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Generalizing and Transferring Mathematical Definitions from Euclidean to Taxicab Geometry

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GENERALIZING AND TRANSFERRING MATHEMATICAL DEFINITIONS FROM EUCLIDEAN TO TAXICAB GEOMETRY

by

AUBREY KEMP

Under the Direction of Draga Vidakovic, PhD

ABSTRACT

Research shows that by observing properties of figures and making conjectures in non-Euclidean geometries, students can better develop their understanding of concepts in Euclidean geometry. It is also known that definitions in mathematics are an integral part of understanding concepts and are often not used correctly in proof or logic courses by students. To further investigate student understanding of mathematical definitions, this dissertation studied students' uses of dynamic geometry software and group work to generalize their understanding of definitions as they completed activities in Taxicab geometry. As a result of the analysis from the group work and use of Geometer's Sketchpad by 18 students in a College Geometry class, suggestions are provided to implement cooperative learning and technology in the classroom. In addition, results are provided from the data analysis of responses to questions pertaining to the definition of *circle* (and its relevant concepts) of 15 students enrolled in the course who volunteered to participate in semi-structured interviews. This dissertation specifically utilizes APOS Theory (Arnon et al., 2014) and the interaction of schema framework provided by Baker et al. (2000) to determine what components of the *circle schema* were evoked by these participating students during these interviews. By adapting and transferring their knowledge of concepts back and forth between Euclidean and Taxicab geometry, these students provided evidence for the relationships they had formed between the components of their *circle schema.* Further, they demonstrated a variety of levels of schema interaction of their evoked *Euclidean geometry schema* and *Taxicab geometry schema*. As a result, a model of schema interaction and suggested pedagogical activities were developed to help facilitate student understanding of the definition of a circle and other relevant concepts.

INDEX WORDS: Definitions, Geometry, Taxicab, APOS Theory, Circle, Cooperative learning, Geometer's Sketchpad

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In memory of

Daddaddy and Grandmama

(Jack and Maureen Spann)

Go raibh céad míle maith agat.

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"If I have seen further it is by standing on ye shoulders of giants."

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LIST OF ABBREVIATIONS

GRED………………………………………….......Geometric Representation of Euclidean distance
ARED…………………………………………....Algebraic Representation of Euclidean distance
GRTD……………………………………….......Geometric Representation of Taxicab distance
ARTD……………………………………….......Algebraic Representation of Taxicab distance
GRD……………………………………….......................Geometric Representation of distance
ARD……………………………………….........................Algebraic Representation of distance
GREC……………………………………….......Geometric Representation of Euclidean circle
AREC………………………………………....... Algebraic Representation of Euclidean circle
GRTC………………………………………..........Geometric Representation of Taxicab circle
ARTC……………………………………….........Algebraic Representation of Taxicab circle
GRC……………………………………….........................Geometric Representation of circle
ARC………………………………………......................... Algebraic Representation of circle
cEg……………………………………….......................circle in Euclidean geometry schema
cTg……………………………………….........................circle in Taxicab geometry schema

1 INTRODUCTION

1.1 Statement of the problem

Definitions are an integral part of understanding concepts and constructing proofs in mathematics. Thus, it is important for students to develop a deep understanding of the content and role of definitions in mathematics. Edwards and Ward (2008) found mathematics majors exist that do not understand the role of definitions in a mathematically acceptable way but have been deemed successful students in advanced mathematical courses. These authors explain that this should be addressed in undergraduate mathematics, and research is needed to determine pedagogical strategies that help facilitate student understanding of the concept of definition. Güner and Gülten (2016) explain that geometry has three dimensions: definitions, images that represent these definitions, and their properties. In addition, with respect to activities that have been designed to deepen the understanding of the concept of definition in mathematics, research is needed to assess their effectiveness (Edwards & Ward, 2008). In this study, I investigate these three dimensions and their relationship to one another to determine how students understand certain definitions and present suggestions for how to develop this understanding.

With respect to this need, the purpose of this study is to identify and analyze students' perceptions and understanding of mathematical definitions in geometry. Further, it is investigated how applying these definitions in an atypical context affects their understanding of the concept of definition. In particular, I explored how being introduced to Taxicab Geometry contributed to students' understanding of mathematical definitions and their roles within geometric reasoning. Also, as Geometer's Sketchpad (GSP) is a dynamic geometry software that participants used in this study, it is investigated how students utilized this program to explore concepts and generalize definitions.

Geometry is a very important area in mathematics because it takes objects and their relationships in the real world and allows students to understand these behaviors. The skills developed through learning geometry, like spatial sense and relationships among figures, is increasingly necessary for many areas of study (Grunbaum, 1981). In college geometry courses, Euclidean geometry and its axiomatic system is deeply studied, but other axiomatic systems receive little consideration (Byrkit, 1971; Hollebrands, Conner, & Smith, 2010), although research shows that by exploring concepts in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands, Conner, & Smith, 2010; Jenkins, 1968). There is also a need for further research to investigate whether students develop deeper insight to Euclidean axioms, concepts, and theorems as a result of a comparison of these ideas in different geometries (Kinach, 2012). In particular, Siegel, Borasi, and Fonzi (1998) encourage the introduction to Taxicab geometry before other non-Euclidean geometries since the simpler space makes it easier for students to reason, and thus abstract concepts. Supporting this, Dreiling (2012) found that "through the exploration of these 'constructions' in Taxicab geometry…[students] gained a deeper understanding of constructions in Euclidean Geometry." (Dreiling, 2012, p. 478). One objective of this study is to determine how students perceive and understand mathematical definitions in Taxicab Geometry in order to facilitate a better understanding of how to use definitions in geometric reasoning, from which concepts are widely used in various areas of study.

There have been many technological tools developed to help students' reason in non-Euclidean geometry, but it is relatively uninvestigated how college students construct arguments in non-Euclidean mathematics when given the ability to use these technological tools. Few research studies have examined how students' uses of such tools affect their mathematical thinking or influence the mathematical arguments they develop (Hollebrands, 2010). There are many advantages to using dynamic software in the classroom. First, according to Glass and Deckert (2001), seeing examples of problems worked out or figures drawn does not help students focus on the relationship and relevant aspects of the material as much as using technology can. Second, by using dynamic technology, students can develop a higher level of geometric reasoning and understanding. Third, with the ability to alter figures and analyze what relationships change or do not change, technology can help students explore these relationships and differentiate between drawings and constructions. Finally, Glass and Deckert (2001) state that research implies students who use this software can generate conjectures better than those who do not use technology, since they are able to visualize patterns and properties. In this study, participants used the dynamic geometry software Geometer's Sketchpad, which will be discussed more about later in Section 2.3.

Despite the expectations held for students enrolled in higher-level mathematics courses, it has been found that commonalities exist regarding students' inability to properly complete tasks involving definitions (Edwards & Ward, 2004). In fact, the authors state that there were misconceptions in students' understanding of "the very nature of mathematical definitions, not just from the content of the definitions," (p. 411). Within the context of geometry, since properties of geometric figures are derived from definitions within an axiomatic system, it is important to note that a figure is "controlled by its definition," (Fischbein, 1993, p. 141).

Schoenfeld (2000) reminds us of the purposes of mathematics education research. The first is in a pure sense, "to understand the nature of mathematical thinking, teaching, and learning. The second, as applied, is "to use such understandings to improve mathematics instruction," (Shoenfeld, 2000, p. 641). This study aims to accomplish both of these with respect to geometry, geometrical thinking, mathematical definitions, and Taxicab geometry. As previously stated, the purpose of this study is to determine how students perceive and understand mathematical definitions in Taxicab Geometry in order to facilitate a better understanding of these definitions and their application in geometric reasoning.

1.2 Research questions

I sought to answer how being introduced to Taxicab Geometry contributes to students' understanding of mathematical definitions with the help of Geometer's Sketchpad (GSP). In particular,

1. In what ways do students use GSP to refine their understanding of mathematical definitions?

   (a) How do students apply their working understanding of a definition in GSP to reason about mathematical problems?

   (b) How does cooperative learning and the use of GSP help students in the abstraction of definitions from Euclidean geometry to axiomatic systems in general?

2. How do students adapt their understanding of concepts in Euclidean geometry in order to apply definitions in Taxicab geometry, a non-Euclidean axiomatic system?

   (a) What activities in Taxicab geometry can aide in the abstraction of a definition?

   (b) How does applying definitions in an atypical context affect the development of student understanding of these definitions?

   (c) How do students transfer their understanding of relationships among concepts in Euclidean geometry to Taxicab geometry?

1.3 Theoretical perspective

Constructivism as an epistemological viewpoint states that an individual constructs his or her knowledge based on his or her experiences with the related concepts. In particular, knowledge is not “passively received, but rather actively constructed by an individual,” (Selden & Selden, 1998, s.1). In regard to mathematics education, the authors continue to explain that “knowledge” refers to the mental structures that allow an individual to interpret the meaning of something, evoke ideas in their mind, or explore new mathematical problems effectively. Further, individuals can use this “old” knowledge in order to construct “new” knowledge. Thompson (1979) discusses a constructivist teaching method within mathematics education research. It is not uncommon for there to be a pattern in the way students learn mathematical concepts, and as such, mathematics educators should strive for a way to incorporate these patterns of construction in their instruction. This study was conducted around a course designed from a constructivist perspective, and relevant data was analyzed with constructivist frameworks.

The method of instruction for this report was based on APOS Theory and the ACE teaching cycle (Arnon et al. 2013) and both APOS Theory and the theory of Concept Image and Concept Definition (Vinner & Hershkowitz, 1980; Tall, 1980; Tall &Vinner, 1981) are used to analyze student perceptions of various mathematical definitions.

1.3.1 APOS Theory

As a constructivist framework, APOS theory is based on Jean Piaget’s theory of reflective abstraction, or the process of constructing mental notions of mathematical knowledge and objects by an individual during cognitive development (Dubinksy, 2002). Piaget believed that reflective abstraction was “critical for the development of more advanced concepts in mathematics,” (Dubinsky, 1991, p. 160). As an extension or adaptation of reflective abstraction, APOS theory, introduced by Dubinksy in 1984, describes how concepts are learned in mathematics. Its aim is to understand how students understand and perceive mathematical concepts, to track the development of concepts, and use this knowledge to develop instructional material to assess the success of this learning (Arnon et al., 2014). APOS is an acronym that stands for Action-Process-Object-Schema, which can be thought of as a non-linear pathway of how an individual learns particular mathematics concepts.

The development of APOS theory by Dubinsky and other researchers in mathematics education is described and can be found in many cases (Asiala et al. 1996; Arnon et al. 2013; Dubinksy & McDonald, 2001). In addition, many examples of how this theory can be used to describe the learning of different concepts within mathematics by students (Asiala et al., 1997; Dubinsky et al., 2005; Weller et al., 2003; Çetįn, 2009; Stalvey et al., 2018; etc.). A general description of APOS Theory is provided below.

APOS theory attempts explain how mathematical concepts might be learned (Arnon et al., 2014). Through reflective abstraction, once an individual has encountered a concept and works with tasks related to this concept, their understanding of the concept moves from the lower level to a higher level of cognitive development with necessary restructuring of their existing knowledge. This continues as needed and helps to build upon the individual's already existing knowledge to construct new relationships between these cognitive structures. In APOS Theory, there are four different stages of cognitive development: Action, Process, Object, and Schema. In addition, APOS Theory includes mechanisms to move between these levels of cognitive development: interiorization, coordination, encapsulation, de-encapsulation, and reversal (Dubinsky, 2010; Weller et al., 2003, Arnon et al., 2014).

An action is being performed when a student is able to transform objects by external stimuli, needing guidance or using memorized rules to perform operations and tasks. As a student reflects on these actions, they are able to *interiorize* them, so they can imagine performing these actions without actually doing so. In this case, we refer to the interiorized action as a process. A student can then *coordinate* this process with others within a schema in order to form connections between concepts. In addition, a process can be *reversed* in order to understand a concept further. Once a student is able to think of this process as a totality to which actions or other processes could be applied, we say that an object is constructed through the *encapsulation* of the process. It is also possible for a student to *de-encapsulate* an object that they have already constructed in order to add new knowledge to their pre-existing knowledge of a mathematical concept. In addition, an individual can de-encapsulate two objects, coordinate these processes, and form a new object from the coordinated process (Arnon et al., 2014). Finally, the entire collection of actions, processes, objects, and other schemas that are connected to the original concept that form a coherent understanding is called a schema (Dubinsky, 2002). A general illustration of the stages of APOS Theory as adapted from Arnon et al. (2014) is summarized in Figure 1.1 below. An example of what it looks like for student to go through the different mental stages of understanding according

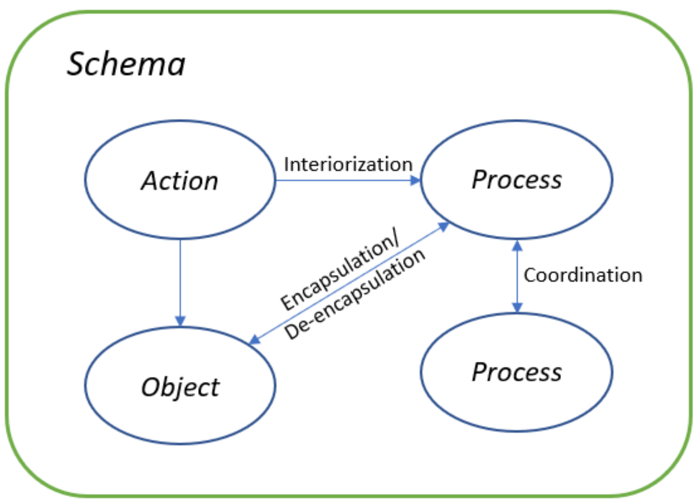


Figure 1.1 Illustration of APOS theory as adapted from Arnon et al. (2014).

to APOS theory is discussed next.

As an example, the different stages of APOS (Action-Process-Object-Schema) Theory are described below in reference to the mathematical concept of **Function**. When a student has an action conception of **Function**, he or she has a need to see a particular expression for a function, say $y = 2x + 4$, and only then are they are able to think of evaluating this function at certain points as an external rule. For example, the student can determine that when $x = 1, y = 6$ and when $x = 20, y = 44$. Once the student performs these actions enough and reflects on these actions, he or she will be able to imagine that for any function, if he or she inputs a value into this function, they will obtain a unique output value. At this point, we say that the student has interiorized these actions, resulting in a process conception of **Function**. In general, a student with a process conception of **Function** does not need to have particular examples of functions to talk about their properties. In addition, the student can imagine inputting values into this function and can describe what will result from this without actually performing these actions. Using this example, such an observation might be that for linear functions, a positive slope indicates that as the x values increase, the y values are also increasing (positive relationship). Eventually the student may reach a conclusion that a function represents a set of inputs and outputs that are related to one another by some expression. A student can then begin to encapsulate this process into an object by classifying functions and describing groups of functions. At this point, the student can perform actions on this object conception, such as composing functions with one another or comparing types of functions to one another. This entire process and collection of actions, processes, and objects is what constructs the schema of the concept of **Function** for an individual student. Specifically, every individual's schema is uniquely formed by his or her experiences and is not necessarily linearly formed in the order of Action-Process-Object-Schema. On the contrary, APOS Theory emphasizes that "the construction of mathematical knowledge is nonlinear," although "the APOS-based description of the mental construction of a mathematical concept is presented in a hierarchal manner," (Arnon et al., 2014, p. 19).

The idea of a *genetic decomposition* in APOS Theory is meant to outline and model the necessary constructions students need to make in order to develop understanding of mathematical concepts (Arnon et al. 2014). The authors define it as a "description of how the concept may be constructed in an individual's mind," (Arnon et al., 2014, p. 17). A genetic decomposition plays an important role in mathematics education research based in APOS Theory, since it provides a necessary theoretical model to aid in the design of instruments to gather and analyze data from students. Based on the researcher's experiences and understanding of the concept, historical development of the concept, and results from relevant research, an initial genetic decomposition is created. This preliminary genetic decomposition is used as a guide in the development of instructional methods. During the analysis of data, the preliminary genetic decomposition is reflected upon to see if the questions and activities asked of students helped to make the mental constructions suggested by the genetic decomposition, or if the data suggests something about students' understanding of the concept that was not included in the initial genetic decomposition. Depending on the reflection, the genetic decomposition or method of instruction may be revised. The repetition of refinement, revision, and data analysis produces a genetic decomposition that will closely mimic the cognitive development of a concept for a large portion of individuals as they learn about the concept. In general, the genetic decomposition can be used to design materials for instruction that will help to better facilitate student learning and understanding of mathematical content (Arnon et al. 2014). In Chapter 3, genetic decompositions are presented for this research study, and in Section 4.2 another genetic decomposition is presented for the interaction of schemata related to the *circle schema*. Another perspective on how students understand definitions is described below.

1.3.2 Concept image and concept definition

Two different ways students can approach a mathematical topic are by a natural or a formal strategy (Tall, 2001). A natural approach is one in which a learner builds on concept imagery to give a personal meaning to a formal definition (Tall, 2001). A student who uses a natural approach is more likely to use examples as references for what a definition or theorem says in order to apply it to another situation. A student who uses a formal approach attempts to avoid using intuition when applying a definition or theorem to another situation. These learners focus on the formal definitions, using formal deductions to build theorems (Tall, 2001). It is very possible that learners use a combination of the two approaches, however it is more typical for them to have a major tendency towards one over the other. With the human mind being a finite entity, it is clear that we cannot hold an infinite number of concepts in it (Tall, 2001). This requires us to compress our knowledge associated with a concept into a manageable form that we can relate to other concepts, build upon to further our understanding, and also apply to situations we have not been made aware of previously.

It is believed amongst many mathematics educators that a mathematical concept within an individual's mind consists of a *concept image* and a *concept definition* (Tall &Vinner, 1981; Edwards &Ward, 2004; Chesler, 2012). The concept definition is the definition that has been assigned to the concept while the concept image is a representation of an individual's understanding of the concept under consideration. In essence, we come to develop a complete concept image in the mind consisting of "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes," (Tall & Vinner, 1981, p. 152). Furthermore, an individual's concept image and concept definition for a particular concept changes and develops over time (Tall, 2008). Note that the idea of concept image is comparable to schema from APOS theory.

By exploring the process and possible learning paths of students, we can improve instruction for students to develop better understanding of definitions which will therefore improve the basis for understanding of proofs and their functionality. In general, a definition is a universal understanding of an object that gives meaning to it. The sophistication that comes about in higher level mathematics is that these definitions are linguistic compressions of phenomena that in turn become "thinkable concepts" themselves (Tall, 2007). Once this is the case, we then use these compressed phenomena to relate to other "thinkable concepts," creating cognitive structures surrounding mathematical concepts and theorems (Tall, 2007).

Tall states that "definitions arise from experience with objects whose properties are described and used as definitions; in formal mathematics, as written in mathematical publications, formal presentations start with set-theoretic definitions and deduce other properties using formal proof" (Tall, 2008, p. 8). According to Pinto (1998), students take two distinct routes in learning concepts in formal mathematical analysis: one where they give meaning to definitions from the information they have compressed into their concept image, and one where they extract meaning from the concept definition. Furthermore, a concept definition generates its own concept image in the mind of each individual student: what we can call a *concept definition image* (Tall & Vinner, 1981). When a student's understanding of the concept of definition is being discussed, it is considering his or her experiences outside of mathematics classrooms as well as what they have developed with the guidance of an instructor. Tall and Vinner (1981) explain that there is a difference between a *personal concept definition* and a *formal concept definition*, since a formal concept definition is accepted by the mathematical community as a whole. Pinto (1998) and Pinto and Tall (1999, 2001) found that students preferred to build on their personal version of a definition of a concept instead of using the formal conception definition assigned to this concept. Some of the concept images evoked by the students in their study were distorted or inadequate.

With a slightly different explanation, Moore-Russo (2008) defines a concept definition to be the "words used to specify a mathematical object", while a concept image consists of the "nonverbal representations, mental pictures, and the associated properties built through a person's experiences…," (p. 408). In addition, when an individual is considering a mathematical concept, often only part of the student's memory, and therefore concept image, is called upon. Tall and Vinner (1981) refer to this induced set of memories and experiences as the *evoked concept image.*

By exploring the process and possible learning paths of students' thinking in relation to these concept images and concept definitions, I hope to develop better instructional materials to aide in the understanding of definitions. Further, I hope an improvement in the understanding of definitions will result in a deeper understanding of how to apply them in proofs and geometric reasoning.

1.4 Outline of the study

In this study, I attempt to answer the research questions presented in Section 1.2, as they relate to student understanding of definitions, geometrical reasoning, and how learning Taxicab geometry has an influence on both of these. In Chapter 2, a review of relevant literature is included to inform the reader of past and current research, needed research, and a general overview of Taxicab geometry and its differences from Euclidean geometry. Chapter 3 will include the research and instructional setting, data collection methods, and instruments used in this research study, as well as a preliminary genetic decomposition. This preliminary genetic decomposition attempts to model the learning of Taxicab geometry and how this affects students' understanding of various mathematical definitions. Chapter 4 will include the analysis of data and the presentation of a genetic decomposition involving the interaction of schema. The conclusions of the study, findings, any adjustments needed for the genetic decomposition, and recommendations for future research related to definitions within geometrical reasoning will be explained in Chapter 5. Further, suggested instructional material, developed as a result of these findings, will be provided in Appendix B.

1.5 Chapter summary

In this chapter, the importance of this research was summarized, as definitions are such an important facet of understanding mathematics. Research shows exposure to non-Euclidean geometry can assist students in the abstraction of meaning of these definitions within the context of geometry. The need for this research was discussed, as many research studies have indicated a need to investigate student understanding of definitions in geometry and how they are used within geometric reasoning, as many proofs rely on definitions. The theoretical frameworks that were used analyze data in an attempt to answer my research questions were presented. These research questions were specified as follows:

1. In what ways do students use GSP to refine their understanding of mathematical definitions?
   (a) How do students apply their working understanding of a definition in GSP to reason about mathematical problems?
   (b) How does cooperative learning and the use of GSP help students in the abstraction of definitions from Euclidean geometry to axiomatic systems in general?

2. How do students adapt their understanding of concepts in Euclidean geometry in order to apply definitions in Taxicab geometry, a non-Euclidean axiomatic system?

(a) What activities in Taxicab geometry can aide in the abstraction of a definition?

(b) How does applying definitions in an atypical context affect the development of student understanding of these definitions?

(c) How do students transfer their understanding of relationships among concepts in Euclidean geometry to Taxicab geometry?

2 LITERATURE REVIEW

2.1 Geometry

Geometry is a very broad area of mathematics that has many applications. In particular, it typically has some reliance on intuition and logic. Semple and Kneebone (1959) give a clear description of geometry,

> Geometry is the study of spatial relations, and in its most elementary form it is conceived as a systematic investigation into the properties of figures subsisting in the space familiar to common sense…even the most abstract geometrical thinking must retain some link, however attenuated, with spatial intuition, for otherwise it would be misleading to call it geometrical; and it is an historical fact that, throughout the long development of mathematics, geometers have again and again arisen who have given a fresh impulse to formal mathematics by going back once more for inspiration to the primitive geometrical sense (p. 1).

Continuing this idea that geometry has to do with intuition, Grunbaum (1981) states that "few people are conscious of the fact that all geometry…is a product of our thinking and represents just one of the ways in which we try to communicate about our surroundings and understand certain aspects of reality," (p. 232). Many students may question why they need to learn geometry, but many non-mathematicians encounter geometric problems regularly but struggle to accurately assess these problems because they are not equipped with the correct mathematical skills (Grunbaum, 1981).

Oladasu (2014) states that "geometry is a central aspect of the school mathematics curriculum and is crucial in the mathematics education of our children from the perspective of providing them with the opportunity to develop spatial awareness and geometric thinking," (p. 2). The author goes on to explain that reasoning activities can strengthen the evaluation of mathematical arguments. It is common for students come into the college geometry classroom without a sufficient formal background, so when exposed to this approach, many students find it unattractive because of the demands imposed by writing formal solutions or using formal logic (Armitage, 1971).

College geometry is difficult for students because they are used to reasoning from intuitive understandings and experiences rather than from axiomatic systems (Hollebrands et al, 2010). Along the same lines, Jenkins (1968) explains that students are surprised to find they have little understanding and insight into the axiomatic systems which are necessary for most mathematics. As Krause (1973) explains, having an understanding of non-Euclidean geometry can deepen a student's insight of Euclidean geometry. According to this author in regard to higher mathematics, non-Euclidean geometry is recognized as a great way to illustrate the nature of axioms and the meaning behind the independency of axioms. At the same time, learning about non-Euclidean geometry is important to the process of using a rigorous axiomatic approach to reasoning. The understanding of geometry for students is significantly modified when they are challenged by different axioms (Hollebrands et al, 2010). As an example, having students construct models in elliptic geometry can reinforce important connections between elementary and higher geometry (Kaisari & Patronis, 2010).

Menger (1971) discusses the importance of learning about deductive geometric systems. Among other benefits, the author explains that the student learns:

1. These systems start from unproven assumptions in terms of undefined concepts.
2. These concepts and assumptions are designed to reflect upon objects and facts of the physical world. In addition, there are many applications of geometry to nature.
3. Definitions in terms of undefined terms can be of great importance.

4. Proving is combining and transforming assumptions according to the rules of logic.
5. Some examples of systems satisfy certain postulates but not certain propositions.
6. One theory may be capable of a variety of interpretations.

Perhaps the most relevant to this dissertation, Menger (1971) specifies that by exploring these systems, "any mystery is dispelled by displaying to the students non-Euclidean geometries in situations with which they are perfectly familiar," (p. 4). In particular, the author provides the example of Taxicab geometry in this context. With such evidence from researchers that emphasizes the instruction of non-Euclidean geometry, I anticipate that during the teaching of Taxicab geometry, students will be able to expand on their understanding for Euclidean geometry in addition to axiomatic systems, which are used throughout mathematics.

2.1.1 Taxicab geometry

Around the end of the 19th century, a Polish-German mathematician by the name of Hermann Minkowski first introduced the Taxicab metric to the world within a collection of proposed metrics (Gardner, 1997; Reynolds, 1980), although the name "Taxicab" was not used until 1952 when Karl Menger established a geometry exhibit in Chicago (Reinhardt, 2005). It is typically first taught in college geometry courses, although many times is ignored in the curriculum. Fortunately, many strides have been taken to encourage the instruction of non-Euclidean geometry in general. In fact, geometry at the university level is no longer strictly Euclidean geometry, and has transitioned to being conceived as geometric topology (Willmore, 1970). As college educators, we should emphasize geometrical discovery and the excitement that accompanies this, along with the idea of several different geometries (Willmore, 1970).

Byrkit (1971) explains that the axiomatic system associated with Euclidean geometry is studied in depth in geometry, while other axiomatic systems receive little attention. The author continues to explain that when non-Euclidean axiomatic systems are studied, often the examples are too difficult, too limited, or too trivial to create interest. Siegel, Borasi, and Fonzi (1998) encourage the introduction to Taxicab geometry before other non-Euclidean geometries since the simpler space makes it easier for students to reason, and thus abstract concepts. One advantage to learning Taxicab geometry is that it can be used as a model for various applications, such as optimizing driving time in cities or laying pipes in a home. Caballero (2006) even explains how it can be used to model the spread of forest fires and discusses how this can be used to improve computer code for these types of simulations. Thus, learning concepts in Taxicab geometry not only can help facilitate geometrical reasoning, but can be applicable to many individuals and their future careers. Taxicab Geometry measures distance only in horizontal and vertical motions, as opposed to Euclidean geometry which measures distance as the length of the straight line between two points. For example, imagine a city where the streets form a perfect grid system. A car can only travel forwards or backwards, with the ability to make left and right turns. Thus, driving 3 blocks straight, making a left, and driving two more blocks is a total of five blocks.

In general, the Taxicab distance between two points is measured as the sum of the change in horizontal and vertical directions between the two points, where Euclidean geometry is measured using the Pythagorean theorem. For simplicity sake, in this report when an object such as Euclidean circle, Taxicab circle, etc. is being referred to, it is intended that I am referring to this object (and associated concepts) as it exists within that particular space, rather than suggesting that object has two distinct concepts (one in Euclidean geometry and one in Taxicab geometry). For example, a “Taxicab circle” is the concept of a circle and its definition within the Taxicab metric system. Figure 2.1 shows visual examples of both metrics. Mathematically, Euclidean distance (d_E) and Taxicab distance (d_T) between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are defined below:

(i) $d_E(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

(ii) $d_T(P, Q) = |x_2 - x_1| + |y_2 - y_1|$

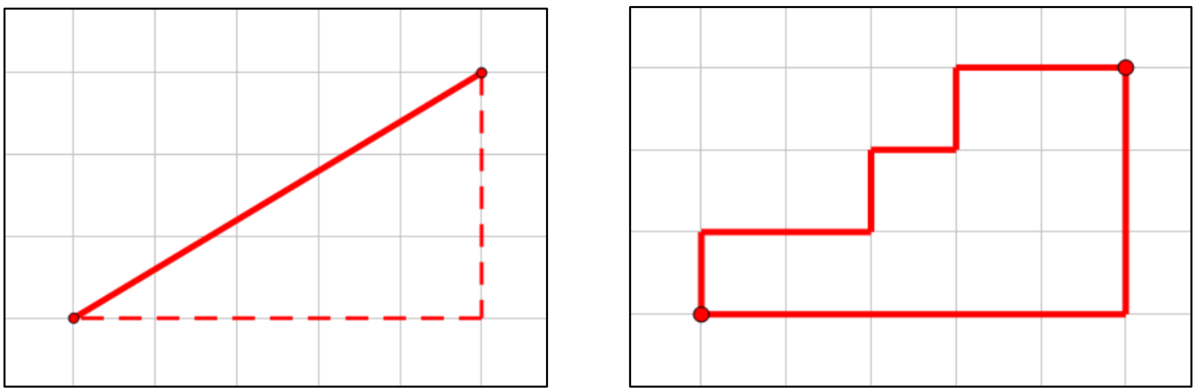


Figure 2.1 Visual representations of Euclidean distance and Taxicab distance, respectively.

Seen in Figure 2.1, in Euclidean geometry, a distance between two points is represented visually as a straight line segment between two given points. In particular, it is calculated as the length of the hypotenuse of a right triangle constructed with legs parallel to the axes, as can also be seen in Figure 2.1. In Taxicab geometry, a distance between two points is represented visually as a path from one point to another by "walking" only over horizontal and vertical blocks. One such path would be along the legs of the right triangle mentioned in the case of Euclidean distance, but Figure 2.1 demonstrates two such paths. It is obvious that, in the case of Euclidean distance, there is a unique geometric representation of a distance between two points, while in Taxicab geometry this is not the case.

Many interesting things occur once we change how distance is measured within an axiomatic system. For example, the triangle inequality does not hold in Taxicab geometry, circles look like squares, and the congruence criteria of Side-Angle-Side, Angle-Side-Angle, and Side-

Side-Side of two triangles does not hold anymore (for more information on Taxicab triangles and trigonometry, please see Thompson & Dray (2000)). These are some examples of why learning Taxicab geometry can be interesting and important for students. By varying assumptions, students can come to the fundamental realization that we can develop new theory and results under certain conditions (Menger, 1971), and begin to abstract and generalize their understanding of particular concepts. As an example, Smith (2013) found that through exploration in Taxicab geometry his students deepened their understanding of a locus of points.

As briefly discussed previously, for this report, when concepts are discussed in each geometry by the convention of Euclidean circle, Taxicab circle, Euclidean perpendicular bisector, etc., I am referring to this object as it exists within this geometry, not that this object is defined differently within these geometries. In other words, the definition of this object is the same, but the properties of this object may be different between the geometries because of the way distance in measured. For example, the Euclidean perpendicular bisector and Taxicab perpendicular bisector of a segment are defined as the locus of points equidistant from the endpoints of this segment in Euclidean geometry and Taxicab geometry, respectively. Thus, it is noted that for the entirety of this report a *perpendicular bisector* of a segment is defined as the set of points that are equidistant from the endpoints of this segment. As a result of this definition, this object has different properties in both geometries. In particular, in Euclidean geometry, this results in a straight line that intersects the segment at its midpoint at a right angle. In Taxicab geometry, this locus of points is not necessarily a straight line, nor does it necessarily intersect this segment at a right angle, depending on the slope of the segment with respect to the axes. Thus, when the concept of **Perpendicular bisector** is discussed in relation to a segment, it is not implied this line intersects the segment at a right angle, because this depends on in what metric space this line is being constructed.

Regarding this idea of properties varying as a result of a definition, Smith (2013) so eloquently discusses that our own assumptions can prevent us from seeing a problem in its full depth. Along these lines, Fujita and Jones (2006, 2007), Okazaki and Fujita (2007), and Turnuklu et al. (2013) talk about prototypical images in geometry and how students use them in their personal concept definitions, which affects how they define or classify figures. Identified as the "prototype phenomenon" (Hershkowitz, 1990), I believe using Taxicab geometry in the classroom can help students to move past this phenomenon and examine definitions and the underlying reasons for the appearance of figures as a result of these definitions, as explained in Berger (2015). This author provides activities and applications of Taxicab geometry, along with Krause (1973), Dreiling (2012), Smith (2013), and Chu and Tran (2017).

2.1.2 Van Hiele levels of Geometric Reasoning

Although mathematicians see proofs and logic as a method for establishing validity, mathematics education researchers have questioned whether students are convinced by proof (Battista & Clements, 1995) or if they perceive proof as "a set of formal rules unconnected to their personal mathematical activity," (Hanna, 1989). In particular, geometrical proofs challenge and encourage students to distinguish the difference between seeing and deducing (Yang & Lin, 2008). In order to construct proofs effectively in geometry courses, it is inferred that appropriate geometric reasoning skills are required. In terms of analyzing student understanding of geometry, it is commonplace for researchers to use van Hiele's levels of geometric reasoning.

As important as proofs are to higher mathematics, it is largely misunderstood by mathematics students. For example, in a large research study conducted by Senk (1985) of over

1500 students enrolled in full-year geometry courses that spent time on proof, only 30 percent of the students achieved a 75 percent-mastery level in proof writing. As an integral part of the formal mathematics world, educators need to be sure their students are actually learning the methods and logic of proof writing since the purpose of a proof is to provide students with mathematical insight (Hannah 1990, Hersh 1993, Thurston, 1994). Mejia-Ramos et al. (2012) state, "exactly what insight is, what it means for a proof to be understood, and how we can tell if students comprehend a given proof remain open questions in mathematics education," (Mejia-Ramos et al., 2012, p. 4).

A theory of geometrical thought, the van Hiele levels of geometric reasoning, will be discussed below. In particular, this theory suggests that students are not able to recognize and appreciate axiomatic systems until they reach the highest level of understanding in this hierarchy. Also, it implies that students must have reached the lower levels of these hierarchies before arriving at the highest level (Battista & Clements, 2004). Thus, when considering this theory in the classroom, activities and instruction must be flexible in order to compensate for students who are at varying levels in these hierarchies before attempting to guide students to the highest levels. Many researchers describe these levels in different ways (Battista & Clements, 2004; Glass & Deckert, 2001; Hansen, 2004), however the following descriptions summarize these levels.

The first van Hiele level, *Visualization*, is when a student is able to recognize figures and relate them to objects they know. The student is not focused on individual parts, but the overall appearance of a figure. Next, stage 2 is *Analysis*, in which a student can identify the properties of figures, but do not know which properties are explicit enough to define the figure itself. The student is aware of properties, but do not comprehend the "significance of sufficiency of conditions," (Glass & Deckert, 2001). Stage 3 is *Abstraction* or Informal Deduction, where a student is able classify figures and use basic logic to justify reasoning. At stage 4, which is *Deduction* (or Formal

Deduction), a student is able to write simple proofs and understand an axiomatic system. At the final stage, *Rigor*, a student is able to understand non-Euclidean geometry and has the ability to construct more difficult proofs.

Since students must traverse the lower levels of these hierarchies before attaining the highest level of understanding, educators should be aware of at what levels their students are thinking when they arrive to their classroom.

2.2 Mathematical Definitions

Piaget (1928) explicitly states that "from the psychological point of view, definition is the conscious realization of the use which one makes of a word or a concept in the course of a process of reasoning," (p. 147). The concept of a definition has historically not been a popular topic of discussion since there is an official view that imposes clarity and order on mathematics, which is apparently unproblematic and completely dominant (Brown, 1998). However, understanding the concept of definition is something that is full of important issues, and many of these issues are central to how we understand mathematics (Brown, 1998). The question of what the concept of a mathematical definition should be has been an ongoing topic of discussion. Krantz says "…a definition must describe the concept being defined in terms of other concepts already known," (2007, p. 5). We can define a term using certain other terms only if these terms are defined previously (Brown, 1998). Reinforcing this idea of an official view on what a definition is, *Principia Mathematica*, written by Whitehead and Russell states:

> A definition is a declaration that a certain newly-introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols of which the meaning is already known. (1910, p. 11)

Krantz (2007) states that "an axiom is a mathematical statement of fact, formulated using the terminology that has been defined in the definitions, that is taken to be self-evident," (p. 6), which implies that definitions precede axioms. Thus, definitions are used to formulate axioms, which can be used in deriving propositions or theorems. As evident, a definition is the only basis on which we can build in mathematics and as a mathematical community, we can decide if a definition is considered valid. Brown (1998) discusses the differences between contextual definitions, explanations, and explicit, directly defined terms. A contextual definition is one in which the axioms are needed in order to find a meaning of the term instead of the term itself (Brown, 1998). In this way, the axioms are not necessarily defined only in terms of previously established defined words and are not explicitly stated. Brown (1998) also discusses the possible differences between an explanation and a definition, assuming there exist any. The mathematical community has varying opinions on whether an explanation could be considered as a definition. In this mathematical debate, Hilbert argues for explanations to be contextual definitions, while Frege's point of view on this is that explanations are helpful with the introduction to a concept but should not be a part of or used as the formal mathematical definition (Brown, 1998).

According to Dormolen and Zaslavsky (2003), there are certain criteria that must be met for a statement to be considered a definition. These are:

- The criterion of hierarchy: The idea that any new concept must be described in terms of previously defined concepts;
- The criterion of existence: The idea that a definition must guarantee that there is a term that satisfies conditions in a way that such a situation exists;
- The criterion of equivalence: If there are two ways to define a mathematical term, these definitions must be equivalent;

- The criterion of axiomatization: The idea that a definition is part of an axiomatic, deductive system.

Other important criterions considered part of the general culture include the criterion of minimality, the criterion of elegance, and the criterion of degenerations (Dormolen & Zaslavsky, 2003; Edwards & Ward, 2008).

When undergraduate students first take upper-level math classes, it is a common occurrence for students to struggle writing mathematical proofs (Edwards & Ward, 2004). Selden and Selden (2015) suggest that helping students understand how to interpret formal mathematical definitions so that they become operable is a good place to start when attempting to teach proof construction. This is because in order for a student to construct an adequate proof, they need the skill of converting definitions into operable interpretations (Selden & Selden, 2015). Conclusions from research that has been conducted suggest that students do not necessarily understand the content of relevant definitions or know how to use them in proof writing, which could be a main cause of these difficulties students are facing (Edwards & Ward, 2004). Menger (1971) explains that when a deductive approach is taken in teaching, a student learns that he or she can carefully choose definitions in terms of undefined terms in order to simplify theorems and proofs. In addition, the student learns that the "deductive geometric system must start from unproven assumptions in terms of undefined concepts," (Menger, 1971, p. 3).

Zandieh and Rasmussen (2010) identified a formal stage of operation of a student when he or she uses definitions without having to "unpack" the meaning of these definitions in order to use them as "links in chains of reasoning," (p.70). Edwards and Ward (2004) identify a few issues students face when understanding the concept of a mathematical definition in this type of formal way:

- Many students do not categorize mathematical definitions the way mathematicians do.
- Many students do not use definitions the way mathematicians do, even when the students can correctly state and explain the difference.
- Many students do not use definitions the way mathematicians do, even in the apparent absence of any other course of action.

So as educators, how do we change this? What pedagogical strategies can be used in order to further the understanding of what a mathematical definition is? Teachers must draw upon a specialized content knowledge in order to interpret, evaluate, choose, or use definitions (Chesler, 2012). By having activities for students that require the construction of definitions for mathematical concepts, students are more likely to develop a deeper understanding of these concepts. In fact, definition construction is an activity just as important as solving problems, making conjectures, specializing, proving, generalizing, etc. (Chesler, 2012; De Villiers, 1998). By conducting this research study, I hope to find more pedagogical tools to be used to help this understanding.

Traditionally in the mathematics classrooms, Freudenthal (2006) explains that students are given definitions and rules and are expected to proceed to show some other concept or idea. However, Pinto and Tall (1999) found that students failed to use the formal definition given to them in order to deduce another idea. Instead, they preferred to build on their own personal version of the definition, sometimes distorted, in order to make these deductions (Tall, 2002). As described by Çetịn (2009), for conflict to occur, "it is needed that conflicting portions are evoked simultaneously in the individuals' mind. Not only portions of concept image can be in conflict [with] each other, but also conflict might happen between [a] concept image and its formal definition." (p. 24)

Thus, the inner conflict of what exactly a mathematical term means in relation to its formal definition influences how students apply these definitions within proofs. Dawkins (2012) suggests that within the context of geometry, students "have a body of experience that facilitates ease of processing and provides basic intuition for assessing the validity of reasoning within the context." This is supported by Alcock and Simpson (2002, 2004), Edwards and Ward (2008) and Vinner (1991). In this sense, there are many different ways individuals can use mathematical terms, properties, and definitions to develop their understanding of concepts. What follows is a summary of these various strategies as a result of a literature review and the researchers' knowledge and experiences. Seen in Figure 2.2, an illustration is provided to model the process individuals may take in three different types of activities related to definitions and mathematical terms.

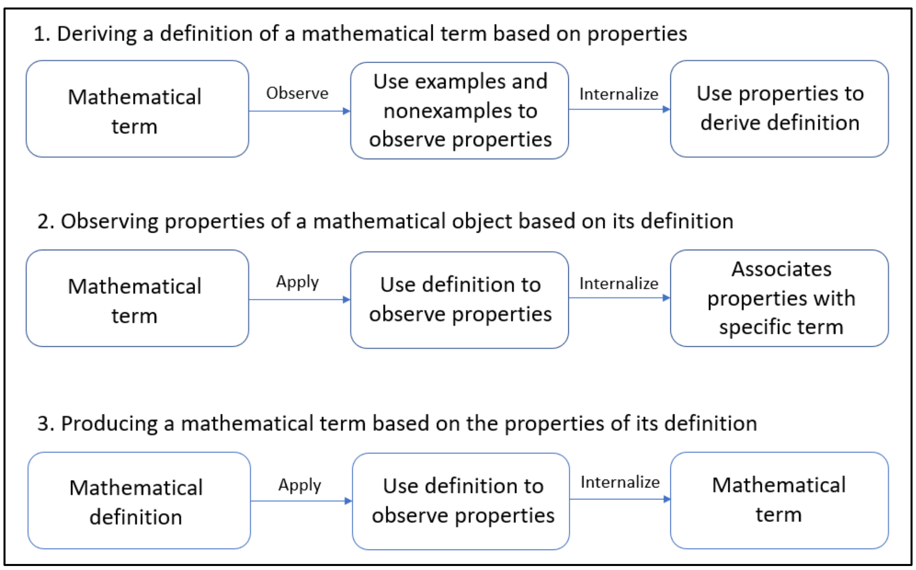


Figure 2.2 Visual representations of ways mathematical terms, properties, and definitions can be utilized.

Deriving a definition of a mathematical term based on properties: By having activities for students of constructing definitions for mathematical concepts, they are more likely to develop a deeper understanding of these concepts. In fact, definition constructions are an activity just as important as solving problems, making conjectures, proving, and generalizing (Chesler, 2012; De

Villiers, 1998). When they are constructing their understanding of a concept, students are usually interacting with examples and nonexamples along with the definition of that concept which will then contribute to and clarify their understanding of that concept and definition along with the roles and features of it (Selden, 2011; Wilson, 1990). Thus, we make meaning through experience.

The first map in Figure 2.2 illustrates that to construct a definition, an individual can first consider examples and nonexamples to observe properties and non-properties of a certain mathematical term. He or she then internalize these properties and derive a personal definition that is unique to the mathematical term. In other words, the individual is attempting to write a definition that captures his or her concept image. Lakatos (1976) emphasizes and provides examples of this type of approach to constructing definitions. Larsen and Zandieh (2008) and Zandieh and Rasmussen (2010) reframe the methods presented by Lakatos (1976) as frameworks to be used in research on the teaching and learning of mathematics. Although there are pros to this method, Dickerson & Pitman (2016) report that participants in their study were "largely unsuccessful at writing definitions that captured their own concept image." The authors claim that when writing their definitions, students failed to consider key examples. An example of this type of activity to help students with their understanding of **Perpendicular bisector** would be providing two endpoints of a segment and giving students a set of points, specifying which points fall on its perpendicular bisector and which do not, making it clear that the mathematical term under consideration is *perpendicular bisector*. Students would begin to conjecture what a perpendicular bisector looks like, and what is required for a point to be on this line. They can then develop or modify their personal definition of this mathematical object.

Observing properties of a mathematical object based on its definition: The second map in Figure 2.2 illustrates how an individual might apply his or her personal definition of a mathematical term to various situations to observe properties about the mathematical object's definition. With the addition of new observations, the individual associates these properties with the mathematical object, and adjusts his or her personal definition and concept image to assimilate them. As this process continues, the individual's personal definition and concept image becomes more aligned with the concept definition of that mathematical term.

Wawro et al. (2011) report that students' descriptions of a particular mathematical concept were substantially different from the language of this concept's formal definition. These authors note this is consistent with other literature. For example, Dickerson & Pitman (2012) state that when attempting to apply mathematical definitions, many students have incomplete concept images from which they reason, resulting in them rejecting given definitions to use their imprecise concept image. An example of this type of activity would be giving students the definition of a *perpendicular bisector* and two arbitrary points. Student would have the task of identifying several points that fall on the perpendicular bisector of the segment connecting the two points and use this to associate properties of a perpendicular bisector with the mathematical object. The extent to which a student varies and generalizes his or her resulting examples and nonexamples affects to what extent it helps to adjust his or her personal definition of **Perpendicular bisector**. To clarify the difference between this method and the previous one, it is accentuated that in this case, a student is strictly using a definition to find his or her own examples and nonexamples and observe properties. In the prior method, a student is given examples and nonexamples and is told to observe properties from these.

Producing a mathematical term based on properties of its definition: Seen in Figure 2.2, the third map shows how an individual can use a definition of a mathematical object in a problem to explore the properties that result from this definition. It is noted in this case, the term that is being defined is not explicitly stated. This “de-labeling” of the mathematical term allows the individual to use problem-solving logic and freely abstract upon this definition without preconceived notions of what that object is. In essence, a student can sort through his or her relevant concept images or schemas to select a mathematical term and object that most closely matches the individual’s understanding of the definition being presented. Otherwise, the individual could involuntarily restrict his or her understanding of the problem to his or her working concept image of the term. This essentially reverses the process of a student being told which mathematical term he or she is operating with in order to discover properties of the definition.

In this case, the individual is less likely to associate an incomplete concept image with the problem which can restrict his or her ability to reason and can explore the problem freely according to his or her own interpretation. It is noted that Godino and Batanero (1998) discuss the meaning of mathematical objects and question whether formal definitions cover the full meaning of concepts. It is with this approach to teaching and learning definitions that we, as educators, can assess what types of meaning students are drawing upon in order to approach real-life mathematical situations. As Fischbein (1993) states, relationships among figures “do not depend on the drawing itself. They are imposed by definitions and theorems,” (p. 142). The author goes on to explain that an individual does not necessarily need to “polish” a figure in order to understand or reason through what it represents. This implies that an individual’s understanding of a figure is “from the beginning, not an ordinary image but an already logically controlled structure,” (p. 143). So, in the exploration a student is conducting in this approach to understanding definitions, while his or her drawing may not be accurate, this does not indicate the student is not making meaning of this mathematical term.

An example of this type of activity would be giving students a definition within another context, like a real-world situation, without mentioning what mathematical term it is defining, and having students identify what mathematical object this definition represents through free exploration. This encourages students to observe properties they may not have initially associated with a mathematical object, expanding their concept image of the mathematical object. An example of this type of activity would be asking students to identify location(s) for an apartment, given that they want it to be equidistant from two buildings in a city that is on a grid system (Taxicab geometry). After reflecting on the task, students will use their evoked schema or concept image to decide how they will approach the problem. Students may draw a representation of where a location could possibly be for this apartment and will begin to notice patterns or properties. Then, once students have a working concept image of this "unlabeled" mathematical object, they are asked to associate some mathematical term with this object. They now can assimilate essentially un-biased properties of this mathematical term in to their existing schema or concept image by the nature of the task.

2.3 Geometer's Sketchpad and group work

As a part of the teaching experiment conducted for this study, the textbook used in the class (Reynolds & Fenton, 2011) made use of the ACE Teaching Cycle and included activities that required the use of Geometer's Sketchpad (GSP). This software allows you to create, manipulate, and analyze the relationships between various figures and concepts in geometry. Many mathematics education researchers encourage the use of dynamic software programs to teach geometry, since this helps students to interact with accurate diagrams which will help to understand abstract properties and relationships of mathematical objects (Hollebrands et al., 2010, Hollebrands, 2003; National Council of Teachers of Mathematics, 1986; Abdullah, 2015;

Contreras, 2001). As students explore theorems using a dynamic geometry software, Chazan (1993) reports some students use their interaction with the computer as an empirical proof and do not understand the need to write a formal proof for theorems. However, Hollebrands et al. (2010) states that “students learning mathematics often engage in mathematical reasoning and sense-making activities prior to constructing a formal proof,” (Hollebrands et al., 2010, p. 236). Further, Edwards (1997) states students are able to make the most use of technological tools within this “territory before proof.” Thus, although this dynamic exploration is of particular figures in GSP, it can help students to generalize relationships and concepts in geometry.

There are many advantages to using dynamic geometry software in the classroom. First, according to Glass and Deckert (2001), seeing examples of problems worked out or figures drawn does not help students focus on the relationship and relevant aspects of the material as much as using technology can. Second, by using dynamic technology, students can develop a higher level of geometric reasoning and understanding. Third, with the ability to alter figures and analyze what relationships change or do not change, technology can help students explore these relationships and differentiate between drawings and constructions. Finally, the authors state that research implies students who use this software can generate conjectures better than those who do not use technology, since they are able to visualize patterns and properties easier than traditional “pencil and paper” exploration.

Many of the studies that have been conducted in relation to technology in the mathematics classroom found that through the use of technology students developed a better understanding of mathematical concepts and were more motivated to learn (Abdullah, 2015; Meng & Idris, 2012; Meng & Sam, 2013; Dogan & İçel, 2010; Cha & Noss 2001; Contreras 2011; Guven, 2012; Hanson, 2004; Lee, 2015; Meng, 2009; Tieng & Leong, 2015). As a counterexample, Tieng &

Leong (2015) and Tieng and Eu (2014) conducted statistical tests to determine there was not a significant difference in students' van Hiele level of geometric reasoning between their control group (traditional method of instruction) and group using GSP to explore concepts, although there was evidence that in general students had improved their understanding. In particular, the authors suggest part of the reason their results were not significant could be because of the students' unfamiliarity with GSP at the beginning of the study, which resulted in many students' inability to use the program well as an exploration tool during the study. Thus, they state future research allows a longer time for students to familiarize themselves with the program.

In regard to this dissertation, since the participants had used GSP for at least the 12 weeks prior to learning concepts in Taxicab geometry as is explained in Section 3.2, these students had the opportunity to achieve a sufficient level of familiarity with GSP. Hull and Brovey (2004) also found no significant difference between their course using GSP from courses from previous years but suggest that instruction that includes technology should be used not only in teacher-led instruction, but in a self-directed way as well. Since the course, textbook, and instruction for this study utilized the ACE Teaching Cycle, this study attempts to account for this, since students were using technology throughout the course leading up to and including the sections on Taxicab geometry.

Many studies that have investigated students' use of technology and geometric reasoning have used the van Hiele levels of geometric reasoning as a framework to do so (Abdullah, 2015; Guven, 2012; Hanson, 2004; Lee, 2015; Meng, 2009; Tieng & Leong, 2015). Further, Hansen (2004) acknowledges there are many studies that have been conducted that investigate the use of dynamic geometry software in middle and secondary school but suggests the investigation of this in the college classroom as well. For this dissertation, APOS Theory is utilized to investigate how undergraduate students use Geometer's Sketchpad to generalize their understanding of particular concepts in Taxicab geometry and Euclidean geometry). Hollebrands (2003) provides a framework based in APOS Theory for studying students' understanding of transformations as their understanding evolves using dynamic geometry software. The author also states that there is a need for further research to investigate the complexities of using technology in the teaching and learning of mathematics, which is something this study seeks to investigate. This framework by Hollebrands (2003) is adapted to analyze data for this dissertation, in addition to the preliminary genetic decomposition, which will be presented in Section 3.1.

