and C =-1. The H meson at 990 $\pm$ 10 MeV has been reported by Barsch et al. (36) but the spin-parity assignment is still unclear. Finally we consider the C=+1 (0+) states. The δ meson at 966 MeV has been identified as I = 1 state, consistent with our model. The

K_N meson with mean mass 1170 MeV is consistent with the spin-parity assignment (0+). For the I=Y=0 state, the situation is not very clear, but we can identify the S* meson, with mass 1070 MeV as having the spin-parity assignment (0+). There are several other candidates for these states but the experimental situation is still very unclear.

D. Remarks

Since our knowledge of the L = 1 nonets is rather incomplete, there are relatively few tests possible for the viewpoint of the quark model discussed here.

Apart from spin-orbit forces, the (1-) and (2+) nonets would be expected to have rather similar features, both having S=1 configurations. For the (1-) nonet, we have m_1 = 799 MeV, m_8 = 777 MeV; for the (2+) nonet, the difference between the octet and singlet masses is larger, and opposite in sign, with m_1 = 1230 MeV, m_8 = 1315 MeV. Since the central forces in these two sets of states are the same, this difference between $(m_1 - m_8)$ should be attributed to an F-dependence in the spin-orbit force, which is effective in the L = 1 state but absent in the L = 0 state. What is known about the symmetry-breaking interaction in the L = 1 nonets appears reasonably consistent with the effects seen in the L = 0 nonets. The situation is only clear for the (2+) nonet, as discussed by Glashow and Socolow (37).

For L = 2, the quark-antiquark model implies nonets for spin-parity values (3-), (2-) with C = ± 1 and (1-). A plausible candidate is the π_A meson of mass 1640 MeV, with the spin-parity assignment (2-). The situation in these higher mass regions is unclear and we will have to wait until complete experimental verification of these higher states becomes possible.

CHAPTER IV

POTENTIAL FOR QUARK - ANTIQUARK COMBINATION

A. q-$\bar{q}$ Potential

The properties of the pseudoscalr and vector mesons have been part of the case made for the physical appropriateness of the larger symmetry of the SU(6) group for the elementary particle interactions, as first proposed by Gursey and Radicati (38) and by Sakata (39). Basically, the statement of SU(6) symmetry is that the q-$\bar{q}$ potential is invariant for simultaneous spin and unitary - spin transformations.

With SU(6) symmetry, the $\bar{q}$-q states are of the type $q^A q_B$. This tensor is reducible into a singlet tensor $q^A q_A$ and a (1,1) tensor $(q^A q_B - \delta^A_B\ q^C q_C/6)$ consisting of the remaining 35 components. The only singlet state available is the S = 0, $\{1\}$ state, so that the S=0, $\{8\}$ and the S = 1, $\{1\}$ and $\{8\}$ states constitute the $\underline{35}$ SU(6) supermultiplet:

$$\underline{35} = 1 \times 8 + 3 \times 1 + 3 \times 8 \qquad (1)$$

It is convenient to introduce the infinitesimal operators F_i (i = 1,...8) of the SU(3) group, which we may call the unitary spin operators. Their commutation relations are given by Gell-Mann (40) and by deSwart (41).

They are completely analogous to the infinitesimal operators σ_i for the SU(2) group and they include the isospin operators τ_i appropriate to the isospin SU(2) subgroup of the SU(3) group. For an SU(3) representation, the eigenvalue of the total unitary spin F^2 is given by

$$F^2 = \sum_{i=1}^{8} F_i^2 = 2\left(p^2 + pq + q^2 + 3(p+q)\right) \tag{2}$$

For the $\bar{q}$-q system, $F_1^2 = F_2^2 = 8$ and we deduce that the scalar product

$$F_1 . F_2 = (F^2 - F_1^2 - F_2^2)/2 \tag{3}$$

has the values + 1 for the $\{8\}$ state, - 8 for the $\{1\}$ state.

Projection operators for the eigenstates of total spin and total unitary spin are then readily constructed and the general form of the S - wave q-$\bar{q}$ potential may be written

$$\begin{aligned} U(\bar{q}q) = \{ & U_{P1}(1-\sigma_1.\sigma_2)(1-F_1.F_2) + U_{P8}(1-\sigma_1.\sigma_2)(8+F_1.F_2) \\ & + U_{V1}(3+\sigma_1.\sigma_2)(1-F_1.F_2) + U_{V8}(3+\sigma_1.\sigma_2)(8+F_1.F_2)\}/36 \end{aligned} \tag{4}$$

Empirically, the interactions U_{v1} U_{v8} and U_{p1} are approximately equal, the interaction U_{p8} being significantly stronger. So, to a good approximation,

$$U(\bar{q}q) = U_0 + \delta U_{P8}(1-\sigma_1.\sigma_2)(8+F_1.F_2)/36 \tag{5}$$

Dalitz gives explicitly the form of this potential as

$$U(\bar{q}q) = \bar{U}(1) + \bar{U}(35)(-\sigma_1.\sigma_2 - F_1.F_2 + \sigma_1.\sigma_2\, F_1.F_2) \tag{6}$$

and with SU(6) symmetry the form expected for U ($\bar{q}$q) is

$$U(\bar{q}q) = \bar{U}_0 + \bar{U}_1(1-\sigma_1.\sigma_2)(1-F_1.F_2)/36 \tag{7}$$

B. Specific Forms of Potentials:

Hydrogenic System Problem

Energy levels and eigenfunctions are given by:

$$W = -Z^2 e^4 m_e / 2\hbar^2 n^2 \tag{8}$$

$$\Psi = N e^{-r/n} r^l L_{n-l-1}^{2l+1}\left(\frac{2r}{n}\right) Y_l^m(\theta, \varphi) \tag{9}$$

as shown by Green (42). If we investigate the hydrogenic system problem for $l \neq o$, we find that an unusual degeneracy occurs in which the energy depends upon the integral combination

$$n = v + 1 + l \tag{10}$$

Harmonic Oscillator:

The three-dimensional harmonic oscillator has been used in many discussions in nuclear physics to furnish a simple reference set of levels. The eigenfunctions, as given by Powell (43) are

$$\Psi_{n,l,m} = N e^{-r^2/2} r^l L_K^{l+1/2}(r^2) Y_l^m(\theta, \phi) \tag{11}$$

where $L_K^{\alpha}(t)$ is the Laguerre polynomial

$$L_K^{\alpha}(t) = \sum_{\nu=0}^{K} \binom{K+\alpha}{K-\nu} \frac{(-t)^{\nu}}{\nu!} \tag{12}$$

The energy levels for this potential are given by

$$W = \left(2v + l + \frac{3}{2}\right)\hbar\omega_c \, , \quad \omega_c = \sqrt{\frac{K}{m}} \tag{13a}$$

Introducing the oscillator number $N = 2v + L$ (13b)

$$W = \left(N + \frac{3}{2}\right)\hbar\omega_c \tag{13c}$$

Spherical Well

Usually one assumes a naive picture of quarks moving nonrelativistically in a very deep flat potential well. For mesons, the form of potential naturally points to infinite spherical well. It is true that there is nothing particularly sacred about either the harmonic oscillator or the Coulomb potential. If one believes the potential picture, one would note that the Coulomb potential with it's singularity at the origin would tend to depress the states of lower angular momentum and therefore pull down the radially excited s - state to make it degenerate with the p - state, whereas the smooth harmonic - oscillator potential has the first radially excited s - state considerably higher. The data would indicate that if a potential has any meaning, the well goes down much more steeply than a harmonic oscillator but may not be quite as singular as the Coulomb potential.

The quark model is sometimes considered to be only a simple representation of an underlying algebraic structure without requiring the existence of physical quarks. With this approach the harmonic and Coulomb potentials can be considered from an algebraic point of view. The accidental degeneracies of these two potentials are characterized by the groups SU(3) and O(4) respectively. Thus, one may attempt to classify the multiplets by using the representations of either of these internal symmetry groups as quantum numbers to label the states, without invoking the physical picture of a harmonic or Coulomb potential. At present, the experimental data are insufficient to provide great support for these approaches.

Particle in a Spherical Box

Green (44) has given the solutions for s and p states. For s states

$$W \simeq -V_0 + (v+1)^2 \frac{\pi^2 \hbar^2}{2ma^2} \tag{14}$$

For p states

$$W \simeq -V_0 + \left(v + \frac{3}{2}\right)^2 \frac{\pi^2 \hbar^2}{2ma^2} \tag{15}$$

Where the field of force is defined by

$$\begin{aligned} V(r) &= -V_0 \qquad 0 < r < a \\ &= \infty \qquad r > a \\ v &= 0, 1, 2, 3, \cdots \cdots \end{aligned} \tag{16}$$

For d, f,....states we consider the general radial wave equation

$$G_i'' + \left[\epsilon_0^2 - \frac{l(l+1)}{\rho^2} - \epsilon_W^2\right] G_i = 0 \tag{17a}$$

Where

$$\epsilon_0^2 = \frac{V_0}{E_0}, \quad \epsilon_W^2 = \mp \frac{W}{E_0}, \quad \rho = r/a \tag{17b}$$

This equation is identical to Bessel's equations and the solutions which vanish at $\rho = 0$ are

$$G_i = \left[\frac{(\epsilon_0^2 - \epsilon_W^2)^{1/2} \rho \pi}{2}\right]^{1/2} J_{l+\frac{1}{2}}\left[(\epsilon_0^2 - \epsilon_W^2)^{1/2} \rho\right] \tag{18}$$

where $J_{l+\frac{1}{2}}$ are Bessel functions of half-integral order. Since the wave function must vanish at $\rho = 1$, the values of ϵ_W must be such that

$$J_{l+\frac{1}{2}}\left[(\epsilon_0^2 - \epsilon_W^2)^{1/2}\right] = 0 \tag{19}$$

Particle in a Spherical Well. We have in this case

$$\begin{aligned} V(r) &= -V_0 \qquad 0 < r < a \\ &= 0 \qquad r > a \\ \rho &= r/a \end{aligned} \tag{20}$$

where a is the radius of the spherical well.

Since V vanishes as $r \to \infty$ the wave function no longer need vanish identically outside the well. The exterior wave function for s - waves must be a well-behaved function

$$G_e'' - \epsilon_W^2 G_e = 0 \tag{21}$$

$$\therefore \quad G_e = C_e \exp(-\epsilon_W \rho)$$

whereas

$$G_i = C_i \sin (\epsilon_0^2 - \epsilon_W^2)^{1/2} \rho \tag{22}$$

Also, since the interior and exterior wave functions must join smoothly at $\rho = 1$,

$$\begin{aligned} G_i(1) &= G_e(1) \\ G_i'(1) &= G_e'(1) \end{aligned} \tag{23}$$

Normalization condition

$$\int_0^a |G_i|^2 dr + \int_a^\infty |G_e|^2 dr = 1 \tag{24}$$

Now,

$$[G_i' / G_i]_{\rho=1} = [G_e' / G_e]_{\rho=1} \tag{25}$$

$$\therefore \quad \tan (\epsilon_0^2 - \epsilon_W^2)^{1/2} = - \frac{(\epsilon_0^2 - \epsilon_W^2)^{1/2}}{\epsilon_W} \tag{26}$$

When the well is shallow, it is impossible to express the energy levels in terms of an explicit formula. In this case, we must find the roots of the above equation by approximate numerical or graphical methods. The no. of ϵ_W roots which exist depends upon the well parameter ϵ_o