In terms of the use of group work in the mathematics classroom, attention is focused toward cooperative learning, which involves students working in groups to complete a common goal (Siegel, 2005) and in this way "students work together to maximize their own and each other's learning," (Johnson & Johnson, 1999, p. 73). The authors also state that in terms of psychological health and social competence, the more individuals participate in cooperative learning the more they value themselves and the more independent they tend to be," (Johnson & Johnson, 1999). In other words, although cooperative learning requires the reliance on others to accomplish a shared goal, it also encourages independence in reasoning.

Because of the potential cooperative learning has to increase academic achievement and social skills in students, there are many researchers who advocate its implementation (Sharan, 2010; Johnson & Johnson, 1978, 1983, 1999; Johnson et al., 1998, 2014; Shimazoe & Aldrich, 2010). Further, many researchers have investigated or report on the use of cooperative learning in the mathematics classroom (Zakaria et al. 2010; Zakaria et al. 2013; Aziz & Hossain 2010; Cavanaugh, 2011; Davidson, 1989; Davidson & Kroll, 1991). In particular, Davidson (1990) provides a handbook for teachers on how to implement cooperative learning in the mathematics classroom. Aziz and Hossain (2010) found a significant improvement in students' mathematics achievement for their group of students who participated in cooperative learning compared to the conventional classroom. Also, Zakaria et al. (2010) describes cooperative learning and the positive effects this has on students' attitudes towards mathematics and state the need for research to investigate this idea over a longer time span.

2.4 APOS Theory

APOS Theory is based on Jean Piaget's constructivist theory of reflective abstraction, or the process of constructing mental notions of mathematical knowledge and objects by an individual during cognitive development (Dubinksy, 2002). As APOS Theory was described in general in Section 1.3.1, what is provided in this section is a description of APOS Theory within the context of geometry.

An action is exhibited when an individual is able to transform objects by external stimuli, performing memorized steps following instructions to complete this transformation. In the context of the concept of **Distance**, for example, an action conception could be demonstrated when a student uses specific points' coordinates to substitute into a formula to calculate the distance between two points. As an individual reflects on an action and has the ability to perform it in his or her head without external stimuli, we refer to that as an *interiorized* action and call it a process. A process conception of **Distance** could be exhibited when an individual is able to imagine in his or her head how to calculate the distance between any two points without actually performing this action. Once an individual is able to think of a process as a whole, viewing it as a totality to which actions or other processes could be applied, we say that an object is constructed through the *encapsulation* of the process. For **Distance**, an example of an object conception could be exhibited by a student comparing two distances to determine if they are equal or not, where the action being applied to this **Distance** object is a comparison. Finally, the entire collection of actions, processes, objects, and other schemas that are connected to the original concept that form a coherent understanding is called a schema (Dubinsky, 2002). In the context of the *circle schema*, for example, other concepts and associated schemas that are involved would be **Distance, Radius, Center,** and **Locus of Points**. It is noted here that a schema can be *thematized* in an object to which actions and processes can applied. The thematization of the *circle schema* will be discussed in detail in Section 3.1.5.

2.4.1 ACE Teaching Cycle

As a pedagogical result of APOS theory, the ACE Teaching Cycle was the method of instruction used in this teaching experiment. In particular, ACE stands for Activities on the computer (A), Classroom Discussion (C), Exercises done outside the class (E) (Asiala et al., 1996). To implement this cycle, first, Activities (A) are conducted in a group setting with guided tasks intended to help students make the mental constructions that the genetic decomposition has suggested. Students are able to explore relationships and form and test conjectures, implementing the process of reflective abstraction. The next step of the cycle, Classroom discussion (C), is primarily an instructor-led discussion, but requires class participation. These discussions are intended to allow students to reflect more formally on the activities from the first step of the cycle. Instructors provide explanations, definitions, and/or theorems in order for students to make connections between material. In the third step, Exercises (E), students are required to complete homework assignments outside of the classroom on selective exercises intended to reinforce the concepts they have learned and to help support the development of mental constructions. This part of the cycle allows students to consider related concepts in mathematics and apply the concepts they have just learned. (Arnon et al., 1996). Labeled as a “cycle,” this method of instruction may move through activities, discussion, and work at home multiple times while learning a single concept, aiming to foster the development of appropriate mental structures for each student in the classroom (Voskoglou, 2013). By this instruction, the teacher is meant to guide students to explore new topics in mathematics by having students reflect on the activities they complete in class. Students typically complete these activities in groups, allowing discussion to facilitate learning among one another. By hearing others' perspectives and explanations, students may be able to reinforce their own knowledge of mathematical concepts. In particular, research shows positive effects of group work and active learning on academic achievement (Freeman et al., 2014; Cavanagh, 2011; Spring et al. 1999).

2.4.2 Triad of Schema development and schema interaction

Although describing the mental structures and relationships a student would need to construct to complete a task is helpful, Arnon et al. (2014) stated that in order to describe certain learning situations, considering the schema structure may be necessary. As a result of the progression of APOS-based research, analyzing this structure may help to explain "why students have difficulty with different aspects of a topic, and may even have different difficulties with the same situation in different encounters," (p. 110). The authors continue to explain that schemas may include a single concept being applied in various situations or can be comprised of multiple concepts that are interrelated. In the development of a schema in the mind of an individual, new relationships can be established among the components of the schema, and new actions, processes, or objects can be *assimilated* into this existing schema through these new relationships. Further, this schema can also relate to other schemas which will result in constructing a new schema which encompasses components of both schemas. For example, a student's *distance schema* and its components are involved in the development of his or her *circle schema*, since the idea of distance is directly involved in the construction of a circle. Recall that a schema can be *thematized* into an object to which actions and processes can applied. If an individual can think of the schema as a total entity, we consider this schema to have been thematized (Clark et al. 1997).

Considering the development of a student's schema has proven to lead to a deep understanding of how he or she reasons when confronted with a mathematical problem situation. In particular, how a student uses certain evoked components of a schema and relates them to one another when presented with these situations can reveal the structure of this schema and its development. Arnon et al. (2014) state that there is a need to investigate the development of schemas and how they are applied in mathematics. In particular, Piaget and García (1989) proposed "the triad" of stages of schema development: Intra-, Inter-, and Trans-, where the hyphen symbol in each of these is followed by the name of the schema under consideration. Each of these stages is described below according to Arnon et al. (2014) and provide descriptions of these stages in relation to this study in Section 3.1.4.

Intra- Stage. This stage of development of the Schema can be identified when a student focuses on individual, isolated components of a schema. The individual can identify a set of common properties among the objects within that schema, where these connections are "local and particular," (p. 112). For example, a student who is able to observe similarities and differences among all circles within a particular geometry is exhibiting evidence of operating in the *Intra-*stage of development of the *circle schema*, or Intra-circle.

Inter- Stage. As knowledge develops within the mind of the individual, "access to necessary connections and the reasoning behind them begins to be developed," (p. 113). This is indicative of a student operating in the *Inter*-stage of development, when the individual constructs relationships among cognitive entities and components of the schema (Dubinksy & McDonald, 2001). Arnon et al. (2014) use the case of geometrical structures as an example to illustrate that algebraic representations in various geometries lead to the introduction of transformations that relate figures under different perspectives. Thus, a student who is aware that circles across various geometries visually appear different because of the way distance is measured but cannot coherently explain why is exhibiting evidence of operating in the *Inter*-stage of development of *circle schema*, or Inter-circle.

Trans- Stage. A student is operating in the *Trans*-stage of development when he or she sees the schema as a whole. At this stage, "the structure is coherent, and the individual can determine whether it is applicable or not to a given situation," (p. 113). Dubinksy and McDonald (2001) describe this stage as the individual constructs an "underlying structure through which the relationships developed in the Inter-stage are understood," (p. 282). For example, a student operating in the Trans-circle stage of schema development if he or she is able to understand the underlying structure in the construction and equation for a circle in a given metric space and how these are a result of the definition of a circle.

As stated previously, in some cases excluding the use of the triad results in an inadequate understanding of schemas involved, although there is a need for more research in using the triad to explain student thinking. Although a relatively new addition, in studies such as Clark et al. (1997), Cotrill (1999), McDonald et al. (2000), Baker et al. (2000), and Trigueros (2000, 2001), researchers found the addition of the triad to their analysis helped to paint a better picture of how the components of schemas work together in certain circumstances. In particular, the reader's attention is focused to Baker et al. (2000) and how the authors define an overall *calculus graphing schema* in terms of the interaction of two schemas, as it is the first model of schema interaction described in detail. It is also noted Trigueros (2004) provides a second model of schema interaction for the solutions of systems of differential equations. Baker et al. (2000) describe the relationship between what are named the *interval schema* and the *property schema* and analyze common student errors when solving "an atypical calculus graphing problem," (p. 558). Further, Cooley et al. (2007) examines this *calculus graphing schema* and the *thematization* of this schema in the context of APOS Theory, providing for the first time a framework for analyzing the thematization of schemata. The authors found that participants in this study supported results found in Baker et al. (2000), indicating a pattern in the existence of "double triad" and found only one of 28 students demonstrated a thematized *calculus graphing schema*. Further, they state that future research needs to consider the thematization of various schemas, and in particular, what it means to thematize a schema for particular concepts and corresponding genetic decompositions. For some of the data I will utilize the idea of schema interaction and will present a genetic decomposition for the development of the *circle schema* within this context in Section 4.2. In particular, this schema is described as its components may be evoked within the interaction of the *Euclidean geometry schema* and *Taxicab geometry schema,* using the framework and genetic decomposition presented in Baker et al. (2000) and Cooley et al. (2007) as models.

3 METHODOLOGY

As a qualitative study, I seek to answer how being introduced to Taxicab Geometry contributes to students' understanding of mathematical definitions with the help of Geometer's Sketchpad (GSP). As a reminder, these research questions are provided below.

1. In what ways do students use GSP to refine their understanding of mathematical definitions?

(a) How do students apply their working understanding of a definition in GSP to reason about mathematical problems?

(b) How does cooperative learning and the use of GSP help students in the abstraction of definitions from Euclidean geometry to axiomatic systems in general?

2. How do students adapt their understanding of concepts in Euclidean geometry in order to apply definitions in Taxicab geometry, a non-Euclidean axiomatic system?

(a) What activities in Taxicab geometry can aide in the abstraction of a definition?

(b) How does applying definitions in an atypical context affect the development of student understanding of these definitions?

(c) How do students transfer their understanding of relationships among concepts in Euclidean geometry to Taxicab geometry?

In this chapter, I will describe in detail the methods of data collection, specifics of recruitment of participants, the overall design of the study, preliminary genetic decompositions, and method of data analysis. Through this analysis I hoped to gain more insight into how students develop their understanding of definitions in geometry and to design activities with the goal to help facilitate this understanding.

3.1 Preliminary Genetic Decompositions

As a facet of APOS Theory, a *genetic decomposition* is constructed by the researcher to outline and model the necessary constructions individuals need to make to develop understanding of mathematical concepts (Arnon et al. 2014). The authors define it as a "description of how the concept may be constructed in an individual's mind," (Arnon et al., 2014, p. 17). A genetic decomposition plays an important role in mathematics education research based in APOS Theory, since it provides a necessary theoretical model to aid in the design of instruments to gather and analyze data from students. Based on the researcher's experiences and understanding of the concept, historical development of the concept, and results from relevant research, the researcher develops an initial genetic decomposition.

This *preliminary genetic decomposition* is used as a guide in the development of instructional methods. During or after this instruction, data is collected and analyzed. Throughout the analysis of data, the preliminary genetic decomposition is reflected upon to see if the questions and activities asked of students helped to make the mental constructions suggested by the genetic decomposition, or if the data suggests something about students' understanding or mental constructions of the concept that was not included in the initial genetic decomposition. Depending on this reflection, the genetic decomposition or method of instruction may be revised. The repetition of refinement, revision, and data analysis produces a genetic decomposition that will closely mimic the cognitive development of a concept for a large portion of the individuals who are learning the concept. In general, the genetic decomposition can be used to design materials for instruction that will help to better facilitate student learning and understanding of mathematical content (Arnon et al., 2014). In addition to identifying relevant concepts, what follows in this section is a description of the mental constructions I suggest can be evoked by a student and are necessary to understand various concepts in geometry. Further, I describe what relationships are formed in a student's mind through the interaction of his or her *Euclidean geometry schema* and *Taxicab geometry schema* as they transfer and adapt definitions between these geometries. I will also elaborate on these relationships that may exist between relevant concepts that are evoked within the *circle schema* in Section 4.2.

3.1.1 *Use of definitions in geometry*

In this section, a very basic preliminary genetic decomposition is proposed based on APOS theory. Figure 3.1 shows a preliminary genetic decomposition for this present study as adapted from Vidakovic, Dubinksy, & Weller (2018) which was developed for the construction of a line in analytic geometry. In this figure, the ways in which students can provide evidence for being at various stages of development are provided. I will use circles in Taxicab geometry as an example to illustrate these various stages.

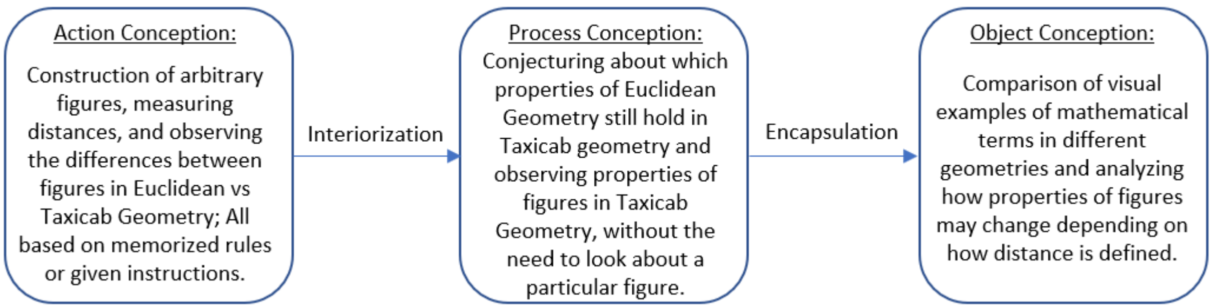


Figure 3.1 Preliminary genetic decomposition for this research study, adapted from a genetic decomposition for analytic geometry from Vidakovic, Dubinsky, & Weller (2018).

As proposed, a student at an action conception for this study will be able to construct figures and observe properties about these figures in Taxicab geometry but will not necessarily be able to conjecture what properties hold in Taxicab geometry in comparison to Euclidean geometry. For this example, students at an action conception of **Taxicab circle** will be able to construct a circle in Taxicab geometry given a center and a radius by counting the measure of the radius out in different directions and connecting these points, but they will not be able to identify any relationship with a circle in Euclidean geometry with the same center and radius. Once a student has interiorized this action conception, they will be able to begin making inferences about properties between Taxicab and Euclidean geometry, and which theorems or characteristics exist in both. They will also be able to investigate and observe how the change in metric affects certain properties. For this example, students with a process conception of **Taxicab circle** would be able to compare the values of π in Taxicab to Euclidean geometry, compare areas of circles in Taxicab to Euclidean geometries, etc. Once a student has encapsulated this process into an object, they will then be able to conjecture as to how changing the metric in a geometry affects properties of circles and will be able to apply this logic to other types of metrics. For the concept of **Taxicab circle**, students with an object conception would be able to construct new figures or lines using Taxi-circles, such as perpendicular bisectors, and analyze how these relate to the same concept in Euclidean geometry.

Once the course was designed and methods of data collection were determined, particular concepts were anticipated to be evoked in the minds of the participants in this study and more specific genetic decompositions were created. In particular, a preliminary genetic decomposition is provided for how a student can move through and exhibit various stages of understanding of particular concepts in Section 3.1.2. In Section 3.1.3, I discuss a genetic decomposition of how students may use GSP to understand these concepts. In Sections 3.1.4, I present a genetic decomposition for the triad of schema development with regard to the *circle schema*. Finally, I discuss the interaction of schema in Section 3.1.5, for which a genetic decomposition is presented in Section 4.2.

3.1.2 Mental constructions in geometry

As a result of a thorough literature review, including historical development of concepts results from relevant research, and the researchers' own experiences and understanding of the concepts, the following genetic decompositions were created. It was anticipated that three main concepts would emerge in the participants' minds as they are presented in this study, based on their understanding of various problems given to them. I define a *subconcept* of another to be a concept that is a main underlying component of the definition of the overall concept. The main concepts are **Midpoint**, **Circle** (with subconcepts of **Radius, Center,** and **Locus of points**), and **Perpendicular bisector** (with subconcepts of **Midpoint** and **Locus of points**), each with the subconcept of **Distance.** When I refer to a subconcept (of another concept), I specify that the subconcept's mental construction and development is directly related to the development of the original concept. For example, students use their concept of **Distance** as it relates to their concept of **Circle** when reasoning through a construction and trying to write the equation of a circle. I also partition each of these concepts and subconcepts into **Geometric Representation** and **Algebraic Representation** to be able to consider the relationship between these different forms of representation in each student's mind.

There is a multitude of literature on the investigation or emphasis on student understanding of various representations of mathematical objects (just to name a few - Ainsworth, 1999; Dreher & Kuntze, 2015; Booth et al., 2017; Boaler et al., 2016; Wilkie, 2016). Thus, the various representations a student can associate with a concept was considered in this genetic decomposition. In the preliminary genetic decomposition I provide the breakdown of **Geometric Representation** and **Algebraic Representation** for the concepts of **Distance** and **Circle**, and omit this detail for the other concepts, but remind the reader it is an implied part of these concepts and the analysis. As another note, when I discuss a student having a process conception of a concept, there is an assumption that the student has coordinated his or her **Geometric representation** and **Algebraic representation** of that concept. For example, students who exhibits a process conception **Radius** has provided evidence that he or she has coordinated his or her **Geometric Representation of Radius** and **Algebraic Representation of Radius** and have constructed a new process called **Radius** from this coordination. From this coordination, they can relate these geometric and algebraic representations to one another independent of the metric space. In

Sections 3.1.3, 3.1.4, and 4.2, it is discussed how these conceptions are used to determine the stages associated with the triad of schema development and levels of schema interaction according to APOS Theory. I remind the reader when I refer to an object such as Euclidean circle, Taxicab circle, Euclidean midpoint, etc., I am referring to this object (and associated concept) as it exists within that particular space, rather than suggesting that object necessarily is defined differently in each geometry.

3.1.2.1 Distance

A student's personal concept definition of distance can be related to a geometrical representation, algebraic representation, or a mixture of these. For example, describing out loud that distance is the 'straight line between two points' exhibits more of a geometric representation, since it elicits an image in an individual's mind. In contrast, describing distance in relation to the Pythagorean theorem is more of an algebraic representation, since this directly relates to the formula for Euclidean distance. If a student were to describe distance as "a measure of the straight line between two points using the Pythagorean theorem," then this clearly is an example of a student who is relating their geometric and algebraic representations of **Distance**.

We note that for each concept of **Euclidean distance** and **Taxicab distance**, there is an understood coordination between a student's **Geometric Representation of Euclidean Distance (GRED)** and **Algebraic Representation of Euclidean Distance (ARED)** processes and **Geometric Representation of Taxicab Distance (GRTD)** and **Algebraic Representation of Taxicab Distance (ARTD)** processes, respectively (see Figure 3.2). In this figure, the double sided arrows indicate a coordination of processes. The arrows coming from the **Euclidean distance** process and the **Taxicab distance** process indicate a coordination of their components, which result in the construction of the **Distance** process. It is noted this is also how each of the additional concepts in this genetic decomposition are believed to develop but omit these descriptions here.

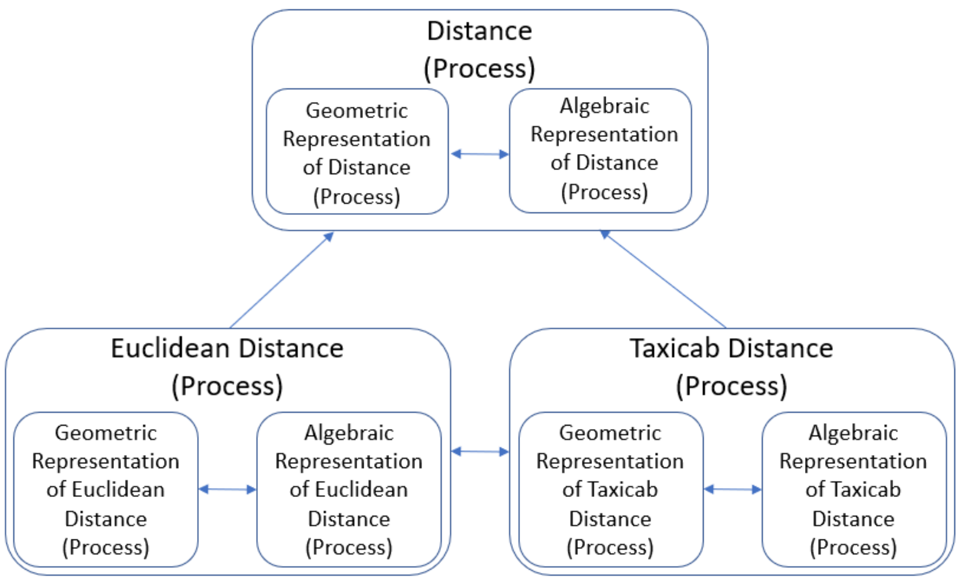


Figure 3.2 Visual representation of the construction of the Distance process.

As illustrated in Figure 3.2, a student must have a process conception of both **Euclidean distance** and **Taxicab distance** to coordinate these processes to form the conflated concept of **Distance.** In other words, a student must have a process conception of each of these metrics in order to make connections between them. What follows is how I have partitioned the concepts relevant to **Distance** (whether that is **Euclidean distance, Taxicab distance,** or the coordination of the two (**Distance**)) into **Geometric Representation of Distance (GRD)** and **Algebraic Representation of Distance (ARD)** in relation to the preliminary genetic decomposition. If a student has not constructed a new process by coordinating their **Euclidean distance** and **Taxicab distance** processes because they do not have a process conception of these concepts, then when we discuss their understanding of **Distance**, we will specify whether we are referring to their understanding of **Euclidean distance** or **Taxicab distance**.

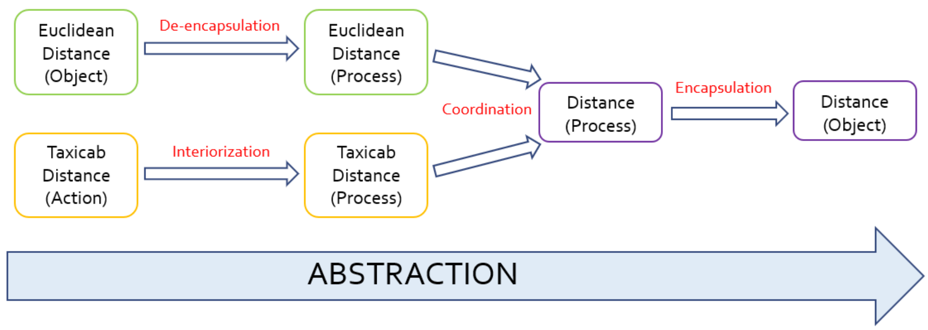


Figure 3.3 Visual representation of a possible learning pathway for a student developing their understanding of Distance.

Geometric Representation of Distance (GRD). Distance can be represented geometrically by graphing or describing out loud what a distance "looks like" in a certain context. Specifically, in Euclidean geometry, distance can be drawn and described as the segment connecting two points. On the other hand, in Taxicab geometry, distance can be drawn and described in various ways. Some of these can be as a step pattern between two points, the vertical and horizontal movements between two points, or the legs of a right triangle whose hypotenuse is the segment connecting two points. Below details of the various levels of conception associated with **GRD** as they could be exhibited given a single metric are provided.

Action: An individual is able to draw a pathway between two points given a metric. He or she may be unable to accurately describe in his or her own words what a distance looks like.

Process: When given a metric to consider, an individual can illustrate the distance between any two points. He or she does not need two specific points to imagine the distance between and can describe this distance geometrically using their own words.

Object: An individual encapsulates this Process into a totality if given a metric, he or she can illustrate and/or describe multiple distances and compare them. Specifically, the individual can successfully illustrate the distance between multiple points and compare them to determine which distances are equal to, greater than, or less than others. The individual can compare these distances using either visual representations or by describing similarities or differences in the shape or length of the distances using his or her own words.

Algebraic Representation of Distance (ARD). In general, distance can be represented algebraically by either stating a formula for distance or by verbally describing an expression or formula in the context of the algebraic representation. For example, if a student references the Pythagorean theorem while explaining Euclidean distance, or describes aloud that in Taxicab geometry, distance is defined by "taking the absolute difference in x-coordinates and adding this to the absolute difference in y-coordinates," this clearly is in reference to the formula for these metrics. What follows are the various levels of conception associated with **ARD**.

Action: Given two specific points and either recalls or is given a distance formula, the individual can correctly identify which values are associated with which variables and substitute these values in to this formula. The individual can then evaluate this expression, resulting in the distance between the two points.

Process: Given a distance formula or told what metric to use, an individual can calculate the distance between any two points. He or she does not need two specific points to imagine how they could calculate distance and can describe what the formula 'says' in their own words.

Object: An individual encapsulates this Process into a totality when they can successfully calculate the distance between any two points and can compare these distances to one another. In this case, the action being performed on this **Distance** object is a comparison.

In general, the addition of a new metric can cause issues with students' understanding of definitions. Overlooking for a moment the partition of these **Distance** concepts (the general **Distance, Taxicab Distance,** and **Euclidean Distance)** into their respective **GRD** and **ARD**, there are many ways a student can assimilate the Taxicab metric into their existing understanding of **Distance.** Figure 3.3 illustrates one way in which this is can take place.

As shown in Figure 3.3, a possible learning path for an individual introduced to Taxicab geometry begins by a student previously exhibiting an object conception of **Distance** in Euclidean geometry and has interiorized his or her action conception of **Distance** in Taxicab geometry into a Process. In this case, the individual must de-encapsulate his or her **Euclidean Distance** object to assimilate the concept of **Taxicab Distance** (a new metric) into this understanding. Once this happens, the processes of **Euclidean Distance** and **Taxicab Distance** can be coordinated, and the individual can construct one Process from this coordination. I will call this transformed concept **Distance**, as it includes more than one metric. A student can then encapsulate this understanding to arrive at an object conception of **Distance**, in which he or she can compare the *definitions* of distance across Euclidean and Taxicab geometry. Another example begins with a student exhibiting an action conception of **Euclidean Distance.** In this case, if a student also has an action conception of **Taxicab Distance,** they must interiorize both of these action conceptions into processes in order to coordinate them to construct the new **Distance** process**.** The individual can then encapsulate this understanding into an object called **Distance** after further reflection. I claim that an individual must have a process conception of both **Euclidean Distance** and **Taxicab Distance** in order to make observations beyond what are call *local observations* of properties between the two metrics. For example, a student making local observations may describe the visual differences in distances between the geometries in reference to particular examples or can identify that one formula for distance has a square root sign while the other does not. If a student has an action conception of either **Euclidean** or **Taxicab Distance,** they are able to illustrate and/or calculate distances in both geometries but cannot begin to describe geometrically or algebraically *in what ways* the distances are different from one another.

Distance. A student's understanding of **GRD** and **ARD** (made up of **GRED/GRTD** and **ARED/ARTD**, respectively) along with his or her verbal communication of these facets makes up their understanding of **Distance.** As noted previously, for a student to have this transformed concept of **Distance** he or she must exhibit a process conception of **Euclidean Distance** and **Taxicab Distance** independently, coordinate these two processes, and construct a new process from this coordination. If a student has an action conception of either **Euclidean Distance** or **Taxicab Distance**, the student will not have the necessary mental constructions to be able to make meaningful connections or comparisons between the two metrics. Below the APOS levels of understanding of the concept of **Distance** in terms of the concepts of **GRD** and **ARD** are described.

Action: Since a student needs a process conception of both **Euclidean** and **Taxicab distance** to construct the transformed concept of **Distance** (that I are describing here), as it is defined it thus far, there is not an action conception of **Distance.** So, instead an action conception of **Distance** is defined to be the combination of the conceptions of **Euclidean distance** and **Taxicab distance**, given that at most one of these is a process conception. In other words, if a student has an action conception of at least one of **Euclidean distance** and **Taxicab distance**, this student is exhibiting an action conception of **Distance**, since he or she would be unable to coordinate these conceptions and construct a **Distance** process. Further, an individual exhibits an action conception of **Distance** if he or she has an action conception of at least one of **GRD** and **ARD**. In other words, the individual is able to graphically represent two or more metrics and/or can use formulas to calculate distances in these geometries but cannot make connections between these representations. The student is able to observe local differences between distances (geometrically or algebraically) in multiple geometries but struggles to verbalize these differences. For example, if a student explains that the difference between the Euclidean distance formula and the Taxicab distance formula is that there is a square root sign for the Euclidean formula but cannot explain why geometrically, this is evidence that the student has not coordinated his or her **GRD** and **ARD.**

Process: Once an individual has interiorized this Action, he or she can describe in his or her own words similarities and differences between various metrics beyond obvious visual differences. Thus, the individual needs to be able to describe the metrics independently in each metric in his or her own words, requiring a process conception of both **GRD** and **ARD**, as shown in Figure 3.4**.** With a process conception of **Distance,** the individual can conjecture about what properties might still hold from Euclidean geometry in a new metric and can begin to explain in their own words the differences between metrics beyond making local observations. For example, if a student explains that the Euclidean distance cannot be more than the Taxicab distance between two points because of the triangle inequality, the individual is coordinating his or her processes of **GRD** (with the visual use of a triangle to consider both distances) and **ARD** (with the algebraic representation of the inequality).

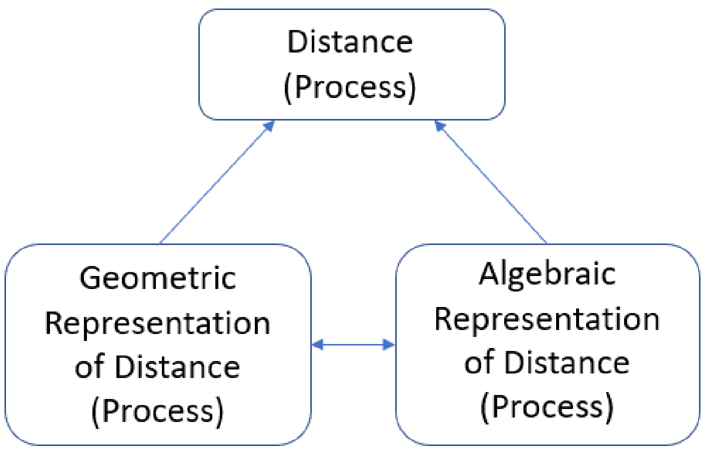


Figure 3.4 The construction of the Distance process.

Object: Once a student has coordinated all of their **Euclidean distance/Taxicab distance** and **GRD/ARD** processes at the same time, he or she can encapsulate this **Distance** process into an object. This Process is encapsulated into an object if an individual can perform actions on this object, such as comparisons or using it as an "input" into a transformation, in both geometries. For example, if a student is given a center point and measure of a radius and can construct a circle in both Euclidean and Taxicab geometry, a student is using his or her **Distance** object as an input to a function or transformation where the output is a locus of points that retain this distance from a center point.

3.1.2.2 Midpoint

A midpoint between two points P and Q is defined as a point M such that the distance from P to M is equal to the distance from M to Q, which is equal to half of the distance from P to Q. An object conception of **Distance** is a necessary prerequisite to identify a point as a midpoint of two others, since a student would need to compare two distances to see that they are equal. In Euclidean geometry, the midpoint between two points is unique and is located on the segment connecting them. Noting that a continuously measured Taxicab metric is being utilized, in Taxicab geometry, any two points have a *midset*, or a possibly infinite set of points that satisfy this definition of a midpoint. In particular, one of the Taxicab midpoints is also the midpoint in Euclidean geometry, i.e.- the formula used to find a midpoint in Euclidean geometry will also provide a midpoint in Taxicab geometry. In the case that the segment connecting two points is parallel to one of the axes or has a slope of 1 or -1, the midset only consists of one point. That is, the Euclidean midpoint is the only point in the Taxicab midset of the two points. Otherwise the Taxicab midset contains an infinite number of points and is represented by a segment intersecting the initial segment at its Euclidean midpoint. Figure 3.5 provides illustrations of the midsets (shown as red segments) of two segments AB with varying slopes. I note that for this study the concept of **Midpoint** includes subconcepts of **Euclidean midpoint** and **Taxicab midpoint** that develop in a similar manner as illustrated in Figure 3.3. How an individual may exhibit each stage of conception associated with **Midpoint** is described below.

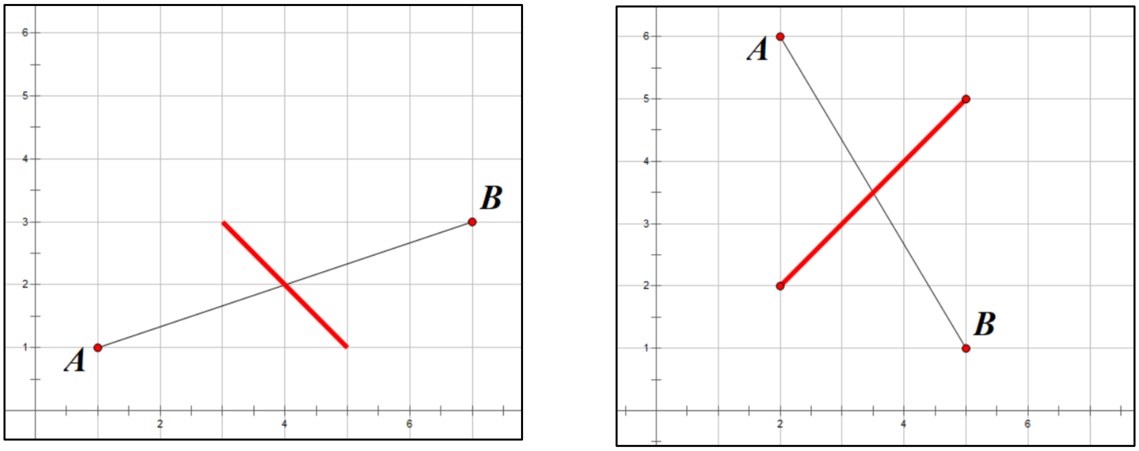


Figure 3.5 Graphical representations of midsets in Taxicab geometry.

Action: To identify a midpoint of a segment connecting two points, the individual can use a formula or expression for finding a midpoint and plug in given values. The individual can evaluate this expression and arrive at an ordered pair that is a midpoint. The individual could also start at two given points and count interchangeably one block at a time from each point until he or she arrives at a point that is equidistant from these points, and this point happens to be a midpoint.

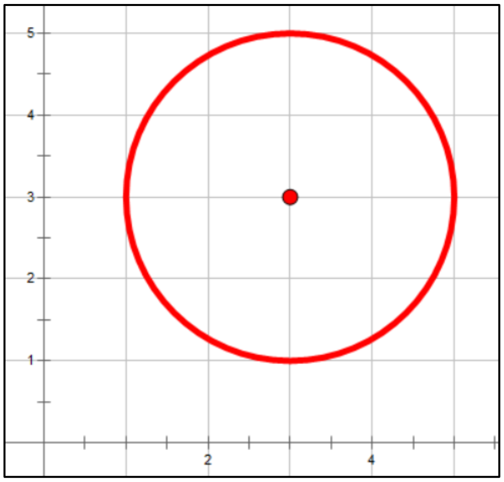
Process: The individual has interiorized a **Midpoint** Action if he or she can imagine the midpoint is located so that the distance to it from the given points is half of the total distance between these points. One way to do this is to find the total distance between the given points, divide by 2, and use this value to identify a point that satisfies this criterion. In this case, a student is able to think of how to find a midpoint of any two given points and can explain in his or her own words how to do so for a general scenario.

Object: The student has encapsulated this Process into an Object if, when given two points, he or she identifies a midpoint of the segment connecting them and can apply an action to it. In addition, the individual can be aware that a midpoint is not unique in

Taxicab geometry when the given segment is not parallel to one of the axes and does not have a slope of 1 or -1. An example of an action that can be applied to this Object is the comparison of locations of multiple midpoints in Taxicab geometry.

3.1.2.3 Circle

Just as a circle is defined in Euclidean geometry, a Taxicab circle is defined as the locus of points that are equidistant from a fixed point. Using the Taxicab metric to measure distance, it is observed that a Taxicab circle resembles a regular diamond, or a square that is "tilted." Based on the definition of a circle, then the Taxicab metric results in the equation of a Taxicab circle to be $r = |x - h| + |y - k|$, where r is the radius and (h, k) is the center of the circle. Figure 3.6 illustrates a Taxicab circle with a center at (3,3) and radius 2. An object conception of **Distance** is a necessary prerequisite to recognize the points on a circle are equidistant from a given fixed point, the center, since a student would need to understand the radii of this circle are all equal in distance, comparing values, i.e. – performing an action on their **Distance** Object. In this study the concept of **Circle** includes subconcepts of **Euclidean Circle** and **Taxicab circle** that develop in a similar manner as illustrated in Figure 3.3. I clarify that a student would need to use their understanding of **Distance** in order to understand what a radius of a circle is, but the concepts of **Distance** and **Radius** are distinguished as they can be used in different ways in regard to a student's overall understand of his or her **Circle** concept. How an individual may exhibit each stage of conception associated with **Circle** are described below.



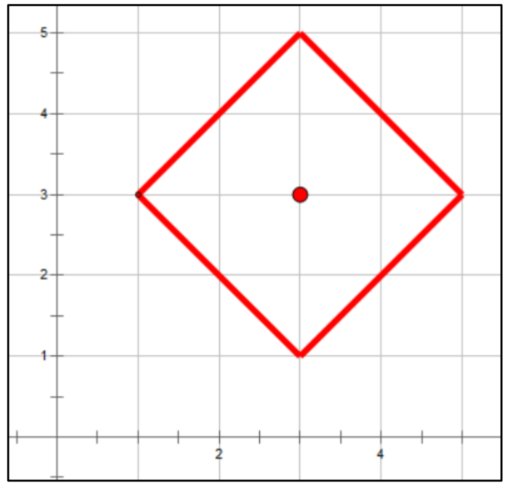


Figure 3.6 Visual representations of a Euclidean and Taxicab circle each with center (3,3) and radius 2.

As with the other concepts in geometry, the concept of **Circle** can be expressed geometrically or by an algebraic representation. For example, describing a Euclidean circle as round or a Taxicab circle as a square is using a visual image, or a geometric representation. On the other hand, writing the equation for one of the circles is clearly a reference to the algebraic representation of a circle. For the sake of not repeating ourselves, almost exactly as illustrated in Figure 3.3 pertaining to **Distance**, a student must have a process conception of both **Euclidean circle** and **Taxicab circle** in order for him or her to make connections between the figures in both geometries, or to coordinate their conceptions in order to form this concept of **Circle.**

We note that for each process of **Euclidean circle** and **Taxicab circle**, there is an understood coordination between a student's **Geometric Representation of Euclidean Circle (GREC)** and **Algebraic Representation of Euclidean Circle (AREC)** processes and **Geometric Representation of Taxicab Circle (GRTC)** and **Algebraic Representation of Taxicab Circle (ARTC)** processes**,** respectively (similar to Figure 3.2). Since the definition of distance is used in the definition of circle, it is noted that **Distance** is a subconcept of **Circle**. Intuitively, this means that **GRD** and **ARD** are subconcepts of both **GRC** and **ARC.** Specifically, it may be that a student utilizes both his or her **GRD** and **ARD** within his or her understanding of **Circle**. Figure 3.7 illustrates in general how this interaction may take place. The red arrows in this figure indicate the interactions that are of most importance within the descriptions of various stages of conception for **Distance** and **Circle.** What follows is how the concepts relevant to **Circle** (whether that is **Euclidean circle, Taxicab circle,** or the coordination of the two – **Circle)** into **Geometric Representation of Circle (GRC)** and **Algebraic Representation of Circle (ARC)** in relation to the preliminary genetic decomposition is partitioned in this study.

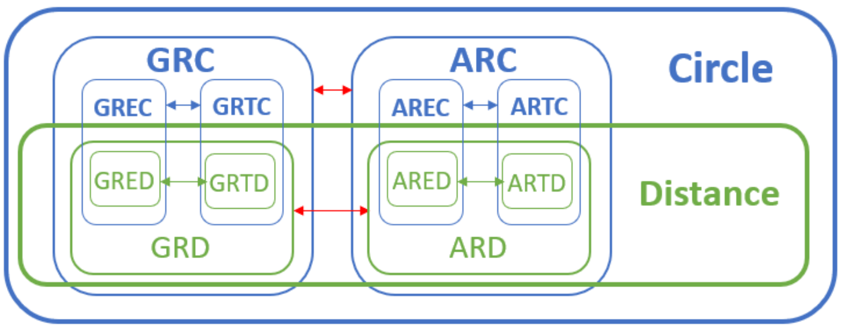


Figure 3.7 Visual representation of the concept of Circle.

Geometric Representation of Circle (GRC). In terms of geometrically representing **Circle**, an individual can do so graphically or by describing various properties of circles in terms of their visual appearance. For example, a student can draw examples of circles or explain how the construction of a circle is related to the definition of a circle. Below details of the various levels of conception associated with **GRC** are described.

Action: An individual is able to draw a circle given a metric, center, and radius. The individual may need specific examples of circles in order to talk about properties of circles within a geometry.

Process: When given a metric to consider, an individual can illustrate a circle with any center and radius. The individual does not need specific values to imagine how to construct the circle and can begin to describe why the circle appears a certain way within a metric using his or her own words.

Object: An individual encapsulates this Process into a totality when he or she can compare multiple circles to one another. Specifically, he or she can successfully illustrate multiple circles within a geometry and can compare the appearance of the circles one another. The individual can compare these circles by describing similarities or differences in the shape or size of the circles using his or her own words. In this case, the action being performed on this object is a comparison.

Algebraic Representation of Circle (ARC). In general, **Circle** can be represented algebraically by either stating an equation for a circle or by verbally describing an expression or formula in the context of the algebraic representation. For example, by describing aloud that for a point to be on a Taxicab circle, "the distance formula would have to be equal to the radius,' this clearly is in reference to how the formula for distance is included in the equation for a Taxicab circle. What follows are the various levels of conception associated with **ARC**.

Action: Given a specific center and radius, the individual can recall/use the equation of a circle, correctly identify which values from the given center and radius are associated with which variables in the equation, and substitute these values into the equation.

Process: Given a metric and any center and radius, an individual can write the equation of the circle associated with them. The individual does not need a specific center and radius to imagine how to write this equation, and can begin to describe in his or her own words how this equation is derived from the definition of a circle.

Object: An individual encapsulates this Process into a totality when he or she can successfully produce equations for different circles in a geometry using his or her definitions of **Distance** and **Circle**. He or she can compare these equations in to one another and explain how they were derived from the definition of a circle. In this case, the action being performed on this **ARC** object is a comparison.

In a similar manner that as described in the development of the **Distance** concept, I proceed by describing the development of the concept of **Circle.** Note that within both **Euclidean** and **Taxicab circle**, there are subconcepts of **GREC/AREC** and **GRTC/ARTC**, respectively. Overlooking for a moment the partition of these **Circle** concepts (the general **Circle, Taxicab Circle,** and **Euclidean Circle)** into **GRC** and **ARC**, there are many ways a student can assimilate the Taxicab circle into their existing understanding of **Circle.** Figure 3.8 illustrates how this is can take place, which is similar to Figure 3.3.

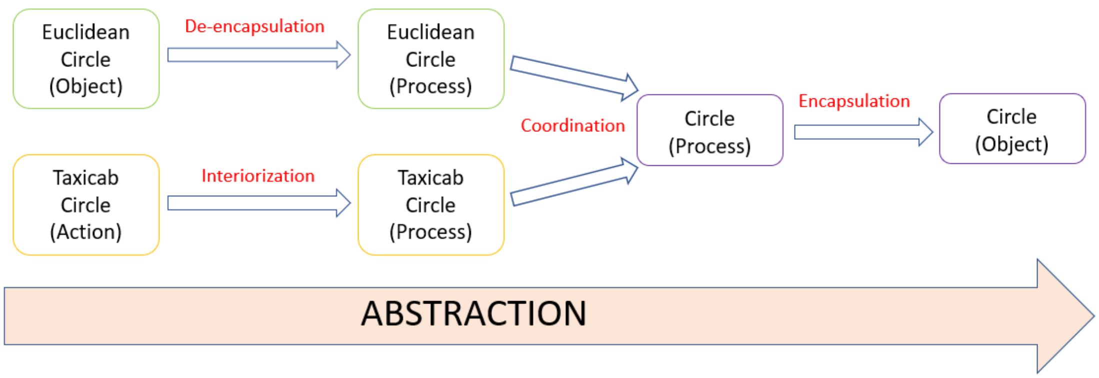


Figure 3.8 Visual representation of a possible learning pathway for a student developing his or her understanding of Circle.

A possible learning path for an individual introduced to Taxicab geometry begins by he or she previously exhibiting an object conception of **Circle** in Euclidean geometry and has interiorized their action conception of **Circle** in Taxicab geometry into a Process. In this case, the individual must de-encapsulate his or her **Euclidean Circle** object to assimilate the new metric, or the concept of **Taxicab Circle**. Once this happens, the processes of **Euclidean Circle** and **Taxicab Circle** can coordinate to form one Process, and conflated concept will be called **Circle**. He or she can then encapsulate this understanding to arrive at an object conception of **Circle.** Another example begins with a student exhibiting a process conception of **Euclidean Circle.** In this case, if a student has an action conception of **Taxicab Circle,** they must interiorize this action conception into a process in order to coordinate these two processes of **Euclidean Circle** and **Taxicab Circle** to form the new **Circle** process**.** The individual can then encapsulate this understanding into an object called **Circle.** An individual must have a process conception of both **Euclidean circle** and **Taxicab circle** in order to make connections between the two metrics. If a student has an action conception of either **Euclidean** or **Taxicab circle,** they are able to draw or write equations of circles in both geometries but cannot accurately describe geometrically or algebraically why the figures appear different.

We illustrate below in Figure 3.9 a possible pathway that a student might take in order to assimilate the subconcept of **Taxicab distance (**outlined in blue in Figure 3.9) into his or her existing *Circle schema*. Compiling figures from the previous sections about these pathways, I suggest that a possible pathway begins with a student de-encapsulating his or her object conceptions of **Euclidean circle** and **Euclidean distance** into processes (red arrow in Figure 3.9) which he or she can then coordinate with his or her process conceptions of **Taxicab circle** (purple arrows in Figure 3.9) and **Taxicab distance** (blue arrows in Figure 3.9), respectively. These coordinations can result in the construction of the **Circle** and **Distance** processes, respectively. Once these new processes are constructed, the student can coordinate any combination of his or her **GRD, ARD, GRC,** and **ARC** to further develop his or her understanding of both **Distance** and **Circle.** The student can then encapsulate his or her processes of **Distance** and **Circle** into objects.

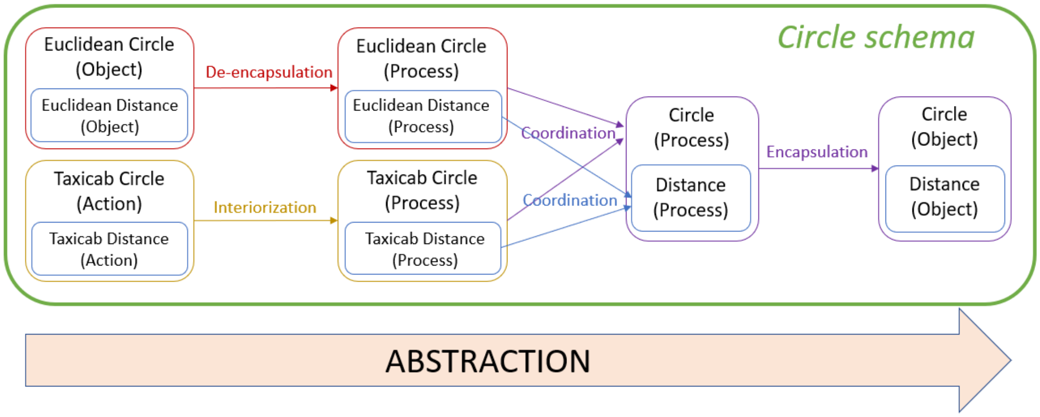


Figure 3.9 Illustration of a possible pathway for a student to take in assimilating the Taxicab metric into his or her circle schema

For a large portion of this dissertation, each student's mental structures are considered and his or her development and understanding of the concepts involved in understanding the general construction of a geometric representation of a circle and the structure of the equation of a circle (algebraic representation). I remind the reader that in addition to **Distance**, the concept of **Circle** also has the subconcepts of **Radius, Center,** and **Locus of points**. All of these develop in a similar manner within the *circle schema* in relation to a student's assimilation of Taxicab geometry into this schema (in a similar manner as illustrated in Figure 3.3) but omit these details for the sake of length.

3.1.2.4 Perpendicular bisector

As stated previously, the concept of **Perpendicular bisector** of a segment in this study is defined to be the set of points that are equidistant from the endpoints of this segment. In Euclidean geometry, this results in a straight line that intersects the segment at its midpoint and does so at a right angle. On the other hand, in Taxicab geometry, the equivalent of the perpendicular bisector as defined in the same manner, is not necessarily a straight line and does not necessarily intersect this segment at a right angle, depending on the slope of the segment with respect to the axes. Thus, when the concept of **Perpendicular bisector** is discussed in relation to a segment, I am not implying this line has to intersect the segment at a right angle, but rather that it is the line defined by the locus of points equidistant from the endpoints of this segment.

To clarify, if a segment is neither parallel to one of the axes nor has a slope of 1 or -1, its Taxicab perpendicular bisector consists of two rays and the midset, which intersects the given segment at the Euclidean midpoint. Figure 3.10 illustrates an example of a perpendicular bisector of a segment AB, where the red portion of this line is the midset of the endpoints of this segment. On the other hand, if the given segment is either parallel to one of the axes or has a slope of 1 or -1, the Euclidean and Taxicab perpendicular bisectors of that segment are the same line.

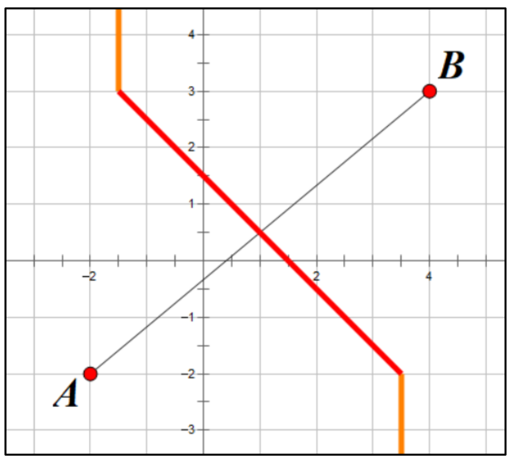


Figure 3.10 The perpendicular bisector of segment AB.

Again, an object conception of **Distance** is necessary for an individual to identify a point that is equidistant from the two endpoints of a segment, and therefore on the perpendicular bisector, since he or she has to be aware that the distances from this point to the endpoints of the segment are equal to one another. Again, it is noted that in this study the concept of **Perpendicular bisector** includes subconcepts of **Euclidean perpendicular bisector** and **Taxicab perpendicular bisector** that develop in a similar manner as illustrated in Figure 3.3**.**

Action: Given two points, the individual identifies a point that is equidistant from both points by guessing and checking for a point that satisfies this property. He or she also can check to be sure the distances from this point to the two given points are equal.

Process: The individual coordinates his or her **Distance, Midpoint,** and **Locus of points** Processes to construct a **Perpendicular bisector** Process (Seen in Figure 3.9). In particular, the individual can imagine in his or her head what it looks like for a point to be on the perpendicular bisector of a segment and can explain in his or her own words how to find such a point. In particular, in Taxicab geometry if an individual can identify a point on the perpendicular bisector that is also in the midset but does not explicitly state it is on the perpendicular bisector, the individual is using his or her concept of **Midpoint**, and not necessarily his or her concept of **Perpendicular bisector**.

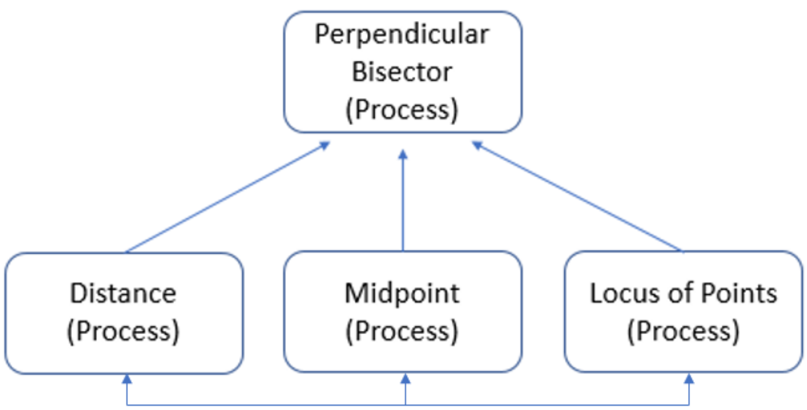


Figure 3.11 The construction of the Perpendicular bisector Process.

Object: The individual has encapsulated this Process into an Object when, if given two points, he or she identifies that the set of points that is equidistant from these points is the perpendicular bisector. Therefore, he or she has recognized that not only is there more than one point that is equidistant from these two points, but there are an infinite number of points that would satisfy this property. In other words, the individual is able to consider all of these distances and points at the same time, and views these points as a totality, or a line, and can apply actions to this object. The student can compare properties of perpendicular bisectors in a single geometry, especially in Taxicab geometry when the slope of the segment directly influences the appearance of the perpendicular bisector.

It is reiterated that if a student is able to make observations and comparisons of a concept across geometries, this indicates that the student is exhibiting at least a process conception of this concept in both Euclidean and Taxicab geometry, since this requires a coordination of processes. If a student has not constructed processes for a concept in both Euclidean and Taxicab geometry, the stage of development of the mental structure is presented as it is exhibited within Euclidean or Taxicab geometry.

3.1.3 Use of GSP in student understanding

We model this preliminary genetic decomposition for the way in which students use GSP in problem solving on the analysis and results presented in Hollebrands (2003), which investigated the use of GSP in how students reason about transformations in geometry in particular reference to the domain of a transformation. The author found that students' understanding of domain was possibly influenced by their interactions with the computer. At first, students perceived the domain only as a single and particular object to which transformations were applied, which may have been reinforced by the requirement within the program to select an object to apply a transformation. Thus, these students were using the computer program to simply perform actions on this static object. Hollebrands (2003) also found that some students had shifted their understanding of a domain to consider all of the points in the plane, indicating that these students were "operating from the theoretical definition of the point…rather than only labeled points on the screen," (p. 70). These students also were able to anticipate the results of a transformation without having to perform the actual transformation in the computer program, indicative of a process conception. Further, when students began to consider the properties and behaviors of transformations rather than rely on the specific image on their screen, this indicated an object conception of the domain of the transformations. These notions of actions, processes, and objects adapted to definitions in GSP in addition to the detailed preliminary genetic decomposition (presented in Section 3.1.2) will be used in order to analyze evidence of the participants' understandings of various concepts in Euclidean and Taxicab geometry in this dissertation.

3.1.4 Triad of schema development

Our focus now shifts to the *circle schema* and what relevant concepts are evoked by a student through the coordination of the *Euclidean geometry* and *Taxicab geometry* schemas during an interview. Below, descriptions are provided of the various stages associated with the triad of development of the *circle schema* in both Euclidean and Taxicab geometries and the mental constructions necessary to achieve these stages. Note that at each stage of development, the *circle schema* is rearranged, or *accommodated*, in order to form new relationships among the components.

After the descriptions of these stages for the *circle schema* in general, descriptions will be provided for what it means for students to accommodate their *circle schema* in order to assimilate Taxicab geometry into their current *circle schema*. It is noted the *circle schema* involves the evoked *Euclidean geometry schema* within the *circle schema* (which will be called *circle in Euclidean geometry schema (cEg)*). In addition, the evoked *Taxicab geometry schema* within the *circle schema* will be referred to as the *circle in Taxicab geometry schema (cTg)*. The genetic decomposition for what will be called the *cEg-cTg schemata interaction* will be presented in Section 4.2. This genetic decomposition will be used in data analysis and possible examples of the various *levels of schema interaction* within the context of the triad of schema development between the *cEg* and *cTg* will be presented. In other words, this interaction considers the coordination of the two schemata of *Euclidean geometry* and *Taxicab geometry* as they are evoked within the *circle schema*, and what relationships are formed from this coordination.

These relationships are predominantly made evident by the verbal explanations and/or written responses (or personal concept definitions) provided by a student about how the definition of a circle relates to various representations of a circle and other relevant concepts/subconcepts. This analysis will focus on what components of each student's *circle schema* were evoked during the interview in order to justify the responses he or she provided on a questionnaire prior to a follow up interview. It will be investigated to what extent students understand how the construction of a circle (geometric representation) and the structure of the equation of a circle (algebraic representation) directly relate to one another and the definition of a circle and what this means in terms of their overall understanding of **Circle**. For the remainder of this report when a *circle schema* is being referenced, I clarify this is in the context of a student that is learning about the Taxicab metric (and other concepts in Taxicab geometry), as the interaction of the *cEg* and *cTg schemata* is a natural result of the introduction of this metric. It is believed that the genetic decomposition of this schema interaction could be used as a model for how a student could assimilate any metric into their *circle schema*, although further research would need to be done to validate this.

As stated previously, Baker et al. (2000) is used as a model for the descriptions of the various stages of schema development for *circle* in addition to the descriptions of the various stages of schema interaction formed from by the coordination of the *cEg* and *cTg schemata*. Recall that the *circle schema* involves the concepts of **Distance (Euclidean distance/Taxicab distance), Radius, Center,** and **Locus of points.** Figure 3.12 shows an illustration of how this schema may be structured in terms of these concepts. As a note, arrows in this figure represent a coordination of processes, possibly across schemata, specifically by connecting various geometric and algebraic representations/properties of circles in both geometries. The blue arrows indicate the coordination of geometric components, while the green arrows indicate the coordination of algebraic components. The red arrows indicate the interaction of the *cEg* and *cTg schemata*, which will be discussed in detail in Section 4.2.

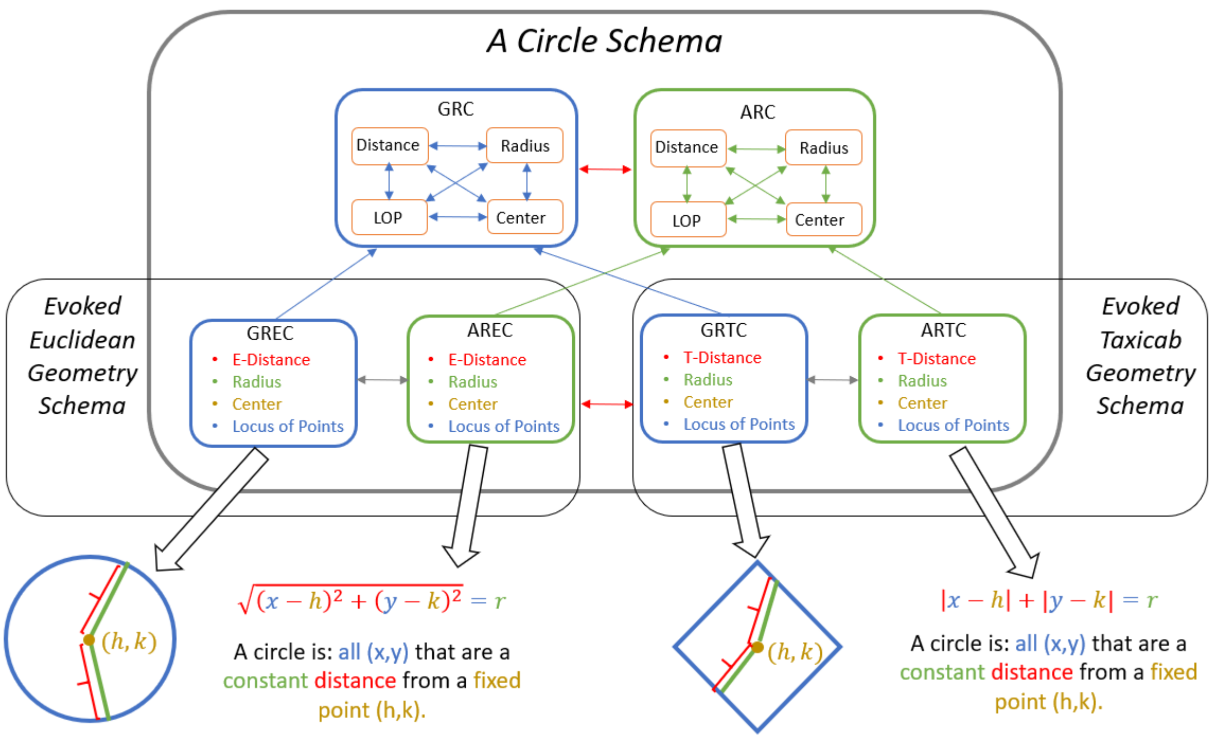


Figure 3.12 Illustration of an example of the underlying structure of a circle schema, including the interaction of schemas, as indicated by red arrows.

The concept of **Distance** in Figure 3.12 as shown in the concepts of **GRC** and **ARC** is a coordination of the **Euclidean** and **Taxicab distance** processes resulting in the construction of the new process of **Distance**, as described previously. Similarly, **Radius, Center,** and **Locus of points** in **GRC** and **ARC** are assumed to be the new processes constructed by the coordination of their respective processes in Euclidean and Taxicab geometry. At the bottom of this figure, a color coordinated breakdown of how each of these concepts is directly involved in the geometric and algebraic representations of a circle in both geometries is provided. It is believed a student who has constructed this underlying structure of the *circle schema* can demonstrate this using his or her personal concept definition and verbal explanations. A student with a coherent understanding of **Circle** can also describe how the construction of a circle and the structure of the equation of a circle are direct results of the definition of a circle.

Below it is described what it looks like for a student to be operating within the various stages of the triad of schema development for the *circle schema* within Euclidean/Taxicab geometry along with the necessary mental constructions that I believe exist at each stage. In particular, the coordination of processes within each schema is paid special attention.

Intra- ***Circle in Euclidean/Taxicab geometry (cEg/cTg)***

At this stage of schema development, a student views the components of his or her *circle schema* as isolated structures. A circle in Euclidean/Taxicab geometry is analyzed in terms of its properties either geometrically or algebraically (e.g. - it is round/square, there is a square root sign/absolute value signs). Explanations of these properties (for example, what a circle looks like) are local and particular, i.e. – a student references specific examples of circles in Euclidean/Taxicab geometry in order to try to explain simple observations. A student may also be able to make simple generalizations about Euclidean/Taxicab circles. He or she talks about a Euclidean/Taxicab circle predominantly either geometrically or algebraically, but not both. In other words, the student’s personal concept definition relates to the visual appearance of the circle or to the algebraic representation of the circle and cannot form or explain connections between the two (specifically, verbally). The student cannot coordinate any of their **Euclidean/Taxicab Distance, Radius, Center,** and **Locus of points** Processes (if they exist) for a circle in Euclidean/Taxicab geometry. A student has a collection of "rules" or properties of a circle in Euclidean/Taxicab geometry but has very limited knowledge about the relationship among these. For example, if a student knows to construct a circle by measuring out a certain distance in multiple directions but cannot explain how this satisfies the definition of a circle, nor how it relates to the equation of the Euclidean/Taxicab circle, the student is exhibiting that he or she is operating at an Intra-*Circle in Euclidean/Taxicab geometry* stage of schema development.

Necessary mental constructions: An action conception of the subconcepts involved in **Euclidean/Taxicab Circle** (**Euclidean/Taxicab Distance, Radius, Center,** and **Locus of points).** If the student has a process conception of any of the four subconcepts in Euclidean/Taxicab geometry, then he or she has not yet been able to coordinate these processes with one another to make necessary connections between the geometric/algebraic representations of a circle.

Inter- *Circle in Euclidean/Taxicab geometry (cEg/cTg)*

The student is able to form relationships among the isolated ideas from the Intra- *Circle in Euclidean/Taxicab geometry* stage. In other words, the student is able to make connections between visual properties of a circle in Euclidean/Taxicab geometry and the algebraic properties (specifically, verbally), and can use either of these representations to talk about a circle. The student can coordinate his or her **Euclidean/Taxicab Distance, Radius, Center,** and **Locus of points** Processes (and corresponding geometric and algebraic representations of each) for a circle in Euclidean/Taxicab geometry, resulting in these explanations. The student can coordinate any number of these processes with one another but does not demonstrate they coherently understand how the construction and the equation for a circle in Euclidean/Taxicab geometry result from the definition of a circle. The student can begin to make more meaningful generalizations about circles in Euclidean/Taxicab geometry by coordinating various representations, and how these representations relate to the definition of a circle. A student begins to group components of Euclidean/Taxicab circles and realizes they are related for all circles in Euclidean/Taxicab geometry but is unaware of a more general relationship among all circles in multiple geometries. For example, if a student can explain that the Euclidean/Taxicab distance formula in the equation for a circle in Euclidean/Taxicab geometry corresponds to the distance from the center of the circle to a point on the circle (coordinating **Euclidean/Taxicab Distance, Radius** and **Center** Processes), but explains this only within the context of Euclidean/Taxicab geometry, and not in general for all circles, then the student is exhibiting that he or she is operating at the inter- *Circle in Euclidean/Taxicab geometry* stage of schema development.

Necessary mental constructions: At least a process conception of some the subconcepts involved in **Euclidean Circle** (**Euclidean Distance, Radius, Center,** and **Locus of points)**, which is necessary in order to make connections between geometric/algebraic representations of the components of a circle. A student has the ability to coordinate at least two of the subconcepts (and some or all of their corresponding geometric/algebraic representations). If a student has at least a process conception of all four of the subconcepts, they have not constructed a new process as a result of the coordination of these four processes.

Trans- *Circle in Euclidean/Taxicab geometry (cEg/cTg)*

A student at this stage constructs an awareness of the completeness of the *circle in Euclidean/Taxicab Geometry schema* and can “perceive new global properties that were inaccessible at the other levels," (Baker et al., 2000, p. 559). The student groups geometric and algebraic representations when presented with a problem involving a circle in Euclidean/Taxicab geometry and can decide when one or both of these representations are needed for reasoning, which indicates a coherence of the schema. The student can coordinate all of his or her **Euclidean/Taxicab Distance, Radius, Center,** and **Locus of points** Processes (and all corresponding geometric and algebraic representations of each) for a circle in Euclidean/Taxicab geometry to coherently understand the construction of a circle and the structure of the equation for a circle in Euclidean/Taxicab geometry. If a student can explain how particular parts of the equation for a circle in Euclidean/Taxicab geometry correspond to parts of a graphical representation, and how they are both a result of the definition of a circle, then the student is exhibiting evidence of operating at the Trans-*Circle in Euclidean/Taxicab geometry* stage of schema development.

Necessary mental constructions: At least a process conception of ALL subconcepts involved in **Euclidean Circle** (**Euclidean Distance, Radius, Center,** and **Locus of points**). A student has coordinated all combinations of these Processes and has constructed a new process (which coordinates all geometric/algebraic representations of all subconcepts) in order to make coherent connections between the geometric and algebraic representations of a circle in Euclidean/Taxicab geometry and how these are a result of the definition of a circle.

Next, the reader's attention is focused to the red arrow shown in Figure 3.12, which represents the interaction of the *cEg* and *cTg schemata.*

3.1.5 Schema interaction and thematization of the circle schema

The genetic decomposition for the schema interaction that occurs within the *circle schema* across the *Euclidean geometry schema* and *Taxicab geometry* is presented in detail in Section 4.2.

In particular, this interaction includes nine levels of schema interaction that result as the two schemata of *circle in Euclidean geometry (cEg)* and *circle in Taxicab geometry (cTg)* interact with one another at various stages. These nine stages are illustrated in Table 3.1 below. It is noted what type of information students are transferring from one geometry to another such as whether a student is transferring local properties of a concept, the definition of the concept, or the definition of concept and how it relates to other concepts. Drawing from relevant literature, accentuating focus to using Cooley et al. (2007) as a framework, a description of what thematizing the *circle schema* may mean and what it involves conceptually follows.

Table 3.1 The nine levels of cEg-cTg schema interaction.

		cTg		
		Intra-	Inter-	Trans-
cEg	Intra-	Intra-intra	Intra-inter	Intra-trans
	Inter-	Inter-intra	Inter-inter	Inter-trans
	Trans-	Trans-intra	Trans-inter	Trans-trans

In APOS Theory, a schema has been thematized into an object when actions and processes can be applied to it (Arnon et al., 2014). Based on Cooley et al. (2007), a student can thematize his or her *circle schema* if the student is sufficiently conscious of it, can reflect on it, and act upon it. In particular, to be recognized as having thematized his or her *circle schema*, a student must be classified as (or exhibit evidence of) operating at the trans-*cEg*, trans-*eTg* stages of their *circle schema*, and also demonstrate that, if given a new metric, they can act upon this understanding to construct a circle and also derive the equation for a circle using this metric. In other words, the student has abstracted the definition of a circle to all metrics known or unknown, can provide geometric and algebraic representations of a circle using any given metric based on the definition of a circle, and understands how these representations are related to one another. It would also be expected the student could verbally communicate all of this information coherently. In general, a student has thematized his or her *circle schema* if the student can demonstrate an awareness of the global definition of a circle over all metrics across all representations (geometric, algebraic, verbal). At this point, the underlying structure of the *circle schema* is a "fundamental part of the understanding and can be viewed in totality as an object conception," (Cooley et al., 2007, p. 7).

3.2 Research and instructional setting

This research study was conducted at a university in a College Geometry course during one Fall semester, which has an introduction to proof course as a prerequisite. Since it is a cross listed course, there were both undergraduate and graduate students enrolled in the course, many of whom were pre-service or in-service secondary mathematics teachers. The study is defined as a teaching experiment, as described by Cobb and Steffe (2010) and Steffe & Thompson (2000), which consisted of sessions of instruction, many of which involved the dynamic geometry software Geometer's Sketchpad, followed up with individual interviews with voluntary participants. During the teaching episodes, there is a method of recording in order to document what occurs during an episode, which can then be used to reflect on and analyze the teaching experiment itself to prepare and develop future methods of instruction (Steffe & Thompson, 2000).

In conducting a research study in this manner, the researcher will be able to implement instructional methods based off of the relevant research and analyze how this instruction has influenced students' mathematical learning, reasoning, and understanding on a first-hand basis, as this is the primary purpose for using teaching experiment methodology (Steffe & Thompson, 2000). Under the assumption that the construction of mathematical knowledge largely is influenced by the experience students have in the classroom and their interactions with classmates and instructors, the emphasis for the researcher as the teacher can be very important in modeling how students come to learn mathematical concepts (Cobb & Steffe, 2010). In fact, the analysis of mathematical reasoning for students has shifted toward conjecturing about the quality of their experience with mathematics, and as a benefit to this type of research, teachers are no longer consumers of findings that are produced outside of the classroom (Cobb, 2000). Many mathematics education research studies are conducted outside of a teaching experiment with theoretical analysis by the researcher.

While this is important to understanding the construction of mathematical knowledge, this analysis can "at best intersect only part of the knowledge gained through experiencing the dynamics of a child doing mathematics." Further, in the past, "classical experimental design inhibited efforts to investigate students' sense-making constructs" and often in these cases, the researchers act as a passive voice, with "conceptual analyses of mathematical understanding and mathematical performance" omitted from the study (Cobb & Steffe, 2010, p. 20). As the dynamics of the classroom can greatly influence student goals, understanding, what constitutes a mathematical explanation, and students' general beliefs as to what mathematics is (Cobb, 2000), it is important to conduct research in this way to gain a deeper understanding of these students and their learning discourse.

3.2.1 Course structure

As a prerequisite for this College Geometry course, students were to have taken and earned a letter grade of C or better in Bridge to Higher Mathematics, which is an introduction to proofs course in which students must use mathematical reasoning and proof to develop their writing and critical thinking skills. In Bridge to Higher Mathematics, students study set theory, algebra, analysis, and real numbers in order to facilitate a formal approach to mathematical concepts and proofs. Although students in College Geometry have been exposed to proof writing, it is noted that students fail to see a need for proof, as found invariably in research (Jones, 2002). In fact, according to Jones (2002),

> "Learning to prove requires the co-ordination of a range of competencies each of which is, individually, far from trivial, that teaching approaches tend to concentrate on verification and devalue or omit exploration and explanation, and that learning to prove involves students making the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures. Further reasons are that students are asked to prove using concepts to which they have just been introduced and to prove things that appear to be so obvious that they cannot distinguish by intuition the given from what is to be proved," (Jones, 2002, p. 132).

Since it is so difficult for students who have even been exposed to proof writing before, it is very important to implement a well thought-out, efficient, and effective instructional approach. For this reason, the textbook chosen for the course is College Geometry Using the Geometer's Sketchpad (2006), written by Barbara E. Reynolds and William E. Fenton. The authors intended the book to aid in showing "the need for developing mathematical proofs in the context on hands-on explorations that help students develop insight into these ideas before they attempt to write rigorous mathematical proofs," (Reynolds & Fenton, 2006, p. xvii). The authors used the following four questions in determining what mathematical topics to include in their text ((Reynolds & Fenton, 2006):

1. Does this topic lend itself to exploration and conjecture with The Geometer's Sketchpad?
2. Does this topic allow us to examine some interesting questions in geometry – and connect the study of geometry to the larger tapestry of mathematics?

3. Does this topic allow explorations that lead students to make and test their own conjectures?
4. Is this topic useful content for future middle-school and high-school mathematics teachers while leading to important ideas that reach considerably beyond the content of a high-school geometry course?

Only topics that answered in the affirmative to all four of these questions were included in the text. The textbook for this course is based on APOS Theory and is explicitly modeled after the ACE teaching cycle. That is, each chapter contains three main sections dedicated towards activities on Geometer's Sketchpad (A), classroom discussion (C), and exercises (E), with a fourth section dedicated as a chapter overview. In particular, the activities on GSP provided by the textbook guide students and are intended for students to explore properties and relationships about various figures and concepts before being formally taught these notions. It is noted that during these activities, students work in groups and are expected to participate in meaningful discussion in order to come up with conjectures.

The classroom discussion section is guided by the instructor, meant to reinforce these explorations by the students. This section of the text and the guided instruction is formalized in order to help students solidify concepts and theorems and be able to present them as proofs. The third section includes many exercises that students are to complete outside of class to strengthen the conceptual understanding they have reached after activities and classroom discussion. These include, but are not limited to, questions about logic, proofs, and may include further constructions in GSP. In this course, students are required to submit their exercises by the respective due dates in Geometer's Sketchpad in order to help facilitate the use of and skill with the program. GSP helps to keep students engaged and guide them to conjecture and think logically through their reasoning in proofs. By allowing students to visually see relationships and then discuss them with their classmates, the authors of this textbook aim to help students understand the underlying mathematical concepts which, in turn, will allow them to gain the ability to correctly write proofs, or in the least improve their proof writing skills. Students in this course will also be required to take online quizzes, written exams, and be able to present their solutions for certain activities, exercises, and proofs in class.

As discussed previously, this study is classified as a teaching experiment. As the class meetings were an hour and fifteen minutes in length, the 18 students enrolled in the course participated in group work for most of the class meetings throughout the semester, and when they were not participating in group work, there were large class-discussions led by the instructor about the theory involved with particular concepts in the class. These students were assigned groups at the beginning of the semester and shortly after the middle of the semester the groups were reassigned based on the level of understanding exhibited by each student and the social interactions the instructor had witnessed throughout the semester. The goal of rearranging the groups was to evenly distribute the level of understanding of students as well as optimize the cooperative learning among groups. The new groups had a few weeks to work together before Taxicab geometry was introduced in the course, so note the interactions presented in this report are not the first interactions these students had with one another within these groups. It is clarified that students learned concepts and theory in Euclidean geometry for the first 12 weeks of the semester, and then the course switched focus to consider some of these concepts and ideas in Taxicab geometry for the remaining two weeks of the semester, resulting in four class days spent on Taxicab geometry.

3.2.2 Taxicab geometry

Before the first day of Taxicab geometry, students completed a few activities from their textbook in GSP pertaining to particular concepts in Euclidean geometry, since they would be introduced to these concepts in Taxicab geometry next. For the class periods designed around Taxicab geometry, the students had four days to explore concepts and complete activities in their groups. On the first day, students were given a "paper and pencil" activity where they were exploring the Taxicab metric in a real-life situation, with the intention that students "get a feel" for the way this metric behaves. To clarify, this activity never mentioned the words "Taxicab metric" or provided a formula for how to calculate this distance (we present this activity in further detail in Section 4.1.1), and instead asked students to try and write this formula on their own. After working in groups on this activity for about 30 minutes, the students began completing activities in GSP from the text about distance in Taxicab geometry and how it is used in the construction of a circle in Taxicab geometry.

On the second day, students continued to complete activities in GSP from the text about other concepts in Taxicab geometry such as ellipses and the distance from a point to a line. On the third day, students began to work on a worksheet designed by the researchers in GSP with more activities that pertained to perpendicular bisectors, the triangle inequality, and congruence of triangles in Taxicab geometry. On the fourth and last day of instruction for Taxicab geometry, about 45 minutes were spent on an instructor-led class discussion to help students reflect on their exploration and talk about some of the theory behind these concepts. The students were then given a different worksheet in GSP designed by the researchers about the construction of an ellipse in Taxicab geometry and proving that all of the points on an arbitrary ellipse with arbitrary foci and axes lengths satisfy the definition of an ellipse. The instructor then went over a few examples in

GSP of how to prove that points on this ellipse in Taxicab geometry satisfied the definition of an ellipse. The last 30 minutes of class were allotted for students to explore and try to prove this for the remainder of the points on this ellipse. It is noted that this activity proved to be somewhat unsuccessful at first, as it was found that almost all of the participants did not understand the definition or construction of an ellipse in Euclidean geometry. This led to many students not appearing to be able to make connections between an ellipse in Euclidean geometry and an ellipse in Taxicab geometry. This was used as an opportunity to go over the definition of an ellipse and how an ellipse is constructed in Euclidean geometry to help deepen this understanding in the participants.

3.3 Participants

For this Fall semester, there were eighteen students enrolled in the College Geometry course, partitioned into seven undergraduate students and eleven graduate students. Of the seven undergraduate students, six were mathematics majors and one was a pre-middle level education major. Of the eleven graduate students, nine were in the Masters of Arts in Teaching (MAT) Mathematics Education program and two are in the Mathematics Education (MED) program. All students that were willing to participate were included in this research study and were given pseudonyms. All students were at varying levels of understanding in mathematics but are all required to have earned a C or better in the transition to proofs course. Provided in Table 3.2 is a summary of these participants names and program they were enrolled in.

Table 3.2 Overview of participants

	Pseudo name	Major/Program
Undergraduate Students	Ally	Math major
	Alicia	Math major
	Eileen	Pre-Middle Ed major
	Samantha	Math major
	Russell	Math major
	Nicole	Math major
	Kristen	Math major
Graduate Students	Felix	MAT Program
	Amy	MAT Program
	Tyra	MAT Program
	Darryl	MAT Program
	Mark	MED Program
	Robin	MAT Program
	Alex	MAT Program
	Parker	MAT Program
	Brianna	MED Program
	Marianne	MAT Program
	Hannah	MAT Program

3.4 Data Collection

In general, data was collected in various ways for this research study. This data can be categorized in three ways.

1. Students' group work (activities) during the Taxicab Geometry lessons along with the guided class discussion were audio and video recorded. Students' work in GSP from these class days were also collected.
2. Students' relevant coursework throughout the semester including homework assignments, quizzes, and exams were collected.
3. Interviews were conducted pertaining to course material and were audio and video recorded; this will include a think-aloud session completed in Geometer's Sketchpad, followed by a group interview conducted based off of a protocol.

All relevant coursework that was used in data analysis was collected and copied before any grading or grade commentary was given, so that student grades were not being used in the analysis. All 18 students consented for their in-class group work and discussion to be audio and video recorded, as well as written work and exams throughout the course to be collected. Semi-structured interviews were conducted after the end of classes but before the Final Exam with 15 of the 18 students enrolled in the course who voluntarily signed up to participate in these interviews. Students who volunteered to participate in the questionnaire and interview sessions received extra credit in the course for doing so. An equivalent extra credit opportunity was given to students who were unwilling to participate in this portion of the study.

Since I was a co-instructor for the course, these interviews were conducted by a colleague in the area of research of Collegiate Mathematics Education, so students would not feel as though their answers affected their final grades in the course. Students formed groups on their own or with the help of the interviewer, and interviews were conducted in groups of one, two, or three, depending on the availability of students. Specifically, two interviews were conducted with individual students, two interviews were conducted in groups of two students, and three interviews were conducted in groups of three students. Students were given approximately 30 minutes to complete a questionnaire prior to the interview, and during the interview each student was asked to elaborate on his or her responses to the questionnaire by the interviewer. To ensure uniform interviews, an interview protocol was provided to the researcher conducting the interviews (part of which is provided in Appendix A). For consistency in group interviews, the interviewer asked that for each new question, the order of responses rotated so each student had an equal opportunity to express his or her original thoughts without an influence from the other students.

3.5 Data Analysis

The data collected from in-class group work during the class sessions on Taxicab geometry was collected in hopes of answering the first research question and sub questions, provided below.

1. In what ways do students use GSP to refine their understanding of mathematical definitions?
   (a) How do students apply their working understanding of a definition in GSP to reason about mathematical problems?
   (b) How does cooperative learning and the use of GSP help students in the development of abstracting definitions from Euclidean geometry.

The second and third methods of data collection which include relevant written coursework and answers from the questionnaire and interview sessions were largely used to answer the second research question and sub questions, provided below.

2. How do students adapt their understanding of concepts in Euclidean geometry in order to apply definitions in Taxicab geometry, a non-Euclidean axiomatic system?
   (a) What activities in Taxicab geometry can aide in the abstraction of a definition?
   (b) How does applying definitions in an atypical context affect the development of student understanding of these definitions?
   (c) How do students transfer their understanding of relationships among concepts in Euclidean geometry to Taxicab geometry?

The data that will be presented in this report includes student work and conversations from certain activities during the in-class group work and portions of each interview that pertained to students' understandings of the components within the *circle schema*.

All of this data was analyzed with relevant research questions in mind, with multiple passes of analysis conducted to ensure consistency and correct interpretation of student work and thoughts. For the first pass of analysis from transcriptions of group work in class, the data was organized by group and included any submitted work in GSP for the members of each group. For the first pass of analysis of the group work, within each group each group member's understanding of particular concepts in terms of APOS Theory were identified based on the parts of conversation in which he or she participated. During the second pass of analysis for the group work data, how each group member interacted with the other members of his or her group and/or instructor during the class period was identified. This allowed me to begin to analyze how cooperative learning and the use of GSP influenced each student's conceptions of different mathematical concepts in Taxicab geometry. On the third pass of this analysis, the results were organized in terms of each activity and identified trends in how students worked on the activity together and in GSP.

For the transcriptions of discussions from the interviews that were conducted, the data was organized by individual student and included their written work on the questionnaire. For the first pass of analysis with these transcriptions, each student's understanding of particular concepts in terms of APOS Theory and concept image/concept definition were identified as they emerged during each interview. For the second pass of data analysis for this aspect of my study, how each student interacted with the interviewer in addition to the other students in his or her interview (if there were any) was identified. In particular, the relationships between the components of each student's *circle schema* that were being evoked were noted and described. It was in this analysis that I noticed the development of these students' *circle schemas* were very complex since they were operating in both Euclidean geometry and Taxicab geometry. Not only was there an interaction of components within the *Euclidean geometry schema* (between **Distance, Radius,**

Center, and **Locus of points**), but there was an interaction of these components from the *Euclidean geometry schema* and the *Taxicab geometry schema* and back. In addition, during this analysis, since the questionnaire protocol asked students to express their thoughts on the algebraic representations of concepts in addition to the geometric representations of concepts, a complex relationship was found with how this fit together and made up each student's understanding of the relevant concepts.

It was after the second pass of this analysis that a genetic decomposition for the schema interaction between each student's *circle in Euclidean geometry (cEg)* and *circle in Taxicab geometry (cTg)* and how the overall *circle schema* could be thematized was developed, as modeled from Baker et al. (2000) and Cooley et al. (2007). For the third and final pass of this data analysis, this genetic decomposition was used to analyze all 15 volunteers' work and responses. From this information, the *level of schema interaction* each student exhibited in terms of the interaction of the *cEg* and *cTg schemata* was determined.

3.6 Chapter summary

In this chapter, genetic decompositions and descriptions as to how APOS Theory would be utilized to analyze each student's understanding of various concepts in geometry were described. In particular, these stages were explained in relation to the concepts of **Distance, Midpoint, Circle,** and **Perpendicular bisector.** In addition, a description of the various stages of schema development for the *circle schema* were provided. A summary of these descriptions is given in the following tables (Tables 3.3 – 3.7).

Table 3.3 Stages of conception for Distance.

Action:	The individual is able to graphically represent a distance and/or can use formulas to calculate distances in these geometries but cannot make connections between these representations. The student is able to observe local differences between distances (geometrically or algebraically) in multiple geometries but struggles to verbalize these differences.
Process:	The individual can conjecture about what properties might still hold from Euclidean geometry in a new metric and can begin to explain in their own words the differences between metrics beyond making local observations.
Object:	The individual can perform actions on this object, such as comparisons or using it as an "input" into a transformation, in both geometries.

Table 3.4 Stages of conception for Midpoint.

Action:	The individual can use a formula or expression for finding a midpoint, substitute given values, and evaluate this expression to arrive at an ordered pair that is a midpoint.
Process:	The individual can imagine the midpoint of two points is located so that the distance to it from the given points is half of the total distance between these points.
Object:	The individual can, when given two points, identify a midpoint of the segment connecting them and can apply an action to it such as a comparison to another midpoint of the same segment in Taxicab geometry.

Table 3.5 Stages of conception for Circle.

Action:	The individual is able to graphically represent a circle in a particular geometry and/or can use equations of a circle in these geometries but cannot make connections between these representations. The student is able to observe local differences between distances (geometrically or algebraically) in multiple geometries but struggles to verbalize these differences.
Process:	The individual can conjecture about what properties of circles might still hold from Euclidean geometry in a new metric, and can understand how various parts of a circle are a part of the definition of a circle.
Object:	The individual can perform actions on this object, such as comparisons or using their understanding of the definition of a circle to draw and write an equation of a circle.

Table 3.6 Stages of conception for Perpendicular bisector.

Action:	Given two points, the individual identifies a point that is equidistant from both points by guessing and checking for a point that satisfies this property.
Process:	The individual can imagine in his or her head what it looks like for a point to be on the perpendicular bisector of a segment, and can explain in his or her own words how to find such a point.
Object:	The individual is able to consider all of the points on the perpendicular bisector at the same time and views these points as a totality, or a line, and can apply actions to this object. The individual can compare properties of perpendicular bisectors in a single geometry, especially in Taxicab geometry when the slope of the segment directly influences the appearance of the perpendicular bisector.

Table 3.7 Stages of schema development for the circle schema.

Intra-:	The individual views the components of his or her *circle schema* as isolated structures. A circle in Euclidean/Taxicab geometry is analyzed in terms of its properties either geometrically or algebraically (e.g. - it is round/square, there is a square root sign/absolute value signs). Explanations of these properties (for example, what a circle looks like) are local and particular, i.e. – a student references specific examples of circles in Euclidean/Taxicab geometry in order to try to explain simple observations.
Inter-:	The individual is able to make connections between visual properties of a circle in Euclidean/Taxicab geometry and the algebraic properties (specifically, verbally), and can use either of these representations to talk about a circle.
Trans-:	The individual can explain how particular parts of the equation for a circle in Euclidean/Taxicab geometry correspond to parts of a graphical representation, and how they are both a result of the definition of a circle.

In this chapter the instructional setting and course structure for the class in which data was collected were described, as well as the participants in this study. In particular, the methods of data collection which included audio and video recordings of group work from class in addition to GSP submissions, written work submitted by students during the course, and student responses to a questionnaire in addition to audio and video recordings of interviews that were conducted which asked questions about these responses were discussed. Further, details were provided of how the data was analyzed and how I arrived at these results. These results are provided in Chapter 4.

4 RESULTS

As details of the method of the analysis were provided in Chapter 3, this analysis will be discussed in further detail for each method of data collection that was also described in Chapter 3. In particular, in Section 4.1, the results from the analysis of the audio and video data collected as well as GSP files that were submitted by students are provided. In Section 4.2, the results from the analysis of the interview data, including the genetic decomposition for the schema interaction that resulted from this analysis, are provided.

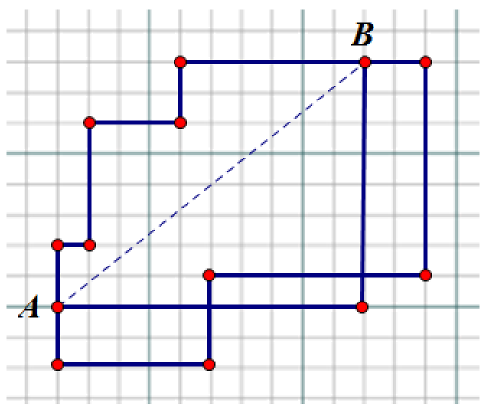
4.1 In class group work with Geometer's Sketchpad

Results from the participants about their understanding of various definitions during certain activities during the four days of class that proved to be motivating to students and provided rich data to help illustrate their understandings are provided below. As there were five groups in the class, five activities are presented (one group per activity), where the group's work and conversations are a representative example of not only how students worked on these activities, but also the various group dynamics that existed in the class. These results also give an idea into students' attitudes towards mathematics as a result of cooperative learning since this research study was conducted as a teaching experiment, and data was collected throughout the course which was built heavily around group work for 14 weeks. In particular, this dissertation sought to investigate how the use of cooperative learning and GSP help students to generalize their understanding of definitions of various concepts in geometry.

4.1.1 "Jason, the Uber driver"

The activity given to students on the first day of class devoted to Taxicab geometry is provided below. The students were given about 30 minutes to work with "paper and pencil" on graph paper to explore this problem. To the instructor's knowledge, students had not seen the formula for distance in Taxicab geometry. In this problem, students were told there was an Uber driver named Jason who had to pick up three different customers from various locations during a shift one night, in each case leaving his apartment and driving along the streets' grid system to arrive at these locations. The task given to students was to explore how many distinct routes of shortest distance there were from Jason's apartment to the pick-up location of each customer that night. Although students did not use GSP for this activity, I believe it is still important to provide this example of students working together with a shared goal.

Activity. Jason is an Uber driver in a major city, where all streets are constructed in a grid system so that, at a bird's eye view, the taxi can only drive vertically or horizontally to get from any two points (in the picture, point A to point B).



Throughout his shift one night, Jason had to pick up three different people, all at different times. Thus, for each trip, he had to leave his apartment to get to each respective pick-up spot. After all the trips were done, he began to wonder how many different routes of shortest distance there were to get to each spot.

(1) Help Jason figure out how many different routes would result in the shortest distance for each pick-up spot. Using the graph paper provided, explore this problem, given that Jason's apartment is located at (0,0) and the three pick-up locations were (2, 3), (5, 4), and (7, 7).

(2) Once you explore this problem, try to come up with a way to calculate the number of shortest routes from Jason's apartment to any given point on the grid. Write any observations here (or on the back of this paper).

(3) The way you were just measuring the distance that Jason traveled is how we measure distance in Taxicab geometry. Write a formula for distance in Taxicab geometry between any two points $A(x_a, y_a)$ and $B(x_b, y_b)$.

This activity was meant to allow students to explore the behavior of the Taxicab metric with limited association of components of their *distance schemas*, which can result in the formation of misconceptions. In particular, when attempting to apply mathematical definitions many students have incomplete concept images from which they reason resulting in them rejecting given definitions to use their imprecise concept image (Dickerson & Pitman, 2012). Thus, by not formally defining the Taxicab metric before this activity, it was hoped students would be able to explore this concept more freely to gain an understanding of Taxicab distance. For part (2) of this problem, although it was not expected for students to reach this conclusion in 30 minutes, the correct solution is that the number of distinct shortest routes from one point to another corresponds to Pascal's triangle. To clarify, the number of shortest routes between two points in Taxicab geometry is equal to the number of ways in which you can choose the change in x (or change in y) from the total distance. Algebraically, the number of distinct routes n from A to B is

$$n = \binom{|x_b - x_a| + |y_b - y_a|}{|x_b - x_a|} = \binom{|x_b - x_a| + |y_b - y_a|}{|y_b - y_a|} = \frac{(|x_b - x_a| + |y_b - y_a|)!}{|x_b - x_a|! * |y_b - y_a|!}$$

As an example, the number of distinct routes of shortest distance from the origin to $(2,3)$ is $n = \frac{5!}{2!3!} = 10$. This activity led to great discussion among all the groups, but one group in particular stood out. This group was made up of Marianne, Hannah, and Eileen. Marianne and

Hannah were graduate students and Eileen was an undergraduate student in the course. Below various portions of their conversations during the group work on this problem are shown. These illustrate these students' reflections on their understanding of Taxicab distance. In particular, these students were working to generalize the number of unique routes of shortest distance as asked in part (2) of this problem.

For the first customer pick up location of (2,3), Marianne and Hannah were counting the number of ways to go a total distance of 5. At first, they were counting each route and trying to keep track of which routes they had already considered, but they decided to organize these by considering how many routes were possible if they only moved a certain distance in the vertical direction initially. This was made evident by Marianne counting these out loud saying, "Starting going up one? So, we got one, one, two one. Then we go one, two, two. I don't think there's another way that we can go if we want to go up one." Hannah spoke up to suggest considering the route composed of single unit movements (like a staircase) and they agreed there were three of these routes that began by moving up one unit first. They then moved on to investigating how many routes there were if they moved two in the vertical direction first and agreed upon two routes. An example of how they were counting these routes can be seen in Figure 4.1, where Marianne appeared to be counting the number of routes between what could have been the origin and (2,2). At this stage of the activity, Hannah and Marianne were collectively exhibiting that they were applying an action conception of **Taxicab distance** to this context by calculating the length of each route from the origin to this particular point.

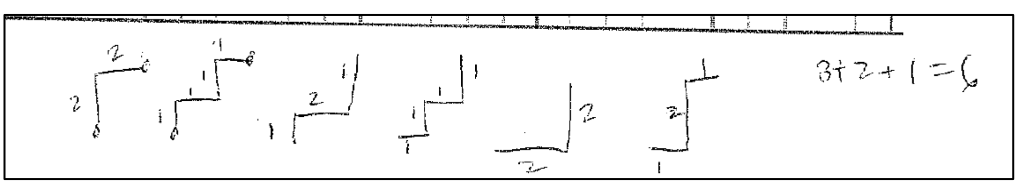


Figure 4.1 Marianne's way of counting routes from (0,0) to (2,2).

Hannah then began to try to generalize what this looks like while still using this particular point, an indication that she may have begun to interiorize these actions. She said, “If you go up to this one, you can go two ways. If you go up to this one you can go three ways. If you don’t go up there, four ways. If you go this way…that’s…four. That’s it.” After a few seconds Hannah said, “I’m mad ‘cause I can’t see the pattern,” indicating that Hannah was trying to interiorize these actions by searching for a pattern among these numbers so that she could apply this pattern to another point. After a moment of working in this way, Hannah asked Marianne if she counted ten routes from the origin to (2,3) and Marianne confirmed this number. Then Hannah stated, “The pattern is just one two three four… So, let’s do the next one and see if there really is a pattern.” Here, Hannah was pushing her group to identify and conjecture a possible pattern among the number of distinct routes between two points. As Hannah began to test another point of (5,4) the following interaction ensued where they related their findings to their previous ones.

Hannah: This one’s crazy. I think the only reason this one was short was because…

Marianne: Yeah, I mean there’s only five spaces and we’re moving…

Hannah: But when we come to this one, something crazy happens.

In this moment, Hannah and Marianne were discovering that the number of distinct routes of shortest distance from the origin went from being ten to the point (2,3) to being much larger for the point (5,4). Marianne was able to conjecture where this difference originated by saying “there’s only five spaces…”, in which she was talking about the total Taxicab distance between the origin and (2,3). After further exploration of using their method of moving a particular distance in a direction and counting the number of routes given this movement, Marianne got slightly distracted by how to calculate the shortest distance between the points (and not necessarily the number of shortest routes between them). In doing so, she said “It better not be you add the numbers together,” which was a conjecture that the distance between the origin and any of these points could be calculated by adding the x and y coordinates of that point to one another. Marianne attempted to come up with a reason as to why this would be the case by saying, “I guess it would make sense because… the shortest distance between one point and another is a straight line, right? So, in order for us to get the height of this one, we have to go here and then…go over…so it will always be the sum of those.”

Here, Marianne was able to coherently explain how to calculate distance in Taxicab geometry by relating it to distance in Euclidean geometry. By saying, “In order for us to get the height of this one…and then…go over,” she was referring to how to obtain the distance in the vertical and horizontal direction between two points, and that the shortest distance in this space would always be the sum of however many units she moved in these directions. In other words, she had generalized that to calculate distance in this way from the origin to any point, she would need to find the change in the x coordinates and add it to the change in y coordinates. This is clear evidence that she was able to interiorize her action conception of this distance and explain how to calculate it in general terms without actually performing these actions.

Eileen then interjected in an attempt to get the group back on track by pointing out that they were supposed to be finding the number of ways to go from the origin to that point and not the distance between these points. Hannah began exploring the problem again by counting different routes and was trying to figure out the pattern they had discussed previously. She made an observation that once she moved further away from the origin, “it gets real crazy, cause you gotta add all the combinations that went ahead before it.” This comment is evidence that in order to reason about the current situation, Hannah had grouped all of the previously considered situations together and was trying to use this group of distances/routes to obtain the total number of routes once she had moved to the next unit out. Here, Hannah was trying to create a recursive function that allowed her to input her existing knowledge (about the number of routes to a point that was closer to the origin) and would output the number of routes she was counting to a different point. This is evidence that she was applying an action to a set of distances. That is, exhibiting an object conception of **Distance** where the action being performed on this object was a transformation from input to output of a mental function.

Although Eileen had been quiet for the majority of the activity, she clearly had been listening to Hannah and Marianne's conversation and excitedly said, "It's like permutations and combinations!" From an initial analysis of the dynamics of this group, it may have appeared that Eileen had been disengaged from the conversation. However, since she was able to make this conjecture that the pattern had something to do with "permutations and combinations," I believe that internally she had been trying to generalize her understanding of the problem along the way. This is evidence that in cooperative learning, some students may take a role that is not as verbal as others, but this may be where they are the most comfortable which enables them to listen and reflect on others' conversation to generalize their own understanding. Shortly after this, Hannah began to reference Bernoulli's triangle (which has a similar construction to Pascal's triangle but is different conceptually). Regardless, Hannah had made a connection between the current activity and prerequisite knowledge from other math courses about combinations. Eileen seemed to be following Hannah's train of thought, while Marianne disagreed with them and continued down a different path, which is when the following conversation took place.

Marianne: If the shortest distance, if the shortest distance is n… max number of ways of shortest distance is equal to $(n-1) + (n-2) + \cdots$ all the way down to 1… When I tried (2,2)… The shortest distance was 4, right? So then if n is 4, then if the shortest distance is 4, then the maximum number of ways of going to shortest distance would be $3 + 2 + 1$ which is 6, and that's what I got. It was the same way when we did it with (2,3), which is 5, and it ended up being $4 + 3 + 2 + 1$. Then when I tried it with this one, when the max was 3 it ended up being $2 + 1$.

Eileen: So, how could we…So would this…

Marianne: So for (5,4)…

Eileen: That's 9 would be the shortest distance.

Marianne: …the shortest distance equals 9, so I think that the max number of ways would equal $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$. whatever that is…

After this Eileen and Hannah showed their slight disagreement that this method of calculation would capture the total number of shortest routes. At this time, the instructor announced to the class that it was time to move on to other activities and that they would discuss details of their solutions later. In general, the interactions and conversation in this group throughout the activity resulted in conjectures pertaining to permutations and combinations, Bernoulli's/Pascal's triangle, Hannah's attempt at writing a recursive formula for the number of shortest routes, and Marianne's attempt to create an explicit formula for the shortest distance between two points. This shows how effective the use of cooperative learning can be. These three students worked together, corrected one another, and challenged each other to generalize and test conjectures throughout the activity, which resulted in this group generalizing the problem more than any other group. Although they did not reach part (3) of this problem to define the Taxicab metric, it is noted that Marianne was able to verbally explain how this distance is defined earlier in this activity.

This example of cooperative learning was thought of as a success, since it could be clearly seen that each of these students working to generalize their understanding of the problem, and thus of distance in Taxicab geometry. Although Hannah and Marianne took the lead roles of this group, Eileen appeared to be engaged the entire time and contributed meaningful observations or conjectures to the conversation. Thus, the dynamics of this group worked well for them and they helped each other to complete this task. It is possible, as suggested by Glass and Deckert (2001), that if these students were given the opportunity to use GSP for this activity, they could have been able to generate conjectures even better than they did in this case, since they would be able to visualize patterns and properties accurately and more quickly.

4.1.2 Constructing a Taxicab circle

The next activity that students worked on came from their textbook for the class. Specifically, this activity is partially summarized in terms of the information and tasks given to students. The activity defines Euclidean distance algebraically, and the text specifies this formula is based on the Pythagorean theorem. The students are informed that in GSP, the **Distance** and **Length** tools calculate the Euclidean distance between two points. The text suggests defining a new rule for measuring distance as it is defined in Taxicab geometry. This part states “you can measure the taxi-distance easily by counting the blocks from one intersection of the grid to the next,” (Reynolds & Fenton, 2011, p. 146). The text then provides the following parts (a)-(c) to guide students through the construction of a Taxicab circle in GSP and the generalization of this construction.

(a) Plot points at $P(3,4), A(2,2), B(3,7), C(2,5)$, and $D(5,5)$. By counting the number of blocks from P to A, we find that the taxi-distance PA is 3 units. Find the taxi-distances $PB, PC,$ and PD. Two of these points are the same taxi-distance from P as A is. Which two?

(b) The set of all points that are at the same taxi-distance from P form a *taxi-circle* centered at P. In part (a), three of the points lie on a taxi-circle of radius 3 centered at P. Find several additional points on this taxi-circle. Describe the set of all points that are at a taxi-distance of 3 units from a fixed point P. How is the shape of a taxi-circle different from (or similar to) the shape of an ordinary Euclidean circle?

(c) If you are given a point $Q(x_Q, y_Q)$ and a radius r, how could you quickly sketch a taxi-circle of radius r centered at Q?

For part (a), they were given an example of the distance measure between P and A and were told how to get this, by "counting blocks." In addition to "counting blocks" to find distance in Taxicab geometry as suggested in the text, students were given a **Taxicab distance** tool in GSP that could measure the distance between two points in Taxicab geometry. It was expected that students would be able to plot the given five points ($P, A, B, C,$ and D) and count the blocks and/or use this tool in GSP to calculate the Taxicab distance between P and the remaining three points and identify which two of these points are the same distance from P as A is. These two points end up being B and D. In part (b), the goal was for students to make a connection that $A, B,$ and D were on the circle centered at P with radius 3. Further, the students would ideally be able to either count blocks or guess and check where several other points on this circle would be located. After finding a few points, students should begin to conjecture what the rest of the locus of points that satisfy this criterion would look like. Thus, when the text asks them to describe the set of points that are 3 units away from P in Taxicab geometry, students can try to imagine all of the points at the same time that satisfy the definition of a circle. The text then asks them to compare this figure to a circle in Euclidean geometry in order to get students to make connections between how the definition of distance affects the appearance of a circle in both geometries while maintaining the same definition of a circle. Part (c) aimed for students to generalize their understanding of the construction of a circle of any radius r by coming up with a general procedure to do so. In Figures 4.2 – 4.4 the completed activity by Ally, Amy, and Brianna are provided. As a note, the position of the text boxes for Amy and Brianna's submissions were slightly altered for the sake of presentation.

Amy and Brianna were both graduate students and Ally was an undergraduate student in the course. These students worked together during this activity, but all completed their own GSP document to submit. The concepts involved with the *circle schema* are **Distance, Radius, Center,** and **Locus of points**, as seen in the preliminary genetic decomposition in Chapter 3. As a part of these students' developing *circle schemas*, their conception of the definitions of these concepts are analyzed as they emerged in conversation. As can be seen in their submitted work, they began by plotting and labeling each of the given points in part (a) of the activity on their grid as shown by the red points in Figures 4.3 and 4.4, and multicolored points in Figure 4.2 (what are labeled as $A, B, C,$ and D). They then each used the **Taxicab distance** tool to calculate each of the distances of $PA, PB, PC,$ and PD, shown in the upper left part of Figures 4.2 – 4.4.

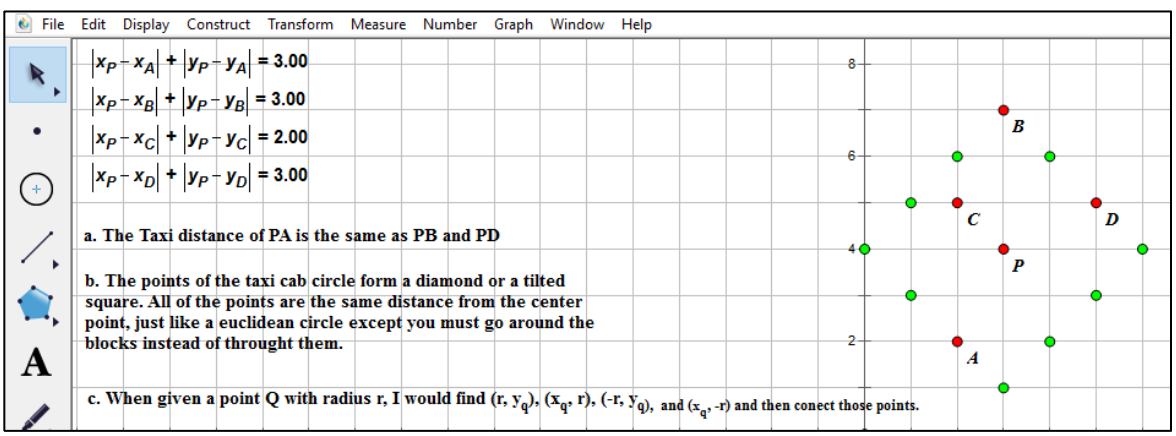


Figure 4.2 Work in GSP for Activity 5 submitted by Ally.