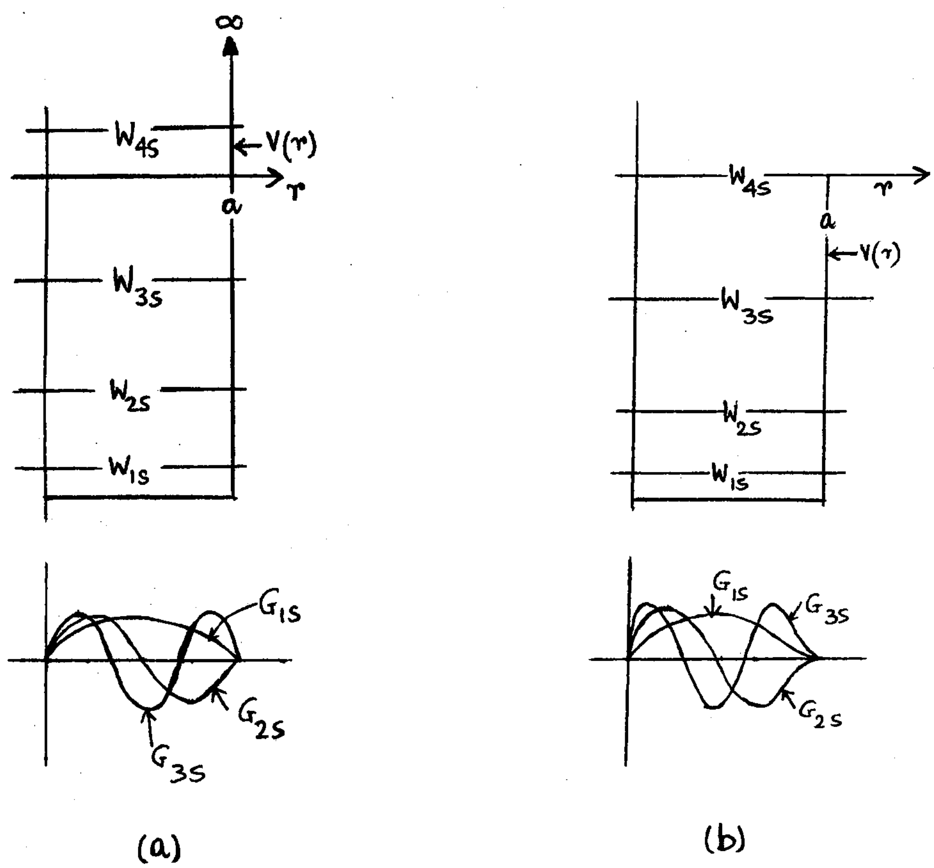

Figure 7. Schematic Diagram Showing Low-lying s States and the Corresponding Wave Functions when $\epsilon_o = 7\pi/2$ in (a) a Spherical Box and (b) a Spherical Well

We note that in each of the latter cases the wave function extends into the external region. This region would be inaccesible to the particle if classical laws were obeyed, since here the classical kinetic energy $T = W-V$ would be negative.

Let's now consider briefly the p states of binding ($L = 1$) for the spherical well. The radial wave equation for the interior region is given by

$$G_i'' + \left(\epsilon_o^2 - \frac{2}{\rho} - \epsilon_W^2\right) G_i = 0 \tag{27}$$

Solution:

$$G_{ip} = C_{ip}\left[\frac{\sin(\epsilon_o^2-\epsilon_W^2)^{1/2}\rho}{(\epsilon_o^2-\epsilon_W^2)^{1/2}\rho} - \cos(\epsilon_o^2-\epsilon_W^2)^{1/2}\rho\right] \tag{28}$$

as can be easily verified by direct differentiation. The wave-equation for the exterior region is given by

$$G_e'' - \left(\frac{2}{\rho^2} + \epsilon_W^2\right) G_e = 0 \tag{29}$$

The solution is:

$$G_{ep} = C_{ep}\left(\epsilon_W + \frac{1}{\rho}\right)\exp(-\epsilon_W\rho) \tag{30}$$

as also can be proved by direct differentiation. The boundary condition is given by:

$$\left.\frac{G_e'}{G_e}\right|_{\rho=1} = \left.\frac{G_i'}{G_i}\right|_{\rho=1} \tag{31}$$

Now,

$$\left.\frac{G_e'}{G_e}\right|_{\rho=1} = -\epsilon_W - \frac{1}{\epsilon_W+1} \tag{32}$$

Also,

$$\left.\frac{G_i'}{G_i}\right|_{\rho=1} = -1 + \frac{(\epsilon_o^2-\epsilon_W^2)^{1/2}}{(\epsilon_o^2-\epsilon_W^2)^{-1/2} - \cot(\epsilon_o^2-\epsilon_W^2)^{1/2}} \tag{33}$$

This yields the energy eigenvalue equation as

$$\frac{\cot(\epsilon_o^2-\epsilon_W^2)^{1/2}}{(\epsilon_o^2-\epsilon_W^2)^{1/2}} - \frac{1}{\epsilon_o^2-\epsilon_W^2} = \frac{1}{\epsilon_W} + \frac{1}{\epsilon_W^2} \tag{34}$$

So, for the case of a shallow well, we find that the critical ϵ_o values, each of which gives rise to a p state of zero energy ($\epsilon_W=0$) are $\pi, 2\pi, \cdots$

corresponding to the relation

$$\tan \epsilon_o = 0 \tag{35}$$

For the solutions for d,f,g,.... states, we need to find the general solutions of the radial equations inside and outside the well for an arbitrary l. These solutions are

$$G_{il} = \left[\frac{(\epsilon_o^2 - \epsilon_W^2)\rho\pi}{2} \right]^{1/2} J_{l+\frac{1}{2}} \left[(\epsilon_o^2 - \epsilon_W^2)^{1/2} \rho \right] \tag{36}$$

$$G_{el} = \left(\frac{\rho\pi}{2} \right)^{1/2} N_{l+\frac{1}{2}}(\rho) \tag{37}$$

where $J_{l+\frac{1}{2}}$ and $N_{l+\frac{1}{2}}$ are Bessel and Neumann functions of half-integral order. On the basis of the properties of these functions the eigenvalues and the eigenfunctions can be determined for any ϵ_o.

Infinite Spherical Well

The eigenfunctions and $\epsilon_o^2 - \epsilon_W^2$ eigenvalues of the spherical well go over to those of the spherical box as $V_o \rightarrow \infty$ So, for d states,

$$J_{5/2}(\theta) = 0 \quad \text{where} \quad \theta = \left[(\epsilon_o^2 - \epsilon_W^2)^{1/2} \right] \tag{38}$$

Now,

$$J_p(x) = \sum_{K=0}^{\infty} \frac{(-)^K \left(\frac{x}{2}\right)^{2K+p}}{K!\,(K+p)!} \tag{39}$$

$$\therefore J_{5/2}(\theta) = \frac{(\theta/2)^{5/2}}{(5/2)!} - \frac{(\theta/2)^{9/2}}{(9/2)!} + \frac{(\theta/2)^{13/2}}{(13/2)!} - \cdots \tag{40}$$

$$\therefore J_{5/2}(\theta) = \frac{(\theta/2)^{5/2}}{\Gamma(7/2)} \prod_{s=1}^{\infty} \left(1 - \frac{\theta^2}{p_s^2}\right) \tag{41}$$

where p_s are the real zero of $J_{5/2}(\theta)$.

Now (45)

$$J_\nu(z) \sim \sqrt{\left(\frac{2}{\pi z}\right)}\left[\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\sum_{s=0}^{\infty}\frac{(-)^s(\nu,2s)}{(2z)^{2s}} - \sin\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\sum_{s=0}^{\infty}\frac{(-)^s(\nu,2s+1)}{(2z)^{2s+1}}\right] \quad (42)$$

Where

$$(\nu,s) = \frac{2^{-2s}}{s!}\left[\left(4\nu^2-1^2\right)\left(4\nu^2-3^2\right)\cdots\cdots\left\{4\nu^2-(2s-1)^2\right\}\right]$$

and $(\nu,0) = 1$ (43)

$$\therefore J_{5/2}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2}\left[P_{5/2}(z)\cos\left(z-\frac{3\pi}{2}\right) - Q_{5/2}(z)\sin\left(z-\frac{3\pi}{2}\right)\right] \quad (44)$$

where

$$P_{5/2}(z) = \frac{\left(\frac{5}{2},0\right)}{1} - \frac{\left(\frac{5}{2},2\right)}{4z^2} + \frac{\left(\frac{5}{2},4\right)}{16z^4} - \cdots\cdots$$

$$= 1 - \frac{3}{z^2} \quad (45)$$

and

$$Q_{5/2}(z) = \frac{\left(\frac{5}{2},1\right)}{2z} - \frac{\left(\frac{5}{2},3\right)}{4z^2}\,\frac{1}{2z} + \frac{\left(\frac{5}{2},5\right)}{(2z)^5} - \cdots$$

$$= \frac{3}{z} \quad (46)$$

$$\therefore \left(\frac{\pi z}{2}\right)^{1/2} J_{5/2}(z) \sim \left\{P_{1/2}^2(z) + Q_{5/2}^2(z)\right\}^{1/2}\cos\left(z-\frac{3\pi}{2}-\theta\right) \quad (47)$$

where

$$\tan\theta = -Q_{5/2}(z)/P_{5/2}(z) = \frac{3z}{3-z^2} \tag{48}$$

So the positive zeros are defined by

$$z - \frac{3\pi}{2} - \theta = \left(s - \frac{1}{2}\right)\pi \quad , \quad s = 1, 2, 3, \cdots \tag{49}$$

$$\therefore \; z = (s+1)\pi + \tan^{-1}\left(\frac{3z}{3-z^2}\right) \tag{50}$$

So for the zeros we have the formula:

$$\left(\epsilon_0^2 - \epsilon_W^2\right)^{1/2} = (s+1)\pi + \tan^{-1}\left(\frac{3\left(\epsilon_0^2 - \epsilon_W^2\right)^{1/2}}{3-\left(\epsilon_0^2 - \epsilon_W^2\right)}\right) \tag{51}$$

For a well which has a large ϵ_0, ϵ_W will be close to ϵ_0 for low-lying state and so

$$\tan^{-1}\left(\frac{3\left(\epsilon_0^2 - \epsilon_W^2\right)^{1/2}}{3 - \epsilon_0^2 + \epsilon_W^2}\right) \cong \tan^{-1} 0 = 0 \tag{52}$$

$$\therefore \; \left(\epsilon_0^2 - \epsilon_W^2\right)^{1/2} = (s+1)\pi \quad ; \quad s = 1, 2, 3, \cdots \tag{53}$$

$$= (\nu+2)\pi \quad ; \quad \nu = 0, 1, 2, \cdots \quad \text{(53a}$$

$$W \simeq -V_0 + \frac{(\nu+2)^2\pi^2\hbar^2}{2ma^2} \tag{54}$$

Similarly for f states,

$$\left(\frac{\pi z}{2}\right)^{1/2} J_{7/2}(z) \sim \left\{P_{7/2}^2(z) + Q_{7/2}^2(z)\right\}^{1/2} \cos(z - 2\pi - \theta) \tag{55}$$

where

$$\tan\theta = -Q_{7/2}(z)/P_{7/2}(z)$$
$$= 3\left(-\frac{2}{z} - \frac{25}{z^3} - \cdots\cdots\right) \quad (56)$$

So the positive zeros are given by

$$z - 2\pi - \theta = \left(s - \frac{1}{2}\right)\pi, \quad s = 1, 2, 3, \cdots \quad (57)$$

$$\therefore z = \left(s + \frac{3}{2}\right)\pi - \frac{6}{z} - \frac{3}{z^3} + \cdots\cdots \quad (58)$$

Writing $\alpha = (s + \frac{3}{2})\pi$ and assuming $z = \alpha + \frac{\lambda}{\alpha} + \frac{\mu}{\alpha^3} + \cdots$ substituting and neglecting α^{-5} etc., we have

$$\alpha + \frac{\lambda}{\alpha} + \frac{\mu}{\alpha^3} + \cdots\cdots = \alpha - 6\left(\frac{1}{\alpha} - \frac{\lambda}{\alpha^3} + \cdots\right)$$
$$-3\left(\frac{1}{\alpha^3} - \cdots\cdots\right) + \cdots\cdots \quad (59)$$

and so that equating the co-efficients of α^{-1} and α^{-3} we have

$$\lambda = -6, \quad \mu = 6\lambda - 3 = -39 \quad (60)$$

$$\therefore z \sim \left(s + \frac{3}{2}\right)\pi - \frac{6}{\left(s + \frac{3}{2}\right)\pi} - \frac{39}{\left(s + \frac{3}{2}\right)^3\pi^3} - \cdots \quad (61)$$

This gives the approximate solution as

$$\left(\epsilon_0^2 - \epsilon_W^2\right)^{1/2} \sim \left(\nu + \frac{5}{2}\right)\pi \quad (62)$$

$$\therefore W \simeq -V_0 + \frac{\left(\nu + \frac{5}{2}\right)^2\pi^2\hbar^2}{2ma^2} \quad (63)$$

Similarly, for arbitrary l

$$\left(\frac{\pi z}{2}\right)^{1/2} J_{l+\frac{1}{2}}(z) = \left\{P^2_{l+\frac{1}{2}}(z) + Q^2_{l+\frac{1}{2}}(z)\right\} \cos\left(z - \frac{(l+1)\pi}{2} - \theta\right) \tag{64}$$

where

$$P_{l+\frac{1}{2}}(z) = \sum_{s=0}^{\infty} (-)^s \frac{\left(l+\frac{1}{2},\ 2s\right)}{(2z)^{2s}}$$

$$= 1 - \frac{l(l+1)(l+2)(l-1)}{2!\ (2z)^2} + \frac{l(l+1)(l+2)(l+3)(l+4)(l-1)(l-2)(l-3)}{4!\ (2z)^4}$$

$$- \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ (-)^s \frac{l(l+1)(l+2) \cdots\cdots (l+2s)(l-1)(l-2) \cdots\cdots (l-2s+1)}{(2s)!\ (2z)^{2s}}$$

and

$$Q_{l+\frac{1}{2}}(z) = \sum_{s=0}^{\infty} (-)^s \frac{\left(l+\frac{1}{2},\ 2s+1\right)}{(2z)^{2s+1}}$$

$$= \frac{l(l+1)}{1!\ 2z} - \frac{l(l+1)(l+2)(l+3)(l-1)(l-2)}{3!\ (2z)^3}$$

$$+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ (-)^s \frac{l(l+1)(l+2) \cdots (l+2s+1)(l-1)(l-2) \cdots (l-2s)}{(2s+1)!\ (2z)^{2s+1}} \tag{65}$$

Also

$$\theta = \tan^{-1} - \frac{Q_{l+\frac{1}{2}}(z)}{P_{l+\frac{1}{2}}(z)} \tag{66}$$

Generalizing the notation,

$$Q_{l+\frac{1}{2}}(z) = \sum_{s=0}^{\left[\frac{l-1}{2}\right]} \frac{(-)^s (l-2s) \cdot \cdots \cdots l(l+1) \cdots (l+2s+1)}{(2s+1)!\,(2z)^{2s+1}} \tag{67}$$

So the positive zeros are given by

$$z - \frac{(l+1)\pi}{2} - \theta = \left(\nu + \frac{1}{2}\right)\pi \tag{68}$$

Assuming that the expansion $\tan^{-1}\left(-Q_{l+\frac{1}{2}}(z)/P_{l+\frac{1}{2}}(z)\right)$ represent small contributions, the approximate energy eigenvalues are given by

$$\left(\epsilon_0^2 - \epsilon_W^2\right)^{1/2} = \left(\nu + \frac{l}{2} + 1\right)\pi \tag{69}$$

$$W \simeq -V_0 + \left(\nu + \frac{l}{2} + 1\right)^2 \frac{\pi^2 \hbar^2}{2ma^2} \tag{70}$$

Schiff (46) has given the solutions for a spherical well as

$$G_i = A_i\, \rho^{1/2}\, J_{l+\frac{1}{2}}(\epsilon' \rho) \tag{71}$$

$$G_e = A_e\, \rho^{1/2}\, K_{l+\frac{1}{2}}(\epsilon_W \rho) \tag{72}$$

The boundary condition yields the energy eigenvalue condition as

$$\epsilon' \frac{j_{l-1}(\epsilon')}{j_l(\epsilon')} = i\epsilon_W \frac{h_{l-1}^{(1)}(i\epsilon_W)}{h_l^{(1)}(i\epsilon_W)} \tag{73}$$

Looking at the asymptotic behavior of the Hankel functions, this simplifies to,

$$\frac{j_l(\epsilon')}{j_{l-1}(\epsilon')} = -\frac{\epsilon'}{\epsilon_W} \tag{74}$$

So for l = 1,

$$\frac{j_1(\epsilon')}{j_0(\epsilon')} = -\frac{\epsilon'}{\epsilon_W} \tag{75}$$

We know that if we put the right hand side equal to zero, the values of ϵ' which satisfy the equation $j_1(\epsilon')=0$ are a trifle smaller than $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, and so on. If we now make a Taylor series expansion of $j_1(\rho)$ about $\rho=\rho_1$ and keep terms upto first order,

$$j_1(\rho) = j_1(\rho_1) + (\rho-\rho_1)\left.\frac{dj_1}{d\rho}\right|_{\rho=\rho_1} \tag{76}$$

Since $j_1(\rho_1) = 0$

$$\therefore\ j_1(\rho) = (\rho-\rho_1)\left[j_0(\rho_1) - \frac{2}{\rho}\, j_1(\rho_1)\right]$$

$$= (\rho-\rho_1)\, j_0(\rho_1) \tag{77}$$

$$\therefore\ \rho = \rho_1 + \frac{j_1(\rho)}{j_0(\rho_1)} \approx \rho_1 \tag{78}$$

For $l = 2$

$$\frac{j_2(\rho)}{j_1(\rho)} = -\frac{\rho}{\epsilon_W}\ ;\ \rho_1 \approx 2\pi, 3\pi, 4\pi, \cdots \tag{79}$$

$$\therefore\ j_2(\rho) = (\rho-\rho_1)\, j_1(\rho_1) \tag{80}$$

$$\therefore\ \rho = \rho_1 + j_2(\rho)\,/\,j_1(\rho_1) \tag{81}$$

In general for $l = n$,

$$\rho = \rho_1 + \frac{j_n(\rho)}{j_{n-1}(\rho_1)} \tag{82}$$

Now for $l = 1$,

$$\rho = \rho_1 - \frac{\rho}{\epsilon_W} \frac{j_0(\rho)}{j_0(\rho_1)} \tag{83}$$

$$\therefore \rho = \frac{\rho_1}{1 + \frac{j_0(\rho)}{j_0(\rho_1)} \frac{1}{\epsilon_W}} \simeq \frac{\rho_1}{1 + \frac{2}{2\epsilon_W}} \tag{84}$$

$$\therefore \rho^2 \simeq \rho_1^2 \left(1 - \frac{2}{\epsilon_W}\right) = K \left(1 - \frac{2}{\epsilon_W}\right) \tag{85}$$

$$\therefore \epsilon_0^2 - \epsilon_W^2 = K\left(1 - \frac{2}{\epsilon_W}\right) \tag{86}$$

$$\therefore \epsilon_W^3 + \left(K - \epsilon_0^2\right)\epsilon_W - 2K = 0 \tag{87}$$

Solutions of cubic equations of this type have been discussed by Cowles (47). Using his notation, the three roots are given by

$$\epsilon_{W1} = y + z \tag{88}$$

$$\epsilon_{W2} = \omega y + \omega^2 z \tag{89}$$

$$\epsilon_{W3} = \omega^2 y + \omega z \tag{90}$$

$$y = \left(\rho_1^2 + \left(\rho_1^4 + \frac{(\rho_1^2 - \epsilon_0^2)^3}{27}\right)^{1/2}\right)^{1/3} \tag{91}$$

$$z = \left(\rho_1^2 - \left(\rho_1^4 + \left(\frac{\rho_1^2 - \epsilon_0^2}{3}\right)^3\right)^{1/2}\right)^{1/3} \tag{92}$$

Now, we note that even though the roots are all real, they can not be reduced to real algebraic form because the square root of the discriminant is imaginary. This is the so called irreducible case of the cubic equation and the roots can only be found by trigonometric methods (48).

Let

$$A = -\frac{r}{2} + i\sqrt{-\left(\frac{r^2}{4} + \frac{q^3}{27}\right)} \tag{93}$$

$$B = -\frac{r}{2} - i\sqrt{-\left(\frac{r^2}{4} + \frac{q^3}{27}\right)} \tag{94}$$

Where

$$q = K - \epsilon_0^2 \tag{95}$$

$$r = -2K \tag{96}$$

$$\therefore \quad A = me^{i\theta} \quad , \quad B = me^{-i\theta} \tag{97}$$

$$m = \left(-\frac{q^3}{27}\right)^{1/2}, \quad \cos\theta = -\frac{r}{2}\left(-\frac{27}{q^3}\right)^{1/2} \tag{98}$$

So the roots of the irreducible case of the cubic are given by,

$$r_1 = 2m^{1/3} \cos\frac{\theta}{3} \tag{99}$$

$$r_2 = 2m^{1/3} \cos\frac{\theta + 2\pi}{3} \tag{100}$$

$$r_3 = 2m^{1/3} \cos\frac{\theta + 4\pi}{3} \tag{101}$$

where

$$m = \left(-\frac{(\rho_1^2 - \epsilon_0^2)^3}{27}\right)^{1/2} \tag{102}$$

$$\cos\theta = \rho_1^2 \left(-\frac{(\rho_1^2 - \epsilon_0^2)^{-3}}{1/27}\right)^{1/2} \tag{103}$$

$$\therefore \quad \epsilon_{W1} = 2\left(-\frac{(\rho_1^2-\epsilon_0^2)^3}{27}\right)^{1/6} \cos\frac{\theta}{3} \tag{104}$$

$$\epsilon_{W2} = 2\left(-\frac{(\rho_1^2-\epsilon_0^2)^3}{27}\right)^{1/6} \cos\frac{\theta+2\pi}{3} \tag{105}$$

$$\epsilon_{W3} = 2\left(-\frac{(\rho_1^2-\epsilon_0^2)^3}{27}\right)^{1/6} \cos\frac{\theta+4\pi}{3} \tag{106}$$

Now,

$$\cos\theta = \rho_1^2\left(-\frac{27}{(\rho_1^2-\epsilon_0^2)^3}\right)^{1/2} \tag{107}$$

So to a first order of approximation,

$$\cos\theta \simeq \left(\frac{27\rho_1^4}{\epsilon_0^6}\right)^{1/2} \simeq 0 \tag{108}$$

$$\therefore \quad \theta \simeq \pi/2 \tag{109}$$

To improve our calculations, let us assume $\theta = \frac{\pi}{2} + \epsilon$ where ϵ represents a small contribution to θ due to the term $3\sqrt{3}\rho_1^2/\epsilon_0^3$ in $\cos\theta$.

$$\therefore \quad \frac{\theta}{3} = \frac{\pi}{6} + \frac{\epsilon}{3} \tag{110}$$

$$\epsilon_{W1} = \sqrt{3}\left(-\frac{(\rho_1^2-\epsilon_0^2)^3}{27}\right)^{1/6} - \frac{\epsilon}{3}\left(-\frac{(\rho_1^2-\epsilon_0^2)^3}{27}\right)^{1/6} \tag{111}$$

since $\cos\frac{\epsilon}{3} \simeq 1$ and $\sin\frac{\epsilon}{3} \simeq \frac{\epsilon}{3}$ (112)

Similarly,

$$\epsilon_{W2} = \sqrt{3}\left(-\frac{(\rho_1^2-\epsilon_0^2)^3}{27}\right)^{1/6} - \frac{\epsilon}{3}\left(-\frac{(\rho_1^2-\epsilon_0^2)^3}{27}\right)^{1/6} \tag{113}$$

$$\epsilon_{W3} = \frac{2\epsilon}{3}\left(-(\rho_1^2-\epsilon_0^2)^3/27\right)^{1/6} \quad . \tag{114}$$

Approximately, since $\theta \simeq \pi/2$

$$\epsilon_{W1} \simeq \sqrt{3} \left(- \frac{(\rho_1^2 - \epsilon_0^2)^3}{27} \right)^{1/6} \tag{115}$$

$$\epsilon_{W2} \simeq -\sqrt{3} \left(- \frac{(\rho_1^2 - \epsilon_0^2)^3}{27} \right)^{1/6} \tag{116}$$

$$\epsilon_{W3} \simeq 0 \tag{117}$$

Another way of improving our order of magnitude calculations is to go back to our original cubic equations and try to improve our roots by algebraic methods:

$$\epsilon_W^3 + (\rho_1^2 - \epsilon_0^2)\epsilon_W - 2\rho_1^2 = 0 \tag{118}$$

$$\therefore (\epsilon_0^2 - \rho^2)^{3/2} + (\rho_1^2 - \epsilon_0^2)(\epsilon_0^2 - \rho^2)^{1/2} - 2\rho_1^2 = 0 \tag{119}$$

Simplification yields

$$-\rho^2\epsilon_0^2 + \rho^4 + \rho_1^2(\epsilon_0^2 - \rho^2) = 2\rho_1^2(\epsilon_0^2 - \rho^2)^{1/2} \tag{120}$$

Let us assume

$$\rho = \rho_1 + c \tag{121}$$

So the right hand side is equal to

$$2\rho_1^2\epsilon_0 - \frac{\rho^2\rho_1^2}{\epsilon_0} \tag{122}$$

and the left hand side is equal to

$$c^4 + 4\rho_1 c^3 + (5\rho_1^2 - \epsilon_0^2)c^2 - 2c\rho_1(\epsilon_0^2 - \rho_1^2) \tag{123}$$