They correctly determined and agreed, after the use of the **Taxicab distance** tool, which of the distances were three units, implying they were aware that B and D were both three units away from P, and indicating at least an action conception of **Taxicab distance**, as they were able to use a tool to measure specific distances. After they read part (b) of the problem which asked students to plot several other points that are three units away from P, Brianna was looking at her graphical representation (seen in Figure 4.3) of the problem and said, "wait, why is PB the same?... Just cause it's three straight up?" to which Amy replied, "'cause it's like the radius, yeah... of that circle." Thus, algebraically they had agreed point B was three units away from P, but geometrically, Brianna did not see how this was the case until she visualized the straight line segment connecting these two points. This indicated she needed this particular example to understand why the point three units directly above the center would fall on the circle, indicative of an action conception of **Taxicab circle**. However, since she was able to visualize the distance between these two points (also a radius of this circle), she appeared to be ready to interiorize her action conception of the geometric representations of her **Distance** and **Radius**. It is possible she was confusing her **Euclidean distance** and **Taxicab distance** concepts (and **Euclidean radius** and **Taxicab radius** concepts), since these distances would be the same between these two points. When Amy explained to Brianna here that the distance from P to B was three because it was the radius of that circle, this provides evidence that Amy had coordinated her **Distance** and **Radius** processes within her **Taxicab circle** concept.

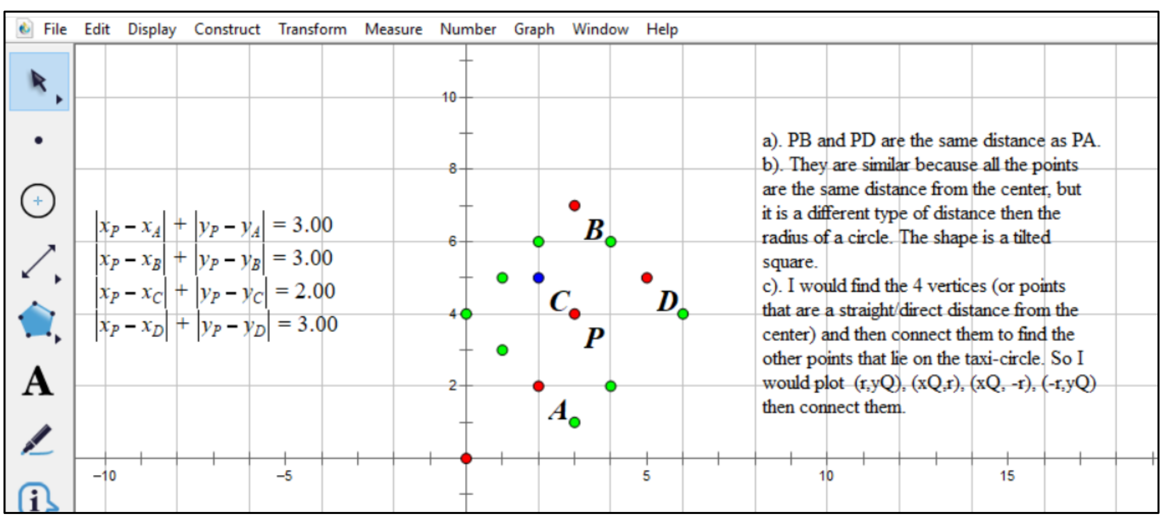


Figure 4.3 Work in GSP for Activity 5 submitted by Brianna.

Amy began to investigate why the distance from P to A (in Figure 4.2, the point A is the blue point to the left of her label) was three and said "'cause if that was a triangle, then the length of that hypotenuse wouldn't be… wait, because PA is like… sides one and two." Here, Amy referred to the components of the Taxicab distance between these two points as the legs of a right triangle whose hypotenuse is the Euclidean distance between P and A. In particular, she provided evidence that her instinct was considering the distance on the hypotenuse of this triangle, but then realized she should be looking at the legs of this triangle. Ally spoke up around this time and said, "so, we just need to form like a Taxi-circle" to which Brianna responded that they are just supposed to find a few points that they think are on this circle in Taxicab geometry.

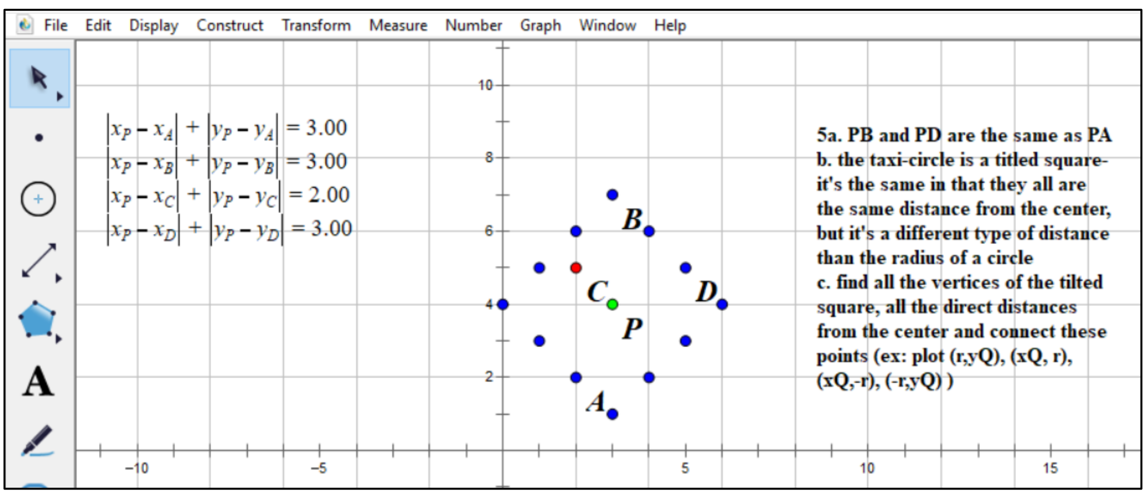


Figure 4.4 Work in GSP for Activity 5 submitted by Amy.

This interaction was especially interesting because of the different stages of conception these students were exhibiting. The way in which Amy was analyzing the geometric representation for the distance between these two points in the form of a triangle is evidence that she was trying to coordinate her process conceptions of **Geometric Representation of Euclidean distance** (**GRED**) and **Geometric Representation of Taxicab distance** (**GRTD**) to try and make connections between them. Ally seemed to have already understood why these points were all three units away from P and fully understood the task given since she said "so, we just need to form like a Taxi-circle." Although this is not explicit evidence of any particular stage of conception of concepts within her *circle schema* by itself, further and supporting evidence of Ally's understanding of **Circle** is discussed later in this activity. Brianna insisting that they just needed to find particular points that were three units away supports the claim that she had an action conception of **Circle** or many concepts within her *circle schema* at this point in the activity, since she wanted to plot particular points on her graph.

Ally listed aloud the point (4,6) that was also three units away from P, and then Brianna said, "I would say like (4,6)," repeating what Ally had said. The reader's attention is drawn to this short statement, because it is possibly in this reflection of Ally's comment that Brianna interiorized her action conceptions of **Radius** and **Distance** and coordinated them, since the next thing she said was, "…wouldn't (3,1)? 'Cause it's the same as PB, just the opposite direction? Like that's the radius, right?" Brianna indicated here that she was able to visualize the vertical distance from P to A as a radius and reflect this distance across the horizontal line through the center of this circle.

Based on the framework presented in Hollebrands (2003), since Brianna was able to anticipate this transformation on the distance/radius of this circle before actually performing it, this is evidence she had interiorized her action conceptions of **Radius** and **Distance** and coordinated them to see that (3,1) would be on the circle. It is possible that she could have still been conflating her conceptions of **Distance** and **Radius** in both her *Euclidean* and *Taxicab schemata.* This is because she had not indicated to this point that she had been able to differentiate the two distances geometrically, since they appeared the same for these points. In any case, she was able to generalize that the point directly three units below the center would also be on the circle because it was the same length of the radius, indicating a coordination of her **Distance** and **Radius** processes within her *circle schema.*

Ally listed aloud another point of (4,2) to which Amy responded, "wait, why (4,2)? 'Cause that's only two away." Recall that Amy had discussed the geometric representation of distance between P and $A(2,2)$ in terms of a right triangle, indicating she was in the state of interiorizing her action conception of **Taxicab distance** to coordinate with her process conception of **Euclidean distance** so that she could coherently make these connections. Since (4,2) is the point (2,2) reflected across the vertical line through the center of the circle, it is believed she most likely was evoking this right triangle again but was still confused or held a misconception about how Taxicab distance was measured since she still thought it was only two units away. Then, Brianna said "So, it's a distance of three though still," and after some thought, Amy replied, "Oh wait, I'm sorry…I understand now," which was the moment it is believed that Amy was able to construct a new process from the coordination of her **Euclidean** and **Taxicab distance** process, which she exhibits later in the conversation.

During this conversation between Brianna and Amy, Ally was plotting all of the additional green points shown in green in Figure 4.4. After Amy said she understood why (4,2) was on the circle, Ally said "it forms a diamond," but Amy and Brianna were still caught up in plotting and labeling additional points they had found that were three units away from P. Ally appeared to be in the state of interiorizing her action conception of **Taxicab circle** (if it was not interiorized already) since she was able to anticipate that the rest of the points on this diamond would also be three units away from the center. A few moments later Ally repeated herself and Amy, surprised, said "Oh, does it really?" Ally showed Amy her work in GSP (which can be seen in Figure 4.4) to which Amy replied, "Oh wow. That's very cool." Amy used this information and verbalized that she wanted to plot more points on her graph so that "you can see that it's a diamond." This is indicative that Amy was working to interiorize her action conception of **Taxicab circle** by focusing on the locus of points of this circle. In particular, Amy was using her conception of **Taxicab distance** and **Radius** to plot several specific points that were equidistant from P (action conception of **Locus of points**) until she could visualize the entire locus of points (process conception of **Locus of points**). This is supported by literature, in that Cha & Noss (2001) describe that it is important for students to have a ‘local’ understanding of locus by seeing properties of individual points on the locus. It is after this that students can generalize the point “into an algebraic form,” (p. 85).

Brianna had caught up to the conversation at this point and said “or, it could make a square,” which led to a lengthy conversation about differences between the definitions of a square and a diamond. But before they began discussing this, Amy said “So, it’s not a circle,” to which Ally responded, “this radius is the same.” This indicates that Ally was aware the diamond they had found indeed was a circle based on the definition of a circle, whereas Amy was still only associating the visual representation of a Euclidean circle to her concept image of **Circle**. This is evidence Ally may have already constructed process conceptions of **Distance, Radius, Center,** and **Locus of points** and was coordinating these processes to justify why this shape satisfied the definition of a circle, whereas Amy provided evidence that she was still in the state of interiorizing some of these concepts, in particular, the concept of **Locus of points**, since she was picturing a round circle in Euclidean geometry. Ally’s statements throughout this conversation about how these circles are similar because the radius is the same despite their shape indicates Ally was coordinating some or all of her **Distance, Radius, Center,** and **Locus of points** processes. As a result of these coordinations, Ally appeared to be forming a coherent understanding of the underlying structure of her *circle schema*, evident by her ability to verbally compare circles in both geometries.

When the problem asked for the students to compare the circle in Taxicab geometry they just constructed with a Euclidean circle, Ally said, “well an ordinary circle is round,” and Brianna said, “an ordinary circle is shaped circular… round.” Both of these comments pertained to the visual appearance of the circles, but then the following conversation occurred, providing great insight to these students’ understandings of these concepts.

Ally: I mean they’re similar because… the radius is always the same.

Brianna: It’s like…they’re all the same distance from the center?

Amy: Well, this is the same distance too, but it’s the same Taxi-distance.

Brianna: But they’re like a different type of distance ‘cause you can go like up, you can move different ways, it’s not like straight to it… like a clock.

Ally continued to demonstrate here that she was coordinating all of her processes of subconcepts of her *circle schema* by explaining that the two circles are similar in the way in which they are constructed. Also, she was using her definition of a circle to compare and contrast a circle in Euclidean geometry and a circle in Taxicab geometry in general terms. Thus, it appeared she had constructed an object in each of Euclidean and Taxicab geometry from the coordination of these processes (**Distance, Radius, Center**, and **Locus of points**), and was performing an action of a comparison on each of these objects. Then Brianna made a comment involving the concepts of **Distance, Center,** and **Locus of points** (when she said “they’re all” it is interpreted she was referring to the points on the circle) which indicated she was attempting to coordinate some processes within her *circle schema*. In her next comment, she provided concrete evidence that she had constructed a **Distance** process from the coordination of her **Euclidean** and **Taxicab distances** by making connections in the visual appearance of the two distances saying Taxicab distance is not like Euclidean distance, which is "straight to it… like a clock."

By this statement it appeared Brianna was using a metaphor of a clock to understand that a clock is round as a result of connecting points with a straight line (by rotating this straight line around one of its endpoints). She went on to reason through the fact that the circles are the same by the definition but are different because Taxicab distance is "like a different type of distance than the radius of a circle." When she said, "radius of a circle," she said this in the context that the radius of a Euclidean circle is a straight line segment (i.e.- the hand of a circular clock), which indicates that Brianna was comparing the appearance of radii between circles in Euclidean and Taxicab geometries. In other words, she had constructed a **Radius** process across her *cEg* and *cTg schemata* and was coordinating this with her **Distance** process.

Ally ended up writing on her submission (Figure 4.4) that the circle in Taxicab geometry was "just like a Euclidean circle except you must go around the blocks instead of [through] them," whereas Amy and Brianna wrote something about the points being the same distance from the center, but that it was "a different type of distance." In other words, they all agreed that the way distance is measured is the cause of the difference in appearance of a circle in Euclidean geometry and in Taxicab geometry. However, Ally gave visual details as to how the change in metric results in the change in appearance of a circle, further indicating a coherence to the structure of her *circle schema*. For part (c) this group began trying to geometrically explain how they would construct a generic Taxicab circle but ended up attempting to use algebraic expressions to explain this construction, indicating they were evoking both of their **GRC**s and **ARC**s. Specifically, they defined four points algebraically, "$(r, y_q), (x_q, r), (-r, y_q)$ and $(x_q, -r)$," and said they would [geometrically] connect those points. It is assumed they were attempting to define the points

$(x_q - r, y_q), (x_q, y_q + r), (x_q - r, y_q)$ and $(x_q, y_q - r)$, as shown in Figure 4.5, but right after they wrote these coordinates, the class period ended. If they had more time to explore their initial attempt at writing these coordinates, I wonder if they would have realized their mistake and adjusted the coordinates.

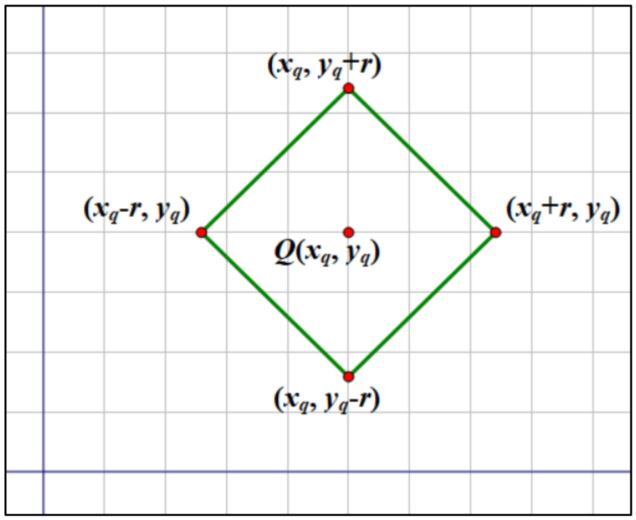


Figure 4.5 Coordinates of the vertices of an arbitrary Taxicab circle with radius r.

Prior to this activity, students had only been exposed to the Taxicab metric for the activity about Jason, the Uber driver, which was presented in Section 4.1.1. It is possible that Ally had interiorized her action conception of **Taxicab distance** during that activity, given the understanding she exhibited at the beginning of this activity. However, the moments that it is believed Amy and Brianna interiorized their action conception of **Distance** by constructing a process from the coordination of their **Euclidean distance** and **Taxicab distances** were able to be pin pointed. Further, the coordination of Amy, Brianna, and Ally's **Distance** processes with their **Radius** processes were observed in these interactions. Further, Ally was able to use the definition of a circle to compare and contrast this concept in both Euclidean and Taxicab geometry, which indicates an object conception of **Circle** in each of Euclidean and Taxicab geometry. It is noted that no particular student held a leadership role in this group and overall, they were able to work together to complete this task of generalizing the concept of a circle in Taxicab geometry by sharing ideas and questioning one another along the way. In fact, by all working separately on their own laptops and then comparing diagrams and observations of their various representations, it appeared to help these students develop their understanding of a circle in Taxicab geometry. This group provided an example of working with dynamic geometry software consistent with Contreras (2011), in that GSP allowed these students to have a firsthand experience with the definition of a circle and to reflect on their actions which allowed for a "more powerful abstraction" of this concept (p. 20).

4.1.3 "The Red Line Investigation"

In this activity, several constructions of perpendicular bisectors of segments with different slopes in Taxicab geometry were given to students on a worksheet in GSP. On the first part of this activity, the construction of the perpendicular bisector of a segment with positive slope (not equal to one) in Taxicab geometry was provided, but the activity did not say what this line was. Rather, it had them investigate and conjecture what this line was based on definitions and what they knew about similar constructions in Euclidean geometry. The first sheet of this activity is provided below in Figure 4.6, with answers from one of the groups of students made up of Robin, Nicole, and Kristen, of which Robin was the only graduate student. Unlike Amy, Brianna, and Ally in the last activity that was presented, these students chose to all work on the same computer (Robin's) while they were discussing the activity and submitted one file for their group at the end of class.

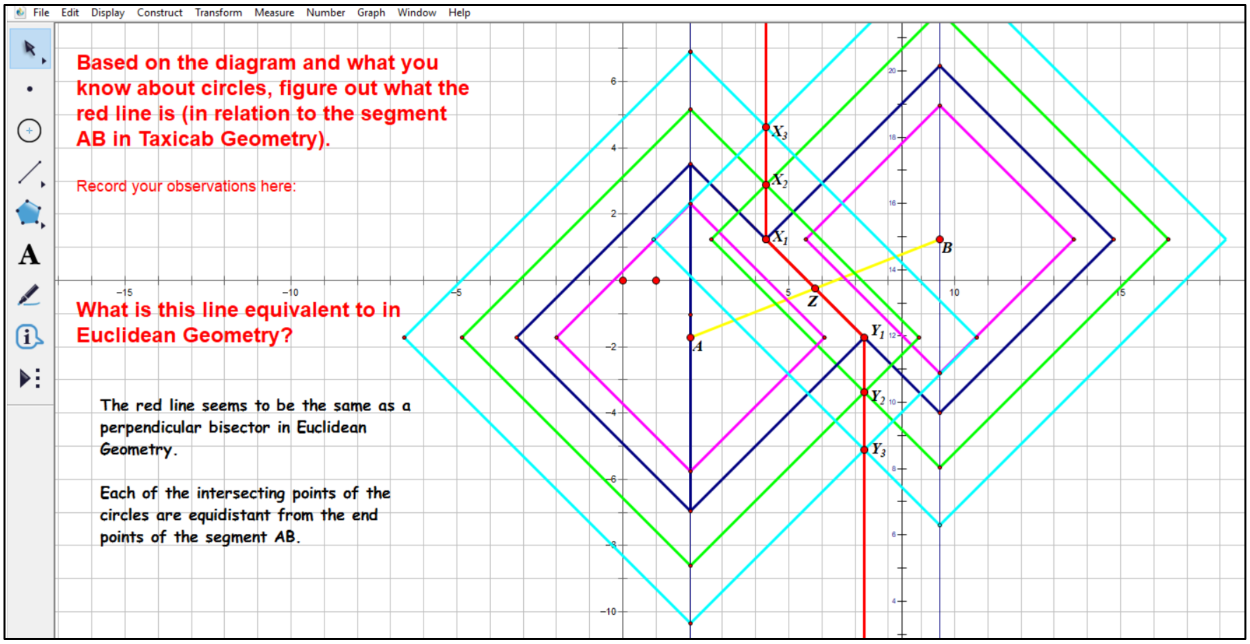


Figure 4.6 Submitted work from Robin, Nicole, and Kristen for the red line investigation.

To clarify, this red line was constructed by the intersecting points of each of the purple, navy blue, green, and light blue Taxicab circles (which are centered at each endpoint and have the same radius respective to their colors). It was intended that students would be able to see that the points $X_1, X_2, X_3, Y_1, Y_2, Y_3$ and Z in Figure 4.6 denote intersections of circles of the same color, i.e. – each of these points is equidistant from both endpoints as the distance between each of the endpoints and each of these points is the length of the radius of corresponding circles. In other words, the line that is being constructed is the set of points that are equidistant from both endpoints, which in Euclidean geometry is called the perpendicular bisector. In the case of the segment on the first part of the activity seen in Figure 4.6, this line bisects the segment but is not perpendicular to it.

As the main concept aimed to be evoked with this activity is **Perpendicular bisector**, the subconcepts that would be evoked as a result would be **Distance, Midpoint,** and **Locus of points**, as seen in the preliminary genetic decomposition presented in Chapter 3. It was also anticipated that the concept of **Circle** would emerge in the students' minds since the construction of this line uses circles. Evidence of Robin's, Nicole's, and Kristen's various understandings of these concepts as they emerged during their group work on this activity follows.

Even just as this group began reading the problem, evidence can be seen of these students' conceptions. Nicole said, "Based on the diagram and what you know about circles… which is that they are round!" to which Robin said, "except in Taxicab geometry." Nicole demonstrated here that she predominantly views circles geometrically and includes the shape of a circle as an extremely important part of her concept image. In particular, she was evoking her *circle in Euclidean geometry schema (cEg)* since she was describing the shape of a Euclidean circle. Robin, was evoking both his *circle in Euclidean geometry* and *circle in Taxicab geometry schemata (cTg)* since he was able to correct Nicole that a circle does not have to be round in order for it to be a circle, as they had seen was the case in Taxicab geometry during the previous activity. In other words, Robin had already begun to assimilate the concept of **Taxicab circle** into his *circle schema* and had expanded his notion of **Circle** to include a circle in Taxicab geometry, while Nicole was still conceptually evoking a geometric representation of a circle in Euclidean geometry.

Nicole then went on to examine the figures in GSP and said, "so would these be the routes?" and motioned in the air with her fingers as though she was drawing different distances in Taxicab geometry. Although it cannot be explicitly determined at what lines she was looking in GSP, Nicole described **Taxicab distance** geometrically by drawing in the air with her fingers. It can be inferred she was drawing arbitrary examples of distances in Taxicab geometry and not specific routes in the figure, since she was not pointing at the screen when she drew them. This indicates that she was able to imagine geometric representations of Taxicab distance (**GRTD**) in her head without using specific examples which is evidence of a process conception of **GRTD**. There was no evidence that Nicole had identified circles in the figure yet, which would imply her **Circle** concept was not being evoked at this time. Next, the following interaction took place.

Robin: So, the red line actually goes like here, here, and here.

Nicole: It goes through all of them!

Robin: But it looks to only intersect… it's like tangent to the blue to the dark blue circle.

Here, Nicole made the connection that the red line was concurrent with the points of intersection of each set of circles but, again, it is not believed she was aware these were circles until Robin said, "it's like tangent to the…dark blue circle." By this statement, Robin provided evidence that he was aware that lines that are tangent to circles in Taxicab geometry can have more than one point in common (whereas in Euclidean geometry, a tangent line to a circle only has one point in common with the circle). Although it is a small distinction, he described this figure as a "circle" instead of a "Taxicab circle," which is further evidence that Robin had expanded his notion of **Circle** to also include a circle in Taxicab geometry. Robin had generalized his personal concept definition of **Circle** to include both Euclidean and Taxicab circles, indicating he had constructed a **Circle** process by coordinating his **Euclidean** and **Taxicab circle** processes. Further he had generalized his understanding of what it meant for a line to be tangent to a circle to include both Euclidean and Taxicab geometries. This provided evidence he may have been ready to encapsulate this **Circle** process, at least geometrically, since he was able to apply this understanding of a circle to his understanding of what a tangent line is.

Shortly after this, Kristen asked what the question was that they were trying to answer, and Robin explained that they were trying to figure out what the red line was equivalent to in Euclidean geometry. After a few seconds of silence, Robin shouted, "Oh, it's the perpendicular bisector of AB!" Kristen replied, "oh yeah," and Nicole said, "that's it?" Nicole began trying to explain that they had been “looking at the big picture and we should have just been looking at this.” Then Robin said, “no…remember how we constructed perpendicular bisectors?” Nicole then replied that she agreed with him but that her “thought process was totally off.”. Through this interaction, it appeared Kristen asking what they were supposed to be doing made Robin reflect on the task at hand. This helped him to identify that this line was constructed the same way as the perpendicular bisector of AB in Euclidean geometry. It was during this moment of silence and reflection that it is believed that Robin began to evoke different concepts within his Euclidean geometry schema in a “guess and check” method to identify what this line could be. The possible trajectory Robin took during this time in regard to his thinking is presented below.

While evoking various concepts, at some point Robin evoked his *Perpendicular bisector schema* and began to reason in his head if this was the correct line. In other words, he was investigating whether or not the construction in Taxicab geometry he was given in GSP was equivalent to the construction of a perpendicular bisector in Euclidean geometry. In doing so, he was exhibiting an action conception of **Perpendicular bisector** in Taxicab geometry since he was referring to the specific example in front of him. He then verified the constructions were the same, which required the coordination of his **Circle** process with his **Perpendicular bisector** processes in both his *Euclidean geometry schema* and his *Taxicab geometry schema.* Thus, very quickly, Robin constructed a **Perpendicular bisector** process that included this concept in both Euclidean and Taxicab geometry by verifying this constructed in Taxicab geometry mirrored the construction of a perpendicular bisector in Euclidean geometry. This is the moment in which he decided the line was indeed a perpendicular bisector in Taxicab geometry. An illustration of this conceptual trajectory as I believe it occurred is provided in Figure 4.7. In particular, the red arrow represents the moment Robin interiorized his action conception of **Perpendicular bisector** in Taxicab geometry, the blue arrows indicate the coordination of a concept (also outlined in blue) from the *Euclidean geometry schema*, the green arrows indicate the coordination of a concept (also outlined in green) from the *Taxicab geometry schema*, and the purple arrows indicate the coordination of a concept (also outlined in purple) in Robin's general *Perpendicular bisector schema*. Thus, his **Perpendicular bisector** process is constructed through the coordination of (1) his **Circle** process in his *Perpendicular bisector schema*, (2) processes within his *Euclidean geometry schema*, and (3) processes in his *Taxicab geometry schema*. Thus, this particular coordination in Figure 4.7 has three different colored arrows.

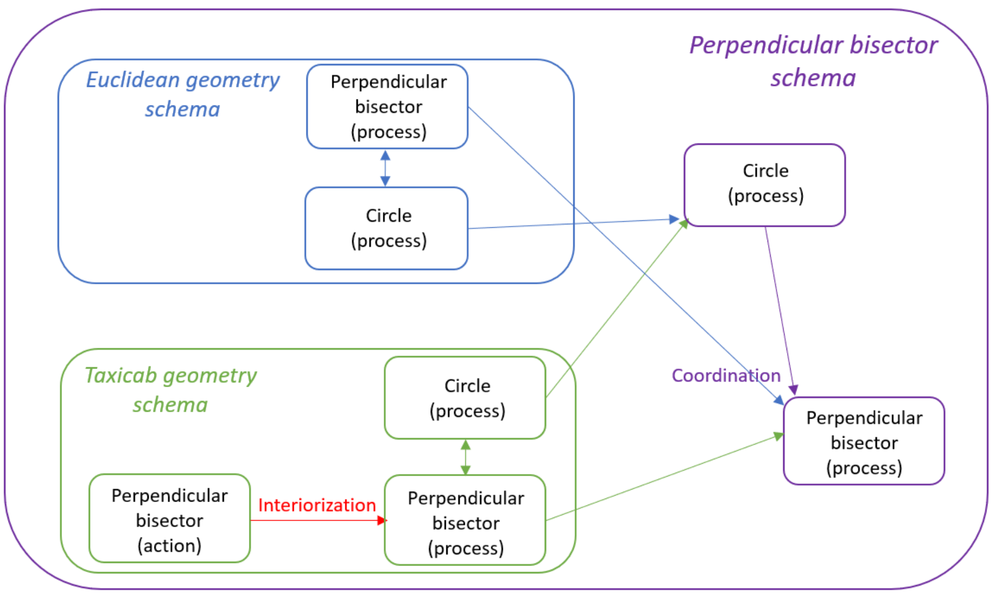


Figure 4.7 Robin's possible trajectory to construct a Perpendicular bisector process.

This type of coordination had not been considered in the preliminary genetic decomposition presented in Chapter 3 as a possibility to guide students to generalize their understanding of **Perpendicular bisector**. However, since **Circle** and **Perpendicular bisector** both have the subconcepts of **Distance** and **Locus of points**, the specific coordination of his **Distance** and **Locus of points** processes across these concepts possibly helped Robin to successfully complete this complex coordination.

Robin then asked his group if they agreed with what he had written in GSP at the time (which is different from what is seen in Figure 4.7 as they edited their work throughout the activity), but they were distracted and talking off topic. Later, the instructor walked over to the group and Robin asked if they were on track with their observations and answer to this first part of the activity. The instructor responded by asking what the definition of a perpendicular bisector was. Robin said, “it cuts the segment” and Kristen, currently engaged in the conversation since the instructor came over, added “in the middle.” The instructor then asked what else they knew about perpendicular bisectors to which Robin replied “perpendicularly.” The instructor asked them if that was still true in this case, referring to their work in GSP on this activity. Kristen immediately said, “yes…that’s what it looks like.” The instructor clarified that angles are measured the same way in Euclidean and Taxicab geometry, and Robin said, “then, no.” The next portion of the conversation was the instructor guiding the students to make a connection between the construction of this line in GSP and how this satisfies the definition of a perpendicular bisector in Euclidean geometry.

In regard to the group’s submitted work for this part of the problem, Robin then edited their explanation and typed that this red line in Taxicab geometry “seems to be the same as a perpendicular bisector in Euclidean geometry” with the justification that “each of the intersecting points of the circles are equidistant from the endpoints of the segment AB,” which can be seen in Figure 4.6. Robin was able to use the instructor’s guidance, his understanding of the construction of a perpendicular bisector in both geometries, and the definitions of a circle and perpendicular bisector to write this justification. Kristen did not seem to be engaged for the majority of this activity, predominantly adding comments such as “oh yeah,” and, “oh!” to the conversation, whereas Nicole was engaged at least at first. However, once Robin determined the line was a perpendicular bisector (i.e. - arrived at an answer), Nicole disengaged and began talking to Kristen about something personal. Robin attempted to bring them back to the activity by asking if they agreed with what he wrote, which ended up reoccurring throughout the activity. For example, a few minutes later Nicole and Kristen got off topic again and Robin tried to get the group back on track again by saying “here we go, here we go… team? Team?” This is unlike Amy, Brianna, and Ally (during the activity presented in Section 4.1.2) since all of the members of their group appeared to be engaged the entire time.

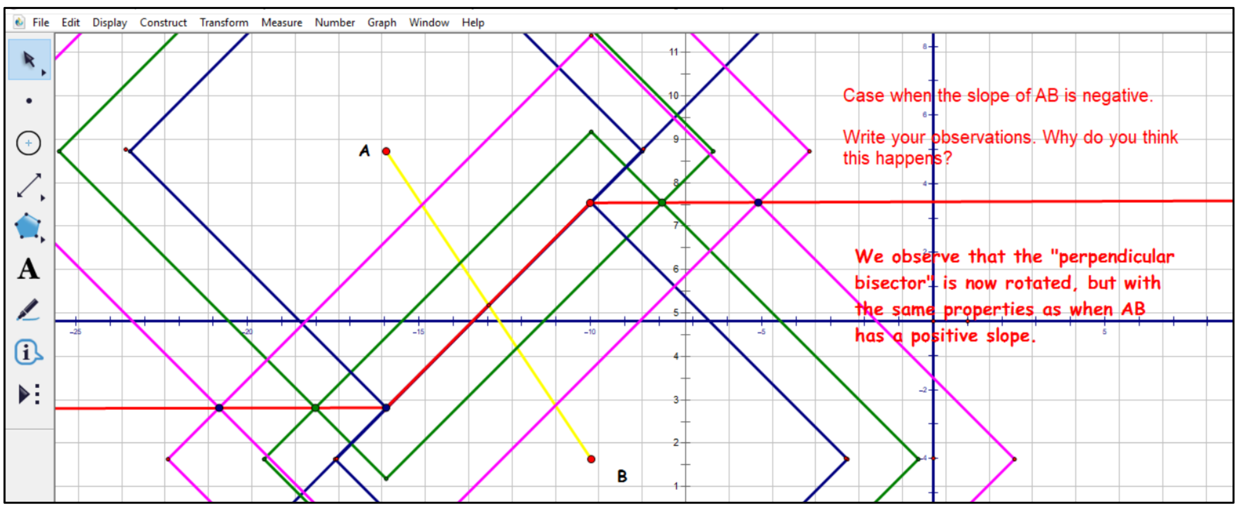


Figure 4.8 Submitted work by Robin, Nicole, and Kristen for the case of a negative slope.

Recall a change in slope of this segment affects the appearance of its perpendicular bisector in Taxicab geometry. The next parts of the activity had the students look at the Taxicab construction of the perpendicular bisector of a segment that has a negative slope and then a slope of positive one. These two constructions can be seen in Figures 4.8 and 4.9, with submitted answers from this group. For the case of segment AB having a positive slope, seen in Figure 4.6, the students began with a misunderstanding as to what the question was asking. In particular, they thought they were supposed to be investigating why AB had a negative slope, and not how the negative slope of AB affects the perpendicular bisector in Taxicab geometry.

After the instructor came over to the group and clarified the task, she helped guide the students through making observations about the appearance of the perpendicular bisector. She asked what the perpendicular bisector from the first part of the activity looked like if they were to trace it with their finger. After they went back to that sheet in GSP to look at its appearance, they all motioned as though they were tracing the locus of points seen in Figure 4.6. She then asked them to go back to the sheet in GSP shown in Figure 4.8 and try to identify any differences. Nicole traced the shape of the perpendicular bisector in this image in the air with her finger and then said, "it flipped!"

By comparing the shape of two Taxicab perpendicular bisectors to determine the difference in shapes, Nicole provides evidence she had at this moment constructed an object out of her **Geometric Representation of Perpendicular bisector** process in Taxicab geometry, where the action being applied to this object is a visual comparison. The instructor confirmed it looked different and asked if it still had the same properties. Nicole said it did, and Robin said, "I mean I guess so." Then Nicole turned her attention to writing an answer in GSP, and said, "so just say it's the same properties? Its flipped…" Robin expressed his hesitation in accepting what the instructor had implied by saying "I mean I guess so" and even went on to say, "I get fearful with observations because I like to make sure my observations are right." This leads to the belief that he wanted to explore this more in GSP when Nicole turned the groups attention to writing an answer and trying to confirm with the instructor that it was correct. This is a situation in cooperative learning where one student seemed to be fixated on obtaining an answer throughout the activity, while the other student in the group wanted to verify these observations and understand why they were true before putting them as part of their answer. This situation may call for a need in "group processing" as defined in Johnson and Johnson (1999) in that they need to discuss how they are working to achieve their goals, so they can identify and solve any issues they are having in working together effectively.

In the next part of the conversation, the members of this group were discussing how to word their observations, which can be seen in Figure 4.8. This excerpt is provided below.

Robin: Observe the perpendicular bisector is now what?

Nicole: It's now…the same?

Robin: I feel like it's rotated.

Nicole: Okay! Well say….

Robin: It's rotated…

Nicole: …but still has the same properties.

Notice in Figure 4.8, Robin wrote "perpendicular bisector" with quotes around it. This indicates he was distinguishing between a perpendicular bisector in Euclidean geometry and Taxicab geometry by putting quotes around this term in the context of Taxicab geometry. This is most likely because he is aware that in this context, the construction of this line no longer resulted in a line perpendicular to the segment, and thus was uncomfortable writing "perpendicular". By Nicole saying here that this perpendicular bisector is "the same," she was talking about the sameness in properties of this line compared to the previous one as a result of their conversation with the instructor. This indicates she may be ready generalize her understanding of the definition of a perpendicular bisector to include her understanding of this term in both Euclidean and Taxicab geometry. The students then went on to the next sheet in this GSP worksheet, which had the construction of a Taxicab perpendicular bisector for a segment AB that had slope equal to one, as seen in Figure 4.9.

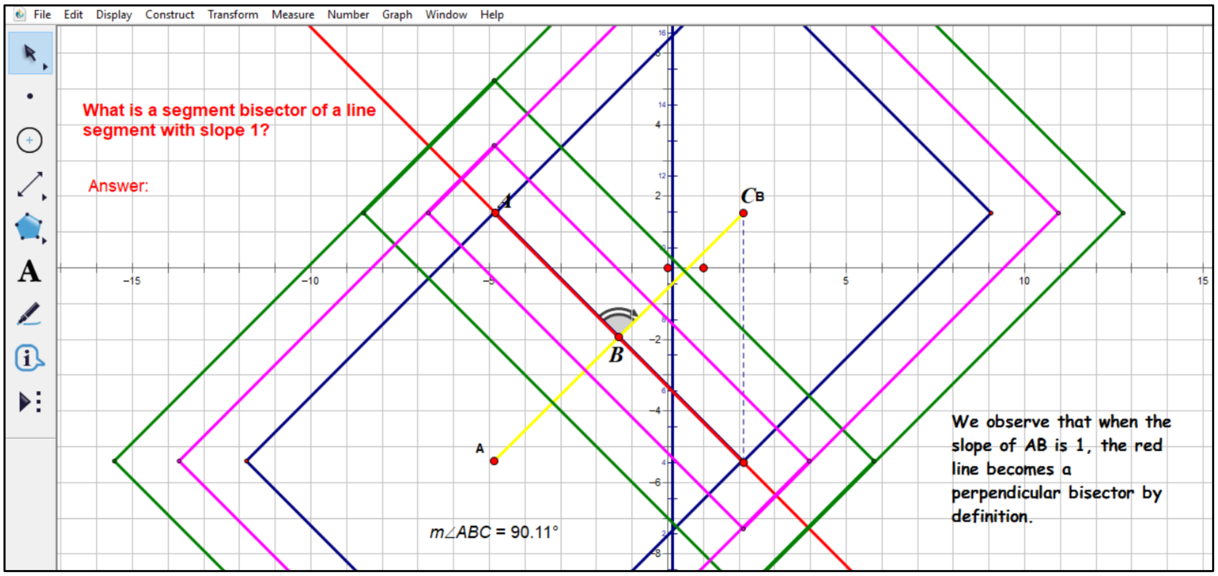


Figure 4.9 Submitted work by Robin, Nicole, and Kristen for the case of slope equal to one.

Robin made the observation that the perpendicular bisector in Taxicab geometry was a [straight] line now, and Kristen said, "it's so strange to be that when the slope equals one. Why does it look that way and not that way?." To clarify, she was referring to the other constructions they had seen. Robin said, "because AB has slope one," to which Kristen acted like she understood this as justification. Kristen had not exhibited evidence that she had the necessary mental constructions in order for her to understand this statement as justification (mainly since she had not participated much in this activity). Although it cannot be said for certain, it is not believed she actually understood the justification that Robin provided in this case. In particular, her observation about this case of the perpendicular bisector and asking why it looked this way appeared to be an attempt to participate in the conversation, although she could not add much in terms of depth.

In other words, her understanding of the involved concepts were not as developed as Nicole's and Robin's. She perhaps felt that she had not participated meaningfully in a while and wanted to try and add to the conversation. However, by making an observation about the change in appearance of the line when that was a given part of the task implies she did not completely understand (in the same way as her group members may have) that this line is defined the same way as it was in the previous constructions, but that the slope of the segment connecting two points affects the appearance of this line.

Nicole read the question from this activity aloud asking what this segment bisector in Taxicab geometry was, and Robin exclaimed, “a perpendicular bisector now!...because it forms the right angles!” This supports the belief that Robin was uncomfortable calling this line in Taxicab geometry a perpendicular bisector since it is not necessarily perpendicular. It is noted that in the submission for this portion of the activity seen in Figure 4.9, Robin typed in the bottom right that “the red line is a perpendicular bisector by definition,” this time writing “perpendicular bisector” without quotes, unlike the way he did in their submission seen in Figure 4.8. Thus, it appeared Robin was referring to the perpendicular bisector in Euclidean geometry here, attempting to clarify that this line is perpendicular to the segment in addition to bisecting the segment (referring to the “perpendicular” part of that term and saying “by definition”) as it would be in Euclidean geometry.

Nicole then agreed and asked if she could see the first construction again. Thus, she wanted to look at particular examples of these different situations and compare their appearances in order to better understand what the perpendicular bisector in Taxicab geometry is. She then said, “I wonder if we can make an angle and then measure it,” and Robin replied, “Do it. Just use this guy and measure from here to here,” in which he was referring to the **Marker** tool in GSP which allowed them to define an angle (and then use another tool to measure this angle). Nicole wanting to explore the idea that this line was perpendicular to the segment and Robin encouraging this is an example of this group conjecturing about a situation and choosing to test this conjecture before believing it as truth. In particular, they were working together to verify or disprove that this line was actually perpendicular to the segment AB. As a note, the measure of the angle they obtained shown in Figure 4.9 was 90.11º because of a slight shift in one of the lines involved in the construction of this perpendicular bisector. The students asked the instructor if it was supposed to measure to be 90º, which was confirmed, and Robin finished typing the rest of their response to this part of the activity.

Overall, Robin took the lead role in this group by working on the computer, writing their observations in GSP most of the time, and trying to keep his group on task. Nicole was engaged for the most part throughout this activity and contributed meaningful observations or conjectures to the conversation. This may have helped her and Robin in generalizing their understandings of concepts in Taxicab geometry. Kristen appeared disengaged much of the time (often distracting Nicole by talking off subject) and contributed "simple" observations and comments to the conversation when she was engaged. There was a "gap" in understanding between each of these students, which may have led to the less cohesive group dynamics as compared to Amy, Brianna, and Ally from Section 4.1.2. Having some sort of structure or instruction for group work while working on the activity could have motivated Kristen to remain engaged and would have encouraged her group mates to make sure she was keeping up with their reasoning. In terms of cooperative learning, this group as a whole did not perceive this activity in the same way as others. They also did not show as much evidence that they were working towards a *shared* goal in the same way that Amy, Brianna, and Ally demonstrated previously.

4.1.4 The Triangle Inequality

After students investigated what a perpendicular bisector was in Taxicab geometry, they worked on an activity that asked them to investigate the triangle inequality. Parker, Darryl, Felix, and Russell were in a group together in the class, although Felix was absent from class this day. Parker, Darryl, and Felix were graduate students and Russell was an undergraduate student in the course. This activity as it was given to the students can be seen in Figure 4.10, and the submission from Parker, Darryl, and Russell, can be seen in Figure 4.11. Since it is hard to differentiate in this figure, it is noted that their written response to the question asking about a strict equality is directly below this question in Figure 4.11. This is where they wrote, “When B is collinear A and C, and is between A and C.” This group, much like Robin, Kristen, and Nicole, chose to work on the same computer (Parker’s) for these activities and submit one file for the group at the end of class. This activity mainly was aiming to get students to become more familiar with the Taxicab metric and how they could use it to analyze various situations. However, one ideal solution here would be for students to figure out that if there is a right triangle that is oriented such that the legs of the triangle are parallel to the axes of the graph, then there would be equality (although this is not the only case there is equality). In particular, it can be proved there is equality in this case in two ways – algebraically and geometrically.

Proposition: If a right triangle ABC exists in Taxicab geometry such that the legs AB and BC are parallel to the x and y axes, then $d_T(A,B) + d_T(B,C) = d_T(A,C)$.

Proof 1: We can show algebraically,

$$d_T(A,B) + d_T(B,C) = |x_B - x_A| + |y_B - y_A| + |x_C - x_B| + |y_C - y_B|$$

$$= (|x_B - x_A| + |x_C - x_B|) + (|y_B - y_A| + |y_C - y_B|)$$

$$= |x_C - x_A| + |y_C - y_A| = d_T(A,C)$$

Where $|x_B - x_A| + |x_C - x_B| = |x_C - x_A|$ and $|y_B - y_A| + |y_C - y_B| = |y_C - y_A|$ as a result of the betweenness axiom, since x_B is collinear and between x_A and x_C, and similarly, y_B is collinear and between y_A and y_C. ■

Proof 2: We can see that for the points A and C, the Taxicab distance between them can be geometrically represented as (and is equivalent to the sum of) the distances between these points horizontally and vertically, which are precisely the legs of triangle ABC. Hence, by the definition of the Taxicab metric, the Taxicab distance between the two points A and C (which is labeled as $d_T(A, C)$) is equal to the sum of Taxicab distances of the legs AB and BC (which is $d_T(A, B) + d_T(B, C)$). ■

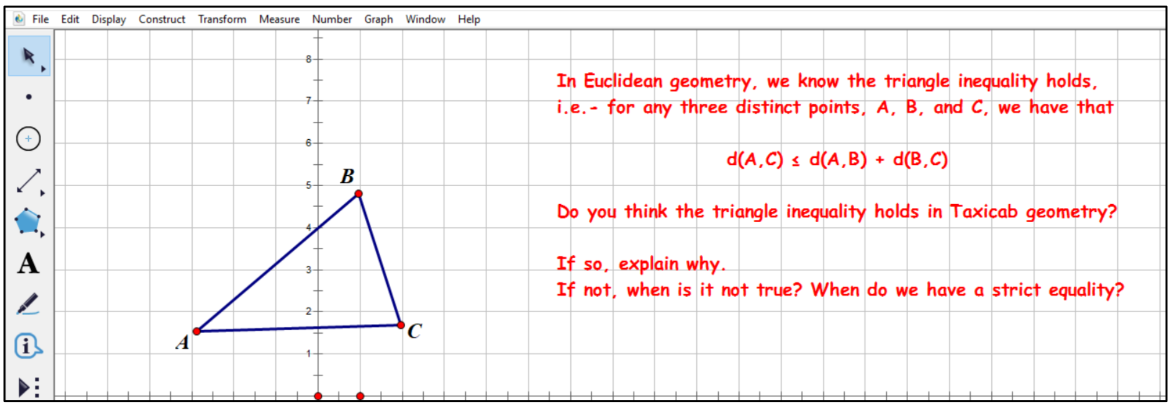


Figure 4.10 Original activity given to students for the triangle inequality.

Darryl began by reading the question, trying to understand the problem along the way. This was evident by Darryl pausing in between reading statements aloud and saying, "okay, that makes sense." When he finished reading the question asking if they thought this inequality holds in Taxicab geometry, Russell immediately said, "Calculate it." Darryl began to try and conjecture whether or not he thought it held, and Russell said again, "Calculate it." Darryl continued to try and think aloud saying, "we know that in Taxicab distances… oh wait." During this time, Russell whispered something to Parker and then Parker said aloud, "let's just see!" Russell most likely told her to calculate the distances of the legs of the arbitrary triangle ABC they had been given on the worksheet initially (as seen in Figure 4.10). Parker then used the **Taxicab distance** tool to find the distances of the sides of this triangle and add the distances AB and BC to compare them to the distance AC, which can be seen near the bottom right of Figure 4.11. In the preliminary genetic decomposition presented in this dissertation, in order to compare distances, it is stated a student must have an object conception of this distance. Thus, a student comparing distances after he or she had calculated them in GSP indicates an object conception of **Distance** in Taxicab geometry. However, in order to understand *why* there is a relationship between these distances requires the de-encapsulation of this object. More about their understanding of this concept in how they used this comparison to talk about the triangle inequality is presented below, in terms of Taxicab (and Euclidean) geometry.

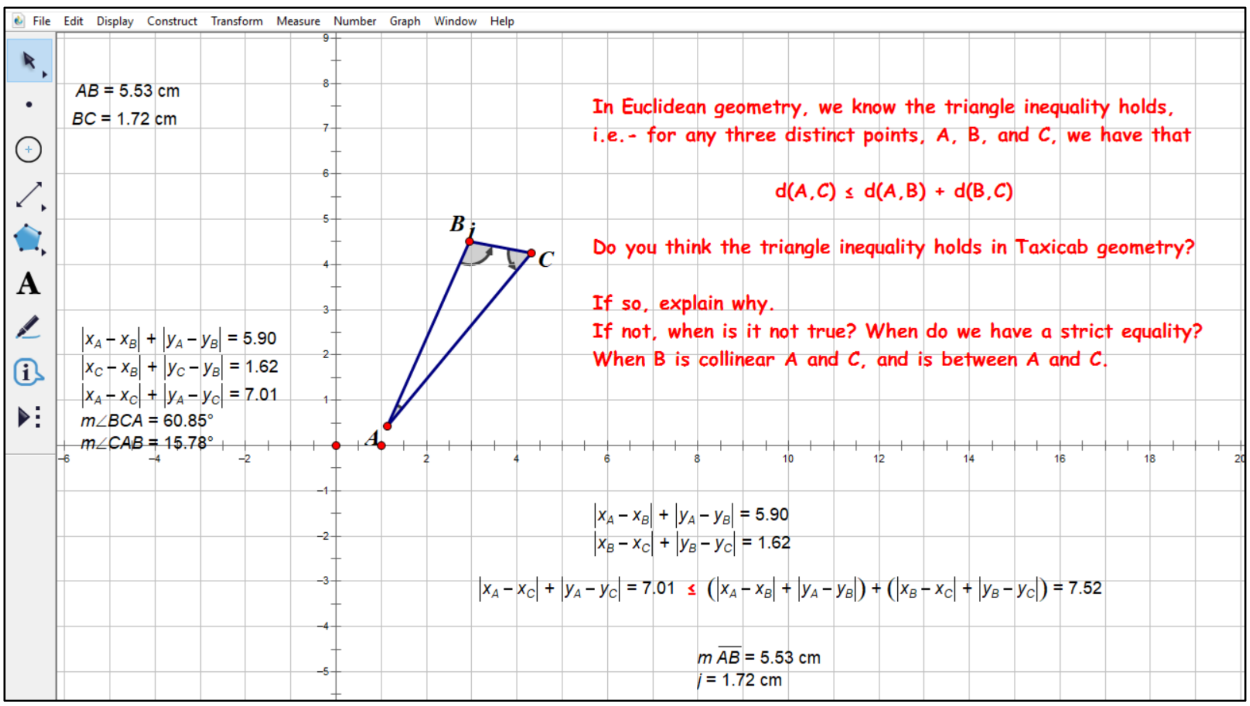


Figure 4.11 Parker, Darryl, and Russell's submission for this activity.

Parker had misinterpreted the inequality given to them which is evident while she was measuring these distances and saying "we're saying the distance from A to B and from B to C, will be less than A to C. Oh no! It's not," where she should have said these two distances "will [sum to] be at least" the distance from A to C. In other words, in her mind she was trying to show $d_T(A,B) + d_T(B,C) < d_T(A,C)$, whereas the inequality as it was given to them was $d_T(A,C) \leq d_T(A,B) + d_T(B,C)$. So, she had noticed the inequality involved a "less than" sign, but switched the sides of the inequality, resulting in a misinterpretation about what the task was. She tried multiple times to convince Darryl that they had found a case where they contradicted the triangle inequality, but Darryl kept trying to explain that they only found a case that supported it. Once Parker realized she had been mistaken, she apologized, and Darryl said, “Don’t worry, I got confused by the wording as well.” This interaction is included as an example of the dynamics of this group. Darryl and Parker felt comfortable enough to have a productive disagreement about the problem but were patient with one another until they came to an agreement. This is when Darryl offered a comment in an attempt to ease any embarrassment Parker may have felt from misinterpreting the inequality. This interaction also may indicate Parker did not fully understand the geometrical representation of the triangle inequality in Euclidean geometry since what she was discussing with Darryl was also not the triangle inequality in Euclidean geometry.

As Parker was adding “≤” sign in red in Figure 4.11, Darryl said, “so it holds for both Taxicab geometry and Euclidean geometry…and this is proof… well through the proof they have it…” In terms of Darryl’s geometric reasoning skills, he took the one example they were given and that they checked to be empirical proof for the idea that this inequality holds in Taxicab geometry. This is supported by Chazan (1993) who found many students did not see writing a formal proof as necessary after interacting with the computer in a similar way. On the contrary, at this time Russell spoke up and said, “but why is that? It says explain why.” This implies Russell did not take this example to be empirical proof and knew they needed to further investigate this conjecture using geometric reasoning. Darryl then began to try and come up with reasons why this inequality would hold, evoking his **ARTD** by bringing up the fact that this distance included absolute values in the formula and if they didn’t take the absolute values of these differences in coordinates, that the “value” would change. It is interpreted that Darryl was referring to the fact that the difference in the coordinates could be negative if they didn't take the absolute value of it, which is when Russel was attempting to explain why the formula had to use the absolute values of these differences so that it would be positive. This portion of the conversation is provided below.