So to a first-order of approximation,

$$c = \frac{2\rho_1^2\epsilon_0 - \frac{\rho_1^4}{\epsilon_0}}{\rho_1^3 - \rho_1\epsilon_0^2 + \frac{\rho_1^3}{\epsilon_0}} \tag{124}$$

$$\rho = \rho_1 + \frac{2\rho_1^2\epsilon_0 - \frac{\rho_1^4}{\epsilon_0}}{\rho_1^3 - \rho_1\epsilon_0^2 + \frac{\rho_1^3}{\epsilon_0}} \tag{125}$$

Also,

$$\rho^4 - \rho^2\left(\rho_1^2 + \epsilon_0^2 - \frac{\rho_1^2}{\epsilon_0}\right) - \rho_1^2\,\epsilon_0\,\left(2 - \epsilon_0\right) = 0 \quad (126)$$

After extracting the root of the equation and simplifying, we have

$$\therefore\ \rho^2 = \left[\left(\rho_1^2 + \epsilon_0^2 - \rho_1^2/\epsilon_0\right) \pm \left(\rho_1^2 - \epsilon_0^2 - \frac{\rho_1^2}{\epsilon_0}\right) \pm \left(\frac{2\rho_1^2\epsilon_0}{\rho_1^2 - \epsilon_0^2 - \frac{\rho_1^2}{\epsilon_0}}\right)\right] \Big/ 2 \quad (127)$$

If we take the positive sign,

$$\rho^2 \simeq \rho_1^2 - \frac{2\rho_1^2}{\epsilon_0} \quad (128)$$

$$\therefore\ \rho \simeq \rho_1 - \frac{\rho_1}{\epsilon_0} \quad (129)$$

If we take the negative sign,

$$\rho^2 \simeq \epsilon_0^2 - \frac{\rho_1^2}{\epsilon_0} \quad (130)$$

which accounts for the root $\epsilon_W \simeq 0$ when to a first approximation we have $\rho^2 \simeq \epsilon_0^2$. So the final formula for an arbitrary state l is given by

$$W \simeq -V_0 + \left(\nu + \frac{l}{2} + 1\right)^2 \frac{\pi^2\hbar^2}{2ma^2} + \frac{\hbar^2}{2ma^2}\tan^{-1}\left(\frac{3\left(\epsilon_0^2 - \epsilon_W^2\right)^{1/2}}{3 - \epsilon_0^2 + \epsilon_W^2}\right) \quad (131)$$

This completes our discussion of the infinite square well with first-order corrections to the binding-energy formula. We note, that if we take an order - of - magnitude approximation for M and V_0, the third term in the binding energy formula is negligible compared to the first two terms.

CHAPTER V

MASS FORMULA CALCULATIONS

Let us consider a $q\bar{q}$ pair bound in a potential of average strength U_o. If the quark velocities in the low lying states are to be non-relativistic and the quark masses very large, then the potential is presumably 'flat-bottomed' in the manner of a square-well or harmonic oscillator. For reasonable and deep potentials of this sort the level structure is roughly independent of the shape and there is no loss in generality in supposing that the potential is a square well.

Referring to equation no. (131) in Chapter V, we can now write the general mass formula as

$$M^2 = M1 + n\Delta + V_1 \vec{L}.\vec{S} + V_2 \vec{S}_1.\vec{S}_2 + (\nu + \tfrac{L}{2} + 1)^2 V_3 \qquad (1)$$

where M1 is the square of the term representing twice the quark mass minus the average well-depth; n is the number of strange quarks making up the boson, S is the spin-operator and L is the orbital angular momentum operator, ν is the total quantum number and Δ, V_1, V_2 and V_3 are constants.

The $\vec{L}.\vec{S}$ term is given by the formula

$$\vec{L}.\vec{S} = (\vec{J}^2 - \vec{L}^2 - \vec{S}^2)/2 \qquad (2)$$

This yields for the triplet:

$$\vec{L}.\vec{S} = L \quad \text{for } J = L+1 \qquad (3)$$

$$= -1 \quad \text{for } J = L \qquad (4)$$

$$= -L-1 \quad \text{for } J = L-1 \qquad (5)$$

For the singlet

$$\vec{L}\cdot\vec{S} = 0 \tag{6}$$

The $\vec{S}_1 . \vec{S}_2$ term is given by

$$\vec{S}_1\cdot\vec{S}_2 = \frac{\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2}{2} \tag{7}$$

This yields

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{4} \quad \text{for} \quad S = 1 \tag{8}$$

$$= -3/4 \quad \text{for} \quad S = 0 \tag{9}$$

The number of excess λ quarks making up the bosons is found by finding the expectation value of the quark content with respect to the Hamiltonian. A complete list is shown in the following table. A list giving the particle masses and the appropriate quantum numbers is also given in Table VI. Two graphs corresponding to this table are given in Pages 56 and 57.

TABLE V

NUMBER OF STRANGE QUARKS FOR MESONS

Particle	Number of Excess λ Quarks
π	0
ρ	0
δ	0
A1	0
A2	0
B	0
π_A	0
K	1
K*	1
K_N	1
K*	1
K**	1
K*	1
η	1.333
ω	0
S*	0
f	0
E	0
Ø	0
D	2
f'	2
X_o	0.666

TABLE VI

MESON MASSES WITH ν, L, L.S, $S_1.S_2$ VALUES

Particle	Mass in MeV	ν	L	L.S	$S_1.S_2$
π	135	0	0	0	-3/4
K	498	0	0	0	-3/4
η	549	0	0	0	-3/4
ρ	765	0	0	0	1/4
K*	892	0	0	0	1/4
ϕ	1019	0	0	0	1/4
ω	784	0	0	0	1/4
δ	966	1	1	-2	1/4
K_N	1170	1	1	-2	1/4
S*	1070	1	1	-2	1/4
A1	1070	1	1	-1	1/4
K*	1230	1	1	-1	1/4
D	1285	1	1	-1	1/4
A2	1300	1	1	1	1/4
K**	1420	1	1	1	1/4
f'	1514	1	1	1	1/4
f	1260	1	1	1	1/4
B	1235	1	1	0	-3/4
K*	1320	1	1	0	-3/4
E	1422	2	0	0	-3/4
π_A	1640	2	2	0	-3/4

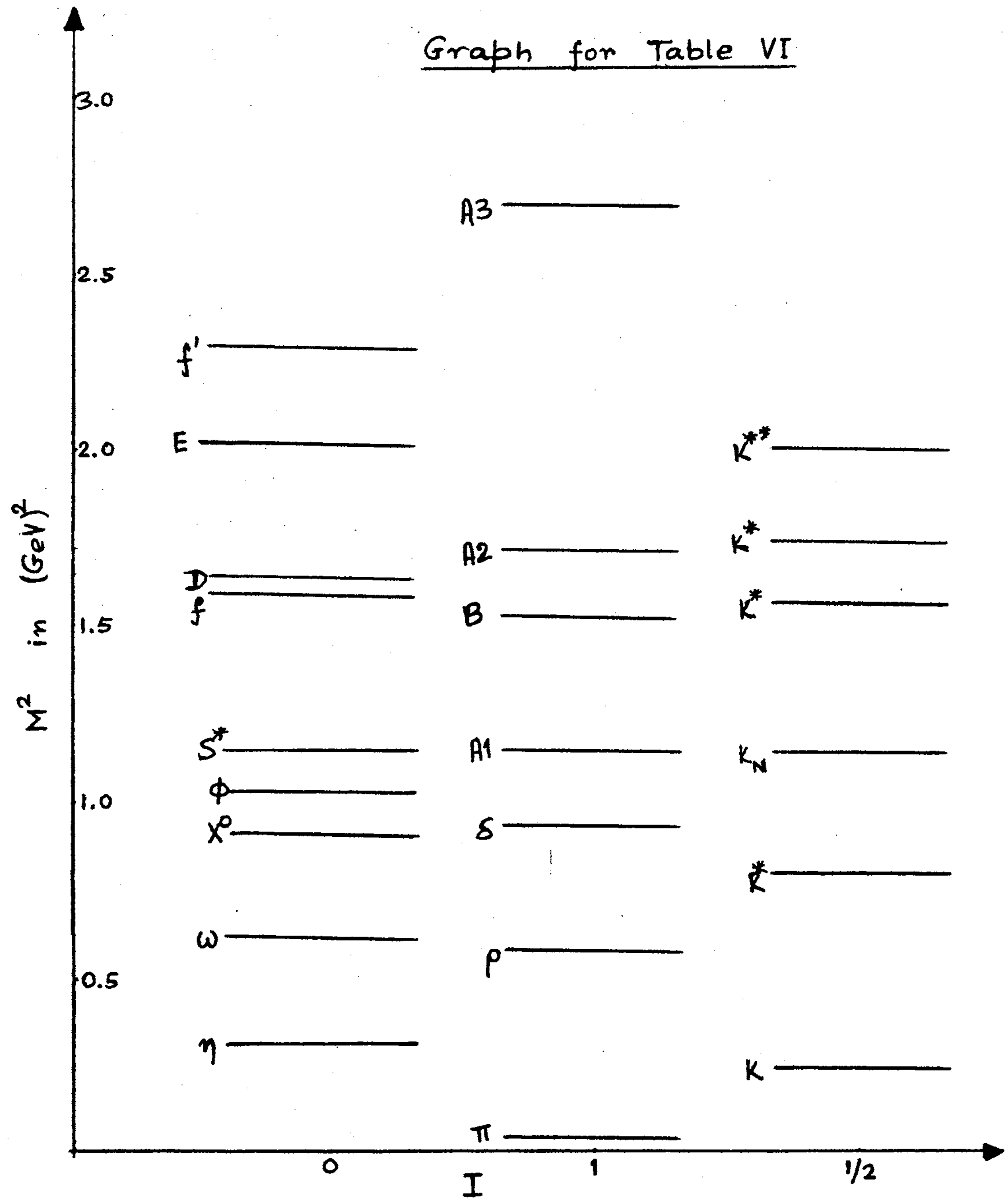

Figure 8. M^2 Versus I-Values for Mesons

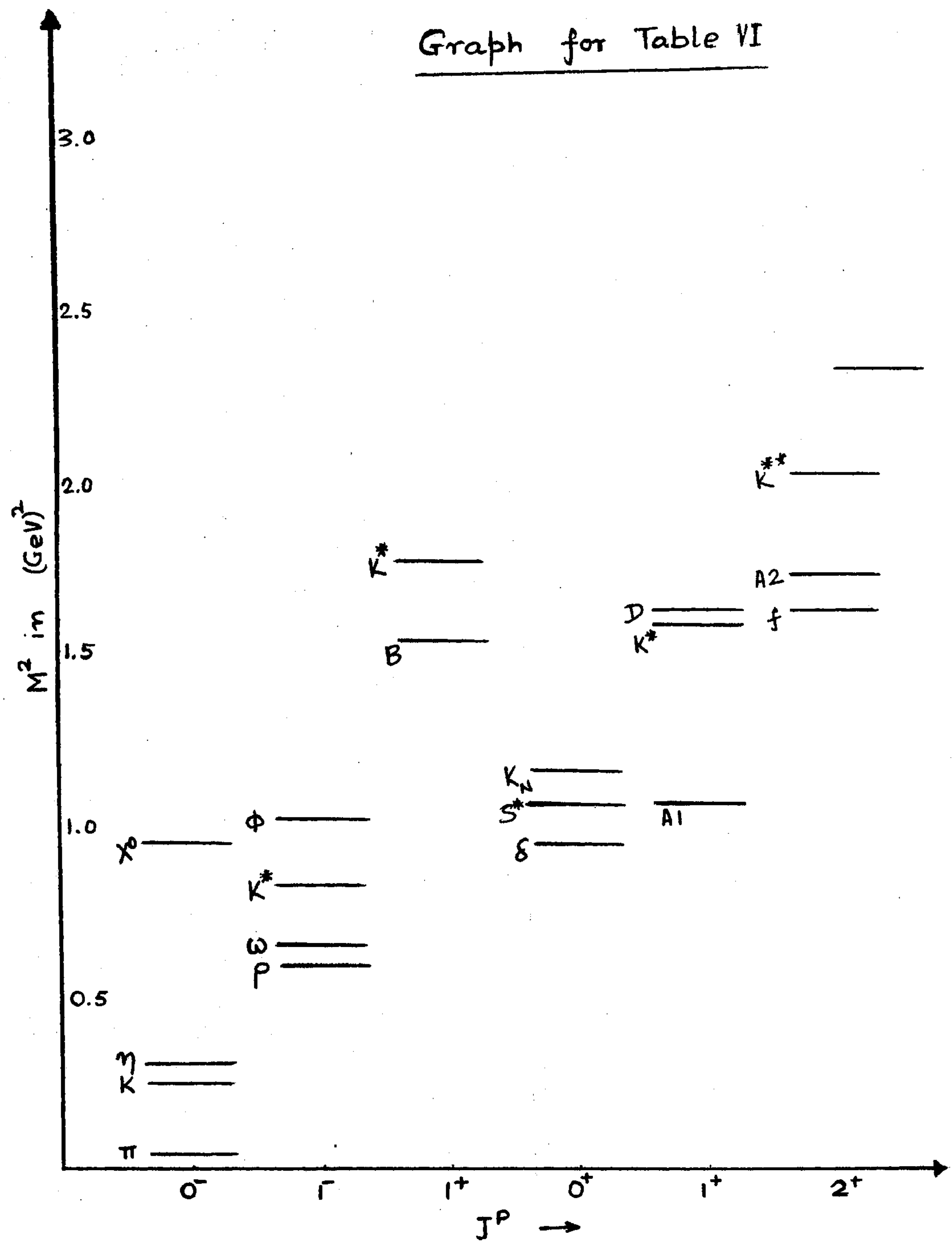

Figure 9. M^2 Versus J^P-Values for Mesons.

So according to our mass formula we have five unknown parameters to fit the experimentally established twenty-one masses. To obtain a best fit, we need to minimize (49)

$$R = \sum_{i=1}^{N=21} \left\{ \frac{t_i - a - bx_i - cz_i - dw_i - ey_i}{t_i} \right\}^2 \qquad (10)$$

where

$$\begin{aligned} t_i &= M^2_{exp} \\ a &= M\,1 \\ b &= \Delta \\ x_i &= n_1 \\ c &= V_1 \\ z_i &= L.S \\ d &= V_2 \\ w_i &= S_1.S_2 \\ e &= V_3 \\ y_i &= (\nu + \frac{L}{2} + 1)^2 \quad \ldots. \end{aligned} \qquad (11)$$

$$\therefore R = \sum_{i=1}^{21} \left[1 - \frac{a}{t_i} - \frac{bx_i}{t_i} - c\frac{z_i}{t_i} - d\frac{\omega_i}{t_i} - e\frac{y_i}{t_i} \right]^2$$

$$\therefore \frac{\partial R}{\partial a} = -2 \sum \left[1 - \frac{a}{t_i} - b\frac{x_i}{t_i} - c\frac{z_i}{t_i} - d\frac{\omega_i}{t_i} - e\frac{y_i}{t_i} \right] \frac{1}{t_i} = 0$$

$$\frac{\partial R}{\partial b} = -2 \sum \left[1 - \frac{a}{t_i} - b\frac{x_i}{t_i} - c\frac{z_i}{t_i} - d\frac{\omega_i}{t_i} - e\frac{y_i}{t_i} \right] \frac{x_i}{t_i} = 0$$

$$\frac{\partial R}{\partial c} = -2 \sum \left[1 - \frac{a}{t_i} - b\frac{x_i}{t_i} - c\frac{z_i}{t_i} - d\frac{\omega_i}{t_i} - e\frac{y_i}{t_i} \right] \frac{z_i}{t_i} = 0$$

(12)

$$\frac{\partial R}{\partial d} = -2\sum\left[1 - \frac{a}{t_i} - b\frac{x_i}{t_i} - c\frac{z_i}{t_i} - d\frac{\omega_i}{t_i} - e\frac{y_i}{t_i}\right]\frac{\omega_i}{t_i} = 0$$

$$\frac{\partial R}{\partial e} = -2\sum\left[1 - \frac{a}{t_i} - b\frac{x_i}{t_i} - c\frac{z_i}{t_i} - d\frac{\omega_i}{t_i} - e\frac{y_i}{t_i}\right]\frac{y_i}{t_i} = 0 \qquad (13)$$

So, we can write these equations in matrix form as:

$$\begin{bmatrix} \sum\frac{1}{t_i} \\ \sum\frac{x_i}{t_i} \\ \sum\frac{z_i}{t_i} \\ \sum\frac{\omega_i}{t_i} \\ \sum\frac{y_i}{t_i} \end{bmatrix} = \begin{bmatrix} \sum\frac{1}{t_i^2} & \sum\frac{x_i}{t_i^2} & \sum\frac{z_i}{t_i^2} & \sum\frac{\omega_i}{t_i^2} & \sum\frac{y_i}{t_i^2} \\ \sum\frac{x_i}{t_i^2} & \sum\frac{x_i^2}{t_i^2} & \sum\frac{x_i z_i}{t_i^2} & \sum\frac{\omega_i x_i}{t_i^2} & \sum\frac{y_i x_i}{t_i^2} \\ \sum\frac{z_i}{t_i^2} & \sum\frac{x_i z_i}{t_i^2} & \sum\frac{z_i^2}{t_i^2} & \sum\frac{\omega_i z_i}{t_i^2} & \sum\frac{y_i z_i}{t_i^2} \\ \sum\frac{\omega_i}{t_i^2} & \sum\frac{x_i \omega_i}{t_i^2} & \sum\frac{z_i \omega_i}{t_i^2} & \sum\frac{\omega_i^2}{t_i^2} & \sum\frac{y_i \omega_i}{t_i^2} \\ \sum\frac{y_i}{t_i^2} & \sum\frac{x_i y_i}{t_i^2} & \sum\frac{z_i y_i}{t_i^2} & \sum\frac{\omega_i y_i}{t_i^2} & \sum\frac{y_i^2}{t_i^2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

Evaluating the constants a, b, c, d and e with the help of a computer program (Appendix A) yields the values as:

$$M1 = a = 215071.5818717349 \quad (MeV)^2$$

$$\Delta = b = 225594.938004106 \quad (MeV)^2$$

$$V_1 = c = 273067.3632776805 \quad (MeV)^2$$

$$V_2 = d = 525934.618160206l \quad (MeV)^2$$

$$V_3 = e = 197697.6403414621 \quad (MeV)^2$$

Using these values we have constructed a complete table of quark-antiquarl meson states, as shown in Table VII. The table is correct up to three significant decimal places. We have shown a comparison between the predicted masses and the experimental masses in Table VIII. We are also able to identify our predicted particles with the following, not fully confirmed, particles in Table IX. We have not taken into account the X^o particle in our calculations. This is due to the appreciable mass difference between the singlet and the octet in the pseudoscalar nonet. If we take into account this mass difference, the X^o can easily be fitted in our table. The average deviation turns out to be 4.7% for all the established mesons to date.

TABLE VII

COMPLETE TABLE OF QUARK-ANTIQUARK MESON STATES

ν	L	S	JPC	I=1	I=½	I=0	I=0	L.S	$S_1.S_2$
0	0	0	0-+	π (135)	K (494)	η (564)	X^o(958)	0	-3/4
		1	1--	ρ (738)	K*(877)	φ (998)	ω (738)	0	1/4
1	1	0	1+-	B (1032)	K*(1138)	X_1(1227)	X_2(1027)	0	-3/4
		1	0++	δ (1022)	K_N(1128)	X_3(1219)	S*(1022)	-2	1/4
		1	1++	A1(1148)	K*(1242)	D (1330)	X_4(1144)	-1	1/4
		1	2++	A2(1365)	K**(1445)	f'(1521)	f (1365)	1	1/4
2	0	0	0-+	X_5(1265)	X_6(1351)	X_7(1432)	E (1265)	0	-3/4
		1	1--	X_8(1458)	X_9(1533)	X_{10}(1605)	X_{11}(1458)	0	1/4
	2	0	2-+	A3(1727)	X_{12}(1791)	X_{13}(1853)	X_{14}(1727)	0	-3/4
		1	1--	X_{15}(1640)	X_{16}(1707)	X_{17}(1772)	X_{18}(1640)	-3	1/4
		1	2--	X_{19}(1799)	X_{20}(1861)	X_{21}(1920)	X_{22}(1799)	-1	1/4
		1	3--	X_{23}(2013)	X_{24}(2069)	X_{25}(2123)	X_{26}(2013)	-2	1/4
3	1	0	1+-	X_{27}(1955)	X_{28}(2012)	X_{29}(2067)	X_{30}(1955)	0	-3/4
		1	0++	X_{31}(1950)	X_{32}(2007)	X_{33}(2063)	X_{34}(1950)	-2	1/4
		1	1++	X_{35}(2019)	X_{36}(2074)	X_{37}(2128	X_{38}(2019)	-1	1/4
		1	2++	X_{39}(2150)	X_{40}(2202)	X_{41}(2253)	X_{42}(2150)	1	1/4
	3	0	3+-	X_{43}(2408)	X_{44}(2455)	X_{45}(2500)	X_{46}(2409)	0	-3/4
		1	2++	X_{47}(2288)	X_{48}(2337)	X_{49}(2384)	X_{50}(2288)	-4	1/4
		1	3++	X_{51}(2460)	X_{52}(2506)	X_{53}(2550)	X_{54}(2460)	-1	1/4
		1	4++	X_{55}(2673)	X_{56}(2715)	X_{57}(2756)	X_{58}(2673)	3	1/4

TABLE VII(Continued)

ν	L	S	JPC	I=1	I=½	1=0	I=0	L.S	$S_1.S_2$
4	0	0	0-+	X_{59}(2182	X_{60}(2233)	X_{61}(2283)	X_{62}(2182)	0	-3/4
		1	1--	X_{63}(2300)	X_{64}(2348)	X_{65}(2396)	X_{66}(2299)	0	1/4
	2	0	2-+	X_{67}(2634)	X_{68}(2676)	X_{69}(2718)	X_{70}(2633)	0	-3/4
		1	1--	X_{71}(2578)	X_{72}(2621)	X_{73}(2664)	X_{74}(2577)	-3	1/4
		1	2--	X_{75}(2682)	X_{76}(2723)	X_{77}(2764)	X_{78}(2681)	-1	1/4
		1	3--	X_{79}(2830)	X_{80}(2869)	X_{81}(2909)	X_{82}(2830)	2	1/4
	4	0	4-+	X_{83}(3083)	X_{84}(3119)	X_{85}(3156)	X_{86}(3083)	0	-3/4
		1	3--	X_{87}(2944)	X_{88}(2982)	X_{89}(3620)	X_{90}(2944)	-5	1/4
		1	4--	X_{91}(3124)	X_{92}(3160)	X_{93}(3195)	X_{94}(3124)	-1	1/4
		1	5--	X_{95}(3336)	X_{96}(3369)	X_{97}(3403)	X_{98}(3335)	4	1/4

TABLE VIII

COMPARISON OF EXPERIMENTAL AND PREDICTED MESONIC MASSES

Particle	Experimental Mass	Predicted Mass	% Error
π	135	135	0%
K	498	494	0.8%
η	549	564	2.8%
ρ	765	738	3.5%
K*	892	877	1.8%
ϕ	1019	998	2.0%
ω	784	738	6.0%
δ	966	1022	5.8%
K_N	1170	1128	3.5%
S*	1070	1022	4.0%
A1	1070	1148	7.0%
K*	1230	1242	0.9%
D	1285	1330	3.4%
A2	1300	1365	5.0%
K**	1420	1445	1.8%
f'	1514	1521	0.4%
f	1260	1365	8.0%
B	1235	1032	15.0%
K*	1320	1138	13.0%
E	1422	1265	10.0%
π_A	1640	1727	5.3%

TABLE IX

COMPARISON OF PREDICTED MESONS WITH NEW MESONS

Predicted JPC	Predicted Mesons	New Mesons	Predicted Mass	Experimental Mass
0-+	X_7	X_o	1432	1430
1--	X_8	X_1	1458	1440
1--	X_{15}	ρ'	1640	1660
1--	X_{18}	ω	1640	1675
2--	X_{19}	X^-	1799	1795
3+-	X_{43}	U	2408	2360
2++	X_{49}	$N\bar{N}$	2384	2375
1--	X_{16}	K_N	1707	1660
2--	X_{20}	K_N	1861	1760

CHAPTER VI

CONCLUSIONS

We have presented in this thesis a model of the bosons in which they are viewed as bound states of a quark and an antiquark moving in a very deep potential. Some degree of symmetry has been implied in the model in two different ways. First, invariance under the isotopic spin transformations has been assumed to hold for two members of the quark triplet. Second, the binding potential has been assumed to be independent of the isotopic spin state of the bound pair. In comparing the model with real life, one finds some comforting successes. The mass difference between the λ quark and nucleon quarks that describes the pseudoscalar and vector mass splittings also works for the tensor (2^+) nonet. The major importance of the non-relativistic quark approach lies in its potential for extrapolations of mass-spectra. Of course, some difficulties have already presented themselves. The B (1235) and K* (1230) mesons seem not to fit very well with our parameters. Though the square-well parameter ma^2 has been determined, it's difficult to quote m separately since the value of a is not known.