Russell: Even in normal geometry… we still square it…[indiscernible]…so isn't the case.

Darryl: Well, if you do think about it. One is making it bigger than another other number, because the absolute value, right?

Russell: I mean wouldn't you say that you should always like…

Darryl: No, no. I see what you're saying, because if you are squaring it or you are making it absolute value, regardless it's going to be positive. I understand that.

By bringing up the formula for distance in Euclidean geometry (or "normal geometry," as Russell called it) to discuss this with Darryl, Russell indicated he was coordinating his **ARED** and **ARTD** processes. This is because he was able to compare these two formulas and explain that the transformations being applied to these terms will result in a positive distance in both cases. In other words, he viewed taking the absolute values of $x_2 - x_1$ and $y_2 - y_1$ in the Taxicab distance formula as a similar action as squaring these terms in the Euclidean distance formula, in that it made these terms positive. By referring to these distances as "numbers," in the next statement, Darryl was exhibiting an action conception of at least **ARTD** (if not also **ARED**) since Russell bringing up the formula for distance in Euclidean geometry could have evoked Darryl's **ARED.** When Russell says "you should always like…" it is interpreted he was trying to express that distance should always be positive, so it is necessary to include the absolute values of these terms in this distance formula (or squaring/taking the square root of the terms in Euclidean distance). Although not clear, this is evidence that Russell had constructed a general **Distance** process since he was using properties of distance to logically explain why the algebraic representation of Taxicab distance made sense to him. Recall he had just brought up the structures for both formulas for Euclidean and Taxicab distance. In the least, he was using his personal concept definition to try and generalize his **ARD**.

Next, Parker dragged point B to make the angle at point B look like a right angle, which is when Russell suggested that maybe the Pythagorean theorem had something to do with why this inequality held in Taxicab geometry. Specifically, he said, "in Taxicab was how we calculate the distance…like no matter the triangle," to which Parker replied, "no matter the points." To clarify, I believe Russell saw this triangle and evoked his **GRTD** and saw the legs of this right triangle (AB and BC) as the Taxicab distance between the points A and C, which fell on the hypothenuse of this triangle. Russell had said, "no matter the triangle," and Parker corrected him by saying, "no matter the points." Although he misspoke, Russell appeared to have been trying to say no matter the orientation of the two points he is measuring the distance between in Taxicab geometry, a right triangle can be constructed with the distance geometrically represented as the legs of this triangle. Parker correctly interpreted what he was saying, which is why she was able to add the clarification that this would be the case for any two points, and not necessarily any triangle. Thus, Parker and Russell exhibited evidence they had interiorized their action conception of **GRD** since they both seemed to understand in general how Taxicab distance is represented for any two points.

As they continued working on the activity, Darryl added "so the absolute value...is always going to be greater than…[indiscernible] and you can say the same thing with the squares in terms of Euclidean geometry." Here, he may have been trying to understand exactly what Russell and Parker's point was by talking about the Pythagorean theorem, which is why he said, "the squares in terms of Euclidean geometry." Because he could not coherently express this thought or connection, Darryl was still exhibiting an action conception of **ARED** and **ARTD** and was trying to make a connection between these two metrics but was unable to do so since he had not interiorized them yet. Parker continued to explore in GSP on her laptop while Russell and Darryl listened and watched as Parker explained she was trying to figure out when the inequality would change to a strict equality. Darryl suggested moving one of the points up to see if the slope of the segments affected the relationship between the distances. Russell then started to suggest things for Parker to do, asking her to move point B to the middle of the segment AC so that it was collinear with them. They concluded that in this case there was a strict equality between $d_T(A, C)$ and $d_T(A, B) + d_T(B, C)$. Darryl then asked, "is it whenever C is between A and B as well?" to which Parker answered, "no, it's only when B is in the middle." Although not explicitly stated, it is interpreted that Darryl was asking if $d_T(A, C)$ would still equal $d_T(A, B) + d_T(B, C)$ if C were collinear with A and B and between them, which is certainly not the case. The fact that Parker was able to quickly respond in the negative makes us believe she was using her process conception of **Distance** to reason through his question and anticipate why this was not true and did not need to see an example to justify her response. At the same time, this is further evidence that Darryl had not interiorized his action conception of **Distance** since he most likely needed to consider a specific example to determine if his conjecture was false.

The group continued to explore in GSP by moving points and measuring angles to see if they could find any patterns about this inequality with different triangles. At one point they found a triangle where they had a strict equality for the expression. Parker conjectured that this may be because this specific triangle appeared to have two angle measures that were equal, so she wanted to measure them and see if once she made them exactly equal they would still have a strict equality for the expression. Once she did this, Darryl observed it was not a strict equality, and said, "it would just follow the rules of an isosceles triangle." Thus, Darryl was able to make a connection between the two angles of this triangle being congruent and the fact that it was an isosceles triangle, although it is not certain what he meant by the “rules of an isosceles triangle.” It is possible that, like Kristen in Section 4.1.3, he was attempting to add a meaningful comment to the discussion since he may have realized Russell and Parker had a more developed understanding of the concept of **Distance.** In this case instead of making a local observation like Kristen did about the appearance of the red line in the previous activity, Darryl made a connection between concepts, but they were just not concepts that were relevant to the task. In either case, he evoked his understanding of what an isosceles triangle was as an attempt to contribute to the conversation when his understanding of **Distance** had hindered him from understanding Russell and Parker’s conversation beginning with Russell mentioning the Pythagorean theorem. Thus, his productive struggle in this activity is attributed to his inability to coordinate his **ARED** and **ARTD** processes (if they existed), since Russell and Parker both exhibited they had already done this, which led to them having this conversation where Darryl was not as engaged verbally.

Overall in this group, Parker seemed to have fallen into a type of leadership role since she had control of the computer, but Russell was extremely engaged and added many meaningful comments to the conversation about distance and connections between Taxicab and Euclidean geometries. Darryl was also engaged for the entirety of the activity and was trying to contribute meaningfully to the conversation, but his conception of **Distance** hindered him from being able to have the same “level” of conversation that Russell and Parker demonstrated.

4.1.5 Congruence of Triangles

The last activity in this worksheet (that is included in this report) had students consider the congruence of triangles in Taxicab geometry. In particular, they were given examples of why each Side-Side-Side (SSS), Side-Angle-Side (SAS), and Angle-Side-Angle (ASA) congruence criterion did not hold in Taxicab geometry. Figures 4.12, 4.15, and 4.16 show these worksheets. The conversations and group work for the group made up of Tyra, Mark, Alex, and Samantha are presented in this section. Out of members of this group, Samantha was the only undergraduate student. This group worked in a similar way as Amy, Brianna, and Ally in that they all worked on their own computers and discussed their findings with one another, showing each other their screens and work throughout the class. Again, the main goal of this activity was to familiarize the students with the Taxicab metric and how to apply it in different situations. The first part of this activity had them investigate the congruence criterion of SSS in Taxicab geometry, as seen in Figure 4.12. Specifically, they were to explain why these two triangles satisfy the SSS criterion but are not congruent. This group proved to work very well together, as is presented in the following analysis of their conversation during this activity.

Mark made a very quick discovery about the example shown on the SSS worksheet in Figure 4.12. This excerpt is included below, as he was trying to explain his understanding to Alex about why he believes the segment AB’ in Figure 4.12 is congruent to segment AB in Taxicab geometry.

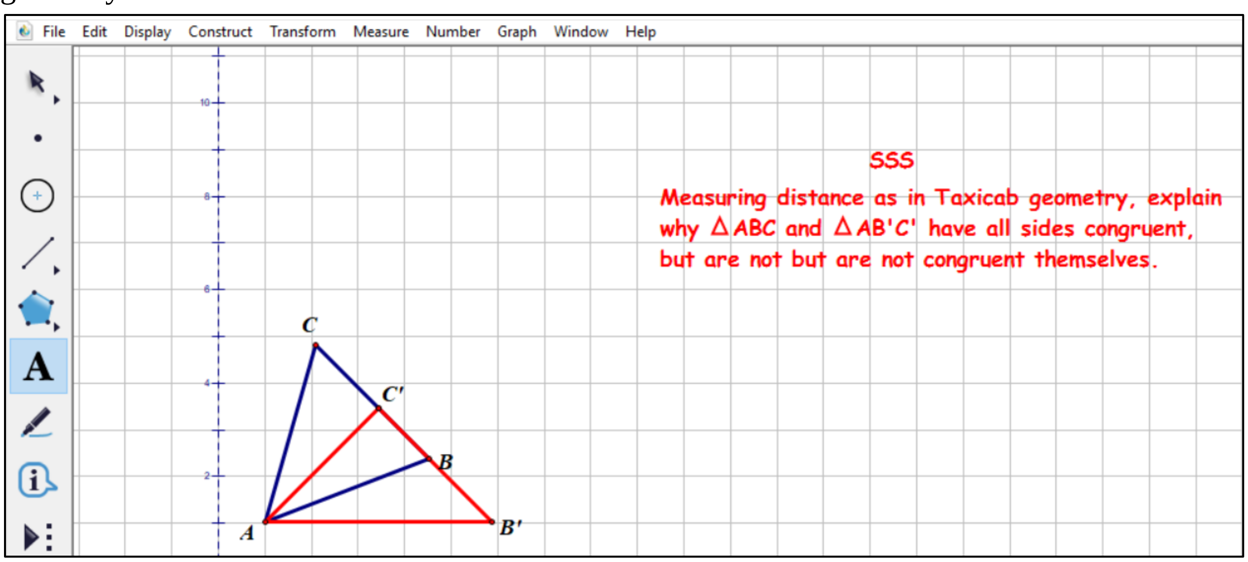


Figure 4.12 GSP worksheet for the SSS criterion.

Mark: AB is actually um, the x… the distance along the x plus the y, right? And there is a point between C'B', such that y…is equal to Euclidean distance A and B…plus this distance right here. Does that make sense?… This distance, the Euclidean distance AB stops… right here at this point… So, you have this distance AB plus this distance AB'…, so there is a point between C'B' such that the distance… between this point [A] and B' is the same distance as the distance between here [A] and here [B]…Which would mean then that AB' is the same as AB in terms of Taxicab.

Mark was able to explain in general terms how it was possible that segment AB' was congruent to segment AB, including Euclidean and Taxicab distance in his reasoning, indicative of a process conception of **Distance** by the coordination of his **GRED** and **GRTD.** It was in the reflection of this exercise that Mark was at a state of encapsulating his process conception of **GRTD** into an object, since he was confident that AB' and AB were congruent in Taxicab distance. In other words, he was performing an action of comparison on his **GRTD** object, although these objects were particular and defined in GSP. At this point, Samantha and Tyra listened as he attempted to explain his thought process again. Before he began explaining, Mark shut his laptop and grabbed a sheet of paper that was on the table to draw a triangle that looked like the triangle from the GSP worksheet (but was arbitrary since he did not have a coordinate grid anymore). A picture of this drawing is provided in Figure 4.13. Numbers are used in the following excerpt that correspond to the numbers in Figure 4.13 to denote in his explanation what portion of this drawing he was referring to. Throughout his explanation, Samantha and Tyra were nodding and saying affirmative comments such as "yes" and "mm hmm" to indicate they were following his logic. After hearing Mark's first explanation, Alex at this time was working on writing his observations on his worksheet in GSP, which will be presented later in this section.

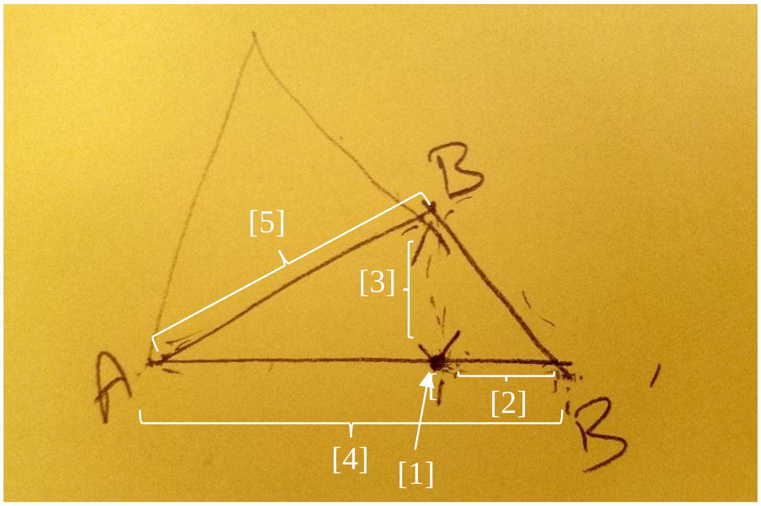


Figure 4.13 Mark's drawing to explain why AB was congruent to AB'.

Mark: We have A, B', and B, right? So by Euclidean um, geometry, this distance AB would be the same as to some point right here[1], right? This distance from here to here[2]…would be the same as from here to here[3]…such that this distance right here[2] is the same as the y value here[3]. So if there is a point along here[4] such that this distance here[2] is this distance here[3], then that means that by Taxi… this distance[4] is the same as this distance[5] when that happens.

Tyra: I agree with that. It looks like that would happen. So, these are congruent by side… by Taxi-Side-Side-Side.

Mark: Right so by Taxi, this distance[4] is the same as this[5], but by Euclidean they're not.

First, it is interesting Mark preferred to use paper and pencil to explain this to his group instead of GSP. It is possible that he felt as though GSP would limit his explanation since the triangle constructed in this worksheet was not necessarily arbitrary. If this were the case, this was an example that is not consistent with Chazan (2003) in that Mark [hypothetically] wanted to construct a formal [geometric] proof for this situation despite the empirical evidence he saw in GSP. As stated previously, Mark exhibited a clear object conception of **GRTD**, but in first part of this excerpt Mark also compares Euclidean distances of the segment AB and the segment formed by A and the point indicated by [1]. This is indicative of an object conception of **GRED** as well. He was able to de-encapsulate his object conception of **GRTD** in order to consider the distance denoted by [3] and how this was a part of the Taxicab distance from A to B, and specifically referred to this as "the y-value." Thus, Mark was evoking his **ARTD** since he seemed to be referencing the y-coordinates of the arbitrary points on this triangle, an indication that he was in the state of encapsulating his **Taxicab distance** process. He seemed to believe the Euclidean distance between A and B would be equal to the Euclidean distance between A and the point he drew on segment AB', which does not fully correspond to the way in which he was trying to describe how the Taxicab distance of segment AB would be measured. In other words, the hypotenuse of the triangle (the segment AB) formed by A, B, and this point he drew would be longer in Euclidean distance than both legs of that triangle (not equal to the leg made up of A and this point like he thought). However, even with this misconception, he seemed to be able to encapsulate his process conception of **GTD**.

Although Tyra indicated she followed his logic, she seemed to think his point was to show these triangles were congruent in Taxicab geometry even though they were not in Euclidean geometry. In actuality, Mark's point was just that the segments AB and AB' were congruent in Taxicab geometry but were not in Euclidean geometry. In any case, Mark's detailed and thought out explanations were helping Alex in the interiorization of his action conception of **GRTD.** In particular, part of his submitted GSP worksheet for this part of the activity can be seen in Figure 4.14 where his answer to the question asking why these triangles were not congruent in Taxicab geometry even though they satisfied the SSS criterion can be seen.

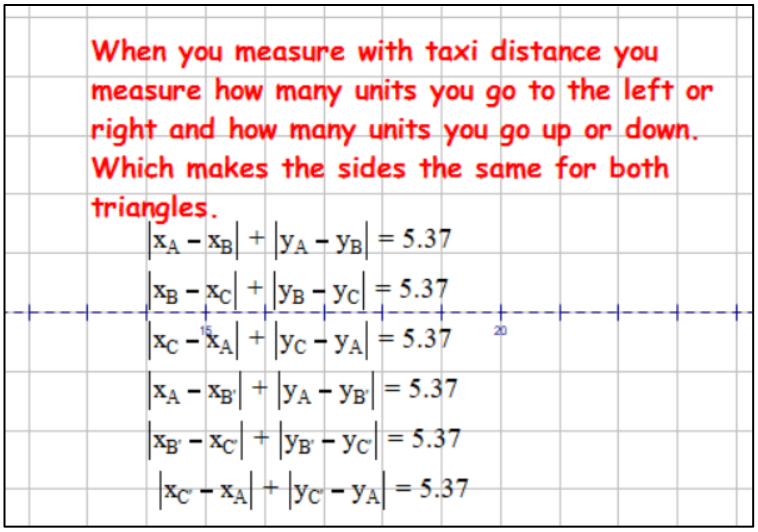


Figure 4.14 Alex's response to the question on the activity for the SSS criterion.

It can be seen from his work that he found the lengths of each of the legs of both triangles to verify that these triangles indeed satisfied the criterion of SSS. Although he did not explain why the triangles were not actually congruent, he explained how Taxicab distance in measured in the red writing seen in Figure 4.14. By describing how to calculate this distance in general terms but also referring to counting units, saying "When you measure with taxi distance you measure how many units you go to the left or right and how many units you go up or down," he provided evidence he had interiorized his action conception of **GRTD.** Specifically, he was able to talk about counting blocks as components of finding the distance ("left or right" and "up or down") in general terms as a process. The students then moved on to the next worksheet, which pertained to the congruence criterion of SAS in Taxicab geometry, which can be seen in Figure 4.15.

Figure 4.15 Worksheet for the SAS criterion.

Once Tyra read the statement on this worksheet asking to show these trianges have "two sides equal" along with congruent enclosed angles, but are not congruent she stated aloud, "I get it now. These are congruent by Side-Angle-Side even though they're not congruent by Euclidean." This indicates, along with her previous comment in the last excerpt, that Tyra had a misconception about the definition of congruence. Further evidence of this misconception appeared on the introduction sheet to this activity which asked students to define what it means for two triangles to be congruent. Tyra's response to this was, "Euclidean: same size and shape." Thus, she had formed a distinction between congruence in Euclidean geometry and congruence in Taxicab geometry. In particular, in Euclidean geometry two triangles had to look the exact same to be congruent, whereas in Taxicab geometry, as long as they satisfied a congruence criterion, they were congruent regardless of their appearance. In other words, she seemed to believe these congruence criteria "trumped" the definition of congruence in Taxicab geometry. This misconception arose despite the fact that each of these worksheets asked the students to observe that the triangles satisfied a congruence criterion in Taxicab geometry but were not actually congruent.

Once Tyra stated that she understood what to do, Mark said, "Oh, cool. Your turn… How does it work?", encouraging Tyra to explain the activity to the group. Tyra was able to explain to her group why the segments AB and ED were congruent in Figure 4.15 by saying, "you go up eight for AB but…E to D is gonna be still four over and four down, that's eight." Mark then said, "so it's basically the same idea…" referring to the last set of triangles in this activity. Tyra replied, "Now it's easy!" and Mark said, "When you realize that once it's at an angle it's just a combination of x and y, then you start to see it." Mark appeared to have understood that if the segment is "at an angle," (**GRD)** the Taxicab distance is measured as "a combination of x and y" (**ARTD**). The fact that he added the "once it's at an angle," implies he had differentiated this case from if the segment was horizontal or vertical. In other words, he knew if the segment was not "at an angle," the length of the segment is the same in Euclidean geometry and Taxicab geometry (comparing **GRED** to **GRTD** implies an object conception of **GRD**). Thus, he may have possibly constructed an object conception of **Distance** since he had apparently started to compare how Taxicab distance is measured and looks like in comparison to Euclidean distance depending on the slope of a segment. His last statement ("combination of x and y") implies he may have de-encapsulated his **GRD** in order to coordinate his **GRTD** and **ARTD** processes. With some reflection on his **ARED** and **ARTD** and how these representations compare to one another, he possibly would have been able to encapsulate his process conception of **ARD** to arrive at an overall object conception of **Distance**. In this case, he would have fully assimilated **Taxicab distance** into his understanding of his *distance schema*.

Samantha then asked for someone to explain to her how these triangles satisfied the SAS congruence criterion, since she had apparently not followed the earlier conversation. Mark broke down the explanation Tyra had provided by asking Samantha in terms of Taxicab distance how many "squares" were between A and B and then how many "squares" were between E and F (as illustrated in Figure 4.15). Samantha replied with an answer of eight both times and then expressed that it made sense to her now. Recall when Mark had been explaining his thought process to his group previously, his explanations exhibited a process and/or object conception of **Taxicab distance**, but in this case seemed to know Samantha most likely needed to hear an explanation that was catered towards an action conception. This is indicated by him saying, "how many squares" were between two points instead of phrasing this as "what is the Taxicab distance" between the points. In this situation, Mark understood and had the social awareness that Samantha had not developed as deep of an understanding of **Taxicab distance** and changed the way in which he explained his thoughts. The students all agreed they understood this part of the activity and moved to the last sheet in GSP about the congruence criterion of ASA seen in Figure 4.16. They all quickly agreed that this "was the same" idea as the last two sheets and moved on to another activity.

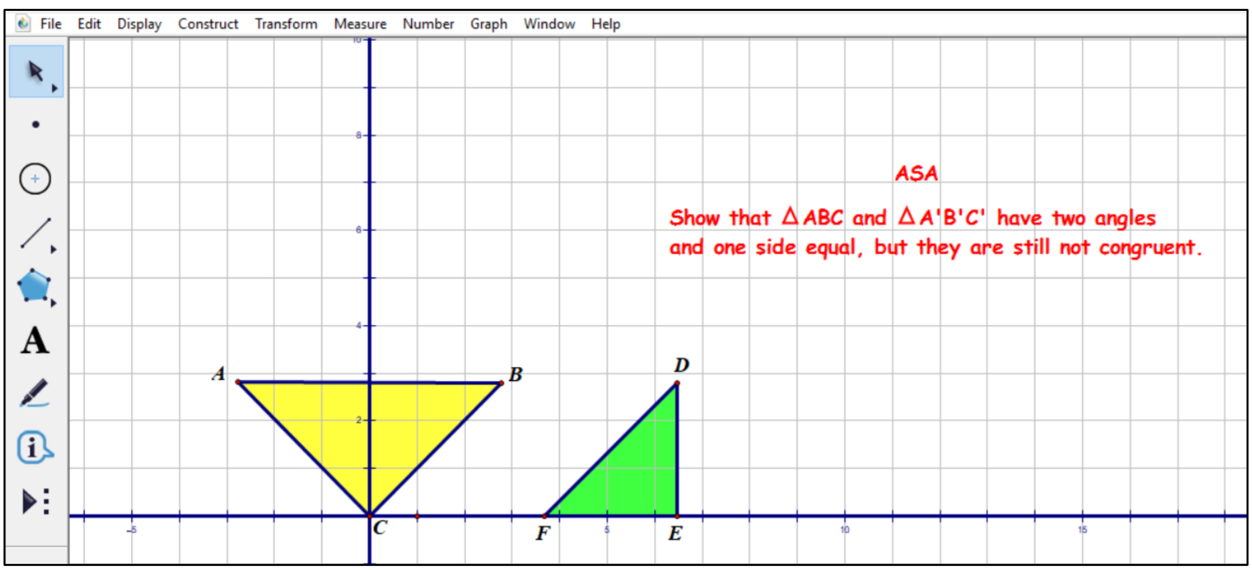


Figure 4.16 Worksheet for ASA criterion.

Overall, this group was successful at cooperatively working together for a shared goal to complete these worksheets. Mark ended up taking a leadership role in this group, which may have happened because he seemed to have the most developed understanding of **Distance**. As with

Amy, Brianna, and Ally, all of the members of this group appeared to be engaged throughout this activity. They each were exploring ideas in their own GSP file and discussing ideas. Not only did they share their computer screens to explain concepts, but Mark even used paper and pencil to do so. This was perhaps because he felt GSP limited him to an environment of working with non-arbitrary triangles and wanted to be able to explain his thoughts in general terms to his group. In this group, Mark had been using "face-to-face promotive interaction" as defined by Johnson & Johnson (1999) in which he was promoting the success of his group members by helping, encouraging, and praising his group member's efforts to achieve, as evident by him making the effort to explain ideas multiple times (in different ways) and encouraging Tyra to explain the ASA worksheet in GSP to the group.

4.1.6 Student opinions of GSP and the course

As part of the interviews that were conducted with the 15 students who volunteered to participate in them (out of the 18 in enrolled in the course), they were asked questions about how the course affected their comfortability with various aspects of mathematics. These questions asked for students to provide their opinions on a Likert scale (1 = Strongly disagree (SD), 2 = Disagree (D), 3 = Neither agree nor disagree (NA), 4 = Agree (A), 5 = Strongly agree (SA)) and are provided below. The summary statistics of each question is presented in Table 4.1, but the question pertaining to technology (Question 1) is discussed in detail in this section. During the interview, the participants elaborated on their answers to these questions and some of their comments about GSP are presented here as well.

1. This course has helped me with my comfortability in using technology to understand mathematical concepts.

2. This course has helped me with my comfortability in reading formal definitions and my ability to apply them in proofs.
3. This course has helped me with my comfortability in reading and understanding proofs in mathematics.
4. This course has helped me with my comfortability in conjecturing and writing my own proofs.
5. This course has helped me with my comfortability in applying my mathematical knowledge to new problems.
6. This course has helped me with my comfortability with mathematics in general.

Table 4.1 Distribution of responses to survey questions.

Question	1	2	3	4	5	6
% SD	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
% D	0.0%	13.3%	20.0%	26.7%	6.7%	6.7%
% NA	20.0%	20.0%	0.0%	6.7%	13.3%	13.3%
% A	40.0%	20.0%	33.3%	26.7%	40.0%	46.7%
% SA	40.0%	20.0%	33.3%	26.7%	40.0%	46.7%
Average	4.20	4.00	4.07	3.80	4.13	4.07
Median	4.00	4.00	4.00	4.00	4.00	4.00
Standard Dev	0.77	1.13	1.16	1.26	0.92	0.88

As can be seen from Table 4.1, the question asking students if the course helped their "comfortability in using technology to understand mathematical concepts" received the highest average of responses with 4.20. This also had the smallest standard deviation, indicating that overall this was the highest scored question out of the six. Representative comments that students made during the interview based on their response to this question about using technology to understand mathematical concepts are presented below.

Neither Agree nor Disagree – "Okay, so question 1- I put neither agree nor disagree because the only thing we used in there was GSP and…I feel that program is pretty limited compared to something like Matlab." – Russell, Undergraduate mathematics major

Neither Agree nor Disagree – "I would say neither of the two, because GSP…I just didn't ever really understand it, you know… I was just doing it…They would tell me to draw a circle and…okay I'm drawing one. Okay, find the radius. We're finding the radius, but … I didn't see what that helped..." – Kristen, Undergraduate mathematics major

Agree – "…a new way of how to look at proofs through different shapes and constructions…with using those constructions you get…a visual aide…of exactly how you are supposed to visualize the image, or how you're supposed to visualize the mathematical concept itself." – Darryl, Graduate MAT student

Agree – " …with GSP, I think it was easier to plot stuff in GSP versus you drawing it on a paper, and trying to figure it out yourself, cause sometimes your drawings aren't always accurate…it definitely helps to like visualize and know that those calculations in there are accurate versus what you conjured up yourself." – Alicia, Undergraduate mathematics major

Strongly Agree – I liked the integration of the GSP, I think it helps strengthen…being able to formally define… anything and apply them to proofs and apply it to new problems, understanding more about the properties so that you can apply it… I just think overall the design of this course was just probably one of my favorites. – Robin, Graduate MAT student

Strongly Agree – "This is the most I had used up to the point I mean other than a calculator it was like excel or something…doing something like GSP… initially was kind of crazy dealing with that - I didn't understand it. But as we went through the class…Something had changed in the classroom when we started working together… groups more and more…our discussions in our groups, and the groups discussing amongst each other in the class … it really got me more engaged into it and I really enjoyed it. I enjoyed the discussion, and I enjoyed the discovery and learning and trying things out in GSP. For me…its opened my eyes significantly with…learning more about geometry and mathematics, and basically looking behind the curtain to see how it works. So, I'm immensely grateful that I managed to take this class…not just this class but this classroom…this group of people that I took it with." – Felix, Graduate MAT student

Overall, the participants in this study appeared to find the use of GSP and group work helpful in their understanding and beneficial for their experience in this class.

4.2 The development of the *circle schema* through schema interaction

In this section, responses from 15 students who volunteered to participate in the interview portion of this study were analyzed in relation to their *circle schema*. Their responses provided good insights about how students' transfer definitions to a new context. The following questions from the questionnaire were relevant to this data analysis, and are a subset of the questions asked before and during the interview:

1. Define and draw an image (or images) that represents each of the following terms however you see fit: Circle, Distance.
2. For any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

   (i) Euclidean distance is given by $d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

   (ii) Taxi distance is given by $d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|$

   (a) Using the grids below, illustrate each of these two distances. Be as detailed as possible in labeling them.

   (b) Is it possible for these two distances to be the same, i.e. $d_E(P,Q) = d_T(P,Q)$? If yes, explain.

3. Is the following definition true in both geometries? Explain. "The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."
4. Using the grids below, sketch the following circle in both geometries: Circle with center at $C(3,3)$ and radius $r = 2$.

In general, there is no unique ideal response for each of these questions since the main goal of these interviews was to get students to elaborate on their thought processes as they responded to these questions. However, Figure 2.1 in Section 2.1.1 shows an example of how a student could illustrate these distances, as asked in Question 2. A student who has formed a coherent understanding of **Distance** should be able to explain exactly how the formula for distance in each geometry is represented by an illustration. For Question 2(b), a possible correct response would be that the Euclidean and Taxicab distances between two points are the same if they lie on a vertical or horizontal line, or if their x-coordinates or y-coordinates are the same. Graphically, the Euclidean and Taxicab distance between two points would then be illustrated in the same way and, algebraically, one of the terms in each equation (either $x_2 - x_1$ or $y_2 - y_1$) would be eliminated, resulting in the same expression for Euclidean and Taxicab distance between the two points. For Question 3, this definition holds in both geometries, but the circles will appear different as a result of the way in which distance is defined. This was anticipated to cause some uneasiness about responding "yes" to this question. For Question 4, an example of how a student could illustrate these circles is shown in Figure 3.6. A student who has formed a coherent understanding of **Circle** would be able to explain exactly how the equation of a circle in each geometry is represented by an illustration in relation to the definition of a circle.

Students' responses to these questions were used in the analysis for this report, since they helped to identify students' understandings of **Circle**. In addition, details would emerge of the possible schema structure for each student associated with this concept and how this would develop in order to transfer his or her definitions from Euclidean to Taxicab geometry. Specifically, how each student uses his or her conceptions of **Distance, Radius, Center,** and **Locus of points** to construct their *circle schema* in his or her reasoning/explanations of how the construction of a circle (geometric representation) and structure of the equation of a circle (algebraic representation) relate to the definition of a circle was considered in this analysis. From this information, the *level of schema interaction* each student exhibited in terms of the coordination of the *cEg* and *cTg schemata* was determined.

It should be noted that the analysis of this data began by identifying mental constructions in terms of APOS Theory without the notion of the triad or schema interaction. However, it was determined that considering the relationships between the Actions, Processes, and Objects associated with these schemata revealed a much richer story in relation to each student's understanding. In particular, a preliminary genetic decomposition for the levels of schema interaction as modeled from Baker et al. (2000) and Cooley et al. (2007) was developed.

From the revised version of this genetic decomposition, all 15 students' work and responses were analyzed. The understanding of each of the subconcepts each student evoked during the interviews and how these concepts interacted with one another across schemata in order to help the student reason through the questions asked of them were analyzed. Although the data from each student was analyzed in this way, the detailed analysis of only one student's level of schema interaction is presented in its entirety here. This breakdown includes details of the components of a participant's schema to illustrate how this information was used to determine at which level of schema interaction a student may have been operating during the interview. For the remaining analysis of work and responses during the interview presented in this section, evidence is provided that a student is operating at each level of schema interaction, but a detailed breakdown is omitted.

What follows are descriptions and representative examples of the levels of schema interaction between the *cEg* and *cTg schemata* associated with the *circle schema.* For these examples, students' answers on a questionnaire and to questions during a follow up interview as they correspond to the genetic decomposition were analyzed.

As noted in Section 3.1, it was anticipated the main concepts to emerge within the *circle schema* to be **Distance, Radius, Center,** and **Locus of Points.** Since the entirety of the distance formula (and concept) is complicated on its own and is used within the equation for a circle for each metric, this was found to be an important facet of a student's *circle schema.* In addition, particular questions on the questionnaire directly asked about this concept, which allowed the analysis of each student's conception of **Distance** to be more in depth (than the other subconcepts of **Circle)** and it was determined how this may have affected his or her understanding of **Circle.**

Recall that in Section 3.1.4, descriptions of the triad of schema development of the *circle schema* as it is evoked in Euclidean and Taxicab geometries were presented. What follows in Sections 4.2.1 – 4.2.10 is the genetic decomposition of schema interaction that includes the nine levels that result as the two schemata of *circle in Euclidean geometry (cEg)* and *circle in Taxicab geometry (cTg)* interact with one another at various stages (as illustrated in Table 3.1 in Section 3.1.5). In particular, the type of information a student was transferring from one geometry to another was of importance during this analysis. It was noted if a student was predominantly observing local properties of a concept, the definition of the concept, or the definition of concept and how it relates to other concepts. For figures presented in this portion of the dissertation (Section 4.2), red ink indicates the student had written or drawn that part during the interview.

4.2.1 Intra-cEg, Intra-cTg

At the intra-*cEg* and intra-*cTg* (intra-intra) level, a student can compare a circle in Euclidean and Taxicab geometries either visually (geometrically) or algebraically but cannot make connections between the geometric and algebraic representations in either geometry. If a student has a process conception of the geometric or algebraic representation of any of **Distance, Radius, Center,** or **Locus of points** (i.e.- has constructed a new process from the coordination of the corresponding processes in his or her **GREC/GRTC** or **AREC/ARTC**), then the student cannot coordinate these concepts across his or her general **GRC** and **ARC**. In general, a student does not understand and cannot make connections about how the construction and equation of a circle within a metric space is a direct result of the definition of a circle. In other words, the student cannot talk about the construction of a circle, the structure of the equation for a circle, or how these two representations are related, other than local geometric/algebraic properties. There were three students, Kristen, Samantha, and Mark, that showed evidence of operating at an intra-intra level of schema interaction. The detailed analysis of Kristen's responses is presented in the next section, but only the summary of analysis of each student following Kristen in relation to his or her level of schema interaction is provided following this for the sake of length.

4.2.1.1 Kristen

Kristen was one of the undergraduate mathematics majors enrolled in the course and was in a group interview with Hannah, a graduate student, and Samantha, an undergraduate mathematics major. As a note, these three students had not worked together in a group in class.

For the first question in this analysis, students were asked to define and illustrate definitions for various mathematical terms. Figure 4.17 shows Kristen's response to this portion of the questionnaire prior to the interview. Specifically, for the definition of distance, she wrote the formula for distance in Euclidean geometry. During the interview, when asked if her definition of distance provided in Figure 4.17 held in both Euclidean and Taxicab geometry, Kristen explicitly says she thinks it does. This implies Kristen was not able to relate her **ARED** and **ARTD.**

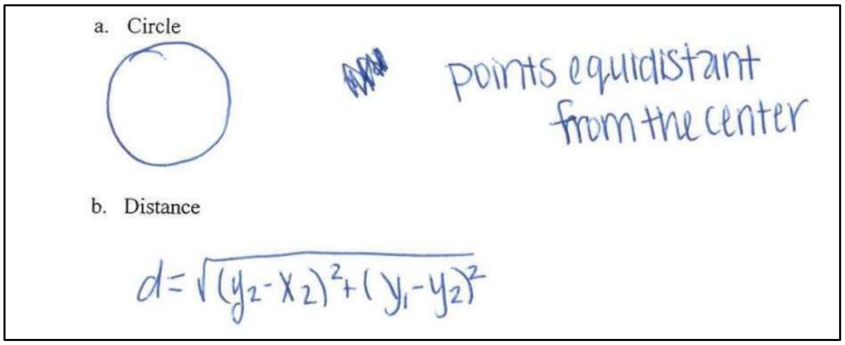


Figure 4.17 Kristen's definitions and illustration of circle and distance.

Later in the questionnaire, participants were asked to draw illustrations of both Euclidean and Taxicab distance without being given specific points. Kristen's drawings are presented in Figure 4.18. For Euclidean distance, Kristen drew the straight line between her arbitrary points P and Q, and for Taxicab distance, she drew various pathways between the same arbitrary points P and Q. Kristen's elaboration during the interview is provided below.

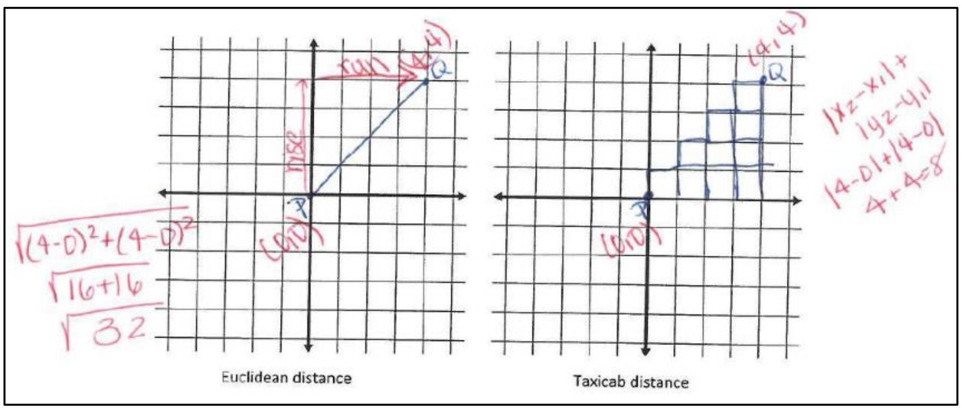


Figure 4.18 Kristen's illustrations of Euclidean and Taxicab distance, respectively.

Interviewer: How do each of your drawings represent each distance? And did you use the definition of each type of metric when you made your illustrations?

Kristen: Okay um yeah so, I guess I'll just show you… um this is just Euclidean so you just do rise and run, rise over run, just normally. And then the Taxicab it's different but you get it's pretty much the same thing but it's different. I don't know how to explain it. It's like you'll get the same answer, but you're using a different process, so this you do, I don't know I don't know how to say it. The whole humpy thing the whole humps up and down…

Kristen went on to describe Taxicab distance as "the zig zag thing," and used phrases like "you can't just go go through the straight line," and "you can just go up and down and left and right…like as if you were in a car," to relate Taxicab distance to driving on roads.

In regard to her **GRED** and **GRTD**, Kristen was able to draw at least one pathway between two points in each metric. When describing her Euclidean distance illustration, she referred to this distance as "rise over run" with no verbal description of what the distance actually looks like. In addition, she struggled to come up with a way to describe Taxicab distance, indicated by her saying Taxicab distance is "different, but you get…I don't know how to explain it." Because Kristen could draw a pathway between points in both metrics, but was unable to give a detailed verbal description of what each metric looks like, she provided evidence of an action conception of both **GRED** and **GRTD.** Thus, Kristen exhibited an action conception of **GRD**.

It is noted that Kristen was able to make a visual distinction between the two drawings by saying "you'll get the same answer, but you're using a different process," in relation to geometrically measuring the distance between two points. By "using a different process," it appeared Kristen was referring to the visual difference in pathways between her two points since at this time in the interview she was evoking her **GRD**. In terms of her schema development, it is concluded that Kristen was not able to make meaningful connections between the geometric representations of distance in both geometries (**GRED** and **GRTD**) other than local properties like the appearance of the path. By saying that the distances use different processes, Kristen was unable to verbally explain what causes the distances to appear different. In other words, if she had process conceptions of **ARD** and **GRD**, she was unable to coordinate them in order to see why the distances appear different.

In regard to her **ARED** and **ARTD**, when the interviewer first asked Kristen to explain how she used the definition of each metric in her illustrations, Kristen struggled to explain the connection between the formula for Euclidean distance and her illustration. She again brought up "rise over run" and drew corresponding segments on her graph (seen in red ink in her Euclidean distance illustration in Figure 4.18). The interviewer then asked where the squared terms in the Euclidean distance formula could be shown on her graph. The following excerpt contains Kristen's response to this question as well other questions related to this formula.

Interviewer: And where do we see that the squares

Kristen: These squares? Oh

Interviewer: No, no in the first one. Because the formula has the square root of…

Kristen: Oh yeah the square root… [indiscernible] because I don't know. Because it's the distance? Between them? So the the distance between this and this, you have to go up and then side to side

Interviewer: Uh huh would that be just x2 or y2 minus y1 without the square…? You go just up…

Kristen: Mm… without the square root?… I'm not sure… I I mean I thought that that was how you do the distance formula so I was just, I just thought that's [indiscernible]… cause I've seen it before.

In this excerpt, Kristen was unable to identify where any part of the equation is seen in her illustration even after prompting from the interviewer. In fact, she said she thought it is this way because she had "seen it before." It seems as though Kristen's strong notion that Euclidean distance is related to "rise over run" prevented her from being able to relate the Pythagorean theorem to this distance. The interiewer later asked Kristen if she thought the Euclidean and Taxicab distance between two points could be the same, and Kristen insisted that they were equal in her illustrations, which is incorrect.

Kristen: Is it possible? I said yes it is, um, because I mean, I thought that's what we were doing up here maybe. So I just thought that… I mean this is equal to this, like they're the same distance. Right? I mean… they're the same distance, PQ here and PQ here, so even if you use this formula and this formula, it should still equal the same thing. It should be equal.

After this, the interviewer asked Kristen how she would be able to check to see if the Euclidean and Taxicab distances between her points were equal, and Kristen continued to count blocks on both of her illustrations, coming to the same conclusion they were equal. So, although Kristen drew the segment connecting her points as her illustration of Euclidean distance, she was calculating the distance between them geometrically by counting blocks, as though she was calculating the distance geometrically in Taxicab geometry. With prompting by the interviewer to consider the formulas provided, Kristen was able to plug in specific values to both of the formulas for distance in Euclidean and Taxicab geometry, and realized these distances were not actually the same in her illustrations (seen in red ink in the left and right margins of Figure 4.18).

Kristen eventually demonstrated after this prompting, that she could use the formulas provided of each metric to calculate the distance between two points in both geometries. In addition, when trying to explain the formula for Euclidean distance, Kristen was unable to provide a connection between the formula and her illustration. By saying that she had "seen it before" in relation to her "rise over run" geometric representation of the Euclidean distance formula, she was relying on memory to produce this representation. By plugging values into formulas to calculate the Euclidean distance and Taxicab distance between two points, Kristen exhibited an action conception of **ARED** and **ARTD**. Although she was not prompted to try and relate her geometric representation and the formula for distance in Taxicab geometry, it is inferred she most likely would not be able to since she was unable to do so for distance in Euclidean geometry, indicating that she had not yet interiorized her action conception of **ARED**. For all of these reasons, Kristen was exhibiting an action conception of **ARD** as she exhibited an action conception of both **ARED** and **ARTD.** In terms of her schema development, Kristen was not able to make connections between the algebraic representations for distance in both geometries (**ARED** and **ARTD**) other than local properties, like that there is a square root sign involved with the Euclidean distance formula. Kristen also believed that the geometric representation of her Eucldiean and Taxicab distances between her points were equal, and eventually concluded that they were not by calculating these distances algebraically after prompting. This further indicates a disconnect between her **GRD** and **ARD**.

By exhibiting an action conception of both **GRD** and **ARD** and no evidence that Kristen was able to make meaningful connections between these representations, Kristen exhibited an overall action conception of **Distance.** In addition, she had not made the necessary mental constructions to successfully begin to assimilate **Taxicab distance** into her existing, working understanding of **Distance.** Without this assimilation, Kristen also struggled to make connections between the geometric and algebraic representations of a circle in both geometries (**GRC** and **ARC**), for which evidence is provided below.

Recall in Figure 4.17, Kristen's personal concept definition of a circle was "points equidistant from the center." When asked if she believed this definition held in both Euclidean and Taxicab geometry, Kristen said that she did not think it did. Later in the interview, she was asked if the definition provided on the questionnaire ("the circle…is a set of points in the plane equally distant from a fixed point") held in both geometries, which was essentially equivalent to her personal concept definition. To this she replied, "I'm not even sure on this, but I said yes." This contradiction of not believing her personal concept definition held in both geometries but then believing the provided definition does indicates a misconception in Kristen's mind about the definition of a circle. Further, when she was asked to explain how she drew her illustrations (which can be seen in Figure 4.19), she indicated she was attempting to work off of memory for her illustrations.

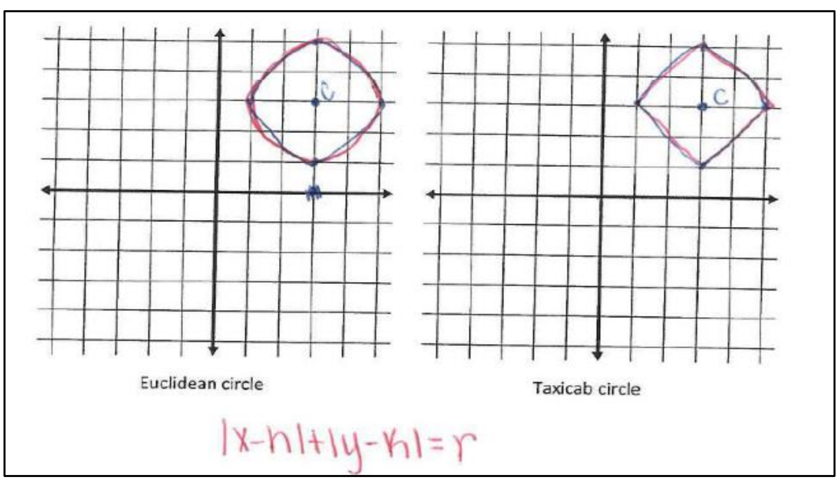


Figure 4.19 Kristen's illustrations of the Euclidean and Taxicab circles, respectively.

Kristen: Well I just drew the two things, um... I know how to draw, sorry, the Euclidean circle easily. It's just Taxicab circle is it's, like I know how to draw it, it's just I

> knew how it's supposed to look, but it's like I don't know why it's supposed to look like that.

By clearly stating that she knew how to draw the Euclidean circle and knew what a Taxicab circle looks like without knowing why it looks this way, Kristen was exhibiting an action conception of **GRC.** The interviewer did not probe for an understanding of her **ARC,** however Kristen wrote the equation for a Taxicab circle after listening to conversation between Hannah and the interviewer (seen in red ink at the bottom of Figure 4.19). For this reason, there is only evidence to claim Kristen exhibited an action conception of **Circle**. In other words, Kristen needed to interiorize her conception of **Euclidean circle** in order to begin assimilating **Taxicab circle** into her existing *circle schema*.

Supporting the preliminary genetic decomposition in this report, Kristen's understanding of **Distance** hindered her ability to make meaningful connections between her **ARC** and **GRC.** There was also not much evidence of Kristen's conception of **Radius, Center,** and **Locus of points** from this data, since she did not elaborate on how she constructed her circles besides indication a reliance on memory. Thus, she was not evoking these concepts within her *circle schema*. In terms of her schema development, having an action conception of both **GRC** and **ARC** implies she was unable to coordinate any processes across her *cEg* and *cTg schemata.* Thus, the only comparisons she was able to make between a circle in Euclidean geometry and a circle in Taxicab geometry were local and particular properties, such as the geometric appearance of each circle. Kristen was unable to talk about how the construction of a circle or the structure of the equation for a circle is a direct result of the definition of a circle. For these reasons, Kristen provided an illustrative example of a student operating at the intra-intra level of schema interaction.

4.2.1.2 Samantha

Samantha was one of the undergraduate mathematics majors enrolled in the course, and was in a group interview with Kristen, an undergraduate student, and Hannah, a graduate student in the course. On the questionnaire, students were asked if a provided definition of a circle was true in both Euclidean and Taxicab geometries. In Figure 4.20, Samantha's response to this question prior to the interview is provided.

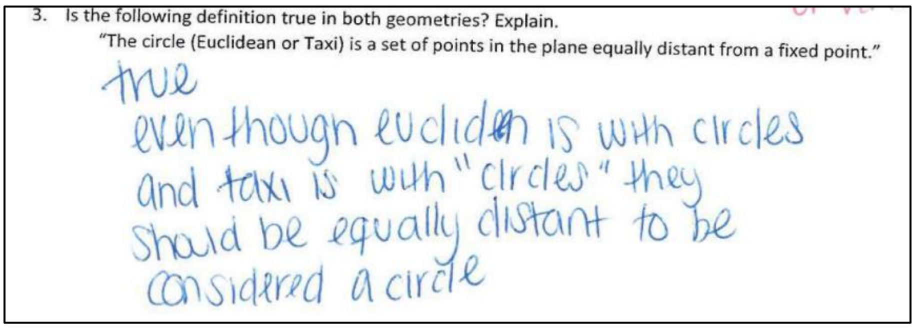

3. Is the following definition true in both geometries? Explain.
"The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."
true
even though euclidean is with circles and taxi is with "circles" they should be equally distant to be considered a circle

Figure 4.20 Samantha's written explanation of why the definition of circle holds in both geometries.

Notice that Samantha wrote the word "circles" without quotes when referring to circles in Euclidean geometry and with quotes when referring to circles in Taxicab geometry. She appeared to distinguish the circles in each geometry in this way because of the shape of the circles in Taxicab geometry. In addition, when she said, "they should be equally distant to be considered a circle," she was not clearly describing what should be equidistant from what, though it is assumed she meant the points on the circle should be equidistant from the center. Below Samantha's verbal explanation of her response to this question is presented.

Samantha: Um, the worded definition I would say holds true for both but also because…if I remember correctly, you know that um, for the Taxi-circle is when you are finding the length [to a] point it's the same in any direction I think? In any direction you go it should be the same and that's how you know that um, it's a Taxi-circle. Because the distance will be kind of like the same from anywhere I think. But I don't know.

Here, Samantha attempted to make it clearer what she meant by her written response. She mentioned "finding a length [to a] point," and that this should be the same in any direction. It is assumed she was referencing the radius of the circle when she was talking about the length to a point and that this length should be the same in any direction from the center. In doing so, she was utilizing her subconcepts of **Radius** and **Locus of Points**. She said that she was trying to remember what she had learned about circles and seemed to be trying to express what it means for a point to be on a circle. In addition, since she referred to a Taxicab circle in her verbal explanation, she provided evidence of attempting to transfer her personal concept definition of *circle* from Euclidean to Taxicab geometry in order to explain how a circle in Taxicab geometry is constructed.

The next question on the questionnaire asked students to draw a circle with center (3,3) and radius two in both Euclidean and Taxicab geometry. Seen in Figure 4.21, Samantha accurately drew the Euclidean circle, but incorrectly drew the Taxicab circle corresponding to these parameters. Note in Figure 4.21, her initial drawing of the circle in Taxicab geometry has been accentuated, seen in blue, since the lines had been lightly drawn and were difficult to see. During the interview, Samantha correctly drew this circle, as seen in red ink in Figure 4.21, after listening to the conversation between the other participants in the interview and the interviewer about their drawings. When asked about her illustrations, Samantha said, "I keep forgetting that the Taxicab circle, it's like a diamond," and "every time I think of Taxicab, I think of actual squares and not like diamond shaped, so I keep drawing just regular squares everywhere." By memorizing that the Taxicab circle resembles a square and using the conversation during the interview to correct her drawing, it seems as though Samantha was only able to talk about local observations about circles in Euclidean and Taxicab geometry.

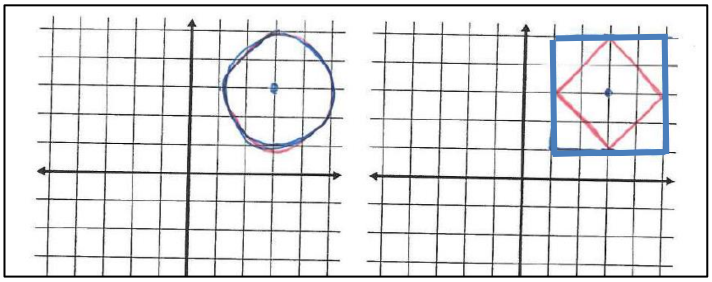


Figure 4.21 Samantha's illustrations of Euclidean and Taxicab circles, respectively.

Although Samantha correctly drew the circle in Euclidean geometry, after detailed analysis of her understanding of subconcepts and the rest of the interview, it is believed that Samantha drew this from memory and would not necessarily be able to explain how she constructed this circle. She was not probed to talk about the equations for each of these circles, as Hannah, one of the other students in the interview, was asked this first and derived the equations. According to the preliminary genetic decomposition, the analysis of Samantha's subconcepts implies she would have been unable to do explain how the construction or structure of the equation of a circle are results of the definition of a circle. Samantha had started to make some connections between her personal concept definition of a Euclidean circle and a Taxicab circle. However, Samantha provided evidence of operating at the intra-intra level because it was not made clear that she was coordinating processes across her **GRC** and **ARC**. Further, her observations about a circle in Taxicab geometry in relation to a circle in Euclidean geometry were local and particular.