At this moment, it is hard to take these difficulties seriously, remembering the uncertain state of our experimental information. We have been able to match our predictions with some new mesons, not yet fully confirmed experimentally. This seems to be a good indication of the success of the model. And, as mentioned before, all mesons, established to date, fit into our formula with errors of the order of a few per cent.

This, then is the quark model for mesons. Inspite of the fact, that no quark has yet been discovered in nature, most of the successes of the model are astonishing and essential features of the mathematical structure of the quark model must survive the test of time.

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APPENDIX

PROGRAM FOR CALCULATION OF MESONIC MASSES

This program, written in the Fortran IV language will calculate the constants a, b, c, d, e, and print out simultaneously the meson masses using the mass formula and the values of the constants. The data cards must give the values of the variables t, x, y, z and w. The masses are then adjusted to the appropriate table.

```
$JOB NOSUBCHK,TIME=30     ASCK     RAY
      IMPLICIT  REAL *8(A-H,O-Z)
      REAL * 4 EPS
      DIMENSION    T(119),X(119),Y(119),Z(119),W(119),R(5),A(25),S(5,5)
      N=21
      T(1)=135.D0**2
      T(2)=498.D0**2
      T(3)=765.D0**2
      T(4)=892.D0**2
      T(5)=1019.D0**2
      T(6)=784.D0**2
      T(7)=966.D0**2
      T(8)=1170.D0**2
      T(9)=1070.D0**2
      T(10)=1230.D0 **2
      T(11)=1300.D0**2
      T(12)=1420.D0**2
      T(13)=1235.D0**2
      T(14)=1320.D0**2
      T(15)=1285.D0**2
      T(16)=1260.D0**2
      T(17)=1514.D0**2
      T(18)=1070.D0**2
      T(19)=1422.D0**2
      T(20)=1640.D0**2
      T(21)=549.D0**2
      X(1)=0.0D0
      X(2)=1.0D0
      X(3)=0.0D0
      X(4)=1.0D0
      X(5)=2.0D0
      X(6)=0.0D0
      X(7)=0.0D0
      X(8)=1.0D0
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      X(11)=0.0D0
      X(12)=1.0D0
      X(13)=0.0D0
      X(14)=1.0D0
      X(15)=2.0D0
      X(16)=0.0D0
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X(101)=0.0D0
X(102)=0.0D0
X(103)=0.0D0
X(104)=0.0D0
X(105)=0.0D0
X(106)=0.0D0
X(107)=0.0D0
X(108)=0.0D0
X(109)=0.0D0
X(110)=0.0D0
X(111)=0.0D0
X(112)=0.0D0
X(113)=0.0D0
X(114)=0.0D0
X(115)=0.0D0
X(116)=0.0D0
X(117)=0.0D0
X(118)=0.0D0
X(119)=0.0D0
Z(1)=0.0D0
Z(2)=0.0D0
Z(3)=0.0D0
Z(4)=0.0D0
Z(5)=0.0D0
Z(6)=0.0D0
Z(7)=-2.0D0
Z(8)=-2.0D0
Z(9)=-1.0D0
Z(10)=-1.0D0
Z(11)=1.0D0
Z(12)=1.0D0
Z(13)=0.0D0
Z(14)=0.0D0
Z(15)=-1.0D0
Z(16)=1.0D0
Z(17)=1.0D0
Z(18)=-2.0D0
Z(19)=0.0D0
Z(20)=0.0D0
Z(21)=0.0D0
Z(22)=0.0D0
Z(23)=0.0D0
Z(24)=2.0D0
Z(25)=-1.0D0
Z(26)=-3.0D0
Z(27)=0.0D0
Z(28)=1.0D0
Z(29)=-1.0D0
Z(30)=-2.0D0
Z(31)=0.0D0
Z(32)=3.0D0
Z(33)=-1.0D0
Z(34)=-4.0D0
Z(35)=0.0D0
```

```
Z(36)=0.0D0
Z(37)=0.0D0
Z(38)=2.0D0
Z(39)=-1.0D0
Z(40)=-3.0D0
Z(41)=0.0D0
Z(42)=4.0D0
Z(43)=-1.0D0
Z(44)=-5.0D0
Z(45)=0.0D0
Z(46)=0.0D0
Z(47)=0.0D0
Z(48)=2.0D0
Z(49)=-1.0D0
Z(50)=-3.0D0
Z(51)=0.0D0
Z(52)=1.0D0
Z(53)=-1.0D0
Z(54)=-2.0D0
Z(55)=0.0D0
Z(56)=3.0D0
Z(57)=-1.0D0
Z(58)=-4.0D0
Z(59)=0.0D0
Z(60)=0.0D0
Z(61)=0.0D0
Z(62)=2.0D0
Z(63)=-1.0D0
Z(64)=-3.0D0
Z(65)=0.0D0
Z(66)=4.0D0
Z(67)=-1.0D0
Z(68)=-5.0D0
Z(69)=-2.0D0
Z(70)=0.0D0
Z(71)=0.0D0
Z(72)=0.0D0
Z(73)=0.0D0
Z(74)=2.0D0
Z(75)=-1.0D0
Z(76)=-3.0D0
Z(77)=0.0D0
Z(78)=1.0D0
Z(79)=-1.0D0
Z(80)=-2.0D0
Z(81)=0.0D0
Z(82)=3.0D0
Z(83)=-1.0D0
Z(84)=-4.0D0
Z(85)=0.0D0
Z(86)=0.0D0
Z(87)=0.0D0
Z(88)=2.0D0
Z(89)=-1.0D0
Z(90)=-3.0D0
Z(91)=0.0D0
Z(92)=4.0D0
Z(93)=-1.0D0
Z(94)=-5.0D0
Z(95)=-1.0D0
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```
      Z(96)=0.0D0
      Z(97)=0.0D0
      Z(98)=0.0D0
      Z(99)=2.0D0
      Z(100)=-1.0D0
      Z(101)=-3.0D0
      Z(102)=0.0D0
      Z(103)=1.0D0
      Z(104)=-1.0D0
      Z(105)=-2.0D0
      Z(106)=0.0D0
      Z(107)=3.0D0
      Z(108)=-1.0D0
      Z(109)=-4.0D0
      Z(110)=0.0D0
      Z(111)=0.0D0
      Z(112)=0.0D0
      Z(113)=2.0D0
      Z(114)=-1.0D0
      Z(115)=-3.0D0
      Z(116)=0.0D0
      Z(117)=4.0D0
      Z(118)=-1.0D0
      Z(119)=-5.0D0
      W(1)=-0.75D0
      W(2)=-0.75D0
      W(3)=0.25D0
      W(4)=0.25D0
      W(5)=0.25D0
      W(6)=0.25D0
      W(7)=0.25D0
      W(8)=0.25D0
      W(9)=0.25D0
      W(10)=0.25D0
      W(11)=0.25D0
      W(12)=0.25D0
      W(13)=-0.75D0
      W(14)=-0.75D0
      W(15)=0.25D0
      W(16)=0.25D0
      W(17)=0.25D0
      W(18)=0.25D0
      W(19)=-0.75D0
      W(20)=-0.75D0
      W(21)=-0.75D0
      W(22)=-0.75D0
      W(23)=0.25D0
      W(24)=0.25D0
      W(25)=0.25D0
      W(26)=0.25D0
      W(27)=-0.75D0
      W(28)=0.25D0
      W(29)=0.25D0
      W(30)=0.25D0
      W(31)=-0.75D0
      W(32)=0.25D0
      W(33)=0.25D0
      W(34)=0.25D0
      W(35)=-0.75D0
      W(36)=0.25D0
```

```
      W(37)=-0.75D0
      W(38)=0.25D0
      W(39)=0.25D0
      W(40)=0.25D0
      W(41)=-0.75D0
      W(42)=0.25D0
      W(43)=0.25D0
      W(44)=0.25D0
      W(45)=-0.75D0
      W(46)=0.25D0
      W(47)=-0.75D0
      W(48)=0.25D0
      W(49)=0.25D0
      W(50)=0.25D0
      W(51)=-0.75D0
      W(52)=0.25D0
      W(53)=0.25D0
      W(54)=0.25D0
      W(55)=-0.75D0
      W(56)=0.25D0
      W(57)=0.25D0
      W(58)=0.25D0
      W(59)=-0.75D0
      W(60)=0.25D0
      W(61)=-0.75D0
      W(62)=0.25D0
      W(63)=0.25D0
      W(64)=0.25D0
      W(65)=-0.75D0
      W(66)=0.25D0
      W(67)=0.25D0
      W(68)=0.25D0
      W(69)=0.25D0
      W(70)=-0.75D0
      W(71)=-0.75D0
      W(72)=0.25D0
      W(73)=-0.75D0
      W(74)=0.25D0
      W(75)=0.25D0
      W(76)=0.25D0
      W(77)=-0.75D0
      W(78)=0.25D0
      W(79)=0.25D0
      W(80)=0.25D0
      W(81)=-0.75D0
      W(82)=0.25D0
      W(83)=0.25D0
      W(84)=0.25D0
      W(85)=-0.75D0
      W(86)=0.25D0
      W(87)=-0.75D0
      W(88)=0.25D0
      W(89)=0.25D0
      W(90)=0.25D0
      W(91)=-0.75D0
      W(92)=0.25D0
      W(93)=0.25D0
      W(94)=0.25D0
      W(95)=0.25D0
      W(96)=-0.75D0
```

```
W(97)=0.25D0
W(98)=-0.75D0
W(99)=0.25D0
W(100)=0.25D0
W(101)=0.25D0
W(102)=-0.75D0
W(103)=0.25D0
W(104)=0.25D0
W(105)=0.25D0
W(106)=-0.75D0
W(107)=0.25D0
W(108)=0.25D0
W(109)=0.25D0
W(110)=-0.75D0
W(111)=0.25D0
W(112)=-0.75D0
W(113)=0.25D0
W(114)=0.25D0
W(115)=0.25D0
W(116)=-0.75D0
W(117)=0.25D0
W(118)=0.25D0
W(119)=0.25D0
Y(1)=1.0D0
Y(2)=1.0D0
Y(3)=1.0D0
Y(4)=1.0D0
Y(5)=1.0D0
Y(6)=1.0D0
Y(7)=6.3D0
Y(8)=6.3D0
Y(9)=6.3D0
Y(10)=6.3D0
Y(11)=6.3D0
Y(12)=6.3D0
Y(13)=6.3D0
Y(14)=6.3D0
Y(15)=6.3D0
Y(16)=6.3D0
Y(17)=6.3D0
Y(18)=6.3D0
Y(19)=9.0D0
Y(20)=16.0D0
Y(21)=1.0D0
Y(22)=9.0D0
Y(23)=9.0D0
Y(24)=16.0D0
Y(25)=16.0D0
Y(26)=16.0D0
Y(27)=20.25D0
Y(28)=20.25D0
Y(29)=20.25D0
Y(30)=20.25D0
Y(31)=30.25D0
Y(32)=30.25D0
Y(33)=30.25D0
Y(34)=30.25D0
Y(35)=25.0D0
Y(36)=25.0D0
Y(37)=36.0D0
```

```
      Y(38)=36.0D0
      Y(39)=36.0D0
      Y(40)=36.0D0
      Y(41)=49.0D0
      Y(42)=49.0D0
      Y(43)=49.0D0
      Y(44)=49.0D0
      Y(45)=9.0D0
      Y(46)=9.0D0
      Y(47)=16.0D0
      Y(48)=16.0D0
      Y(49)=16.0D0
      Y(50)=16.0D0
      Y(51)=20.25D0
      Y(52)=20.25D0
      Y(53)=20.25D0
      Y(54)=20.25D0
      Y(55)=30.25D0
      Y(56)=30.25D0
      Y(57)=30.25D0
      Y(58)=30.25D0
      Y(59)=25.0D0
      Y(60)=25.0D0
      Y(61)=36.0D0
      Y(62)=36.0D0
      Y(63)=36.0D0
      Y(64)=36.0D0
      Y(65)=49.0D0
      Y(66)=49.0D0
      Y(67)=49.0D0
      Y(68)=49.0D0
      Y(69)=6.25D0
      Y(70)=6.25D0
      Y(71)=9.0D0
      Y(72)=9.0D0
      Y(73)=16.0D0
      Y(74)=16.0D0
      Y(75)=16.0D0
      Y(76)=16.0D0
      Y(77)=20.25D0
      Y(78)=20.25D0
      Y(79)=20.25D0
      Y(80)=20.25D0
      Y(81)=30.25D0
      Y(82)=30.25D0
      Y(83)=30.25D0
      Y(84)=30.25D0
      Y(85)=25.0D0
      Y(86)=25.0D0
      Y(87)=36.0D0
      Y(88)=36.0D0
      Y(89)=36.0D0
      Y(90)=36.0D0
      Y(91)=49.0D0
      Y(92)=49.0D0
      Y(93)=49.0D0
      Y(94)=49.0D0
      Y(95)=6.25D0
      Y(96)=6.25D0
      Y(97)=9.0D0
```

```
      Y(98)=16.0D0
      Y(99)=16.0D0
      Y(100)=16.0D0
      Y(101)=16.0D0
      Y(102)=20.25D0
      Y(103)=20.25D0
      Y(104)=20.25D0
      Y(105)=20.25D0
      Y(106)=30.25D0
      Y(107)=30.25D0
      Y(108)=30.25D0
      Y(109)=30.25D0
      Y(110)=25.0D0
      Y(111)=25.0D0
      Y(112)=36.0D0
      Y(113)=36.0D0
      Y(114)=36.0D0
      Y(115)=36.0D0
      Y(116)=49.0D0
      Y(117)=49.0D0
      Y(118)=49.0D0
      Y(119)=49.0D0
C     TO  FIND  S(1,1)
      TSUM=0.D0
      DO  10  I=1,21
   10 TSUM=1.D0/(T(I)**2)+TSUM
      S(1,1)=TSUM
C     TO  FIND  S(1,2)
      XTSUM=0.D0
      DO  20  I=1,21
   20 XTSUM=X(I)/(T(I)**2)+XTSUM
      S(1,2)=XTSUM
C     TO  FIND  S(1,3)
      ZTSUM=0.D0
      DO  30  I=1,21
   30 ZTSUM=Z(I)/(T(I)**2)+ZTSUM
      S(1,3)=ZTSUM
C     TO  FIND  S(1,4)
      WTSUM=0.D0
      DO  40  I=1,21
   40 WTSUM=W(I)/(T(I)**2)+WTSUM
      S(1,4)=WTSUM
C     TO    FIND     S(1,5)
      YTSUM=0.D0
      DO  50  I=1,21
   50 YTSUM=Y(I)/(T(I)**2)+YTSUM
      S(1,5)=YTSUM
C     TO    FIND    S(2,1)
      S(2,1)=S(1,2)
C     TO    FIND    S(2,2)
      XXTSUM=0.D0
      DO      60  I=1,21
   60 XXTSUM=(X(I)*X(I))/(T(I)**2)+XXTSUM
      S(2,2)=XXTSUM
C     TO    FIND    S(2,3)
      XZTSUM=0.D0
      DO      70  I=1,21
   70 XZTSUM=(X(I)*Z(I))/(T(I)**2)+XZTSUM
      S(2,3)=XZTSUM
C     TO    FIND    S(2,4)
```

```
      XWTSUM=0.D0
      DO    80  I=1,21
   80 XWTSUM=(X(I)*W(I))/(T(I)**2)+XWTSUM
      S(2,4)=XWTSUM
C     TO   FIND    S(2,5)
      XYTSUM=0.D0
      DO    90  I=1,21
   90 XYTSUM=(X(I)*Y(I))/(T(I)**2)+XYTSUM
      S(2,5)=XYTSUM
C     TO  FIND   S(3,1)
      S(3,1)=S(1,3)
C     TO  FIND   S(3,2)
      S(3,2)=S(2,3)
C     TO  FIND   S(3,3)
      ZZTSUM=0.D0
      DO   100   I=1,21
  100 ZZTSUM=(Z(I)*Z(I))/(T(I)**2)+ZZTSUM
      S(3,3)=ZZTSUM
C     TO  FIND   S(3,4)
      WZTSUM=0.D0
      DO   110   I=1,21
  110 WZTSUM=(W(I)*Z(I))/(T(I)**2)+WZTSUM
      S(3,4)=WZTSUM
C     TO   FIND   S(3,5)
      YZTSUM=0.D0
      DO   120   I=1,21
  120 YZTSUM=(Y(I)*Z(I))/(T(I)**2)+YZTSUM
      S(3,5)=YZTSUM
C     TO   FIND   S(4,1)
      S(4,1)=S(1,4)
C     TO   FIND   S(4,2)
      S(4,2)=S(2,4)
C     TO   FIND   S(4,3)
      S(4,3)=S(3,4)
C     TO   FIND    S(4,4)
      WWTSUM=0.D0
      DO    130  I=1,21
  130 WWTSUM=(W(I)*W(I))/(T(I)**2)+WWTSUM
      S(4,4)=WWTSUM
C     TO   FIND   S(4,5)
      YWTSUM=0.D0
      DO    140  I=1,21
  140 YWTSUM=(Y(I)*W(I))/(T(I)**2)+YWTSUM
      S(4,5)=YWTSUM
C     TO     FIND    S(5,1)
      S(5,1)=S(1,5)
C     TO     FIND    S(5,2)
      S(5,2)=S(2,5)
C     TO     FIND    S(5,3)
      S(5,3)=S(3,5)
C     TO     FIND    S(5,4)
      S(5,4)=S(4,5)
C     TO     FIND    S(5,5)
      YYTSUM=0.D0
      DO    150   I=1,21
  150 YYTSUM=(Y(I)*Y(I))/(T(I)**2)+YYTSUM
      S(5,5)=YYTSUM
C     TO   FIND   R(1)
      SUM=0.D0
      DO   160   I=1,21
```

```
  160 SUM=1.D0/T(I)+SUM
      R(1)=SUM
C     TO    FIND     R(2)
      XSUM=0.D0
      DO   170   I=1,21
  170 XSUM=X(I)/T(I)+XSUM
      R(2)=XSUM
C     TO   FIND     R(3)
      ZSUM=0.D0
      DO   180   I=1,21
  180 ZSUM=Z(I)/T(I)+ZSUM
      R(3)=ZSUM
C     TO   FIND     R(4)
      WSUM=0.D0
      DO   190   I=1,21
  190 WSUM=W(I)/T(I)+WSUM
      R(4)=WSUM
C     TO  FIND   R(5)
      YSUM=0.D0
      DO   200   I=1,21
  200 YSUM=Y(I)/T(I)+YSUM
      R(5)=YSUM
      DO   1000  J=1,5
      DO   1000  I=1,5
      L=5*(J-1)+I
 1000 A(L)=S(I,J)
      EPS = 1.E-16
      CALL SYSTEM(R,A,5,1,EPS,IER)
      DO 500 K = 1,5
  500 WRITE(6,600) R(K)
  600 FORMAT(D26.16)
      RM1=R(1)
      DEL=R(2)
      Y1=R(3)
      Y2=R(4)
      VO=R(5)
      DO    2000  K=1,21
      RMASS2=RM1+X(K)*DEL+Y1*Z(K)+Y2*W(K)+VO*Y(K)
 2000 WRITE(6,5)K,RMASS2,T(K),DSQRT(RMASS2),DSQRT(T(K))
*EXTENSION*  OTHER COMPILERS MAY NOT ALLOW EXPRESSIONS IN OUTPUT LISTS
*EXTENSION*  OTHER COMPILERS MAY NOT ALLOW EXPRESSIONS IN OUTPUT LISTS
    5 FORMAT(I5,4D26.16)
      DO    5000    K=22,119
      RMASS3=RM1+X(K)*DEL+Y1*Z(K)+Y2*W(K)+VO*Y(K)
 5000 WRITE(6,7)K,RMASS3,DSQRT(RMASS3)
*EXTENSION*  OTHER COMPILERS MAY NOT ALLOW EXPRESSIONS IN OUTPUT LISTS
    7 FORMAT(I5,2D26.16)
      STOP
      END

      SUBROUTINE  SYSTEM(R,A,M,N,EPS,IER)
      IMPLICIT REAL * 8 (A-H,O-Z)
      REAL * 4 EPS
      DIMENSION A(1),R(1)
C
C     ..............................................................
C
C        SUBROUTINE DGELG
C
C        PURPOSE
```

```
C           TO SOLVE A GENERAL SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS. DELG  70
C                                                                       DELG  80
C        USAGE                                                          DELG  90
C           CALL DGELG(R,A,M,N,EPS,IER)                                 DELG 100
C                                                                       DELG 110
C        DESCRIPTION OF PARAMETERS                                      DELG 120
C           R      - DOUBLE PRECISION M BY N RIGHT HAND SIDE MATRIX     DELG 130
C                    (DESTROYED). ON RETURN R CONTAINS THE SOLUTIONS    DELG 140
C                    OF THE EQUATIONS.                                  DELG 150
C           A      - DOUBLE PRECISION M BY M COEFFICIENT MATRIX         DELG 160
C                    (DESTROYED).                                       DELG 170
C           M      - THE NUMBER OF EQUATIONS IN THE SYSTEM.             DELG 180
C           N      - THE NUMBER OF RIGHT HAND SIDE VECTORS.             DELG 190
C           EPS    - SINGLE PRECISION INPUT CONSTANT WHICH IS USED AS   DELG 200