4.2.1.3 Mark

Mark was a graduate student in the course who participated in an interview with Eileen, an undergraduate student, and Felix, another graduate student in the course. Mark only drew images for his personal concept definitions on the questionnaire, but verbally elaborated on his definition of a circle. His explanation is provided below.

Mark: In terms of the definition for a circle I said it's a figure that is um, every locus from the figure is equidistant from the same, from a particular point…In Taxicab geometry, it would pretty much be the same thing, right? Where um, every point along the circle in Taxicab is actually equidistant from the same point. And um, as Felix said, it is a square right? But…the square is actually oriented in such a manner where each vertex lies on…the coordinate axis, the coordinate grid, the x-y axis…When you use Pythagorean theorem, you realize that every point in distance…is actually equal at every point along the square.

By saying "every locus from the figure is equidistant from…a particular point," Mark was elaborating on his personal concept definition of a circle. He went on to talk about this definition within Taxicab geometry explaining that "every point along the circle in Taxicab is actually equidistant from the same point." In addition, he said, "In Taxicab geometry, it would pretty much be the same thing, right?," which further provides evidence he had generalized the definition of a circle across these two geometries. He talked about the shape of the circle in Taxicab geometry, explaining that each vertex of this square lies on "the coordinate axis." This is interpreted to be his way of saying to construct the square so that its diagonals are parallel to the coordinate axes. Then he said that by using the Pythagorean theorem, you can verify that "every point along the square" is an equal distance from the center. Although he never explicitly said "center", he appeared to be implying this is the "same point" he referred to in this excerpt. Since the Pythagorean theorem would be involved in the calculation of the radius of a circle in Euclidean geometry, Mark demonstrates that he may have had a misconception associated with his **Distance** and/or **Radius** concept, since he was trying to explain how to measure the radius of a Taxicab circle at this time in the interview. In particular, he was initially trying to use the Euclidean concept of **Distance** to justify that every point on the square is equidistant from the center.

Mark actually brought up this idea about the Pythagorean theorem in Taxicab geometry again later but ended up correcting himself. This can be seen at the end of the next excerpt where Mark was explaining his answer to the question that asked if the definition provided on the questionnaire held in both geometries. He began to explain why he believed circles appear differently in Euclidean and Taxicab geometry. To simplify the presentation of the analysis of this lengthy excerpt, what follows is a partition of this excerpt as he was verbally explaining his thought process. The full excerpt has been broken in this way to explain how each set of comments has been interpreted in the analysis, but it is noted they were all said sequentially.

Mark: What the difference from Euclidean to Taxicab is that um Euclidean makes use of every possible angle. And so when we're looking at... a figure that is equidistance from a particular point, we can take into consideration every single possible angle, and because of that a circle in terms of a round figure…is um created…

In this portion, it appears Mark may have interpreted that the "roundness" of a circle in Euclidean geometry to be a result of considering the 360 degrees of the circle. The beginning of the next excerpt is how he transfers this interpretation to Taxicab geometry.

Mark: …Now in Taxicab, however, when we draw a circle, essentially, we can only consider 90 degree angles because we are using the coordinate grid which is created in 90-degree angles, and so… the only direction that we can consider is up and down or left and right…

Mark was evoking his concept of **Distance** by saying "the only direction that we can consider is up and down or left and right," and that these directions make 90-degree angles, like the coordinate grid on which he had drawn circles, shown in Figure 4.22. Because he said this in relation to "in Taxicab…when we draw a circle," he seemed to have been evoking his **GRTD** concept and was trying to explain why circles in Taxicab geometry appear square instead of round. He continued to elaborate on his thinking about how to draw a Taxicab circle in the next two portions of this excerpt.

Mark: …the question is then okay is it possible to create a figure within these limitations that is actually equidistant from a point?... And the answer is yes um, because if we create a square that has its vertices along those x and y values, then when we consider the Pythagorean theorem, right? Then each point between these vertices will actually be um, be they will actually be equal from the center right? When you do the math. Short answer, when you do the math, every single point is actually…

By saying "based on these limitations," Mark was referring to the way we calculate distance in Taxicab as a limitation, or rather that he meant "based on these conditions" instead of "based on these limitations." Further, he explains that in order to construct a Taxicab circle, he would first plot points as the vertices ("along those x and y values") and connect them, with the understanding that each of the points on these lines between the vertices are also equidistant from the center of the circle. Thus, he was continuing to transfer his personal concept definition of a circle to his **GRTD** but indicated that he was following a specific procedure based on memory to draw this circle. This portion of his explanation is interpreted to be his attempt to explain that the property of equidistance in this figure is a result of the shape of the figure, and not the other way around. In other words, Mark believed that in Taxicab geometry, this square happens to form the locus of points that are equidistant from a center point, and not that the shape was a result of the construction of an infinite number of radii resulting in this locus of points. The next excerpt was when Mark corrected himself about using the Pythagorean theorem to measure the radius of a circle in Taxicab geometry.

Mark: …Okay the math is based on the Pythagorean theorem. Or not even necessarily Pythagorean… no… actually it's not the Pythagorean theorem, it's simply adding the x value and the y value, that's what it is, it's not the Pythagorean theorem. And so um when you draw a square with vertices along the x axis and along the y axis…as long as the radius of the circle along the x axis and the y axis the same from that particular point, what we will find is if we were to draw in the vertices, any point along that line that draw in the vertices, when you when you add the x and y value to get the point, it actually still is the same value as the radius along the x or the radius along the y…

Mark was able to fix his misconception that the radius of a circle in Taxicab geometry was measured using the Pythagorean theorem, and began explaining it is measured "along the x-axis and along the y-axis" or "when you add the x and y value to get the point." In this case, Mark was ready to or had interiorized his action conception of **Radius** within his *cTg* during this part of the conversation. In general, from this long excerpt, it is deduced that Mark was attempting to make connections between his **GRC** and **ARC** by trying to explain what formula was used to measure the radius of a circle while he was evoking his **GRC,** as he said things like "when you draw your square." However, later in the interview there was evidence that he was not able to fully coordinate these processes across his **GRC** and **ARC** to explain how the construction of a circle and the equation of a circle are results of the definition of a circle, possibly as a result of the misconception he had just encountered. As part of this evidence, Mark was discussing how he constructed his illustrations of the circles in both geometries, seen in Figure 4.22. Mark explained that he plotted the four points on the vertical and horizontal from the center for both circles, and eventually said "the truth is, I drew it from memory. I mean I understand um, the rules behind the circle in… Euclidean and Taxicab, but…I just purely did it from memory because I knew that's what it would look like."

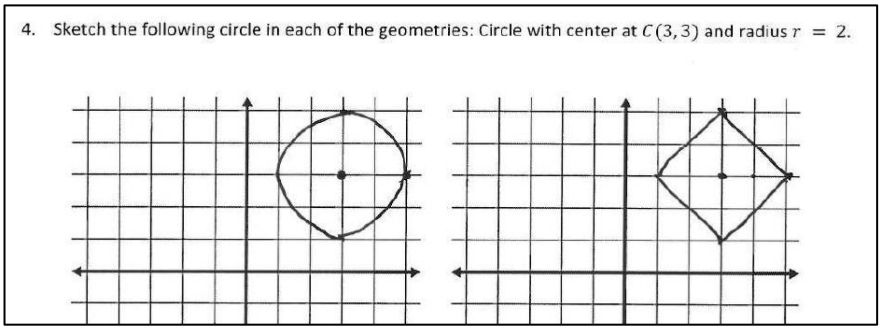


Figure 4.22 Mark's illustrations of a circle in Euclidean and Taxicab geometry.

When asked if he would be able to write the equations of these circles, he said that he could after hearing the conversation between the interviewer and the other students about them, but that had not remembered these equations until then. Since Mark was able to make local observations about the differences between these circles, like the shape, but was not able to accurately explain why the shapes would be different, he provided evidence that he was operating at the intra-intra level. In particular, it did seem that he was close to making certain connections between his **ARC** and **GRC** in terms of his **ARD** and his concept of **Radius** within his **GRC** but did not coherently explain these connections with regard to how the construction of the radius leads to this change in shape of a circle. Instead, his comments implied he believed the shape of the circle was inherently defined, so it had this property of equidistance. In addition, he struggled to explain why the shapes of these circles were different other than connecting it to the way distance is measured in these two geometries. Thus, the connections he was making between his *cEg* and *cTg* did not appear to be the coordination of processes, but rather that he was considering these concepts in a more isolated way about which he could make local observations. That is, in Taxicab geometry he was relying primarily on memory to draw and talk about a circle.

Mark provided some insight about his misconceptions in the conclusion of this portion of the interview, where he demonstrated that he was aware he did not coherently understand the definition of a circle. Mark talked about this after he was asked if he would be able to draw a circle using a new metric if he were given one. He quickly replied, "based on what happened last week… I'd say no," and continued to elaborate about the quiz the students had taken the week before in class where they were asked to use a new metric (metric is defined in Appendix B, Activity 6) to try and sketch a circle.

Mark: ...That asked us to draw a circle um, using a differen.. using a metric I had never seen before, and I had no clue, right? And so… I guess what I discovered last week is I have… a difficulty, or okay. This. I don't quite get how metrics work, apparently, in the general of sense. Because um, Taxicab, I get it. Euclidean, I get it. Give me another one…unless it's actually broken down to me and explained to me, then I don't understand, which means then that there is a fundamental concept that governs them all that I don't get. You know. So the answer, the honest to goodness answer is no.

This last excerpt showed that his reflection on the problem he had the week prior and this questionnaire led him to be aware that if he had a better understanding of the concept of **Distance** ("I don't quite get how metrics work, apparently, in the general sense"), he would be able to complete these activities on his own without external guidance. This is evident by him saying "unless it's actually broken down to me…then I don't understand…there is a fundamental concept that governs them all that I don't get." In other words, he was aware that there is some structure for how a metric would be applied in order to construct a circle using that metric, but he did not have a coherent understanding of this structure.

Since most of the data for Mark pertained to his understanding of a circle in Taxicab geometry, it is possible that he may have exhibited an inter-*cEg* stage of schema development. However, from the interview data and evidence that was obtained, the level at which Mark evidence of operating was the intra-intra level.

To summarize, Mark, Samantha, and Kristen all ended up relying on memory to draw the circles in both geometries (as well as write the equation of a circle, if attempted). This supports the literature, as Akkurt (2010) described that when a student cannot make a connection between his or her prior knowledge and new information about a concept, the student will attempt to memorize the new concept.

4.2.2 Intra-cEg, Inter-cTg

At the intra-*cEg* and inter-*cTg* (intra-inter) level, a student has begun to make connections in Taxicab geometry about how the equation and appearance of a circle is a direct result of the definition of a circle but cannot transfer this knowledge back to Euclidean geometry to make the same connections about a circle. In other words, the student is able to coordinate some of his or her processes of **Distance, Radius, Center,** and **Locus of points** with one another across his or her **GRTC** and **ARTC** to make non-local comparisons between these representations but cannot coordinate these processes (if they exist) with his or her **GREC** and **AREC**, respectively. There were no students that exhibited evidence they were operating at the intra-inter level of schema interaction.

4.2.3 Intra-cEg, Trans-cTg

At the intra-*cEg* and trans-*cTg* (intra-trans) level, a student has formed a coherent understanding in Taxicab geometry of how the construction of a circle and the equation of a circle are a direct result of the definition of a circle but cannot transfer this knowledge back to Euclidean geometry to begin to make the same connections between these representations. In other words, the student is able to coordinate all of his or her processes of **Distance, Radius, Center,** and **Locus of points** with one another across his or her **GRTC** and **ARTC** to understand the underlying structure of a circle within Taxicab geometry but cannot coordinate any of these processes with his or her **GREC** and **AREC,** respectively. The student is able to compare local properties of circles across his or her *cEg* and *cTg schemata* but cannot discuss any deeper connections. There were no students that exhibited evidence they were operating at the intra-trans level of schema interaction.

4.2.4 Inter-cEg, Intra-cTg

At the inter-*cEg* and intra-*cTg* (inter-intra) level, a student has begun to make connections in Euclidean geometry about how the equation and appearance of a circle is a direct result of the definition of a circle but cannot transfer this knowledge to Taxicab geometry to make the same connections about a circle in this space. In other words, the student is able to coordinate some of his or her processes of **Distance, Radius, Center,** and **Locus of points** with one another across his or her **GREC** and **AREC** to make non-local comparisons and observations between these representations but cannot coordinate these processes (if they exist) with his or her **GRTC** and **ARTC**, respectively. There were three students, Alicia, Felix, and Darryl, that exhibited evidence of operating at an inter-intra level.

4.2.4.1 Alicia

Alicia was an undergraduate student enrolled in the course who participated in an interview with Darryl, a graduate student. Alicia first said that she struggled to write a definition of a circle as the questionnaire asked to do. However, she saw the definition written on the next page (as "a set of points in the plane equally distance from a fixed point") and decided to write it in her own words on the first page. As a result, her definition for a circle was "a shape that is centered at a fixed point with all points being equidistance from that fixed point." By re-wording this definition, Alicia made evident what facets she found important about the definition of a circle within her schema/concept image. In particular, the fact that she specified it is a shape centered at a point "with all points being equidistance…" implies that she viewed a circle as a figure with these properties, and not necessarily that these properties define the shape.

Alicia provided more insight in to her evoked *circle schema* for this questionnaire as she discussed the construction of her circles, which can be seen in Figure 4.23. She stated, "we know like in a Euclidean circle, is centered at a fixed point and every point on the circle is equal distance from that fixed point. And then at first in Taxi geometry, I didn't see how they were equal distant." Thus, geometrically, Alicia was able to visualize how her Euclidean circle satisfied the definition of a circle but struggled to see this for her Taxicab circle. Next, Alicia elaborated on these circles she drew in each geometry.

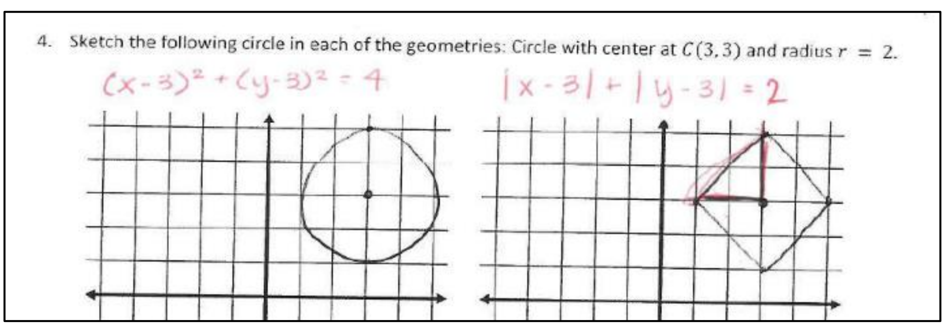


Figure 4.23 Alicia's illustrations of a circle in Euclidean and Taxicab geometry, respectively.

Alicia: Okay, so um, I drew my, well I plotted my center at (3,3) at first um and then we know that the radius is 2, so I just did it like… it's easiest to do 2 to here, in four sides and then I drew my circle. This one, um for my Taxi-circle... um… I know it's whenever I did it in GSP it always gave me like a diamond shape, um, so I drew my center at (3,3) again and I did the radius at 2 and then I connected all the points.

Regarding her geometric representations of a circle in both geometries, Alicia implied here, in addition the last presented excerpt, that she had made some connections in Euclidean geometry between the definition of a circle and her geometric representation. This is evident by statements such as "we know the radius is 2…it's easiest to do 2 here, in four sides, and then I drew my circle." At the same time, she was unable to make these same connections in Taxicab geometry, evidenced by her expressing in the last excerpt that she was unsure of how the points on her Taxicab circle were all equidistant from the center. It seemed as though Alicia was trying to transfer her understanding of the construction of a circle to Taxicab geometry by following the same procedure as she had done in Euclidean geometry. However, she reached a point where she was confused and relied on memory to complete the figure. This was evident by her saying "GSP always gave me like a diamond shape…and then I connected all the points."

Although she did not understand how her circle in Taxicab geometry satisfied the definition of a circle, she pinpointed her confusion to the concepts of **Distance** and/or **Radius**. This was demonstrated as she said she "didn't see how they were equal distant," in reference to the points on the circle being equidistant from the center of the circle. In other words, she knew what the figure should look like, but could not explain why it looked that way. This implies she was having trouble transferring her understanding of a circle from Euclidean to Taxicab geometry but was able to differentiate the shapes of these circles, implying she could make local observations and comparisons between her **GRC** across her *cEg* and *cTg schemata*. Similar evidence regarding her **ARC** was found by her explanations regarding the algebraic representations of these circles, provided below.

Alicia was able to produce the equation for her circle in Euclidean geometry (shown in Figure 4.23), verbally explaining, "I know a circle is x minus h squared plus y minus k squared…is equal to r squared." When asked what the equation of her circle would be in Taxicab geometry, Alicia explained the following about her thought process.

Alicia: Umm... just based off the distance formula I think it would be the absolute value of these, so the absolute value of x minus 3 plus the absolute value of y minus 3 but I wouldn't know if it would be equal to just 2 or 2 squared… I think it would just be equal to 2…Because in the distance formula for Taxi…nothing is squared, like how it is in Euclidean…so I think it equals 2.

This excerpt showed that Alicia understood certain algebraic aspects of the equation for a circle, specifically since she referred to the formula for distance in both geometries being involved with these equations. However, since she was relying on local properties of these algebraic representations by saying things such as "because in the distance formula for Taxi… nothing is squared… so I think it equals 2." Thus, she had made meaningful connections between the algebraic representations for **Distance** and **Circle** within her *circle schema* in both geometries, but she could not use these connections to explain how the algebraic representation for a circle in either geometry is a result from the definition of a circle. Alicia showed evidence that she was trying to write the equation for a circle in Taxicab geometry by following the same format of the equation of a circle in Taxicab geometry. She eventually relied on patterns she saw in the Euclidean circle equation instead of connecting this equation to her geometric representation or definition of a circle. Alicia appeared to have transferred some knowledge (the notion of using the distance formula in the equation of a circle) from her **AREC** to her **ARTC** and then relied on copying a format to finish writing her equation.

Alicia was able to make some connections between her definition of a circle in Euclidean geometry and her geometric representation for this circle and was able to easily produce the equation of a circle in Euclidean geometry. However, when Alicia tried to transfer this knowledge to Taxicab geometry, she relied on transferring and copying a process or format instead of transferring a conceptual understanding from one space to the other regarding both her **GRC** and **ARC**. Alicia provided evidence she was operating at the inter-intra level of schema interaction because she was making these connections both geometrically and algebraically in Euclidean geometry, but could only make local observations between these representations in Taxicab geometry.

4.2.4.2 Felix

Felix was a graduate student in the course and participated in an interview with Eileen, an undergraduate student, and Mark, a graduate student in the course. In his written work before elaborating on questions during the interview, Felix defined a circle as "a locus of points equidistant from a central point," and drew a Euclidean circle with three radii, indicating they were congruent, as seen in Figure 4.24. This indicated he was aware of how a circle in Euclidean geometry is constructed but stated during the interview that this written definition of a circle would be different in Taxicab geometry. He explained they would be different because a "Taxicircle is a square, uh around a central point,". The interviewer probed him further about this by asking Felix what his definition of a circle in Taxicab geometry would be. He replied, "I believe it would be the set of points…it can't be tangent because…I'm not sure how to find it." Thus, he was focused on the differences in the visual appearance of the circles which led him to believe that the definition of a circle itself would be different between Euclidean and Taxicab geometry. Further, he was not able to write the definition of a circle in Taxicab geometry and was trying to evoke other concepts or properties, such as tangency to try and explain his understanding.

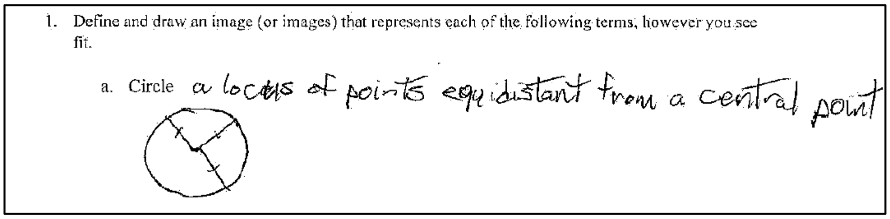


Figure 4.24 Felix's personal concept definition and illustration of a circle on the questionnaire.

His written explanation for why the definition provided on the questionnaire would not hold in both geometries is shown in Figure 4.25. He said it was "because the "circle" in Taxi is a square." The use of quotations around the word "circle" in this response further showed his belief that the geometric representation of a circle in Taxicab geometry was pre-defined by shape and not by the construction using the definition of a circle. This further indicates Felix was able to make local observations about the difference in geometric representations of a circle in Euclidean and Taxicab geometry, but that he may have a misconception associated with his understanding of the definition of a circle in Taxicab geometry.

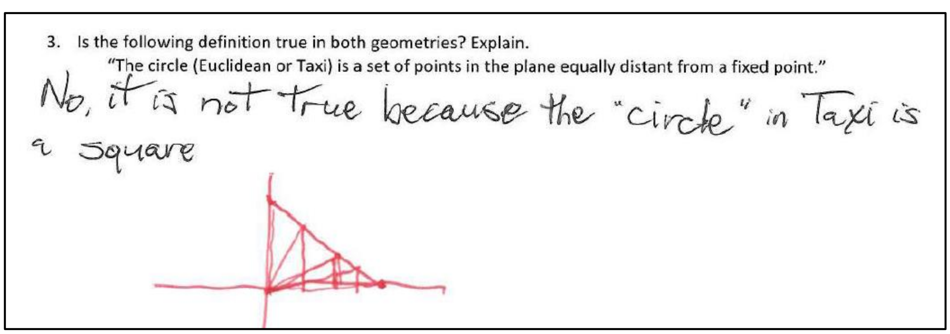


Figure 4.25 Felix's response to whether the provided definition of circle held in both geometries.

Recall the red portion of Figure 4.25 was drawn during the interview when he was asked to elaborate on this written response. The excerpt of this portion of the interview which pertains to this drawing is provided below.

Felix: I actually said I don't think that that statement is true um, because I'll, I'll go with this. With the Euclidean geometry, yes it's true because it's loc…all the points everything is equidistance forms a circle so everything is equally distanced. With a circle in Taxi, uh we have uh the radius, defined on… the axis of the coordinate plane…but the other points are a go... just a straight line that's identified by a right triangle created with the axes, and it connects directly. Now, the relationship between the points moving across to get to various points a long those distance between uh, x, 0, and 0, y, um, their relationship will always remain constant in how they change with the x and the y moving across.

Interviewer: Can you, can you just illustrate that? And you can use that space below.

Felix: …The distance here, and [indiscernible] and this to measure this measure is equal to this measure, but

Interviewer: From the x…

Felix: Yeah all the x and y, but likewise the measure, the relationship between the edges will always remain the same cause they have to be… this is a right triangle so like this, relationship here between these two sides… will be equal to the relationship between these two sides, which is equal to the relationship between this side because it has to maintain along the hypotenuse of this triangle of the side, of that Taxicircle. So that relationship is true, but the points are not equidistant from a fixed point. At least I don't I don't see that, so that's why that's why my answer was no… that the definitions aren't the same between Euclidean and Taxi.

Felix said a lot of very interesting things within this excerpt. First, he drew a quarter of a Taxicab circle and called it a right triangle. This could be due to some misconception in Felix's mind that a Taxicab circle is defined by these triangles, instead of constructed by the radii that happened to be legs of the triangles Felix was referring to. This is evidence he had mentally partitioned these circles into four right triangles, specifying the hypotenuse of these right triangles as "edges". Second, he established that there is a relationship between the x and y coordinates of the points ("relationship here between these two sides") that are on this edge. As he was saying "between these two sides…these two sides…" he was drawing the straight lines from the right angle of this bigger triangle to its hypotenuse and was referring to the two legs that made up the newly formed [smaller] right triangles that can be seen in Figure 4.25.

Third, Felix talked about "the relationship between these two sides" being equal to "the relationship between this side because it has to maintain…" In other words, he was saying these two relationships must be the same but does not specify what exactly this relationship was. That is, until he brought up equidistance again saying, "but the points are not equidistant from a fixed point." Felix implied he understood there should be a relationship between the x and y coordinates of points on a circle in Taxicab geometry (referring to the distance of the radius of a Taxicab circle) but did not understand how these x and y coordinates related to this radius or showed equidistance, and how the radius would be represented geometrically. For these reasons, any misconception he was having with regard to his **GRC** was most likely due to either his concept of **Distance** or **Radius** in his *cTg* and the inability to coordinate these processes, if they exist, with the other necessary concepts involved with his *circle schema.*

Felix went on to describe how he correctly constructed the circles requested in the questionnaire by mainly using visual descriptions of the figures. For example, he explained that he “did the round sides equidistant” for the Euclidean circle and drew the Taxicab circle “by creating straight edges…connecting the points on the vertices…have a square standing on one of its vertices.” From his statements, it seemed as though Felix understood how the geometric representation of a circle in Euclidean geometry is constructed since he referenced equidistance with his construction of that circle. However, he could only transfer knowledge of visual differences in terms of his **GRC** from his *cEg* to his *cTg schemata*, as evident by him saying, “connected the points” and continuously referring to the shape of the Taxicab circle as to why the definition of a circle would not hold in both geometries. When Felix was asked to write the equations for both of his circles during the interview, he was able to produce the equation for his Euclidean circle based on memory. He then used properties of this equation to attempt to write the equation for this circle in Taxicab geometry. These equations can be seen in Figure 4.26.

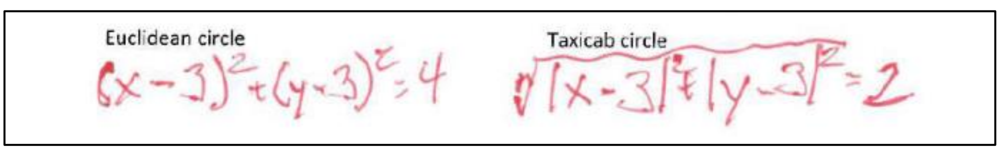


Figure 4.26 Felix’s equations for his circles in Euclidean and Taxicab geometry.

When he was writing the equation for the Taxicab circle, he explained, “I think it would be the uh, the distance formula, cause that’s what we used so that would be… so I don’t have to necessarily have to do square root…” and ended up squaring some terms and using a square root over part of his equation. Recall in Section 4.2.4.1, Alicia also noted that the distance formula appeared in the equation of a circle but did not indicate she knew why. Because of this, she reproduced patterns she noticed in the equation of a circle in Euclidean geometry. Similarly, Felix did not provide much verbal evidence of understanding the derivation of the equation for the circle in Euclidean geometry besides stating that it involved the formula for distance. He resorted to identifying local properties of that equation and attempted to transfer them to write the equation for his Taxicab circle without deep consideration of why the distance formula is involved in these equations. Felix is a clear example of a student who was operating at an inter-intra level since he had formed some connections between representations of a circle in Euclidean geometry, but was unable to transfer his understanding of a circle to Taxicab geometry in order to make the same connections.

4.2.4.3 Darryl

Darryl was a graduate student who participated in an interview with Alicia, an undergraduate student in the course. Darryl was extremely open about his thinking and, as a result, provided very detailed insight to his geometric reasoning. Darryl's personal concept definition of a circle was "a locus of points that are equidistant from the center," which can be seen written and accompanied by an illustration and equation of a circle in Euclidean geometry in Figure 4.27. During the interview, Darryl explained this definition in further detail. An excerpt from this portion of the interview is provided below when he was asked to explain his definition and if it would hold in Taxicab geometry.

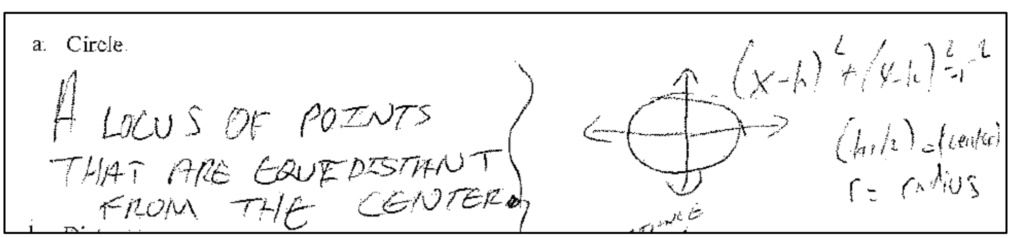


Figure 4.27 Darryl's definition of circle written on the questionnaire prior to the interview.

Darryl: Okay so my definition for circle was a locus of points that are equidistant from the center…So I thought about the definition because I thought about if you have a center, and you have your radius, your radius has to be equal for each direction that you have your circle… The locus of points are all of the points on the circle itself. And so, uh if it applies to Taxicab geometry, yes. It does.

Here, Darryl demonstrated that he had evoked all of his **Distance, Center, Radius,** and **Locus of points** conceptions. Later in the interview, Darryl was asked if the definition of a circle provided on the questionnaire held in both geometries, to which he elaborated even further about his personal concept definition of a circle. An excerpt of this conversation is presented below Figure 4.28, which shows his written answer to this question on the questionnaire prior to the interview, confirming that he had a coherent understanding of the definition of a circle.

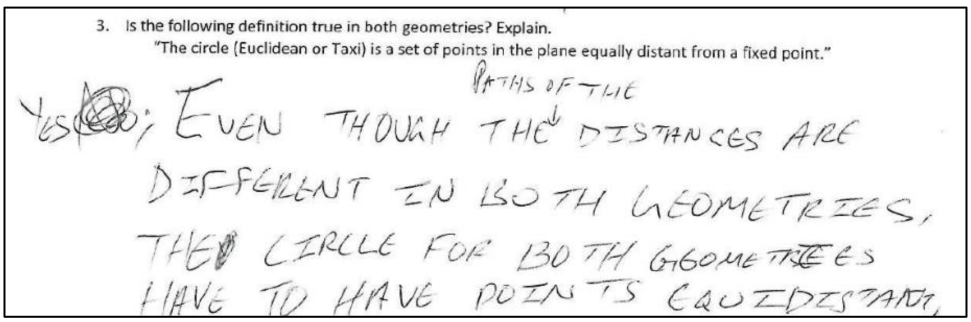

3. Is the following definition true in both geometries? Explain.
"The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."
YES; EVEN THOUGH THE PATHS OF THE DISTANCES ARE DIFFERENT IN BOTH GEOMETRIES, THE CIRCLE FOR BOTH GEOMETRIES HAVE TO HAVE POINTS EQUIDISTANT,

Figure 4.28 Darryl's written response to whether the provided definition of a circle held in both Euclidean and Taxicab geometry.

Darryl: Yeah it would be the same. The definition holds for both Taxicab and Euclidean and I said the same thing for the previous question too because I was really thinking about the radius between uh the points that were on the circle regardless of Euclidean or Taxicab…That distance from uh…the point on the circle to the center and uhh regardless of whether it's going to be in Taxicab or in Euclidean, the distance is gonna hold the same so it's gonna be equidistant to each other. So if your radius was 2, uh, if your center was at some point and your radius was 2 all the way around the circle, it's going to be 2 regardless of what metric you're in.

From the portion of his written answer when he said "even though the paths of the distances are different in both geometries…" Darryl provided evidence he was evoking his **GRC,** since the word choice of "paths" comes with imagery. Further, his explanation that a radius would be the same "all the way around the circle…regardless of what metric you're in," showed Darryl had formed a fairly coherent understanding of the definition of a circle, and how this related to the construction of a circle (**GRC**).

Further information about this understanding is evident in his illustrations, shown in Figure 4.29, and his explanation of how he constructed these circles. When talking about the circle in Euclidean geometry, he stated that he plotted the center, "went whatever two radius from the center is," counted two units up, down, left, and right (points are plotted in illustrations), and drew "my circle and with the center at that point." He correctly drew the circle in Euclidean geometry but oriented his circle in Taxicab geometry incorrectly. The following excerpt from the interview was part of a conversation between Darryl and the interviewer in which Darryl explains his reasoning for orienting his circle in Taxicab geometry in this way.

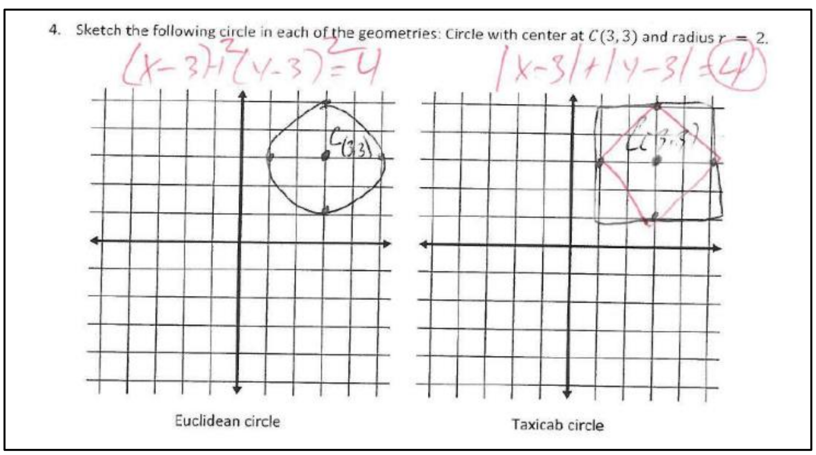


Figure 4.29 Darryl's illustrations of a Euclidean and Taxicab circle.

Darryl: Now through Taxicab geometry, again I did the same thing except when I think about Taxicab I think since you can't uh go through the fence…it's that you can't go through the graph um, which I drew mine in this, the form of a square for Taxi…

> essentially in Taxicab geometry like the circle…it kinda looks similar to a square, I could be wrong but….that's what I thought the circle would look like.

His explanation for why he oriented the circle in Taxicab geometry this way illustrated a misconception about the manner in which distance can be drawn or measured in Taxicab geometry. Recall in Section 4.2.1.3, Mark said regarding Taxicab distance, "we are using the coordinate grid which is created in 90 degree angles." This was interpreted to mean he felt restricted in Taxicab geometry to drawing lines only on the coordinate grid. Mark ended up relying on his memory of the shape and orientation of a Taxicab circle to draw one on the questionnaire. It is interpreted that by Darryl stating, "you can't go through the graph" he had a very similar idea to what Mark explained and to what Nicole described, whose work is presented later in 4.2.1.3.

These types of statements are attributed to Mark, Darryl, and Nicole believing that Taxicab distances, and therefore lines, can only be drawn on the coordinate grid formed by the set of integers on each axis. In other words, they have discretized the Taxicab metric and do not view it as a continuous measure. This is consistent with research, in that Smith (2013) discusses having conversations with students about how it was possible to draw line segments through the blocks even though a taxicab or car would not be able to drive through the blocks. This misconception led Darryl and Nicole to disregard their understanding of the definition of a circle to draw the circle in Taxicab geometry, resulting in an incorrectly oriented circle. This affirms Fischbein (1993), as it is explained that in geometrical reasoning, many students decide to neglect a definition if they are presented with constraints on a figure. In these cases, the figural constraint is the manner in which distance is defined. In contrast to Darryl and Nicole's drawings, Mark encountered the same misconception, but relied on the memory of the shape and orientation to finish constructing the Taxicab circle.

In general, Darryl demonstrated that he understood how the construction of a circle is a result of the definition in Euclidean geometry and was trying to transfer part of this understanding (**GREC**) to Taxicab geometry (**GRTC**). This was evident by his statements about the "paths" or the way in which radii of a circle in Taxicab geometry were constructed being different than Euclidean geometry. This implies he understood why the circles appear different but was not able to use this understanding to correctly construct this circle. Darryl said he constructed his Taxicab circle "in the same way" as he constructed his Euclidean circle and had previously described in the interview how he would do this in Taxicab geometry. When he was explaining how he illustrated this specific circle, he gave the impression he had copied the procedure of plotting the four points on the vertical and horizontal from the center.

With prompting, Darryl was able to recall the equation for a circle in Euclidean geometry and wrote what he thought the equation for a circle in Taxicab geometry would be, shown at the top of Figure 4.29. The interviewer asked him how the equation he wrote compared to the distance formula, and the following interaction ensued.

Darryl: Well… if we're saying that the definition of both circles hold for both Euclidean and Taxicab geometry, essentially

Interviewer: The definitions… uh how things are calculated… metric

Darryl: Right, cause the form would be a little different.

This excerpt showed that Darryl implied he understood that the definition of the circle coupled with the defined metric resulted in the difference in appearances of these equations, connecting his definition of a circle to his **ARC**. He also specified, with help from the interviewer, that since the definition of a circle holds in both geometries, that the difference in metric is causing the equations to appear different, referring to the "form" of these equations. He went on in the interview to say “I would think that if you add the absolute values that would still equal the radius squared. So, that’s how I think you could form it in Taxicab.” This implies he was actively trying to transfer knowledge from his **AREC** to **ARTC** but ended up reproducing patterns he saw in the equation for a circle in Euclidean geometry, rather than referring to his understanding of the definition of a circle to see if the radius term should be squared. Recall that Alicia and Felix (presented in Sections 4.2.4.1 and 4.2.4.2, respectively) also verbally stated that the distance formula was involved in these equations and ended up relying on reproducing patterns to finish writing the equation of the Taxicab circle.

Darryl showed he had transferred some knowledge from his *cEg* to his *cTg* about the definition of a circle in his **GRC** and **ARC** independently. However, he was only able to talk about the appearance of his Taxicab circle and how the format of the equations for a circle were related. He did not exhibit evidence of using his definition of a circle to construct and derive the equation of his circle in Taxicab geometry. For these reasons, Darryl provided evidence he was operating at an inter-intra level of schema interaction since he was only able to transfer and make local observations about both of these representations in Taxicab geometry.

To summarize, Felix, Alicia, and Darryl were all able to make observations about local or visual properties of the geometric representation of a circle in Taxicab geometry. Two out of these three students correctly oriented the Taxicab circle on the questionnaire but indicated they did so based on memory. All three students used patterns they noticed from the respective distance formulas to try and write their equation of a circle in Taxicab geometry. This resulted in two of the three students writing the incorrect equation.

4.2.5 Inter-cEg, Inter-cTg

A student operating at the inter-*cEg* and inter-*cTg* (inter-inter) level can begin to generalize and make connections about how the equation and/or appearance of a circle within a metric is a direct result of the definition of a circle. In other words, the student can begin to talk about the construction of a circle and the structure of the equation of a circle as a result of this definition other than local, particular properties in either geometry. At this level, the student is able to coordinate some of his or her processes of **Distance, Radius, Center,** and **Locus of points** with one another across his or her **GREC** and **AREC** to make non-local comparisons between these representations. The student can also coordinate some of these processes across his or her **GRTC** and **ARTC**. Further, the student can coordinate these processes across their **GREC** and **GRTC** in an attempt to construct a general **GRC,** resulting in the interaction of his or her *cEg* and *cTg* *schemata***.** Similarly, these processes can be coordinated across their **AREC** and **ARTC** to construct a general **ARC**. The student can also coordinate some of these processes across his or her **GRC** and **ARC** and can make meaningful connections between these representations in both geometries at this level but have not formed a coherent underlying structure for how these representations are related. There were six students that showed evidence of operating at the inter-inter level of schema interaction. Specifically, these students were Hannah, Nicole, Brianna, Robin, Marianne, and Eileen, of which Nicole and Eileen were the only undergraduate students.

4.2.5.1 Hannah

Hannah, a graduate student in the class, provided some insightful responses to the questions posed on the questionnaire and by the interviewer. Hannah was in an interview with Kristen and Samantha, both undergraduate math majors. When asked to define and draw images for both concepts of distance and circle prior to the interview, Hannah defined a circle as "all points equidistant from [a] specific point." Later, Hannah was asked if the definition of a circle provided on the questionnaire held in both Euclidean and Taxicab geometry. She explained that the formulas were different, but she believed this "worded" definition holds in both geometries. In particular, the question asked for an explanation of why or why not the definition for circle held in both geometries. To this, Hannah stated yes, and only provided the equations for a general circle in each geometry as her justification, seen in Figure 4.30. This indicated that Hannah may have made a connection between this "worded" definition and how it related to the structure of the equation of a circle in both geometries. However, later in the interview, she seemed to be unsure of herself when talking about the equation for a circle in Taxicab geometry, for which evidence will be provided.

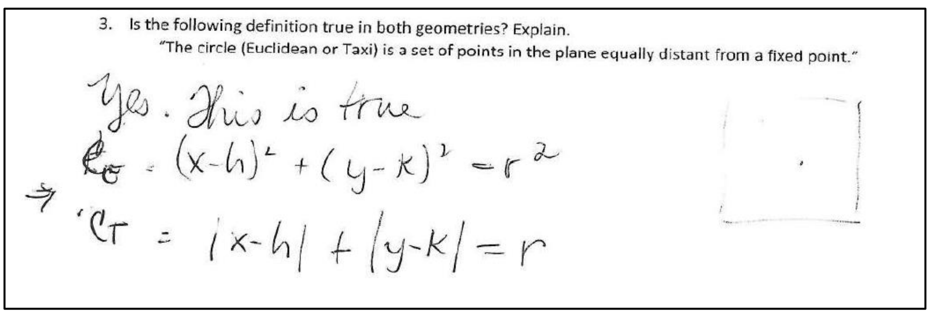


Figure 4.30 Hannah's written response to whether the definition of circle held in both geometries.

Notice in the right side of Figure 4.30, Hannah seemed to have erased an incorrect attempt to draw a Taxicab circle, possibly recalling it resembles a square. This indicated that at this point in the questionnaire, Hannah was attempting to rely on memory to sketch a circle in Taxicab geometry but may have realized it was not oriented correctly at some point. In fact, Figure 4.31 shows Hannah's correct illustrations for a circle in both geometries, and excerpts are provided from her conversation with the interviewer that indicated she could use more than just her memory to draw this circle. This portion of the interview ensued after the interviewer asked Hannah how she (1) went about drawing her circles, (2) how she used the definition of circle in each scenario, and (3) if she could write the equation of each circle. Recall that red ink in Figure 4.31 indicates writing Hannah did during the interview.

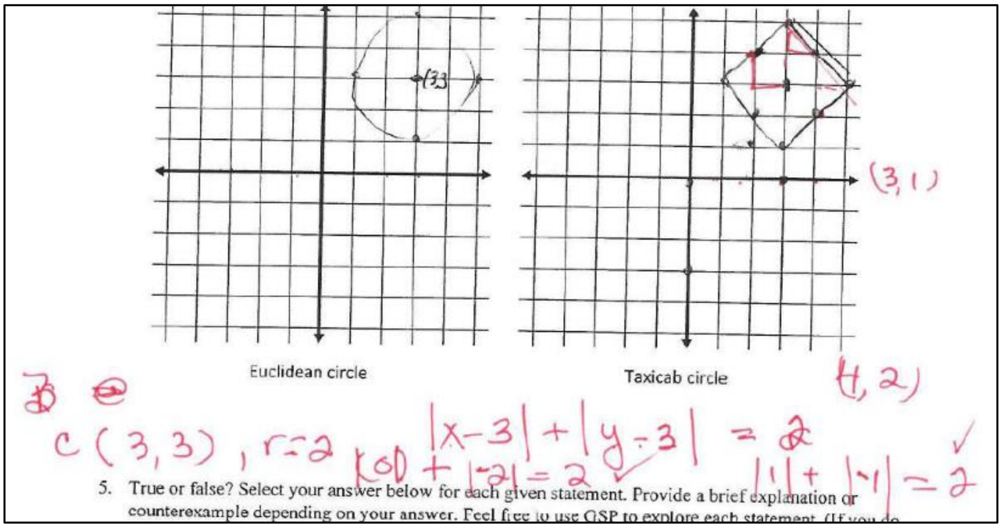


Figure 4.31 Hannah's illustration of a Euclidean and Taxicab circle, respectively.

Hannah: Well I started with the equation, and I don't know if I'm right or not [indiscernible]… We know with um Euclidean geometry it looks like that one, x minus h squared plus y minus k squared equal r squared…and kinda based on the distance formula, so I was thinking this might be the one for Taxicab I don't know if that's right or wrong.

Interviewer: So how would you check?

Hannah: Okay, so when I drew it, [indiscernible] so I just did the paths thing, so I just I knew okay when you go to 2,2 if the radius is 2 I knew that's going to be a point on that and I can go out 2 or up 2 or either one, but then, you know those distance things, so I just did like 1,2 and put a point there… you know and just started count[ing]... checked it and that's how I got that.

In this excerpt, Hannah first explained how she understood the distance formula was involved with the equation for each circle and how this influenced her derivation of the equation for her circle in Taxicab geometry. Specifically, she said she wrote it "based on the distance formula," but wasn't sure if she had written it correctly, indicating she was transferring local observations from Euclidean to Taxicab geometry in terms of her **ARC**. She also explained how she constructed her circle in Taxicab geometry based on her understanding of the definition. She seemed to evoke all of her **Distance, Radius, Center,** processes geometrically here by explaining that a point would be on this circle if it was at a distance of two away from the center. Further, she described this by saying things like "if the radius is 2, I knew that's going to be a point on that."

The interviewer then asked Hannah to show her that any other point (that was not one of the vertices of the circle in Taxicab geometry) was also two units away from the center. Although Hannah struggled with this initially, with prompting she was able to show this was true for a couple of the midpoints on the edges of the circle by counting blocks and then stated, "oh yeah… in fact that's how I got the point." It is possible that Hannah had not fully coordinated some of the processes within her **GRC** prior to the interview (which could explain the light sketch of the incorrect Taxicab circle in Figure 4.30) and was beginning to coordinate them during the interview. Kristen, one of the other students in this interview, was asked to explain her illustrations after Hannah. As Kristen appeared to be struggling to explain how she constructed her circle in Taxicab geometry, Hannah jumped in and stated, "well it's cause you gotta be able to… finding every point with the path of 2." This demonstrates that Hannah was able to imagine how to construct this circle without performing actions of drawing specific radii, and that she was able to describe this process in her own words geometrically.

Pertaining to her **ARC**, recall that in Figure 4.31, Hannah was able to produce the equations for an arbitrary circle in both Euclidean and Taxicab and geometry. When Hannah was asked to write the equations for the particular circles she had drawn, Hannah stated that the equation of a circle was based on the distance formula, although she did not explain how this related to the definition of a circle. As a result, she wanted to use the formula for Taxicab distance within her equation for a Taxicab circle, but said she was not sure that what she had written was correct for this equation, shown towards the bottom of Figure 4.31. The interviewer asked Hannah how she would be able to check to see if this equation was correct, and Hannah immediately wanted use points on her circle in Taxicab geometry to substitute them into her written equation for the circle in Taxicab geometry and verify that she would get two when she evaluated this expression. In other words, Hannah was using the definition of a circle and her concept of **GRC** to verify that the equation she had written (**ARC**) was correct. The following excerpt is part of her going through this process, where she specifically relates her equation to the center and radius of the circle.

Hannah: So if the center is at uh 3,3 and then r is equal to 2. Then our formula this guy to look something like um square, the absolute value of x minus 3 plus the absolute value of y minus 3, 'cause that's our h, is equal to 2…So let's take a sample of a point, (x,y)… this (x, y) is um 3.. (3,1). Let's see if (3,1) is on the circle as a test point. So for (3,1) I would have 3 minus 3 which is 0, k? Plus um, y which is 1 minus 3…Right so, is equal to… our answer, 2…. Cha ching! That's true.

By saying things like "…absolute value of y minus 3, 'cause that's our h," Hannah seemed to be identifying and substituting values for variables in her equation to see if, after evaluating the expression, she would get the value of two. It was as though she was performing an action by substituting values into an equation with little interpretation rather than confirming the equation satisfied the definition of a circle. In other words, she did not explain that she was checking to see if a point was on the circle by finding the distance between it and the center and comparing it to the size of the radius. Specifically, in terms of her concept of **Radius**, she was checking to see if this expression was equal to the radius as a *number* instead of a *concept*, since she referred to the expression being "equal to…our answer, 2" instead of something like, "equal to the radius, which is 2." Once she verified that substituting the coordinates of two of the points on her circle into her equation both result in a value of 2, she explained that her equation made sense to her.

Hannah stated that she "derived the formula. I didn't do it by memory, I was deriving it, that's what makes sense to me. I'm just glad it kind of worked." This statement implied that Hannah believed she had derived this equation for a circle in Taxicab geometry by using the definition of a circle, despite evidence throughout the interview implying she was unsure of the equation she had written until this point. While looking at her illustration Hannah was able to verify that the equation of a circle in Taxicab geometry was correct by substituting points on the circle into this equation. That is, since she was evoking her **GRC**, Hannah was able to coordinate some of her processes across her **GRC** and **ARC** by referencing her graph and the equation at the same time. Thus, she was able to make some connections between the algebraic representation and geometric representation for a circle in both geometries but did not exhibit a coherent understanding of how these are a direct result of the definition of a circle. However, the connections she was making between her *cEg* and *cTg schemata* were not local properties, since she knew how to test to see if her equations of a circle satisfied the definition of a circle.

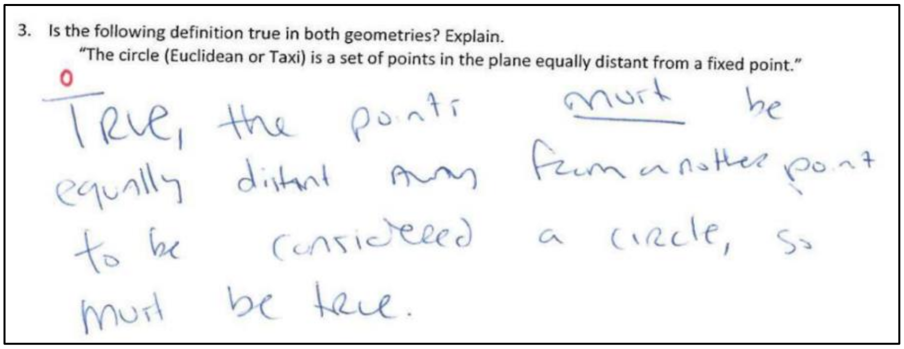
Hannah was able to understand how the definition of a circle resulted in the construction of a circle geometrically but seemed to still be in the process of forming this coherent understanding for her **ARC**. She knew the distance formula was involved with each equation but did not explain why in relation to the definition of a circle. Further, when she substituted a point into her equation, she explained how to check to see if her equation was correct, she simply substituted the values for her center and radius, evaluated the expression, and says "is equal to…

our answer 2." She said, "let's see if (3,1) is on the circle as a test point," but lacked reference to any subconcepts of **Circle.** This implies she may have just been performing an action with little interpretation of how it related to these components of the *circle schema*. Her reference to "our answer 2" instead of phrasing this in context of the radius is a small, but meaningful difference in the interpretation of these comments.

There was not specific evidence referring to Hannah's ability to explain the structure of the equation of a circle in Euclidean geometry, however she appeared to have been forming this understanding in Taxicab geometry, as evident by these excerpts. She was most likely transferring knowledge back and forth between Euclidean and Taxicab geometry during the interview to reflect on her understanding of the definition of a circle. Thus, Hannah exhibits evidence of operating at the inter-inter level of interaction of her *cEg* and *cTg schemata.*

4.2.5.2 Nicole

Nicole was an undergraduate student in the course and participated in an individual interview. Nicole originally defined a circle on the questionnaire as a "figure of equidistant points from another point" and stated that she had written this as a "Euclidean definition," and continued to say that when she thought of a circle in Taxicab geometry, "the figure won't be the same… visually it won't be the same, but I do think the definition it would be the same, because it has to be equidistant to be a circle." Further, on the questionnaire when asked if the provided definition of a circle held in both geometries, Nicole said "the points <u>must</u> be equally distant away from another point to be considered a circle," as seen in Figure 4.33. Note the word "must" is underlined in her explanation, implying she wanted to emphasize that this was important regarding the definition of a circle.


3. Is the following definition true in both geometries? Explain.
"The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."

0

True, the points must be equally distant away from another point to be considered a circle, so must be true.

Figure 4.33 Nicole's written response to whether the provided definition of circle held in both geometries.

In all of these statements, Nicole clearly showed that her personal concept definition of a circle was coherent. She was aware that the definition would hold across geometries, but with this comes a change in the shape of a circle Taxicab geometry. Later, Nicole was discussing how she constructed the circles in both Euclidean and Taxicab geometry, seen in Figure 4.32. She drew the circle in Euclidean geometry and in Taxicab geometry drew a square that was oriented incorrectly. Nicole's explanations of her illustrations are provided below Figure 4.32.