C                    RELATIVE TOLERANCE FOR TEST ON LOSS OF             DELG 210
C                    SIGNIFICANCE.                                      DELG 220
C           IER    - RESULTING ERROR PARAMETER CODED AS FOLLOWS         DELG 230
C                    IER=0  - NO ERROR,                                 DELG 240
C                    IER=-1 - NO RESULT BECAUSE OF M LESS THAN 1 OR     DELG 250
C                             PIVOT ELEMENT AT ANY ELIMINATION STEP     DELG 260
C                             EQUAL TO 0,                               DELG 270
C                    IER=K  - WARNING DUE TO POSSIBLE LOSS OF SIGNIFI-  DELG 280
C                             CANCE INDICATED AT ELIMINATION STEP K+1,  DELG 290
C                             WHERE PIVOT ELEMENT WAS LESS THAN OR      DELG 300
C                             EQUAL TO THE INTERNAL TOLERANCE EPS TIMES DELG 310
C                             ABSOLUTELY GREATEST ELEMENT OF MATRIX A.  DELG 320
C                                                                       DELG 330
C        REMARKS                                                        DELG 340
C           INPUT MATRICES R AND A ARE ASSUMED TO BE STORED COLUMNWISE  DELG 350
C           IN M*N RESP. M*M SUCCESSIVE STORAGE LOCATIONS. ON RETURN    DELG 360
C           SOLUTION MATRIX R IS STORED COLUMNWISE TOO.                 DELG 370
C           THE PROCEDURE GIVES RESULTS IF THE NUMBER OF EQUATIONS M IS DELG 380
C           GREATER THAN 0 AND PIVOT ELEMENTS AT ALL ELIMINATION STEPS  DELG 390
C           ARE DIFFERENT FROM 0. HOWEVER WARNING IER=K - IF GIVEN -    DELG 400
C           INDICATES POSSIBLE LOSS OF SIGNIFICANCE. IN CASE OF A WELL  DELG 410
C           SCALED MATRIX A AND APPROPRIATE TOLERANCE EPS, IER=K MAY BE DELG 420
C           INTERPRETED THAT MATRIX A HAS THE RANK K. NO WARNING IS     DELG 430
C           GIVEN IN CASE M=1.                                          DELG 440
C                                                                       DELG 450
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  DELG 460
C           NONE                                                        DELG 470
C                                                                       DELG 480
C        METHOD                                                         DELG 490
C           SOLUTION IS DONE BY MEANS OF GAUSS-ELIMINATION WITH         DELG 500
C           COMPLETE PIVOTING.                                          DELG 510
C                                                                       DELG 520
C     ..................................................................DELG 530
C                                                                       DELG 540
      IF(M)23,23,1                                                      DELG 600
C                                                                       DELG 610
C     SEARCH FOR GREATEST ELEMENT IN MATRIX A                           DELG 620
    1 IER=0                                                             DELG 630
      PIV=0.D0                                                          DELG 640
      MM=M*M                                                            DELG 650
      NM=N*M                                                            DELG 660
      DO 3 L=1,MM                                                       DELG 670
      TB=DABS(A(L))                                                     DELG 680
      IF(TB-PIV)3,3,2                                                   DELG 690
    2 PIV=TB                                                            DELG 700
      I=L                                                               DELG 710
```

```
      3 CONTINUE                                                        DELG 720
        TOL=EPS*PIV                                                     DELG 730
C       A(I) IS PIVOT ELEMENT. PIV CONTAINS THE ABSOLUTE VALUE OF A(I). DELG 740
C                                                                       DELG 750
C                                                                       DELG 760
C       START ELIMINATION LOOP                                          DELG 770
        LST=1                                                           DELG 780
        DO 17 K=1,M                                                     DELG 790
C                                                                       DELG 800
C       TEST ON SINGULARITY                                             DELG 810
        IF(PIV)23,23,4                                                  DELG 820
      4 IF(IER)7,5,7                                                    DELG 830
      5 IF(PIV-TOL)6,6,7                                                DELG 840
      6 IER=K-1                                                         DELG 850
      7 PIVI=1.DO/A(I)                                                  DELG 860
        J=(I-1)/M                                                       DELG 870
        I=I-J*M-K                                                       DELG 880
        J=J+1-K                                                         DELG 890
C       I+K IS ROW-INDEX, J+K COLUMN-INDEX OF PIVOT ELEMENT             DELG 900
C                                                                       DELG 910
C       PIVOT ROW REDUCTION AND ROW INTERCHANGE IN RIGHT HAND SIDE R    DELG 920
        DO 8 L=K,NM,M                                                   DELG 930
        LL=L+I                                                          DELG 940
        TB=PIVI*R(LL)                                                   DELG 950
        R(LL)=R(L)                                                      DELG 960
      8 R(L)=TB                                                         DELG 970
C                                                                       DELG 980
C       IS ELIMINATION TERMINATED                                       DELG 990
        IF(K-M)9,18,18                                                  DELG1000
C                                                                       DELG1010
C       COLUMN INTERCHANGE IN MATRIX A                                  DELG1020
      9 LEND=LST+M-K                                                    DELG1030
        IF(J)12,12,10                                                   DELG1040
     10 II=J*M                                                          DELG1050
        DO 11 L=LST,LEND                                                DELG1060
        TB=A(L)                                                         DELG1070
        LL=L+II                                                         DELG1080
        A(L)=A(LL)                                                      DELG1090
     11 A(LL)=TB                                                        DELG1100
C                                                                       DELG1110
C       ROW INTERCHANGE AND PIVOT ROW REDUCTION IN MATRIX A             DELG1120
     12 DO 13 L=LST,MM,M                                                DELG1130
        LL=L+I                                                          DELG1140
        TB=PIVI*A(LL)                                                   DELG1150
        A(LL)=A(L)                                                      DELG1160
     13 A(L)=TB                                                         DELG1170
C                                                                       DELG1180
C       SAVE COLUMN INTERCHANGE INFORMATION                             DELG1190
        A(LST)=J                                                        DELG1200
C                                                                       DELG1210
C       ELEMENT REDUCTION AND NEXT PIVOT SEARCH                         DELG1220
        PIV=0.DO                                                        DELG1230
        LST=LST+1                                                       DELG1240
        J=0                                                             DELG1250
        DO 16 II=LST,LEND                                               DELG1260
        PIVI=-A(II)                                                     DELG1270
        IST=II+M                                                        DELG1280
        J=J+1                                                           DELG1290
        DO 15 L=IST,MM,M                                                DELG1300
        LL=L-J                                                          DELG1310
```

```
      A(L)=A(L)+PIVI*A(LL)                                              DELG1320
      TB=DABS(A(L))                                                     DELG1330
      IF(TB-PIV)15,15,14                                                DELG1340
   14 PIV=TB                                                            DELG1350
      I=L                                                               DELG1360
   15 CONTINUE                                                          DELG1370
      DO 16 L=K,NM,M                                                    DELG1380
      LL=L+J                                                            DELG1390
   16 R(LL)=R(LL)+PIVI*R(L)                                             DELG1400
   17 LST=LST+M                                                         DELG1410
C     END OF ELIMINATION LOOP                                           DELG1420
C                                                                       DELG1430
C                                                                       DELG1440
C     BACK SUBSTITUTION AND BACK INTERCHANGE                            DELG1450
   18 IF(M-1)23,22,19                                                   DELG1460
   19 IST=MM+M                                                          DELG1470
      LST=M+1                                                           DELG1480
      DO 21 I=2,M                                                       DELG1490
      II=LST-I                                                          DELG1500
      IST=IST-LST                                                       DELG1510
      L=IST-M                                                           DELG1520
      L=A(L)+.5D0                                                       DELG1530
      DO 21 J=II,NM,M                                                   DELG1540
      TB=R(J)                                                           DELG1550
      LL=J                                                              DELG1560
      DO 20 K=IST,MM,M                                                  DELG1570
      LL=LL+1                                                           DELG1580
   20 TB=TB-A(K)*R(LL)                                                  DELG1590
      K=J+L                                                             DELG1600
      R(J)=R(K)                                                         DELG1610
   21 R(K)=TB                                                           DELG1620
   22 RETURN                                                            DELG1630
C                                                                       DELG1640
C                                                                       DELG1650
C     ERROR RETURN                                                      DELG1660
   23 IER=-1                                                            DELG1670
      RETURN                                                            DELG1680
      END                                                               DELG1690

    $ENTRY
0.2150715818717349D 06
0.2255949380004106D 06
0.2730673632776805D 06
0.5259346181602061D 06
0.1976976403414621D 06
 1   0.1831825859304240D 05   0.18225CC0C0000000D 05   0.1353449614616015D 03   0.1350000000000000D 03
 2   0.2439131965934530D 06   0.2480040000000000D 06   0.4938756894132905D 03   0.4980000000000000D 03
 3   0.5442528767532485D 06   0.585225C000000000D 06   0.7377349637595118D 03   0.7650000000000000D 03
 4   0.7698478147536590D 06   0.795664C000C00000D 06   0.8774097188620941D 03   0.8920000000000000D 03
 5   0.9954427527540698D 06   0.1038361000C00000D 07   0.9977187743818744D 03   0.1019000000000000D 04
 6   0.5442528767532485D 06   0.614656CCC0000000D 06   0.7377349637595118D 03   0.7840000000000000D 03
 7   0.1045915644007636D 07   0.933156CCC0C00000D 06   0.1022700173C75000D 04   0.9660000000000000D 03
 8   0.1271510582C08047D 07   0.13689CC000000000D 07   0.1127612780172363D 04   0.1170000000000000D 04
 9   0.1318983007285317D 07   0.11449CCC0CC00000D 07   0.1148469854756892D 04   0.1070000000000000D 04
10   0.1544577945285727D 07   0.1512900000000000D 07   0.1242810502564943D 04   0.1230000000000000D 04
11   0.1865117733840678D 07   0.169C0CC000000000D 07   0.1365693133116176D 04   0.1300000000000000D 04
12   0.2090712671841088D 07   0.2C164CC0C0000000D 07   0.1445929691181797D 04   0.1420000000000000D 04
13   0.1066115752402791D 07   0.1525225000000000D 07   0.1032528814320836D 04   0.1235000000000000D 04
14   0.1291710690403202D 07   0.17424CCC0CCC0000D 07   0.1136534509112328D 04   0.1320000000000000D 04
15   0.1770172883286138D 07   0.1651225000000000D 07   0.1330478441496193D 04   0.1285000000000000D 04
```