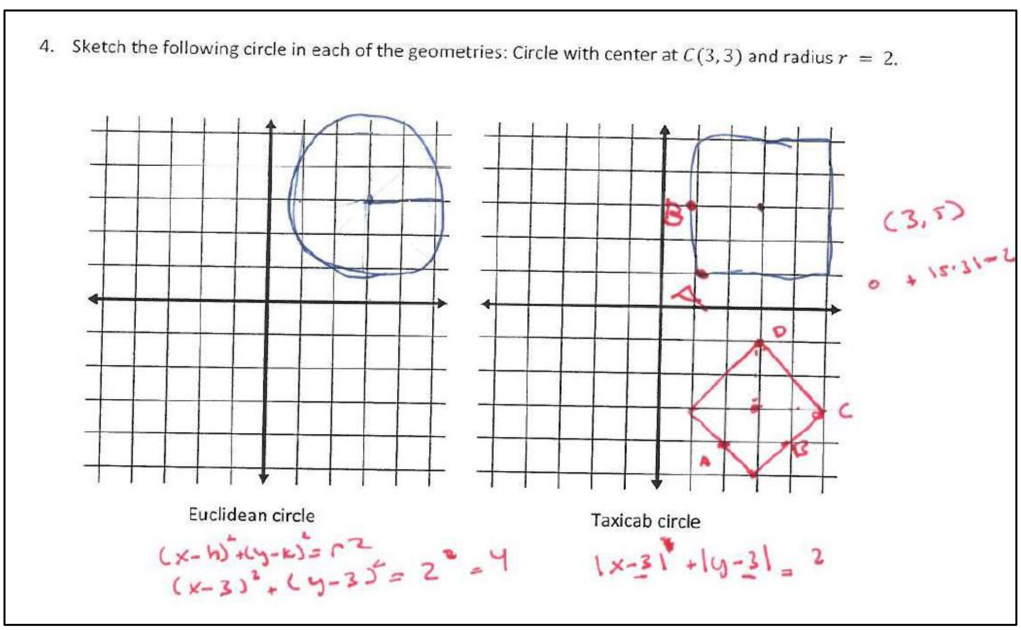


Figure 4.32 Nicole's illustrations of a Euclidean and Taxicab circle.

Nicole: For my Euclidean circle I made my point and you can see my lines, I was trying to get a... to where these two [indiscernible] and try to make it equidistant from that

> point (3,3). However, when I went in Taxicab, I wasn't able to get such a curved line, so I had to follow the ways... and I keep wanting to say routes, I don't know why…the ways that you're able to move in Taxicab. So, my Taxicab circle was way different from my Euclidean because I had to follow strict routes, making my radius…

With prompting, Nicole was able to recall that a circle in Taxicab geometry looked more like a "diamond" and was able to draw this circle (seen in red ink in Figure 4.32). In general, this excerpt shows that Nicole was able to coherently explain how she constructed her circle in Euclidean geometry. Further, she indicated how she was trying to transfer her understanding of how to do this to Taxicab geometry, by using Taxicab distance to construct radii for this circle ("follow strict routes, making my radius"). By saying "my Taxicab circle was way different from my Euclidean because I had to follow strict routes, making my radius," this provided clear evidence that Nicole did not just see that the shapes were different, but understood that the way distance was measured led to this difference.

Although she did not draw the circle correctly initially, with little prompting during the interview, she demonstrated she understood why the diamond shape satisfied the definition of a circle geometrically. This is evidence that Nicole had formed an underlying structure to the construction of a circle in Euclidean geometry that was based off of her personal concept definition and was transferring this understanding to Taxicab geometry. She explained later in the interview why she had been confused about the orientation of the circle in Taxicab geometry saying, "I was thinking from each point it had to be on a strict grid, and not go in between… but I can go in between." In other words, Nicole thought that it "wasn't allowed" to draw lines such that they crossed the "grid" at places other than the intersections formed by the integers on each axis.

This misconception, similar to Mark's and Darryl's (as discussed in Section 4.2.4.3), hindered Nicole from successfully constructing her circle in Taxicab geometry initially, although she indicated she was trying to construct radii in order to sketch her circle. It is most likely the case that Nicole got confused because she thought she could not draw the circle in this way and then relied on memory of the shape of the circle to finish her drawing. Nicole's explanations differed from Darryl's and Mark's in a meaningful way. Recall Darryl explained how he would use the concept of **Radius** to construct a circle when describing his personal concept definition but did not talk about this at all when describing how he actually constructed his circle in Taxicab geometry. On the other hand, Nicole specifically referred to the construction of radii in relation to her geometric representation and how she tried to construct it. Further, when Nicole drew the correct circle, she was able to use the definition of a circle to verify this was correct.

Although Nicole did not construct this Taxicab circle correctly at first, she explained that she was constructing her circle in the same way she did in Euclidean by constructing radii. This implies she was transferring her understanding of the definition of a circle in **GRC** from Euclidean to Taxicab geometry. When she arrived at a situation that did not adhere to her understanding of Taxicab distance, she resorted to memory. As described by Çetịn (2009), there was a conflict within portions of her concept image, perhaps between her **GRTD** and other components in her **GRTC**. Still, Nicole showed she could talk about how she was using radii to construct both of her circles, which provided evidence that she had made meaningful connections within her **GRC** across her *cEg-cTg schemata* other than observing local properties of the appearances or equations of a circle.

With prompting, Nicole was able to piece together the equation for a circle in Euclidean geometry. She was unable to explain how any of these pieces related to the definition of a circle and indicated she was trying to recall a memorized equation. When the interviewer asked her what the equation for a circle in Taxicab geometry would look like, Nicole responded "I'm wanting to use… absolute values… simply because we use absolute values for the distance? But that could be wrong." Like Alicia, Felix, and Darryl, she identified the formula for distance was used in the equation for a circle in Euclidean geometry and wanted to copy this pattern for her equation of a circle in Taxicab geometry. This was further shown when she wrote the equation for the circle in Taxicab geometry (seen in Figure 4.32) and explained that she decided not to square the radius value of 2 because nothing was squared in the formula for Taxicab distance.

In terms of her **ARC**, Nicole did not provide evidence that she had made meaningful connections between the definition of a circle and the equation for a circle and was observing local properties about these equations. However, it was clear that she was making meaningful connections in her **GRC** across her *cEg* and *cTg schemata* by explaining the construction of both of her circles and why the geometric representations would visually appear different as result of the definition of a circle. This implies Nicole was operating at an inter-inter level of schema interaction.

4.2.5.3 Brianna

Brianna was a graduate student in the course who participated in an interview with Amy, another graduate student. The results of Brianna's work begins by the presentation of her written response (shown in Figure 4.35) followed by her verbal explanation of why the definition of a circle provided in the questionnaire held in both geometries. Brianna believed this definition was true in both geometries and said "but the distances are different, like the formulas…"

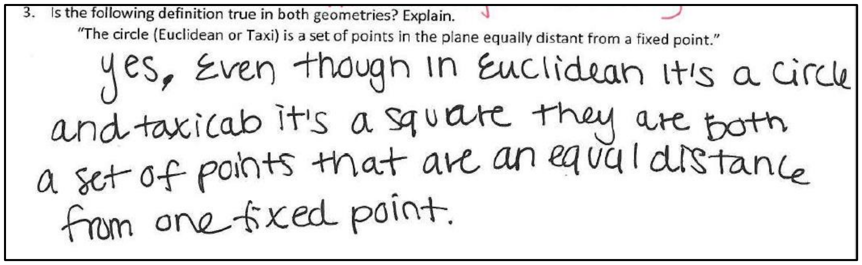

3. Is the following definition true in both geometries? Explain.
"The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."
yes, Even though in Euclidean it's a circle and taxicab it's a square they are both a set of points that are an equal distance from one fixed point.

Figure 4.35 Brianna's written response to whether the provided definition of a circle held in both geometries.

Here Brianna verbally brought up the algebraic representation of distance as a distinguishing factor of a circle between these geometries. Seen in Figure 4.35, she also described the differences in geometric representation between geometries by observing local properties, such as shape. She also added the comment about the different formulas for distance, which indicates she was using her evoked **ARD** within her **ARC** to justify why the shape would be different, which is a part of her **GRC**. Her explanations to this question illustrate that she had made some connections between the algebraic representation and geometric representations of these circles. Brianna went on to explain how she drew her circles in Euclidean and Taxicab geometry, which can be seen in Figure 4.34. A descriptive portion of the conversation between her and the interviewer during this part of the interview is provided below.

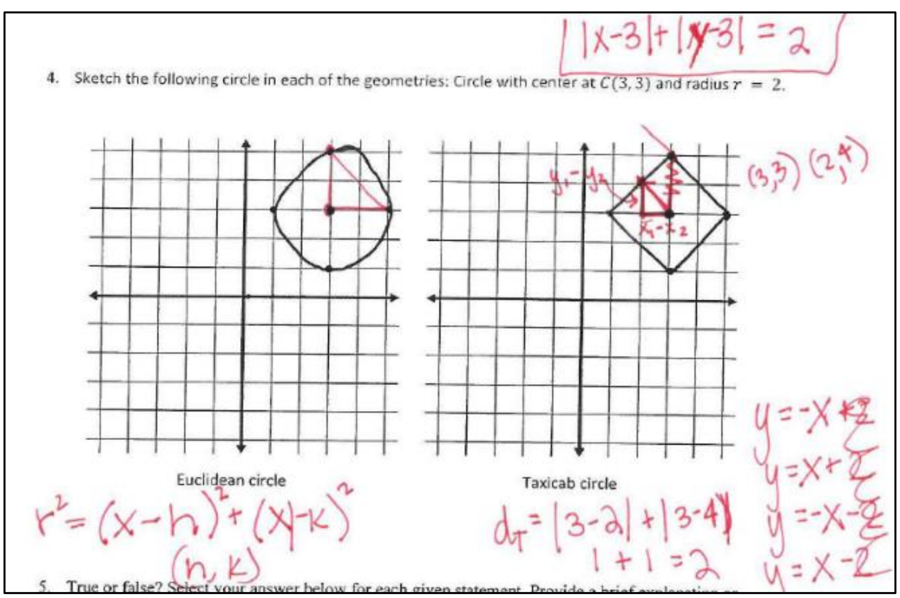


Figure 4.34 Brianna's illustrations of a Euclidean and Taxicab circle.

Brianna: I first placed the point um (3,3) so that's the center point… and then I found you know of radius 2, four points, made a circle for Euclidean. And then the Taxicab geometry though… or circle… um... I found the radius, but there it's a square. And that's how I connected them.

Interviewer: So how did you use a definition of a circle in each of these two drawings?

Brianna: Well, since it's centered at (3,3), you know that's the fixed point, and then since we have the radius 2, you know that means um, uh all the points that lie on the circle, since they are the same distance from the fixed point here… that I could just go over two to find points to help me make the circle. Even though all the points that uh lie on here are too... I just did the easy ones.

Brianna clearly explained here that she found the four points directly horizontally and vertically that were two units away (seen as plotted points in Figure 4.34) and then drew the rest of her circle in each geometry with the understanding that all of those points were also two units away from the center. This, again, shows that her **GRC** was fairly developed since she could explain this idea coherently and in general terms. Further, after this excerpt, the interviewer asked Brianna to show on her illustration in Taxicab geometry that one of the other points (other than the vertices) on her circle was also two units away from the center. Without prompting, Brianna used the formula for distance in Taxicab geometry to calculate the distance between a point on her circle (2,4) and the center (3,3), seen in red ink below her Taxicab circle in Figure 4.34. This indicates that while she was evoking concepts within her **GRC** (since she was looking at her drawing), she also evoked her **ARD** to verify this point was on her circle instead of her **GRD** to do so (by counting blocks).

Without prompting right after this, she explained and labeled what part of the Taxicab distance formula corresponded to each part of a radius she constructed between (2,4) and (3,3). In doing so, she labeled parts of a radius in small red ink on her illustration, writing "$x_1 - x_2$" and "$y_1 - y_2$." This is further evidence that she was coordinating her **GRD** and **ARD** with her evoked **Radius** concept within her *circle schema.* In other words, Brianna was demonstrating that she had formed a coherent structure for the geometric representation of a circle (**GRC**) in both geometries while also evoking concepts from her **ARC**. As Akarsu & Yilmaz (2015) suggest, the reflection and interpretation of diagrams in Brianna's mind helped her to formulate connections and define her knowledge.

When asked to write the equations for each of these circles, Brianna tried to recall a memorized equation for Euclidean geometry and even said "I should remember this." At this point, Amy stepped in and began to explain these equations to Brianna, which was when Brianna wrote these equations seen in Figure 4.34. Although Brianna had made coherent observations about the geometric representation of a circle in both geometries, she had only been able to coordinate some of her processes (like **Radius**) across her **ARC** and **GRC** in both geometries. Thus, Brianna shows evidence of operating at an inter-inter level of schema interaction.

4.2.5.4 Robin

Robin was a graduate student in the class who participated in an interview with Marianne and Parker, two other graduate students in the course. Robin's personal concept definition of circle that he wrote on the questionnaire prior to the interview was "an infinite set of points equidistant from a central point." When he was asked during the interview to elaborate on this, he said, "I really don't know how to explain my thinking behind what a circle is…it just kinda makes sense to me that it's a bunch of points around one point in the middle." Further evidence of his understanding of the definition of a circle is exhibited in his explanation of why he said the definition of a circle provided on the questionnaire would hold in both Euclidean and Taxicab geometry. This response on the questionnaire prior to the interview can be seen in Figure 4.36.

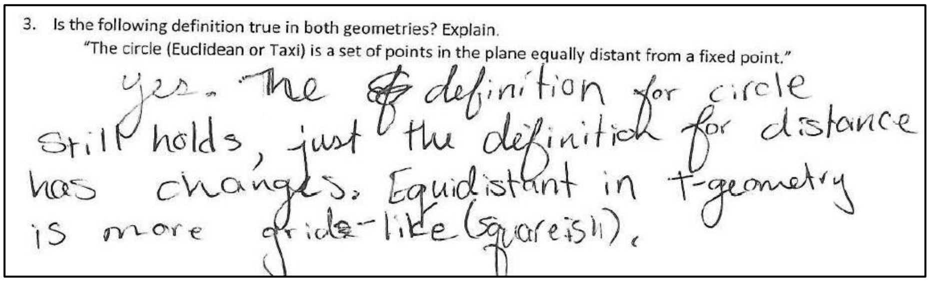

3. Is the following definition true in both geometries? Explain.
"The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."
yes. The definition for circle still holds, just the definition for distance has changes. Equidistant in t-geometry is more grid-like (squareish).

Figure 4.36 Robin's written response to whether the provided definition of a circle held in both geometries.

Robin specified that the definition of distance had changed from Euclidean to Taxicab geometry, and then wrote "equidistant in t-geometry is more grid-like (squarish)." Here he was evoking his **GRC** since he mentioned the shape of a circle in Taxicab geometry and inferred that this shape is different because of the way distance looks. In other words, this is evidence that Robin had made a connection between the definition of a circle (by bringing up equidistance) and how the choice of metric affects the shape of a circle based on this definition. Thus, he provided evidence he had generalized the construction of a circle (**GRC**) across his *cEg* and *cTg schemata*, which was supported further by his illustrations shown in Figure 4.37. Although he did not have a chance to explain how he constructed these circles verbally during the interview, the detail in his illustrations provided a good idea of how he did this.

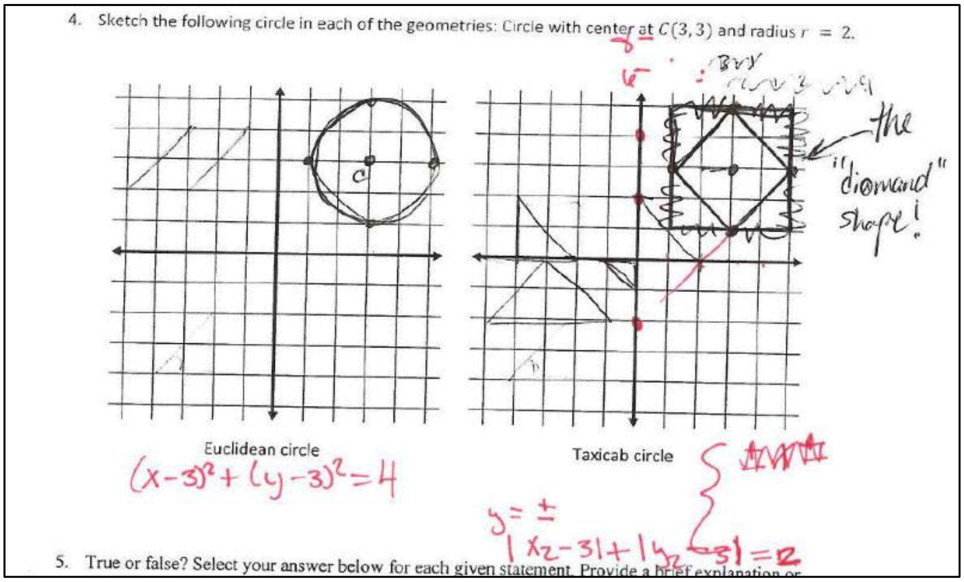


Figure 4.37 Robin's illustrations of a Euclidean and Taxicab circle.

In his drawing of the circle in Euclidean geometry, Robin plotted the four points on the vertical and horizontal that were two units away from his center point and then connected them by a curved line. It then appears he went back to check that his circle was accurately drawn, in that all of the points on his circle were exactly two units away from his center (in Euclidean distance). This is evident by the lines indicating he went over portions of his figure multiple times to refine the exact location of his curved edges. It is assumed he took this same approach in his drawing for a circle in Taxicab geometry, except that he initially drew his circle as a square with an incorrect orientation. Again, he went back to check and make sure all of the points on the figure drew were actually two units away from the center (in Taxicab distance) and discovered that he had drawn it incorrectly. He then realized the correct orientation of his Taxicab circle and noted this by writing "the "diamond" shape!"

Here Robin was clearly transferring his understanding of the construction of a circle in Euclidean geometry to the construction of a circle in Taxicab geometry and was able to fix his initial misconception of the orientation of the circle using this understanding. Thus, Robin had a coherent understanding of the construction of a circle using both of these metrics and how this was a result of the definition of a circle. Robin's discourse was different from others that drew the circle in Taxicab geometry incorrectly oriented. In particular, he drew it from memory but then used the definition of a circle to make sure this figure satisfied this definition. When he realized it did not, he adjusted the orientation. As a note, his incorrect orientation was a result of operating from memory (which he was able to correct), whereas Nicole's and Darryl's incorrect orientation was a result from a misconception about Taxicab distance.

We do not have much evidence of Robin's understanding of **ARC** since Parker, one of the other students in the interview, and the interviewer had a lengthy conversation about this topic. As such, Robin did not say too much about these equations besides a few comments that implied he was trying to recall the format of the equation of a circle in Taxicab geometry. Regardless, Robin had clearly made meaningful connections across his *cEg* and *cTg schemata* regarding his **GRC**, implying he was coordinating processes across these schemata, and was operating at the inter-inter level of schema interaction.

4.2.5.5 Marianne

Marianne was a graduate student in the course and participated in an interview with Robin and Parker, two other graduate students in the course. Marianne defined a circle on the questionnaire before the interview as "a set of points equidistance from center, x." Her written response to whether the definition of a circle provided by the questionnaire held in both geometries, which is accompanied by illustrations, is provided in Figure 4.38. Specifically, she wrote that every point on both circles would be "the same distance from the center", but that the Euclidean distance may not be equivalent to the Taxicab distance.

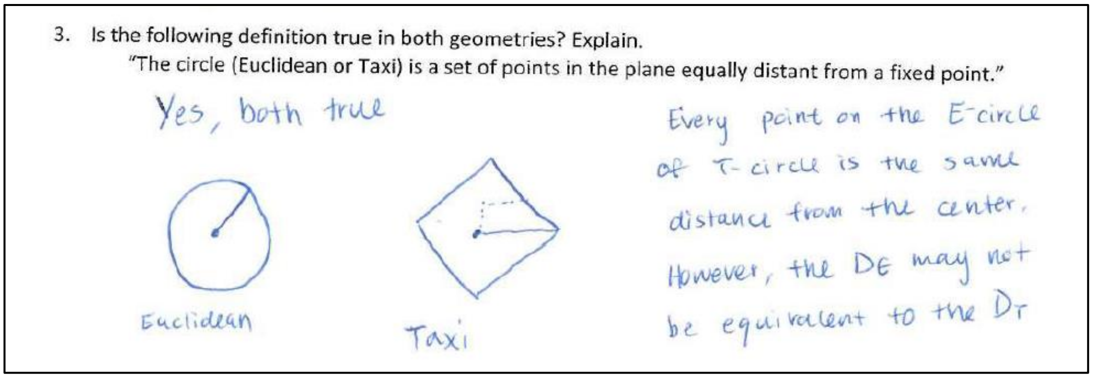


Figure 4.38 Marianne's written response to whether the provided definition of a circle held in both geometries.

As she brought up the use of Euclidean distance versus Taxicab distance, it is interpreted that she was referring to the construction of radii for each of the circles. In particular, in her illustrations in Figure 4.38, she clearly drew a radius in the Euclidean circle and drew two radii in the Taxicab circle. One reason she may have drawn two different radii in the circle in Taxicab geometry was to demonstrate that she understood the points that were not vertices of this square were the same distance away from the center. She did not explicitly state this was why she drew these two radii on her circle in Taxicab geometry, but she drew the horizontal radius in her Taxicab circle as a solid line just as she did for the radius drawn in her Euclidean circle. Thus, this indicated she may have understood these radii would be measured the same way (or at least would appear the same).

Further, the fact that the other radius on the Taxicab circle was drawn as a dotted line further implies Marianne wanted to distinguish the appearance of this radius from the other, in that it was not measured in a straight line. In any case, Marianne was clearly making a connection between the definition of a circle and how the construction of a circle using different metrics results in different shapes. In other words, she was aware that the change in metric is what is causing the figures to look different and had formed a coherent understanding of her **GRC.** This is illustrated even further in her next illustrations, shown in Figure 4.39, of a circle centered at (3,3) with a radius of 2 in both Euclidean and Taxicab geometry.

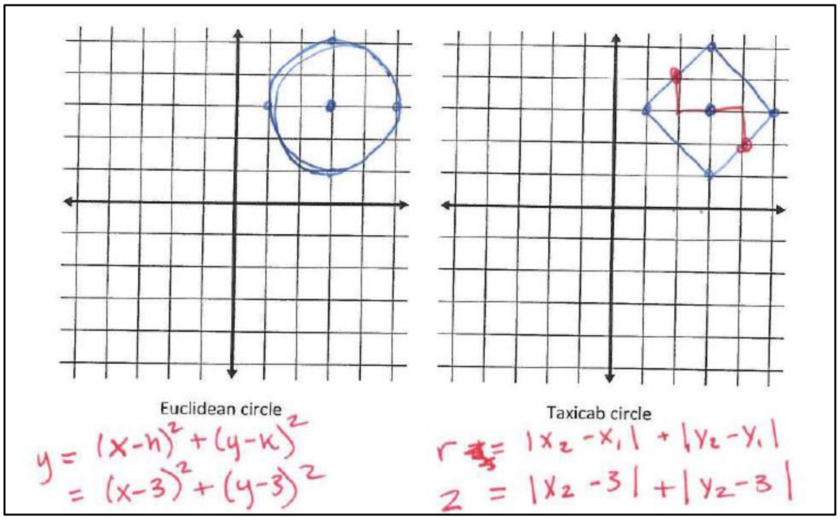


Figure 4.39 Marianne's illustrations of a Euclidean and Taxicab circle.

Marianne explained that she began constructing her circles by plotting the four points on the vertical and horizontal from the center points. She stated she knew what each circle would look like so finished the drawing in this way. When asked how she used the definition of a circle in her illustrations, she stated she knew they had to be equidistant. Further, she explained for the circle in Taxicab geometry she "went back to check that…the points that weren't directly vertical or horizontal…to see if it was still the same distance." The interviewer asked her to explain how she checked, and Marianne drew the two radii on her Taxicab circle in red ink, shown in Figure 4.39 while explaining this how she verified her circle was correct. This is further evidence that Marianne had formed a coherent understanding of the underlying structure associated with her **GRC** in both geometries within her *circle schema*.

In terms of her **ARC**, Marianne was able to reproduce the majority of the equation of a circle in Euclidean geometry but wrote "$y =$" in her equation shown in Figure 4.39 instead of something in terms of the radius. Further, once she substituted her center into the right side of this equation, she never set this expression equal to a value. Thus, Marianne demonstrated she had a limited understanding of how the definition of a circle is related to the algebraic representation of a circle in Euclidean geometry. Later, Parker, one of the other students in the interview, was talking about how she was able to write the equation for a circle in Euclidean geometry but forgot what it looked like in Taxicab geometry. The interviewer then began to prompt all three students in this interview about this equation and how it would be derived. This excerpt of the interview is included below as evidence of Marianne's understanding of **ARC** in addition to an example of an effective set of leading statements/questions that were used to get these students to reflect on their understanding of this equation. Note that during this part of the conversation, Marianne, Robin, and Parker were all attempting to write this equation at the same time, so when the interviewer says "no, no," she was referring to something one of the students had written.

Interviewer: Can you use… the definition or distance formula to write them? Again, just uh, keep in mind what's your definition of um

Robin: Oh I see what you are saying

Interviewer: Yeah what is the definition of a circle. You said it's a set of points that is equally distant from the center. And you have distance formula. And so, you have um, you have when this distance formula, applies for every point on the circle… and so what do we usually want the distance for the uh for the circles...No, no. That's the distance, right? And so you have the distance for every point. And so that distance is…what is the name of the distance?

Marianne: Oh the radius?

After this interaction, (plus some more comments made by Parker which will be presented in Section 4.2.9.3), Marianne verbally indicated that she understood how this equation was derived. Overall, Marianne appeared to have formed a coherent understanding of the underlying structure of her **GRC** (entirely coordinated her **GREC** and **GRTC**). Further, she provided evidence she was at least coordinating a process conception of **Radius** with her process conception of **Distance** since she knew was able to "name the distance" that the interviewer was describing. Thus, she exhibited she had a process conception of **ARTC**. From the data, it was determined Marianne exhibited evidence of operating at an inter-inter level of schema interaction.

4.2.5.6 Eileen

Eileen was an undergraduate student participating in an interview with Felix and Mark, two graduate students in the course. Eileen originally defined a circle on the questionnaire as "points that are an equal distant from a given point," without indicating this was an infinite number of points. Eileen wrote on the questionnaire prior to the interview that the definition provided on the questionnaire was true in both geometries. When the interviewer asked Eileen to elaborate on her response, Eileen replied with the following explanation.

Eileen: Um, I know that it's equal distance from based... 'cause when I did my picture, I knew that like um, given…the center, the radius is equal distance from each side in a circle as well as in like uh…Taxicab geometry. But it's not like uh…but it's not the same… at all points. It's only like vertical and horizontals, and the radius in Taxicab is the absolute value of the x plus the absolute value of the y and then, um, in um, Euclidean, you can use the um, Pythagorean theorem too… I didn't go through I just said they were equal.

This excerpt illustrated how Eileen differentiated a circle in Euclidean and Taxicab geometry. Specifically, she understood that a circle is defined the same way in both geometries by saying "given…the center, the radius is equal distance from each side in a circle." It is interpreted she was trying to say that given a center, each point on the circle ("each side in a circle") is a certain distance ("radius is equal distance") from that center. Notice to begin this excerpt, Eileen said, "when I did my picture," which indicates she was evoking her **GRC** and then went on to explain how the radius is calculated in both geometries ("…radius in Taxicab is the absolute value…" and "Euclidean, you can use the um, Pythagorean theorem"). In other words, she had made a connection between the algebraic and geometric representations for radius in both geometries and how this radius is used to construct a circle. Her misconception that not all of the points on a Taxicab circle are equidistant from the center will be addressed after a few more excerpts. The next passage is from conversation between Eileen and the interviewer about how she constructed each of the circles, which can be seen in Figure 4.40.

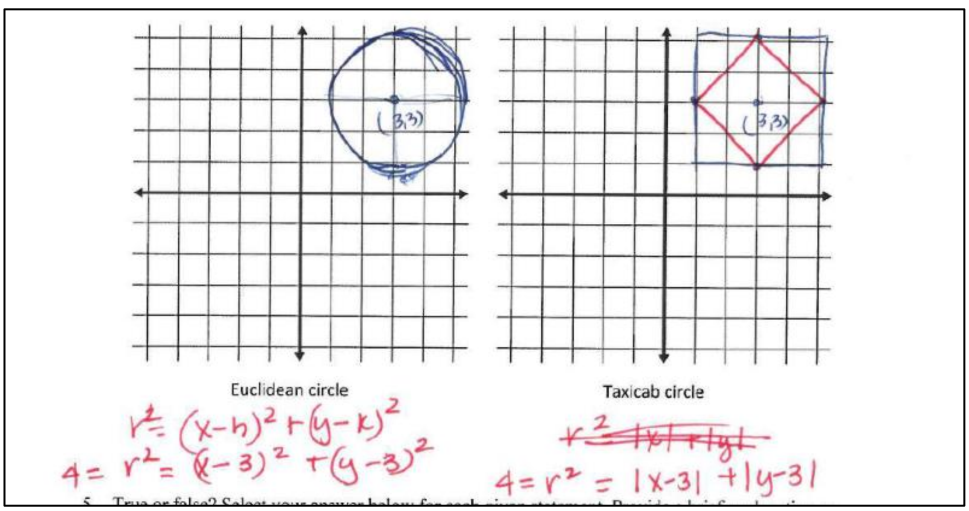


Figure 4.40 Eileen's illustrations of a Euclidean and Taxicab circle.

Eileen: Okay, with mine and I did, I found the point (3,3) and then um, with the circle well with Euclidean I…went over 2 horizontally both sides and then vertically both sides and then I drew a circle to get…my construction of [the] Euclidean circle. And then with the Taxicab circle, I did pretty much the same thing, found the point (3,3) and I went over two with each… like horizontal as well as vertical and then I um, I drew the um, square to get the um Taxicab circle.

⋮

Eileen: I did it like the where it's equal, where the radius…the circle is equidistance from the radius, so that's how I did… I didn't use a formula or anything to draw my picture, I just knew it that it would be equidistance and I know my Taxi-circle is incorrect.

At first, it seemed like Eileen only drew her illustrations from memory. In general, this is still possible, but in the second portion of this excerpt Eileen demonstrated that she knew how to construct her circles. This is evident from the fact that she could talk about how to do this, despite some ambiguity in her choice of words. In fact, based on her drawing of the circle in Euclidean geometry, it appears Eileen went back to make sure all of the points on her circle were actually two units away from the center, which could be why the lines have been drawn multiple times. In other words, she wanted to refine the locus of points of her circle that appeared to be more or less than two units away from the center.

Eileen demonstrated she understood how she should construct a circle based on the definition in Euclidean geometry but did not provide evidence that she had successfully transferred her understanding of this construction from Euclidean to Taxicab geometry. Thus, she most likely decided to rely on memory for the shape of a circle in Taxicab geometry, which is why she drew a square. However, she had apparently forgotten how the square should be orientated. This could also explain why Eileen stated earlier that the points on the vertical and horizontal from the center are the only ones in Taxicab geometry that are equidistant from the center. If she had gone back to check this drawing of a circle in Taxicab geometry to make sure all of the points were two units away from the center (as she appeared to have done in Euclidean geometry), then she would have discovered they were not. In other words, it is interpreted that Eileen demonstrated she coherently understood how the construction of a circle in Euclidean geometry is a result of the definition of a circle (**GREC**). She seemed to attempt to transfer this understanding to the square she had drawn as a circle in Taxicab geometry (**GRTC**). In doing so, she ended up confusing herself on how her illustration satisfied the definition of a circle but was not sure how to fix it. Baker et al. (2000) found similar confusion between the visual expectation a student has of a figure versus reality. In other words, this type of confusion can be a result of students who have images in their mind of what a circle should look like (a Euclidean circle), and the conditions they were given (Taxicab metric) conflicted with these images.

Regarding her algebraic representations, recall Eileen was able to specify how the radii of both circles would be calculated and talked about this while evoking her **GRC**. Further, when talking about the equations for a circle in Euclidean, she said and wrote (seen in Figure 4.40), "r…equals to x minus h squared plus y minus k squared," then corrected herself that it should be "r squared." She substituted values in to this equation saying, "h would be 3, and then k would be 3." These statements imply she was relying on memory and performing actions to write the equation of this circle. She almost derived the equation for a circle in Taxicab geometry, seen in Figure 4.40, with the only incorrect facet being that she squared the radius. Her first statement regarding this was, "I think it would be r equals to absolute value of x plus absolute value of y," then again changed her mind and decided it should be "r squared." After substituting the center in to the appropriate places in this equation, she said, "right, so I would say x minus 3 absolute value plus y minus 3 absolute value…equals r squared." There was no further evidence of her understanding of **ARC**, but Eileen clearly made a connection between the geometric and algebraic representations of a circle in both geometries in regard to **Radius** and **Distance**. In particular, she discussed how to construct a circle using a radius and how to calculate the length of this radius in each metric. Thus, Eileen demonstrated she was operating at the inter-inter level of schema interaction.

Overall, the students who exhibited evidence of operating at the inter-inter level of schema interaction identified that the choice of metric, or the way distance is defined, affected either their geometric or algebraic representations of a circle. Although these students could not coherently explain how both representations were results of the definition of a circle, many of the students demonstrated they could do so geometrically. Thus, it is possible that further reflection on the structure of the equation of a circle using a metric and how this relates to the definition of a circle would benefit these students in the development of their *circle schema.*

4.2.6 Inter-cEg, Trans-cTg

A student at the inter-*cEg* and trans-*cTg* (inter-trans) level coherently understands how the construction of a circle and the equation for a circle in Taxicab geometry is a direct result of the definition of circle. At the same time, in Euclidean geometry, the student has made non-local connections between the geometric and algebraic representations of a circle. In other words, the student has successfully coordinated all of the concepts within his or her **ARTC** and **GRTC** and some of the concepts within his or her **AREC** and **GREC**. At this point, the student can coordinate some of their **Radius, Center,** and **Locus of points** processes across their *cEg* and *cTg schemata*, resulting in the interaction of schemata. In general, the student begins to transfer knowledge from Taxicab geometry back to Euclidean geometry to try and make the same coherent understanding of a circle that they have in Taxicab geometry. There were no students that exhibited evidence of operating at the inter-trans level of schema interaction.

4.2.7 Trans-cEg, Intra-cTg

A student at the trans-*cEg*, intra-*cTg* (trans-intra) level coherently understands how the construction of a circle and the structure of the equation of a circle in Euclidean geometry is a direct result of the definition of a circle. However, they cannot transfer this knowledge to Taxicab geometry to begin to make any connections between representations. In other words, the student has successfully coordinated all processes across his or her **GREC** and **AREC.** If the student has a process conception of any concept within his or her *cTg,* he or she cannot coordinate them with the corresponding process in his or her *cEg.* The student cannot make connections other than local properties between circles in Euclidean and Taxicab geometry. There were no students that exhibited evidence of operating at a trans-intra level of schema interaction.

4.2.8 Trans-cEg, Inter-cTg

A student operating at the trans-*cEg* and inter-*cTg* (trans-inter) level of schema interaction is able to understand in Euclidean geometry, the construction of a circle and the structure of the equation of a circle is a result of the definition of a circle. The student is able to transfer this knowledge from Euclidean geometry to Taxicab geometry in order to begin making similar connections between the geometric/algebraic representations of a circle in Taxicab geometry but has not been able to generalize this for Taxicab geometry the same way that they have in Euclidean geometry. In other words, the student has successfully coordinated all of the concepts within his or her **AREC** and **GREC** and some of the concepts within his or her **ARTC** and **GRTC**. At this point, the student can coordinate some of their **Radius, Center,** and **Locus of points** processes across their *cEg* and *cTg schemata,* resulting in the interaction of schemata. In general, the student begins to transfer knowledge from Euclidean geometry to Taxicab geometry to try and make the same coherent understanding of a circle that they have in Euclidean geometry.

4.2.9 Trans-cEg, Trans-cTg

A student at the trans-*cEg*, trans-*cTg* (trans-trans) level when the student has generalized the concept of **Circle** completely and has a coherent understanding of how the construction of a circle and the equation of a circle for any metric is the direct result of the definition of a circle. That is, the student has generalized the structure of the equation for a circle geometrically and algebraically. In terms of APOS Theory, the student has successfully constructed new processes of **GRC** and **ARC** by coordinating all of their **Distance, Radius, Center,** and **Locus of points** processes. There is a complete coordination of the student's *cEg* and *cTg schemata,* and the student would be able to evoke any necessary components of these schemata to coherently talk about a circle in both geometries. There were three students, Russell, Amy, and Parker who exhibited evidence of operating at the trans-trans level of schema interaction, of which Russell was the only undergraduate student.

4.2.9.1 Russell

Russell was an undergraduate student enrolled in the course who participated in an interview by himself. Russell seemed to have a coherent understanding of most of the concepts involved with his *circle schema,* although throughout his interview he hesitated to be open about his thinking. However, from his written work and explanations, Russell provided sufficient evidence he had begun operating at the trans-trans level of schema interaction by the end of the interview.

Russell originally defined a circle on the questionnaire prior to the interview as "a center with equidistance [sic] points," and specified during the interview that he drew a "circle in

Euclidean geometry" to accompany this definition. Russell was asked if his definition held in both geometries and the following conversation took place, which provided insight to Russell's thought process in determining what classifies a figure as a circle.

Russell: I know that a circle in Taxicab is square, but you know like those diagonal right? Uh, I'm not sure if the diagonal from the center to the point that is diagonal...

Interviewer: Okay, so how would you know… how would you check that the… it is a center

Russell: You calculate the distance of those points around?

Interviewer: Uh huh…

Russell: Okay, then it probably would be the same as Taxicab.

In this excerpt, Russell was clearly expressing his lack of comfortability with assuming what he has learned is a circle in Taxicab geometry satisfies the definition of a circle, implying he was thinking critically about the definition of a circle. With prompting, he was able to verbally express that if he followed the same logic that he did in Euclidean geometry, then the figure he had learned is a circle in Taxicab geometry would "probably" satisfy this definition. Thus, Russell was making connections between the definition of a circle in both Euclidean and Taxicab geometries. Russell went on to describe that he constructed his circle in Euclidean geometry (seen in Figure 4.41) by plotting the four points directly vertical and horizontal from the center that were two units away because "the r is 2,"and connected these points "by drawing the curve." For his circle in Taxicab geometry, Russell explained that "I pin point the point where…with the radius equal to 2, the same way, right?... but I draw a straight line to connect them." This statement demonstrated that Russell had transferred knowledge from Euclidean geometry to Taxicab geometry by saying "the same way, right?" Also, he specified that he was counting two units because the two units represented the radius of the circle.

The interviewer asked Russell how he would explain that his circle in Taxicab geometry represented the definition of a circle. She specifically asked Russell to pick another point on his circle that was not one of the vertices of the square, and to “show me how that definition applies.” Russell immediately said “we can take this point, which is (2,4), x is 2, y is 4. And the center is given, (3,3), so the difference in x, 3 minus 2, plus the difference in y, 3 minus 4. And that would give us [2].” While he was saying this out loud, he was writing this out in the right margin of his paper, shown in red ink in Figure 4.41.

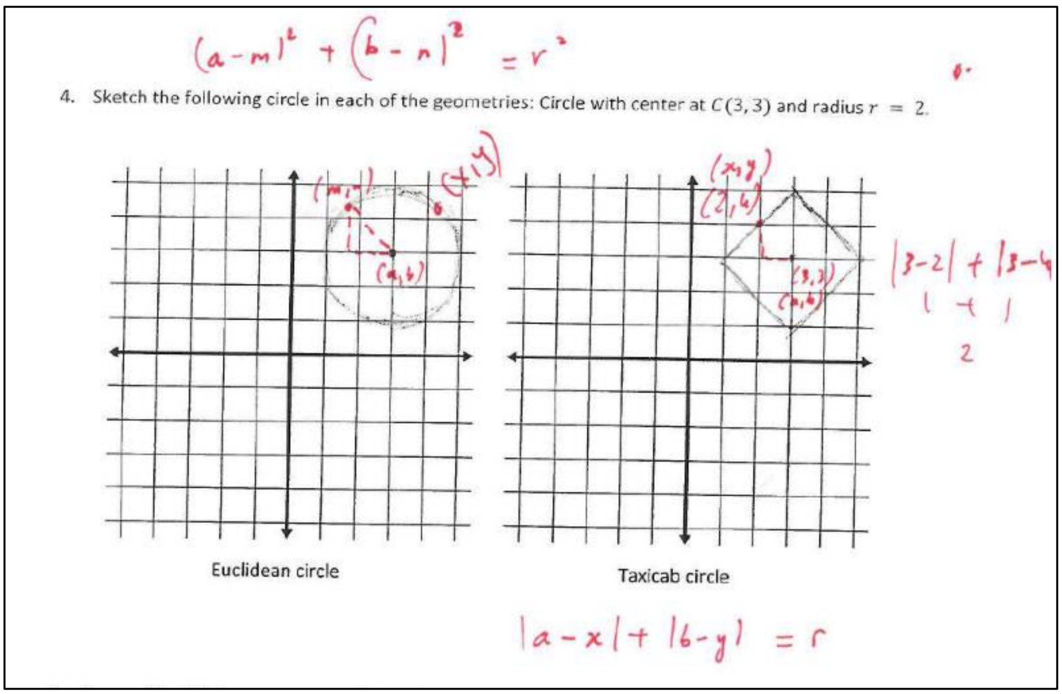


Figure 4.41 Russell’s illustrations of a Euclidean and Taxicab circle.

The interviewer asked Russell if he could illustrate this on his graph, which is when he drew the red dotted lines that form a radius of the circle connecting the center and the point (2,4). Thus, Russell knew how to justify geometrically and algebraically that an arbitrary point was on his circle. This conversation, along with analysis of the rest of the interview implies Russell had formed a coherent understanding of his **GRC** and understood how the construction of a circle is a result of the definition of a circle.

Russell and the interviewer began discussing the equations for a circle in both geometries, which he eventually wrote at the top and bottom of his work, shown in Figure 4.41. Before he wrote these equations, he had told the interviewer he would not be able to write the equation for either of these circles. The interviewer was able to push Russell to try anyway, which is when he had a "lightbulb" go off about the equation for a circle in Euclidean geometry. The passage in which this happened is provided below.

Interviewer: So... that definition or formulas for distance would not be helpful?

Russell: Uh the distance for Euclidean… or the distance… Pythagorean and I would illustrate the… may I?

Interviewer: Yeah, try.

Russell: [Drawing]

Interviewer: So if you pick any point on the circle and label that x and y.

Russell: Oh, I get it! Oh so that would equal to r, and r would be just equal to…

Interviewer: Okay so… now to put together.

Russell: Okay, so the first point, the center, say a and b. the second point, a point on the circle, say m and n. and that's r, the distance is r is the distance between center and that point. So, [smiling].

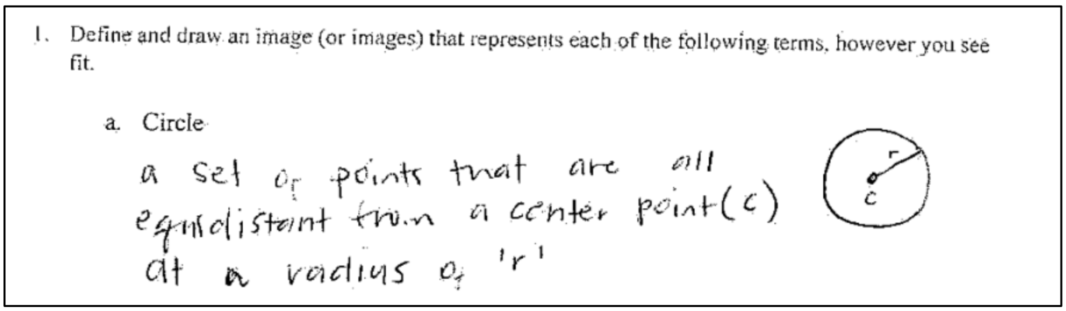
Here, it is clear that Russell has coordinated all of his processes within his *cEg* since he took the algebraic representation for a circle in Euclidean geometry and was able to graphically represent where each portion of this equation was represented on his illustration (see Figure 4.41). Further, he was able to do this with all general variables and explanations. In other words, he was able to coordinate his **Euclidean Distance, Radius, Center,** and **Locus of points** processes across his **GREC** and **AREC** in order to form a coherent structure to the equation of a circle. As such, he was able to verbally express, coherently and in his own words, that the equation is defining all points that are the measure of the radius away from the center in Euclidean distance as a result of the definition of a circle. Further, when asked if he could write an equation of a circle in Taxicab geometry, Russell was able to take this structure from Euclidean geometry and was successful in writing the equation for a general circle in Taxicab geometry, although he was never probed to explain his thought process in doing so.

Russell was able to transfer his knowledge from Euclidean geometry to Taxicab geometry to make similar connections among the concepts that he observed in Euclidean geometry. In particular, he used the definition of a circle to explain the construction of both circles and the derivation of the equation of the circle in Euclidean geometry. Further, he appeared to have transferred all of these connections to Taxicab geometry to write the equation of this circle. Thus, he was able to accommodate his *circle schema* during the interview (possibly pin pointed to when he smiled during his explanation), to fully assimilate his *cTg schema* into his existing *circle schema*. Thus, Russell provided evidence he was operating at a trans-trans level of schema interaction by the end of the interview.

The interviewer proceeded to ask him if he would be able to sketch a circle using a new metric if it were to be defined for him. Russell quickly replied that he wouldn't know what to do. This provided evidence that Russell would need to reflect further on his *circle schema* in order to generalize his understanding and be able to transfer this knowledge to a new metric. Thus, he had not thematized this schema in order to construct a circle and derive the equation of a circle with another metric.

4.2.9.2 Amy

Amy was a graduate student who participated in the interview with another graduate student, Brianna. For her personal concept definition of circle written on the questionnaire, she addressed multiple parts of the definition of a circle carefully and related them to a graphical representation of a Euclidean circle, as seen in Figure 4.42.


1. Define and draw an image (or images) that represents each of the following terms, however you see fit.

a. Circle

a set of points that are all equidistant from a center point (c) at a radius of 'r'

Figure 4.42 Amy's written definition of a circle on the questionnaire prior to the interview.

Before the interviewer could ask Amy about the construction of her circles, Brianna, the other student in the interview, had been asked about the equation of a circle in Euclidean geometry. While Brianna was trying to recall the format of this equation, Amy stepped in and asked to explain. Figure 4.43 shows Amy's illustrations and equations of the given circle in both geometries and excerpts from Amy's explanations from the interview during this interaction are provided. It is noted that in the equation she wrote for her circle in Euclidean geometry seen in Figure 4.43, she wrote the value of the radius as 3, but this seemed to be a simple mistake and the analysis of her work disregards this error. In addition, during the course, the students did an activity where they found the linear equations of the four edges of a Taxicab circle, which was most likely the reason Amy wrote these equations next to each edge of her circle. The next excerpt illustrated Amy's thought process in deriving a different form of the equation of her Taxicab circle (instead of the linear equations of the edges). In this passage Amy was verbally explaining in detail how the equation for a circle in Euclidean geometry is derived and used the same logic to derive the equation for a circle in Taxicab geometry.

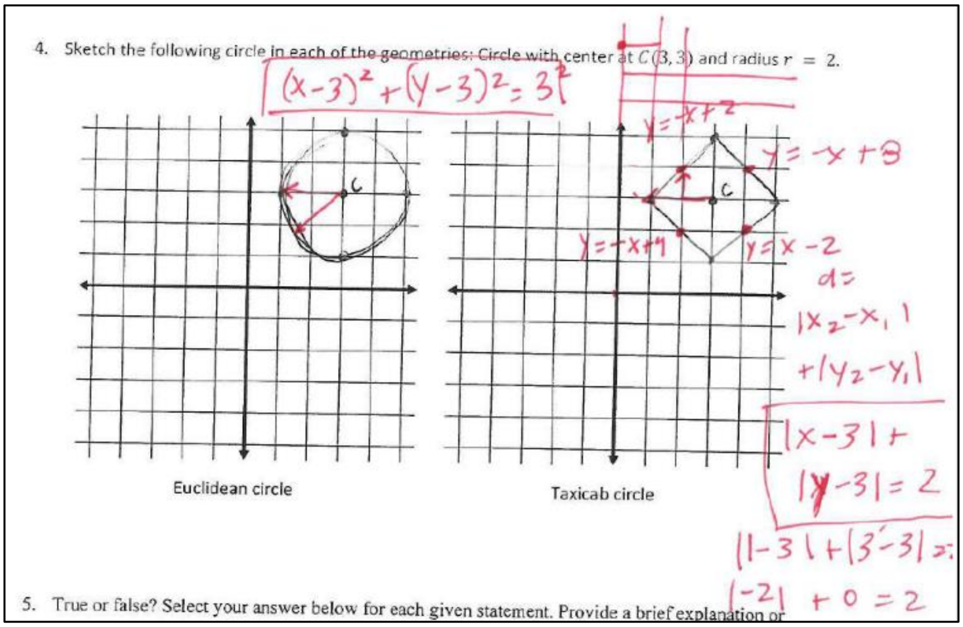


Figure 4.43 Amy's illustrations of a Euclidean and Taxicab circle.

Amy: Oh, okay. Cause I think that the formula for the circle is the same… I didn't notice that… yeah… so it would be x minus the h and then squared plus y minus k squared, and that equals… usually that would equal the distance… if you're using the distance formula, but that equals r squared… that is the distance between… the center and any point…

Interviewer: Exactly. And so what would be for Taxicab?

⋮

Amy: So that? I guess that relates to the distance as well? The distance formula… right okay… yeah… so x minus 3, plus [y] minus 3 equals… equals… umm… so equals the radius which is 2? This is y… so like if we had… if we plugged in one of these points it should equal 2? Yeah… okay. Yeah. So that's the formula. Final answer ha.

Coupled with Amy's personal concept definition provided in the first question, her explanation of the derivation of the equation of a circle in Euclidean geometry indicated that she had formed a coherent understanding of both **AREC** and **GREC**. This is made evident by the connections she was making between various concepts such as the use of the distance formula in this equation and the radius of a circle. Her statement about the equation for a circle in Taxicab geometry, "I guess that relates to the distance as well?" was an indication that she was transferring knowledge from her *cEg schema* to her *cTg schema.* She then explained articulately in Taxicab geometry that the distance formula (with her center substituted in) should be equal to the value of the radius. Thus, she demonstrated she had successfully transferred her understanding of a circle from Euclidean geometry to Taxicab geometry, which indicates a coherent understanding of both her **GRC** and **ARC**. This was also made evident through her comment that "if we plugged in one of these points it should equal 2." This statement indicates she understood any of the points she had drawn were on the circle by definition and could anticipate this was the truth through reasoning.

Although Amy never had a chance to explain how she constructed her circles, in Figure 4.43 it can be seen that during the interview, she drew multiple radii on each of her drawings. By the analysis of her conceptions of various components within her *circle schema* throughout the rest of the interview, Amy provided more evidence she had constructed a coherent understanding of her **GRC** and the underlying structure that connects this to her **ARC.** Thus, Amy demonstrated she was operating at a trans-trans level. Later, the interviewer asked Amy if she were given a new metric if she would be able to sketch a circle using this new metric, and Amy's response was "I guess uh yeah… applying the same logic…" Neither the questionnaire nor the protocol probed to see if she would actually be able to do this or derive the equation for a circle using another metric.

However, the fact that Amy thought she could by "applying the same logic" is evidence that Amy may have been ready to thematize her *circle schema.*

4.2.9.3 Parker

Parker was a graduate student who participated in an interview with Robin and Marianne, who were also graduate students. Parker's definition of a circle written on the questionnaire was the "set of all points equidistant from a given point known as the center." On the part of the questionnaire that asked if the definition of a circle provided held in both Euclidean and Taxicab geometry, Parker responded "Yes, the given distance just looks different in reference from the center." This provided clear evidence that while evoking her **GRC** and saying, "distance just looks different," Parker was aware that the choice of metric is the distinguishing factor between the visual appearance of a circle in Euclidean and a circle in Taxicab geometry. Although she was not asked to explain how she constructed her geometric representations of the circles shown in Figure 4.44, many details can be deduced from her illustrations.

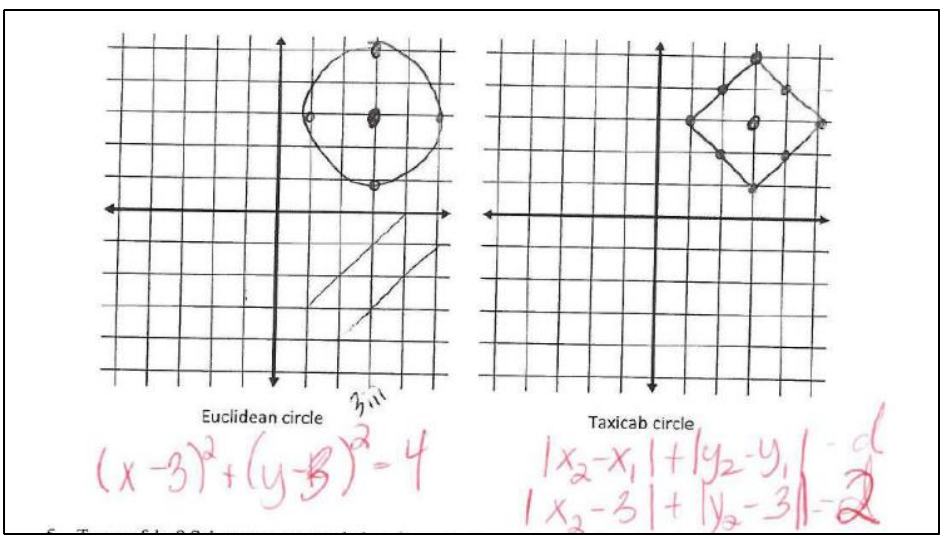


Figure 4.44 Parker's illustrations of a Euclidean and Taxicab circle.