16	0.1865117733840678D 07	0.1587600000000000D 07	0.1365693133116176D 04	0.1260000000000000D 04
17	0.2316307609841499D 07	0.2292196000000000D 07	0.1521942052064236D 04	0.1514000000000000D 04
18	0.1045915644007636D 07	0.1144900000000000D 07	0.1022700173075000D 04	0.1070000000000000D 04
19	0.1599399381324739D 07	0.2022084000000000D 07	0.1264871290418412D 04	0.1422000000000000D 04
20	0.2983782863714973D 07	0.2689600000000000D 07	0.1727362979722262D 04	0.1640000000000000D 04
21	0.3190363109475898D 06	0.3014010000000000D 06	0.5648329938553428D 03	0.5490000000000000D 03
22	0.1599899381324739D 07	0.1264871290418412D 04		
23	0.2125833999484945D 07	0.1458024005112723D 04		
24	0.4055852208430540D 07	0.2013914647752119D 04		
25	0.3236650118597499D 07	0.1799069236743683D 04		
26	0.2690515392042138D 07	0.1640279059197592D 04		
27	0.3823997835166187D 07	0.1955504496329831D 04		
28	0.4622999816604074D 07	0.2150116233277651D 04		
29	0.4076865090048713D 07	0.2019124832705673D 04		
30	0.3803797726771032D 07	0.1950332722068476D 04		
31	0.5800974238580808D 07	0.2408521172541526D 04		
32	0.7146110946574055D 07	0.2673221080751469D 04		
33	0.6053841493463333D 07	0.2460455545923017D 04		
34	0.5234639403630292D 07	0.2287933435139731D 04		
35	0.4763061626788132D 07	0.2182443957307525D 04		
36	0.5288996244948338D 07	0.2299781782028099D 04		
37	0.6937735670544215D 07	0.2633958175549531D 04		
38	0.8009805015259781D 07	0.2830159892172133D 04		
39	0.7190602925426740D 07	0.2681529959822702D 04		
40	0.6644468198871379D 07	0.2577686598264300D 04		
41	0.9507804994983221D 07	0.3083472878911572D 04		
42	0.1112600906625415D 08	0.3335567278028454D 04		
43	0.9760672249865746D 07	0.3124207459479243D 04		
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45	0.1825494319325149D 07	0.1351108552013919D 04		
46	0.2351428937485355D 07	0.1533436968866133D 04		
47	0.3209377801715384D 07	0.1791473639693139D 04		
48	0.4281447146430951D 07	0.2069165809313248D 04		
49	0.3462245056597910D 07	0.1860710900865019D 04		
50	0.2916110330042549D 07	0.1707662241206541D 04		
51	0.4049592773166597D 07	0.2012360000886173D 04		
52	0.4848594754604484D 07	0.2201952486908944D 04		
53	0.4302460028049123D 07	0.2074237215954126D 04		
54	0.4029392664771443D 07	0.2007334716675682D 04		
55	0.6026569176581218D 07	0.2454907162517804D 04		
56	0.7371705884574466D 07	0.2715088559250778D 04		
57	0.6279436431463744D 07	0.2505880370541208D 04		
58	0.5460234341630702D 07	0.2336714433051395D 04		
59	0.4988656564788542D 07	0.2233530068028756D 04		
60	0.5514591182948749D 07	0.2348316670074279D 04		
61	0.7163330608544625D 07	0.2676439913120529D 04		
62	0.8235399953260192D 07	0.2869738655916282D 04		
63	0.7416197863427150D 07	0.2723269700824204D 04		
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66	0.1135160400425456D 08	0.3369214152329080D 04		
67	0.9986267187866157D 07	0.3160105565937024D 04		
68	0.8893997734755435D 07	0.2982280626425930D 04		
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77	0.4275187711167008D 07	0.2067652705646431D 04
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87	0.7389325546545035D 07	0.2718257814583642D 04
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117	0.1112600906625415D 08	0.3335567278028454D 04
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CORE USAGE OBJECT CODE= 19406 BYTES,ARRAY AREA= 5200 BYTES,TOTAL AREA AVAILABLE= 88160 BYTES

DIAGNOSTICS NUMBER OF ERRORS= 0, NUMBER OF WARNINGS= 0, NUMBER OF EXTENSIONS= 3

COMPILE TIME= 5.15 SEC,EXECUTION TIME= 0.66 SEC, WATFIV - VERSION 1 LEVEL 3 MARCH 1971 DATE= 73/158

VITA

Asok Kumar Ray

Candidate for the Degree of

Master of Science

Thesis: NON-RELATIVISTIC QUARK MODEL FOR MESONS

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