In her circle in Euclidean geometry, it can be seen that she plotted the four points on the vertical and horizonal from the center that were two units away from the center and then most likely “connected” these points based on how she knew the circle would look. For her circle in Taxicab geometry, in addition to plotting the same four points on the vertical and horizontal from the center, she also plotted four more points that were two units away from the center. She then used these eight points to construct her circle. This is a clear indication that she used her understanding of the definition of a circle and the Taxicab metric to construct this circle, and that she had formed a coherent underlying structure for her **GRC.** Again, she did not have an opportunity to verbally explain how she constructed these circles, but her illustrations and the detailed analysis of her understanding of concepts involved in her *circle schema* indicated this coherence.

Recall in Section 4.2.5.5 when results for Marianne were presented, it was mentioned that Parker made comments about the derivation of the equation of a circle in Taxicab geometry. During the prompting from the interviewer presented at the end of Section 4.2.5.5, Parker was writing down the equation for distance under her illustration of a circle in Taxicab geometry, seen in Figure 4.44. Note that she used the variable d, and then substituted the value of 2 (the radius) for this variable, which is what indicated she understood how distance was involved in the equation of a circle. She interrupted the conversation between the interviewer and the other students and said the following.

Parker: This might be wrong… I wrote the formula for…calculating distance for Taxicab, and then I know…this has to be compared to the center which is (3,3) so… if my distance is 2, doesn’t that make it?...Is equal to 2. So let me change that…’Cause then like this point up here is [(3,5)] and if we plug in 3 and 5…3 minus 3 is 0, 5 minus 3 is two, so that would mean our distance is 2, which means it’s on the circle.

Parker implied here verbally that she knew the distance formula was related to the radius. Specifically, if the distance from the center to all of the points on this circle is the same, then this distance should be equal to the radius of the circle. She then used an arbitrary point to verify on her own that a point she knew geometrically was on the circle, would also satisfy her algebraic representation of the circle. In this moment, it appeared Parker had generalized her understanding of a circle geometrically and algebraically, and how these representations are a result of the definition of circle. In general, Parker did not talk about her equation in Euclidean geometry other than the fact that she was able to write it. However, from the detailed analysis of her conceptions of relevant concepts involved with her **AREC**, at this point in the interview she would have been able to explain how this equation was derived. For these reasons, Parker provided evidence that by the end of the interview she had formed a complete, coherent understanding of the underlying structure of the *circle schema* and was operating at the trans-trans level of schema interaction.

4.2.10 Thematization of the circle schema

As stated previously, a student has thematized his or her *circle schema* if given a new metric, they can act upon this understanding to construct a circle and also derive the equation for a circle in this "new" space. In other words, the student has abstracted the definition of a circle to all metrics, can provide geometric and algebraic representations of a circle using any given metric, and understands how these representations are related to one another. It would also be expected that the student to be able to verbally communicate all of this information coherently. In general, a student has thematized their *circle schema* if he or she demonstrates an awareness of the global definition of a circle over all metrics across all representations (geometric, algebraic, verbal). At this point, the structure of the *circle schema* is a "fundamental part of the understanding and can be viewed in totality as an object conception," (Cooley et al., 2007, p. 7).

With regard to the possible thematization of her *circle schema*, the interviewer asked Parker if she was given a different metric, if she could draw a circle using that metric and how she would approach that task. Parker responded confidently, "yeah… figure out what the metric is asking…and what it's… well we eventually get our radius, and then we can…" During this explanation of how she would approach this problem, she began moving her finger in the air like she was starting at a center point and drawing outward. It is interpreted that she was constructing multiple radii using some arbitrary metric.

It was previously mentioned in the presentation of Mark's responses in Section 4.2.1.3 that the students were given a new metric on a quiz the week prior to this and were asked to sketch a circle using this metric. On the last day of class during the semester prior to this interview, the instructor went over this problem and explained how to construct the unit circle using this metric. This is what Parker began discussing at the end of this portion of the interview. In particular, she said "I mean we did it in class. The unit circle…and the radius needs to be one, so we have to figure out from the metric…what would yield an answer of one and if we do that then we can sketch the circle." Parker demonstrated here that she was completely aware of how she would be able to construct a circle if she were given a new metric. Without explicitly stating she would construct multiply radii in order to sketch the circle, she implied this by her motions in the air with her finger. Further, she described that for a unit circle, she would use the new metric to find the points that would be one unit away from the center ("would yield an answer of one") when we calculated the distance from that point to the center using that metric. Once she had done this, she implied she would be able to sketch the circle from these points.

Although she was not presented with the opportunity to actually use a new metric in order to construct a circle (besides the week before, which was prior to when Parker most likely made these generalizations), Parker was able to imagine how she would do so. In other words, Parker gave explicit directions as to how one could use any metric to construct an arbitrary circle. She was not asked about deriving the equation for a circle using this metric but based on her generalization of the equation for an equation in Taxicab geometry, Parker may have been able to transfer this knowledge and generalization to a new metric. Without this evidence, it cannot be said for certain that Parker had completed the thematization of her *circle schema* but was at least in the stage of thematizing it.

None of the other students that volunteered for these interviews demonstrated such a coherent understanding of the underlying structure of the *circle schema*, however, the questionnaire did not necessarily probe for this type of understanding. In Chapter 5, these findings are summarized in regard to the research questions for this report, including common obstacles that emerged during these tasks. In addition, in Appendix B suggestions for how to help guide students towards forming a coherent understanding of the structure (and thematization) of their *circle schema* are discussed.

In this part of the results, the analysis of 15 students' *circle schema* was presented. Specifically, how each student represented a circle geometrically and algebraically and how he or she used the definition of a circle in each of these representations (and to make connections between them) was described. The main focus of this analysis was on how each student coordinated processes (if they existed) between his or her geometric and algebraic representations in each geometry and how he or she transferred knowledge from Euclidean to Taxicab geometry (and back, in some cases). Four out of the 15 students incorrectly drew their circle in Taxicab geometry, all of which were a square just oriented in the wrong manner. This does not include Robin, who initially drew his circle oriented incorrectly, but was able to adjust his drawing using reasoning prior to the interview. In other words, the four students who did not correctly draw this circle were aware that circles in Taxicab geometry are squares but relied on their memory to finish their drawing instead of using their understanding of **GRC** to adjust their sketch. It is noted that Eileen checked her illustration and knew most of the points were not actually equidistant but could not use her definition of a circle and **GRC** to fix it. Of the nine students who actually wrote an equation for the circle in Taxicab geometry on their own (not as a result of others' conversations), three of them wrote an incorrect equation after trying to copy the format of the equation of a circle in Euclidean geometry. Further, two [arguably, three] students wrote the equation correctly by copying this format but could not explain why the format would change in relation to the use of distance in the definition of a circle.

These students demonstrated a strong tendency of relying on memory and particular properties when their reasoning and logic "failed" them. As Fischbein (1993) explains, in geometrical reasoning, a major obstacle is the tendency to "neglect the definition under the pressure of figural constraints," (p. 155). In other words, many of these students somewhat disregarded their understanding of the definition of a circle once they were faced with a new metric, and relied on memory to complete tasks. This happened even in cases when his or her personal concept definition was well developed, as was the case with Darryl. At the same time, many students who demonstrated they had at least a fairly coherent understanding of their **GRC** and/or **ARC** struggled to make connections between individual components of these representations as well as make connections between these in the "big picture" that is the *circle schema.*

5 DISCUSSION AND CONCLUSIONS

As a result of the analysis of the data presented, the research questions for this dissertation were answered. In addition, some revisions have been identified to the preliminary genetic decomposition for many mental structures that exist within a student's *geometry schema*. Also, ways in which students used GSP to help generalize their understanding of various concepts were identified. Lastly, a genetic decomposition for the interaction of two schemata and how this influences students' understandings of definitions was developed. Below, details about how each of the research questions have been answered are given provided these results.

5.1 Discussion of results

5.1.1 Research question 1

The first research questions pertained to students' use of GSP and group work to better understand mathematical definitions in geometry. Specifically, in what ways do students use Geometer's Sketchpad to refine their understanding of mathematical definitions? This included the following two sub questions.

(a) How do students apply their working understanding of a definition in Geometer's sketchpad to reason about mathematical problems?

(b) How does cooperative learning and the use of Geometer's Sketchpad help students in the abstraction of definitions from Euclidean geometry to axiomatic systems in general?

In order to answer these questions, the results from the in-class group work data and some of the data from the interviews provided insight as to student opinions about the course. For Research Question 1(a), these results support the framework provided by Hollebrands (2003) in terms of analyzing students' use of dynamic geometry software using APOS Theory. In particular, if a student was using GSP to measure specific lengths or in a guess and check method in relation to a particular concept, he or she also verbally exhibited an action conception of that concept. An example of this conception was presented in Section 4.1.5 when Alex measured the lengths of each leg of both triangles in GSP using the **Taxicab distance** tool. If a student was able to anticipate a transformation of an object (in relation to a particular concept) in GSP without actually performing it, he or she also exhibited a process conception of this concept verbally. An example of this was Brianna's process conceptions of **Radius** and **Distance**, as she suggested another point on her Taxicab circle based on the reflection of a known radius and point on the circle, presented in Section 4.1.2. Finally, if a student was able to consider the properties and behaviors of a concept rather than rely on the specific image on their screen in GSP, they were also exhibiting an object conception verbally. An example of this conception was presented in Section 4.1.2 when Ally was able to consider general circles in both Euclidean and Taxicab geometry to compare their constructions without the use of the specific circle she had constructed in GSP.

For Research Question 1(b), of the five groups presented in Section 4.1, three of the groups tended to work on their own computers, and two of the groups tended to use one shared computer to explore concepts in GSP. All of the groups had discussions about their exploration, but it was found that the members of groups that worked on individual computers tended to be more engaged in activities and discussion, whereas some members of the groups that worked on one computer were distracted or disengaged for portions of class. In terms of cooperative learning, this may imply that having students be responsible for their own work for submission may encourage more active discussion and cooperative learning. Although students would be individually working on their own computer, they may be more likely to remain on task and participate in discussion or ask questions for things they may not understand about the task at hand which may result in a deeper understanding of the material. This supports results found by Abdullah et al. (2015) (and constructivism, in general) in the sense that “any knowledge of an individual is the result of activities undertaken by the individual,” (p.110).

In general, it was found that by conjecturing about the behaviors of objects in GSP and listening to group members’ thoughts about these conjectures, the participants in this study were able to generalize their understanding of concepts such as **Circle** and **Perpendicular bisector**. In addition, students were able to make observations about similarities and differences between objects and relationships in Euclidean and Taxicab geometry. As evident by the data provided in Table 4.1 in Section 4.1.6, students appeared to have had a positive experience with the design of the course and the ways in which GSP and group work contributed to their understanding of the course material. Thus, for Research Question 1 it is determined that students use GSP in order to observe patterns, make conjectures about these patterns and relationships, and test these conjectures with the software. They can then use results of this “testing” to either justify why their conjecture is correct or why it is incorrect and how they can adjust their understanding to compensate for this. As an example, in Section 4.1.3, when Robin, Nicole, and Kristen were exploring the concept of **Perpendicular bisector** in Taxicab geometry, they conjectured that when the slope of the segment connecting two points was equal to 1 that this segment’s perpendicular bisector in Taxicab geometry would intersect the segment at a right angle, resulting in the same line as the Euclidean perpendicular bisector of this segment. Instead of just observing this phenomenon on their screen, Nicole wanted to use GSP to mark and measure this angle to be sure they had made a correct observation. Another example was presented in Section 4.1.4 when Parker, Darryl, and Russell were exploring the triangle inequality in Taxicab geometry, and they made several measurements in GSP to compare distances and try to find patterns that existed when these distances had a certain relationship with one another.

Overall, students appeared to have benefitted from and found significance in using GSP in the classroom to explore concepts, even some students who did not express this. For example, Samantha was one of the students who chose "Neither Agree nor Disagree" for the question that asked if the course helped with her comfortability in using technology in mathematics. Throughout the interview she continued to say how she wished she had GSP in order to answer some of the questions on the questionnaire, implying she found GSP to be useful for exploration. Based on the analysis of the videos of in class group work, there also appeared to be a positive relationship between the level in which a student engaged with GSP and the level of understanding of concepts and material in the class. Further, Felix pointed out his appreciation for the group work and cooperative learning in the classroom, which supports Cavanagh (2011), who found that students greatly valued these opportunities to improve their understanding and to remain engaged and focused in the course. Felix's comment about group work is also consistent with Zakaria (2010) in how the use of cooperative learning in the classroom affected his attitude towards math. His comment was among many that confirmed this notion, which is also supported by the average scores for Question 6 on the survey which asked about how this course affected their comfortability with mathematics in general, which can be seen in Table 4.1.

5.1.2 Research question 2

To answer the second research question, which asked how students adapt their understanding of concepts in Euclidean geometry to apply definitions in Taxicab geometry, the results from all three methods of data analysis are used. In particular, the reader's attention is shifted to how students relate concepts in Euclidean and Taxicab geometry and how they use these observations to generalize their understanding. In general, it was found students can better adapt their understanding of a definition of a conception from Euclidean to Taxicab geometry if they can make connections between the algebraic and geometric representations for these concepts. This was not initially anticipated that this aspect of a student's understanding would be so influential to their transferal of knowledge to Taxicab geometry from Euclidean geometry. In order to better answer this overall research question, the following sub questions that are also part of the second research question are answered in Sections 5.1.2.1 and 5.1.2.2.

(a) What activities in Taxicab geometry can aide in the abstraction of a definition?

(b) How does applying definitions in an atypical context affect the development of student understanding of these definitions?

(c) How do students transfer their understanding of relationships among concepts in Euclidean geometry to Taxicab geometry?

What follows is how each method of data analysis helped to answer these sub questions.

5.1.2.1 Group work and GSP

Through the data that was collected from the group work in class, results can provide insight into these sub questions. For Question 2(a), some activities that students worked on helped them to familiarize themselves with the Taxicab metric. This also allowed students to make connections between concepts in Euclidean and Taxicab geometry in order to make generalizations about definitions. The five activities presented in this report are five examples of activities that were very productive in this regard. One example of an activity that helped students to generalize definitions was the activity in which the students were asked to construct a circle in Taxicab geometry. In this activity presented in Section 4.1.2, Ally, Brianna, and Amy were able to generalize their understanding of the definition of a circle by comparing and contrasting properties of circles in Euclidean and Taxicab geometry. In particular, this construction led Ally to see very quickly that the definition of a circle controlled the construction of a circle, which can lead to a different shape of a circle by using different metrics. She specified that she understood the way to construct the radii of circles in both geometries is the same, but this results in different appearances of these circles by the way distance is measured. Amy and Brianna were also able to generalize why this was the case.

As another example, recall that by seeing the construction of a perpendicular bisector of a segment by using circles in Taxicab geometry, Robin was able to identify that this line was indeed a perpendicular bisector and explicitly stated that he understood how this construction was related to the same construction in Euclidean geometry. It is believed that in the moment that he made this connection, he was able to generalize his understanding of the definition of a perpendicular bisector is. Thus, Robin would most likely be able to make a distinction between a definition of this object versus a description of the properties of this object, such as “the straight line that intersects the segment at a right angle,” since he was aware how this construction leads to a line that does not necessarily satisfy this description.

In general, by either having students construct figures in Taxicab geometry that mirror the construction in Euclidean geometry, students were able to generalize these definitions or properties they observed. In particular, by comparing geometrical or algebraic representations of concepts between Euclidean and Taxicab geometry, they were able to understand what aspects of a figure part of the definition of that object and what aspects were results of the definition using a different metric. Thus, as Dreiling (2012) found, by having students follow procedures based on definitions in Euclidean geometry in order to construct figures in Taxicab geometry, these students were able to make meaningful generalizations about the definition of various mathematical objects.

5.1.2.2 The development of the circle schema through schema interaction

Recall the other sub questions of the second research question: (b) How does applying definitions in an atypical context affect the development of student understanding of these definitions? and (c) How do students transfer their understanding of relationships among concepts in Euclidean geometry to Taxicab geometry? To answer these questions, I ended up having to alter my perspective by considering the interaction of schema that exists between a student's *Euclidean geometry* and *Taxicab geometry schemata.* In particular, initially the interview data was analyzed by identifying student understanding of various concepts, but I realized the relationships that existed were too complex to be described only using this idea.

Using the idea of schema interaction allowed for a deeper analysis of each student's make up, or structure, of the *circle schema*. In particular, considering each of the subconcepts within this schema (**Distance, Radius, Center,** and **Locus of points)** helped to describe these structures. There was also a need to consider each of these concepts within both Euclidean geometry and Taxicab geometry, how each student understood these concepts within these geometries both algebraically and geometrically, and how this resulted in their overall understanding of the definition of **Circle.** In other words, for each concept each student's Euclidean algebraic representation, Taxicab algebraic representation, Euclidean geometric representation, Taxicab geometric representation, the relationship between all of these aspects, and how this influenced his or her personal concept definition was considered. It was determined that a student's personal concept definition, ability to verbalize coherent statements about the definition of a circle, and how these relate to the algebraic and geometric representation depended on how developed the student's *circle schema* structure was. It is summarized below how each level of schema interaction (that there was data for) apparently resulted in how students were able to discuss the concept of **Circle** during the interviews.

By asking questions about both Euclidean and Taxicab geometry on the questionnaire as well as probing from the interviewer, students were presented with an atypical task, just as the participants in Baker et al. (2000). In this dissertation, the participants had been accustomed to talking about a circle within Euclidean geometry (for a good part of their lives), and then were asked to compare and contrast ideas in both geometries about various concepts. A breakdown of the number of students that displayed evidence of operating at each level of schema interaction within their *circle schema* is provided below in Table 5.1. The various descriptions of levels of schema interaction are also summarized in table format in Appendix A.

Table 5.1 Distribution of students operating at various levels of schema interaction.

		cTg		
		Intra-	Inter-	Trans-
cEg	Intra-	3	0	0
	Inter-	3	6	0
	Trans-	0	0	3

In particular reference to this table and Table 5.3 (which summarizes descriptions of the levels of schema interaction), the levels outlined in blue (intra-inter, intra-trans, and inter-trans) are levels of that students primarily would be transferring knowledge from Taxicab to Euclidean geometry. In other words, a student would be working to make connections in Euclidean geometry that they have observed in Taxicab geometry but are not able to fully transfer this knowledge necessary to do so. One possible explanation for none of the students exhibiting evidence of operating at a higher stage of understanding of their *cTg schema* than their *cEg schema* could be since [most of] these students have only learned concepts within Euclidean geometry for their entire lives until about two weeks prior to these interviews. In other words, it is much more likely that students would have a better understanding of the concepts and relationships within their *circle schema* as they are situated in their evoked *Euclidean geometry schema* than how they understand these concepts in their evoked *Taxicab geometry schema.* Perhaps it is because of this that there were not any students that exhibited evidence of operating at the intra-*cEg* and inter-*cTg* (intra-inter), intra-*cEg* and trans-*cTg* (intra-trans), or inter-*cEg* and trans-*cTg* (inter-trans) levels of schema interaction during these interviews.

Similarly, the levels outlined in orange (inter-intra, trans-intra, and trans-inter) are levels where I believe students would primarily be attempting to transfer knowledge from Euclidean to Taxicab geometry. In particular, if they have a "higher" stage of schema development for their *cEg* than their *cTg*, then it is intuitive that in order to continue to form an understanding of the underlying structure of their *circle schema*, students would need to make the same connections in Taxicab geometry as they see in Euclidean geometry, so they can generalize properties. Any of the other levels that have a combination of orange and blue outlines indicate that a student could be transferring knowledge in either direction discussed previously, or back and forth between their *cEg* and *cTg* in order to make connections between the various representations of circles across geometries. In general, the results support this notion.

There were many trends that emerged in the analysis of the various levels of schema interaction for which there was data. What is described below are the most common ways in which students appeared to transfer their knowledge of various concepts from one geometry to another. By summarizing each student's attempt at drawing a circle and writing the equation of this circle in Euclidean and Taxicab geometry and partitioning these summaries by the particular level of schema interaction each student was operating at, particular trends in the data were identified. Geometrically, the majority of these trends were found in how students tried to transfer their understanding of **Distance** and **Radius** from Euclidean to Taxicab geometry in their illustration. Algebraically, the majority of these trends were found as the students were deriving the equation for the circle in Taxicab geometry and what particular components they drew upon from their understanding of a circle in Euclidean geometry. In particular, to write the equation of a circle, some students tried to reproduce patterns, some tried to adapt their understanding of a definition, and some tried to transfer their understanding of the relationships between various concepts in Euclidean geometry. Below is a summary of these findings and descriptions of some of the examples from the participants in the study that exhibited these particular trends in their thinking. In addition, how these trends relate to the genetic decomposition for schema interaction and descriptions of these levels are provided. In particular, more details about the trends that emerged for the students operating at the inter-inter level are presented, since this proved to be the broadest classification of understanding.

Intra-intra.

Kristen, Samantha, and Mark all demonstrated they were operating at the intra-intra level of schema interaction. In terms of their **GRC**s, all three of these students demonstrated that they had memorized the shapes of the circles, particularly in Taxicab geometry, by saying things like "I knew what it was supposed to look like" (Kristen), "I keep forgetting that the Taxicab circle it's like a diamond…I keep drawing regular squares" (Samantha), and "I drew it from memory…I knew that's what it would look like," (Mark). Regarding their **ARC**s, Kristen and Samantha were not probed to write the equations of the circle in both geometries since another student in their interview, Hannah, was asked this first and ended up deriving the equations. Mark was probed about writing these equations but stated that he didn't remember them until the other students in his interview (Eileen and Felix) talked about them.

From the evidence provided from the participants, that students operating at the intra-intra level of schema interaction rely on memory for both their geometric and algebraic representations of a circle, and although they may have a coherent understanding of the definition of a circle as evidenced by their personal concept definition, they do not know how to apply this definition in the construction or equation of a circle. In other words, their *circle schema* is made up of isolated components, and their explanations usually involve visual observations or local properties about a specific circle or formula/equation.

Inter-intra

Three students, Alicia, Felix, and Darryl, provided evidence that they were operating at the inter-intra level of schema interaction. In the construction of the circles in Taxicab geometry, these students tried to transfer their understanding of **Circle** from Euclidean to Taxicab geometry but ended up relying on their memory once they reached a point where they got "stuck". For example, Alicia explained that for her Euclidean circle she constructed several radii and then finished drawing her circle. However, for her Taxicab circle she said, "whenever I did it in GSP it always gave me like a diamond shape," and had also said previously that for a circle in Taxicab geometry that she "didn't see how they were equal distant." Regarding their **ARC**s, these students identified that the distance formula was used in the equation for the Euclidean circle and tried to then copy the "format" of this for the Taxicab equation. For example, Alicia said, "based off the distance formula" and wrote part of the equation and then verbally explained that she did not want to square the radius in this formula "because in the distance formula for Taxi…nothing is squared like how it is in Euclidean." In other words, the only information she was able to transfer from her *cEg* to her *cTg* were local properties of the distance formulas, without knowing why one was squared and the other was not. Similarly, Felix explained he thought the distance formula would be involved "cause that's what we used…so I don't think I have to necessarily…square root." Meaning, he saw the distance formula in the equation for Euclidean circle and wanted to use this again in the equation for the circle in Taxicab geometry and tried to copy the format of this equation. He ended up using a radical in his equation for a circle in Taxicab geometry and squaring the terms with absolute values, which visually is similar to the formula for distance in Euclidean geometry.

From the evidence provided by the students in these interviews, students operating at the inter-intra level of schema interaction can make connections between their **AREC** and **GREC**, but when they attempt to transfer these connections to Taxicab geometry, they end up only being able to transfer the visual or local properties from their **AREC** to their **ARTC**, and **GREC** to **GRTC**, separately, without the connections. In other words, they could "copy" the construction of a circle in Taxicab geometry, but do not understand why it worked, and they tried to "copy" the format of the equation of a circle but cannot describe why these equations are different in relation to the definition of a circle and a particular metric. In general, students operating at the inter-intra level of schema interaction in this study often fall into the issue of the "prototype phenomenon" (Hershkowitz, 1990) for the concept of **Circle**. This affirms studies that discuss students in geometry having similar issues with other concepts (Fujita and Jones, 2006, 2007; Okazaki and Fujita, 2007; Turnuklu et al., 2013).

Inter-inter

There were six students, Hannah, Nicole, Brianna, Robin, Marianne, and Eileen that demonstrated they were operating at the inter-inter level of schema interaction. It is believed that a relatively high number many of the participants in this study were classified as operating at this level because this level can be "achieved" in a wide variety of ways by making connections between concepts and schemas. For example, a student could have a coherent understanding of their **GRC**, but not show evidence of the ability to transfer meaningful connections from his or her **AREC** to their **ARTC** and how these relate to their **GREC** and **GRTC**, respectively. However, another student may not have a coherent understanding of their **GRC** but were more successful in making connections between his or her **GRC** and **ARC** in both geometries. In both cases, the inter-inter level of schema interaction is being exhibited since students are making connections between representations between Euclidean and Taxicab geometry.

In terms of the **GRC**s of the participants classified as operating at the inter-inter level, all six students began by transferring the definition and concept of a circle from Euclidean geometry to Taxicab geometry in the construction of their circles. Four of the students, Hannah, Brianna, Robin, and Marianne, were successful with this. Hannah and Brianna both explained how they constructed particular radii to draw their circles, but also showed that they were aware why all of the other points on the circle also satisfied the definition of a circle. Robin had constructed particular points that satisfied the definition of a circle, tried to draw the circle based off of memory and did so incorrectly, but then used the definition of a circle to go back and check to make sure he drew it correctly. When he realized he had not, he changed the orientation of his circle to be correct. Marianne explained how she constructed her circle and also went back to make sure multiple points off the vertical and horizontal satisfied the definition of a circle.

In general, all of these students were able to use the definition of a circle to verify that their drawings were accurate. Eileen used her definition of a circle to first try and draw her Taxicab circle, and then ended up relying on memory for the shape. As this resulted in an incorrect orientation of her circle, she stated that she didn’t think all of the points were equidistant from the center expressing a clear understanding that this shape should satisfy that definition and the figure she had drawn did not. Nicole attempted to use the definition of a circle in her drawing of the circle in Taxicab geometry but had a misconception similar to Darryl's about how the way of measuring distance affects how you can draw lines in Taxicab geometry. In particular, they seemed to have discretized Taxicab distance which influenced them to think this is the reason why a circle in Taxicab geometry is a square, stating things like "I was thinking from each point it had to be on a strict grid, and not go in between" (Nicole), and "since you can't uh go through the fence...you can't go through the graph…I drew mine in…the form of a square," (Darryl). Once Nicole and Darryl overcome this misconception (which Nicole may have accomplished during the interview), it may be the case they will have formed a coherent understanding of their **GRC**s.

For the **ARC**s of these students, three of the students, Brianna, Robin, and Marianne, did not write equations for the circles on their own and/or were not [heavily] probed about this understanding. However, Robin made a few comments leading us to believe he was trying to remember the equations, and Brianna indicated she had coordinated her **Distance** and **Radius** processes across her **AREC** and **GREC** but then resorted to trying to remember the equations. Marianne did not derive the equation on her own, but after being involved in conversation between the other participants in her interview, indicated that she understood this derivation. The other three students, Hannah, Nicole, and Eileen, first recalled the equation for a circle in Euclidean geometry, and then identified that the formula for distance would be involved with the equation for a circle in Taxicab geometry based on the equation for a circle in Euclidean geometry (we assume Eileen did this, although she did explicitly state it). In attempting to write her equations, Eileen ended up relying on trying to copy the format of the Euclidean circle equation and incorrectly squared the radius term in her Taxicab circle equation. Nicole also attempted to copy the format but decided that since nothing was squared in the formula for distance in Taxicab geometry, that she would not square the radius. Hannah wrote out her equation for a circle in

Taxicab geometry, and then checked to be sure it was correct by substituting in a point from her circle to verify the expression would simplify to two, although she did not indicate she understood what this meant in terms of the definition of a circle.

To summarize further, for the six students in this study that exhibited they were operating at the inter-inter level of schema interaction, at least four of the six had a coherent understanding of the geometric representation of a circle and how this relates to the definition in both geometries. At the same time, at least three of the six students exhibited evidence that they were relying on memory when they were trying to think about the equations for the circle in Euclidean and Taxicab geometry. The other three attempted to copy the *format* of the equation of a Euclidean circle to produce the equation for a Taxicab circle. In general, the students at this level in the study have made meaningful connections between their **GREC** and **GRTC** (if not formed a coherent understanding of **GRC)** but struggled to make meaningful connections between their **AREC** and **ARTC**. Of the students who attempted to use the definition of a circle in writing their equation of a circle in Taxicab geometry, Brianna, Hannah, and Eileen clearly demonstrated they had made some connections between their **GRC** and **ARC**, while the others either did not exhibit this, or tried to make these connections but struggled to do so. Perhaps guiding these students to make [more of] these connections would be beneficial in the development of their *circle schema.*

In Table 5.2 the distribution of students in this level of schema interaction is summarized as it pertains to their **ARC** and **GRC** with regard to the interaction between their *cEg* and *cTg schemata*. The data is summarized in this way as this was the distinguishing factor among these students' development of their particular *circle schema.* To clarify, the title of the rows/columns indicate what each student relied on to construct their circles (**GRC**) and to write the equation of their circles (**ARC**) in terms of their *cEg* and *cTg schemata*. "Memory" means they predominantly relied on memory to construct/write the equation of their circles, "Connections" means they were able to make some connections between their evoked *cEg* and *cTg schemata* in order to do so. "Coherent" means they had formed a coherent understanding of the construction/equation of a circle in general terms. It is noted that if probed, Robin may have been able to verbally explain some connections between his **GRC** and **ARC**, but this evidence did not exist. Table 5.2 shows that all six of the students operating at this level would significantly benefit from the reflection of their **ARC**s and how they relate to their **GRC**s.

Table 5.2 Distribution of the students operating at the inter-inter level in terms of GRC/ARC

inter-inter		***ARC***		
		Memory	Connections	Coherent
GRC	Memory			
	Connections		Eileen Nicole	
	Coherent	Robin	Hannah Marianne Brianna	

Trans-trans

There were three students, Russell, Amy, and Parker, who showed evidence of operating at the trans-trans level of schema interaction by the end of their respective interview. All three of these students were able to form a coherent understanding of the construction and equation of a circle and how these are a result of the definition of a circle. In other words, they formed this understanding in Euclidean and Taxicab geometry separately and together, requiring the coordination all of their processes across their **GREC** and **GRTC** to construct the corresponding more general processes (**Distance, Radius, Center,** and **Locus of points**) for their **GRC,** and the coordination of all of their processes across their **AREC** and **ARTC** to construct the corresponding more general processes for their **ARC.** They were then able to coordinate all of these general processes (**Distance, Radius, Center,** and **Locus of points**) across their **GRC** and **ARC** in order to identify the underlying structure of their *circle schema*. They can coherently explain how the construction of a circle and the equation of a circle is a result of the definition of a circle and how these different representations relate to one another.

In terms of these students' **GRC**s, all three students demonstrated throughout the interviews that they were aware all of the points on the circle in each geometry satisfied the definition of a circle, in that they are all a particular distance away from the center. Parker was the only student to plot more than four points on the circle in Taxicab geometry by constructing radii (while others plotted the four points on the horizontal and vertical and used these to complete the sketch). Although Russell was the only student of the three to specifically be asked about his constructions, the evidence of this understanding for all three students actually emerged in the discussion of their equations for their circles. This indicates that they had evoked both their **GRC** and **ARC** at the same time (coordination) to talk about these equations.

For their **ARC**s, all three students were able to explain the structure of the equations of a circle in both geometries in general terms and how it related to their illustrations and the definition of a circle. This is the main difference between the students operating at the inter-inter level and the trans-trans level. As stated previously, some of the students operating at the inter-inter level of schema interaction could recall the Euclidean circle equation and could write the equation of a circle in Taxicab geometry, but mainly relied on "copying" the format of distance formulas. In contrast, the students operating at the trans-trans level understood how the distance formulas were involved with this equation as a result of the definition of a circle. For example, Amy explained in so many words that for a circle in Euclidean geometry, the formula for distance represents "the distance…between…the center and any point" and goes on to say that in Taxicab geometry, the distance formula “equals the radius which is 2.” As another example, Parker said “I wrote the formula for…calculating distance for Taxicab…this has to be compared to the center which is (3,3) so… my distance…is equal to 2.” Both Amy and Parker explicitly stated that if they substituted an arbitrary point on their circle into their equation, that they should get two. In particular, Parker even said, “that would mean our distance is 2, which means it’s on the circle,” indicating that in writing and explaining the equation of a circle in Taxicab geometry, Parker relied on its definition.

To summarize, by making connections between the algebraic and geometric representations of concepts in both Euclidean and Taxicab geometry, students were able to generalize their understanding of this definition. There appeared to be a positive relationship between the amount and depth of these connections and the extent to which students were able to generalize the definition of a circle. By visualizing and algebraically defining a circle in an atypical context, such as Taxicab geometry, these students continued to construct relationships between components within their *circle schema*, which once fully developed, will assist students in the generalization of axiomatic systems in geometry and to fight the urge to transfer local observations and properties of objects between geometries, since they are aware these are essentially side effects of a definition and not part of the definition. In other words, with a coherent schema structure for this concept, when students evoke their concept image of **Circle**, they are able to identify what parts of this concept image are part of the formal definition and what parts are not.

Fischbein (1993) explains that in geometrical reasoning, a major obstacle is the tendency to “neglect the definition under the pressure of figural constraints,” (p. 155). The results presented in this paper are consistent with this notion. Moore-Russo (2008) specified that participants in their study did not seem to understand the concept of slope deeply when they thought about slope outside of a common scenario. Something similar seemed to occur with many of the participants in this dissertation.

Since geometry courses examine the Euclidean geometry axiomatic system in depth (Byrkit, 1971), it follows that some of these students would face obstacles in thinking about the concepts presented in this report in Taxicab geometry. However, it is possible that with this further experience, students can make more meaning of these concepts. As Kaisari and Patronis (2010) suggest, “the meaning of mathematical concepts cannot be grasped or produced only by definitions and/or formal mathematical explanations. It needs an awareness of human action and depends on the use of concepts within a particular concept…,” (p. 255).

5.2 Revised Genetic Decompositions

5.2.1 Mental Constructions in geometry

A revision to the preliminary genetic decomposition pertains to students’ conception of **Distance.** It was stated that a student should have an object conception of **Euclidean distance** and de-encapsulate this object before successfully coordinating the process of **Taxicab distance** with his or her **Euclidean distance** process. Through this analysis, it was found that students do not necessarily have to exhibit an object conception of concepts in Euclidean geometry to assimilate the corresponding Taxicab geometry concept into their understanding. For example, an individual can begin with an action conception of **Euclidean distance** and interiorize this to be able to coordinate it with his or her **Taxicab distance** Process.

Figure 5.1 provides an illustration of this revision to the genetic decomposition, in that a student can begin at various stages of conception of **Distance** in Euclidean geometry and still have the ability to understand **Taxicab distance**. However, students must exhibit a process conception of both **Euclidean** and **Taxicab distance** in order to make connections between properties and figures in these geometries. This idea translates to all of the emergent concepts of **Midpoint, Circle,** and **Perpendicular bisector.** In other words, regardless of the conception a student has of this concept in both Euclidean or Taxicab geometries independently, he or she should have at least a process conception of that concept in both geometries in order to coordinate these processes and make connections or distinctions between them. Below two revisions to the descriptions of mental structures within the preliminary genetic decomposition based on these results are given.

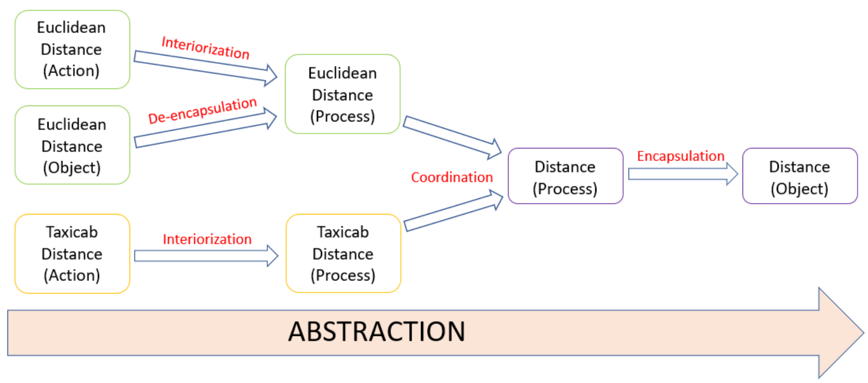


Figure 5.1 Revised illustration of a possible way for a student to assimilate Taxicab distance into his/her understanding of Distance.

One revision of the preliminary genetic decomposition arises from how several students compared objects in Euclidean and Taxicab geometries. As an example, recall that in Section 4.1.2 Amy compared Euclidean and Taxicab distances between two points as a triangle, where the hypotenuse of this triangle was the Euclidean distance between the points, and the legs of the triangle made up the Taxicab distance between them. Within the context of the genetic decomposition for the mental structures in geometry, it had not been considered that comparing the geometric and algebraic representations of an object in multiple geometries required the coordination of these processes. It had also not been considered that the way in which a student describes an object as it relates to another within a definition contributes to evidence of his or her conception of that object. For example, In Section 4.1.2 when Brianna explained that a Euclidean circle is round because the radius is a straight line because of how distance was measured, she was exhibiting a process conception of **Euclidean Circle** because she was coordinating her **Radius** and **Distance** processes to explain in general terms why this circle is round.

In other words, I had not considered to what extent relationships among components were necessary in order to make these connections, until I began to consider the interaction of schemas presented in this report. I rephrase this portion of the preliminary genetic decomposition for the concept of **Distance** below, but note that similar statements should be applied to all process conceptions of **Midpoint, Circle,** and **Perpendicular bisector**.

Process: Given a distance formula or told what geometry to consider, an individual can calculate the distance between any two points. He or she does not need two specific points to imagine the distance between and can describe this distance using his or her own words. If a student can consider the Euclidean and Taxicab distances between two points at the same time and describe this relationship algebraically and/or geometrically, the student is exhibiting a process conception of **Distance**, since this requires the coordination of processes across their *Euclidean* and *Taxicab geometry schemata*. As an example, a student can describe these distances between two points as a right triangle whose hypotenuse represents the Euclidean distance, and the legs of which comprise the Taxicab distance between the points. A student can also exhibit a process conception of **Distance** by providing evidence that they are coordinating this process with another within a schema. For example, if a student is able to explain why the way distance is measured affects the appearance of a circle, they are coordinating their **Distance** process with at least one of **Radius, Center,** and **Locus of points**, depending on how he or she chooses to explain this.

To summarize, a student exhibits a process conception of a concept in geometry if he or she provide evidence of any of the following:

(1) Can talk about the concept in their own words or using arbitrary examples.
(2) Can talk about the concept in general terms as it exists across geometries (algebraically or geometrically).
(3) Can talk about the concept in general terms as it relates to other concepts.

The next revision is a result of Tyra's solution to a problem on the final exam with regard to **Midpoint.** Although this solution was not presented, Tyra found two Taxicab midpoints of a given segment and used these to find the equation of the line containing the midset. In the preliminary genetic decomposition, this possibility of using midpoints as an input into a function to determine an equation that represents all midpoints of two points in Taxicab geometry had not been considered. Thus, regarding an object conception of **Midpoint**, this portion of the preliminary genetic decomposition is rephrased below.

Object: The student has encapsulated this Process into an Object if, when given two points, he or she identifies a midpoint of the segment connecting them and can apply an action to it. An example of an action that can be applied to this Object is the comparison of locations of multiple midpoints in Taxicab geometry. In addition, the individual can be aware that a midpoint is not unique in Taxicab geometry when the given segment is not parallel to one of the axes and does not have a slope of 1 or -1. Another way a student can perform an action on his or her **Midpoint** Object is by using two midpoints of a segment in Taxicab geometry to derive an equation that represents the midset of the two given points. The action being applied to this Object is a transformation of the Object from two points to an equation of a line.

Results from this question on the Final Exam which had students investigate the idea of equidistance in Taxicab geometry were not provided in this report. However, in Appendix B, a suggested wording of this problem to better assess students' understandings of each of **Midpoint, Taxicab circle,** and **Perpendicular bisector** is provided as a result of this analysis**.**

The magnitude of which students' conception of **Radius** would have on their overall understanding of **Circle** was also not anticipated. In particular, it was certainly anticipated that to form a coherent understanding of the definition of a circle a student would need to coordinate his or her **Distance** and **Radius** processes, but I did not consider the complexity of this relationship until I began using the framework of schema interaction. Thus, in the preliminary genetic decomposition that was created for the various mental structures within the *circle schema*, details had been provided for the concept of **Distance,** but not for the concept of **Radius** and the relationship that exists between these concepts in a student's mind.

Below I provide the descriptions of the various stages of conception a student can have exhibit of **Radius**, how this concept interacts with the concept of **Distance**, and what this means in terms of the overall *circle schema.* Some examples are inspired by some of the responses of the participants in this dissertation. For example, the idea Brianna suggested by using the metaphor of a clock influence the description of a process conception of **Radius**.

Radius. The radius of a circle is defined as the distance from the center of a circle to any point on the circle or the length of a segment connected the center to any point on the circle. Thus, the concept of **Radius** inherently has the subconcepts of **Distance, Center,** and **Locus of points,** and as these concepts would be evoked by reading this definition. What follows is a description of the various stages of conception associated with the concept of **Radius** and how a student operating at each stage by relate the radius to these other concepts within their *circle schema*.

Action: A student exhibiting an action conception of **Radius** is able to find the length of a radius of a circle either by looked at an equation of a circle or by looking at a drawing of a circle. A student could also draw a circle using a particular radius length (and center) but cannot explain how this relates to the definition of a circle in general.

Process: A student has interiorized his or her action conception of **Radius** if he or she can understand how to find the length of a radius of any circle in any metric through understanding the role a radius has in the definition of a circle. A student can exhibit a process conception of **Radius** as well if he or she is able to make connections between how this concept relates to others in the definition of a circle. For example, a student explaining that if the radius of a circle is measured as a straight line, then the circle will be round is exhibiting evidence that the student is coordinating his or her **Radius** process with at least one of **Distance, Center,** and **Locus of points,** depending on exactly how this is explained.

Object: A student has encapsulated their **Radius** process into an object if he or she can perform an action on this object, such as using it as an input into a function. For example, if a student uses his or her understanding of the concept of **Radius** to write the equation of a circle, then the student is using his or her **Radius** object as an input into a mental transformation, where the output is the equation for a circle with that radius.

Following this idea for an object conception of **Radius,** it had not explicitly been identified what mental structures were necessary in order to understand the equation of a circle until the interaction of schema framework was introduced. In particular, for a student to be able to write and talk about the equation of a circle, he or she needs to have an object conception of (at least) the algebraic representation of each of **Distance, Radius, Center,** and **Locus of points**, as they are being used as inputs into a mental function where the output is an equation. However, these objects can be de-encapsulated in order to coordinate the resulting process with other processes within his or her *circle schema*. This would help a student to better understand the derivation of this equation. For example, a student can write the equation for a circle but not understand how each portion of this equation is a result of the definition of a circle. In this situation, the student has an object conception of all of the subconcepts of **Circle**, since he or she is using them as inputs, but this student does not have a coherent understanding of the underlying structure of the *circle schema*. This understanding can be gained by the student de-encapsulating his or her object conceptions of each of **Distance, Radius, Center,** and **Locus of points** and coordinating them with one another in order to observe these relationships. An illustration of this is provided in Figure 5.2. In particular, the blue arrows indicate the de-encapsulation of all of these objects into processes. The red arrows in this figure indicate the possible coordinations that could then occur among these processes.

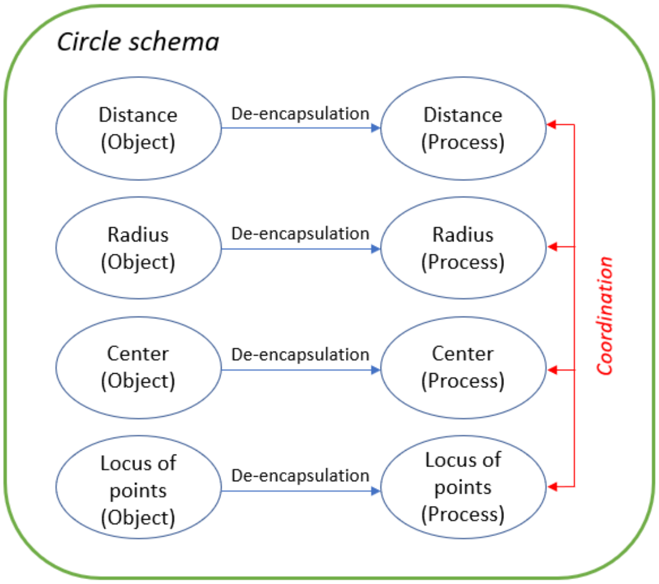


Figure 5.2 De-encapsulation of components in order to coordinate processes within the circle schema.

Kaisari and Patronis (2010) suggest there are three types of use of geometrical concepts: (1) as elements of representation of spatial experience, (2) as objects of traditional School practice, and (3) as constituents of an abstract mathematical theory (pp. 255-256). The first focuses on geometry as a model separated from formal definitions and proofs, the second focuses on defaulting to Euclidean geometry, and the third focuses on abstraction and formalization. The students in this report illustrated a combination of geometry of these types of uses. For example, Parker demonstrated in Section 4.2.9.3, that she was using spatial sense to first understand the definition of a circle, then used abstraction and formalization to generalize her personal concept definition of a circle. An example of a student using geometry as an object of traditional School practice was presented in Section 4.1.3 when Nicole seemed to become disengaged once her group found the "desired" answer of the instructor, as she interpreted it to be.

Many participants in this report follow the findings of Kinach and Fostering (2012) that their students demonstrated an understanding of concepts in geometry mainly through algebraic and numerical methods, absent of a spatial understanding. For example, many students in this report some students relied solely on their algebraic and numerical scratch work when finding the equation of a circle. On the other hand, many of the solutions and descriptions provided by these participants were very complex, which is why there was a need to incorporate the framework for schema interaction in the analysis. In general, these results affirm the claim that there is a need for further research in determining whether comparing ideas and concepts in different geometries result in a deeper insight into Euclidean axioms (Kinach & Fostering, 2012).

5.2.2 Schema interaction

There is not a revision to the genetic decomposition for the interaction of these schemata. In lieu of a revised genetic decomposition for this section, below in Table 5.3 is a summary table of the various levels of schema interaction as they were exhibited and believed to have existed within these participants. Recall that the levels outlined in blue (intra-inter, intra-trans, and inter-trans) indicate a level where a student would primarily be transferring knowledge from Taxicab to Euclidean geometry. Similarly, the levels outlined in orange (inter-intra, trans-intra, and trans-inter) are levels where students would primarily be attempting to transfer knowledge from Euclidean to Taxicab geometry. As stated previously, these students were much more familiar with Euclidean geometry than they were with Taxicab geometry, so the natural flow of knowledge would be in this direction from their *cEg schema* to their *cTg schema*. Any of the other levels that have a combination of orange and blue outlines indicate that a student at that level could be primarily transferring knowledge in either direction discussed previously, or back and forth between their *cEg* and *cTg* in order to make connections between the various representations of circles across geometries.

Table 5.3 Genetic decomposition for the cEg-cTg schemata interaction within the circle schema.

CIRCLE	Intra-Taxicab geometry	Inter-Taxicab geometry	Trans-Taxicab geometry
Intra-Euclidean geometry	In general, a student cannot make connections about how the construction and equation of a circle are direct results of the definition of a circle (and how the geometric/algebraic representations are related). The student cannot describe the construction or structure of the equation for a circle other than local properties of the representations (such as shape or the use of certain operators in the equation).	A student has begun to make connections in Taxicab geometry about how the construction and equation of a circle are direct results of the definition of a circle (and how the geometric/algebraic representations are related), but cannot transfer this knowledge to make the same connections about a circle in Euclidean geometry and can only make local observations between representations across geometries.	In Taxicab geometry, a student understands how the construction and equation of a circle are direct results of the definition of a circle (and how the geometric/algebraic representations are related). However, they cannot transfer this knowledge to Euclidean geometry and can only make local observations between representations across geometries.
Inter-Euclidean geometry	A student has begun to make connections in Euclidean geometry about how the construction and equation of a circle are direct results of the definition of a circle (and how the geometric/algebraic representations are related), but cannot transfer this knowledge to make the same connections about a circle in Taxicab geometry and can only make local observations between representations across geometries.	In general, a student begins to make connections about how the construction and equation of a circle are direct results of the definition of a circle (and how the geometric/algebraic representations are related). The student can begin to describe the construction or structure of the equation for a circle other than local properties of the representations.	In Taxicab geometry, a student understands how the construction and equation of a circle are direct results of the definition of a circle (and how the geometric/algebraic representations are related). The student begins to transfer knowledge from Taxicab geometry back to Euclidean geometry in order to try and make the same connections they see in Taxicab geometry.
Trans-Euclidean geometry	In Euclidean geometry, a student understands how the construction and equation of a circle are direct results of the definition of a circle (and how the geometric/algebraic representations are related). However, they cannot transfer this knowledge to Taxicab geometry and can only make local observations between representations across geometries.	In Euclidean geometry, a student understands how the construction and equation of a circle are direct results of the definition of a circle (and how the geometric/algebraic representations are related). The student begins to transfer knowledge from Euclidean geometry to Taxicab geometry in order to try and make the same connections they see in Euclidean geometry.	In general, the student has generalized a circle completely, and has a coherent understanding of how the construction and equation of a circle within a metric space is a direct result of the definition of a circle. In other words, the student has generalized the structure of the equation for a circle geometrically and algebraically.

5.3 Implications for instruction

In this study, illustrations of various stages of conception in APOS Theory were presented as they were exhibited by students enrolled in a college geometry for many different concepts. Many students could recite and explain the definitions of various mathematical objects but struggled to correctly draw or algebraically define these objects in Taxicab geometry based on this definition. By using APOS Theory to analyze these participants' work and responses to questions, I was able to identify some common misconceptions about Taxicab distance, circles, midpoints, and perpendicular bisectors. For example, multiple participants believed one could not travel in non-integer increments in Taxicab geometry (i.e. – "split" units or measure distance continuously) since a car would not be able to drive through blocks in a city. This did lead to students attempting to calculate and imagine distance under a certain constraint, which resulted in misconceptions in Taxicab geometry. This is consistent with Smith (2013), as he had to explain to students that we have to think of the streets and the blocks in between as "negligible width, allowing nonlattice points to be generated," (p. 619).

This study has several implications with regard to the teaching and learning of geometry in the college classroom. First, it was found that through group work and the use of a dynamic geometry software, students were generally successful in accomplishing a shared goal and deepening their understanding of various definitions. However, in the cases where students all chose to share one computer in order to use this software, many times members of the group were off task and distracted, as they perhaps did not feel as much accountability for the group's work. Thus, with the use of cooperative learning and technology in the classroom, these results are consistent with results from Cavanagh (2011) in that students positively react to the use of cooperative learning and Zakaria (2010) in that students' attitude towards math had improved [possibly] as a result of the structure of this course. It is noted, that as many factors affect student success and attitude, it cannot be said for certain the cooperative learning and use of GSP led to the overall positive experience these students had. However, these results further support Hull and Brovey (2004) that instruction that includes technology should be used not only in teacher-led instruction, but in a self-directed way as well, since there were many cases of students discovering and generalizing without the explicit guidance of the instructor. Thus, the use of well-prepared activities that each student is required to complete on their own for submission is suggested, but that they complete these activities with one another. In this way, each student is responsible for their own understanding within their submission, but the group as a whole is still working towards a shared goal. In other words, as they work together, students are not only using self-guidance but are helping guide one another as well. This also can lead to a better sense of community within the classroom and a comfortability that students can find with each other to talk about their understanding and ask questions.

Second, this study led to the creation of a model for schema interaction within the context of geometry which in turn leads to another teaching implication. In particular, as students tried to transfer knowledge from Euclidean to the Taxicab geometry context, the connections between algebraic and geometric representations played a much bigger role than anticipated. In the context of APOS Theory, the more processes a student had coordinated between his or her **Algebraic Representation of Circle** and **Geometric Representation of Circle** (across Euclidean and Taxicab geometry), the more the student was able to coherently talk about a circle and its properties. In particular, the more coherent this structure was, the more students were able to identify the difference between a definition in mathematics and the properties that are a consequence of the setting in which this definition was being considered. Further, the more connections between these algebraic and geometric representations students made, the more likely they were to be able to construct figures and identify relationships in new metric spaces.

Based on these results, in Appendix B suggested activities are provided that will help in the development of the *circle schema* in relation to a student's **GRTC**, **AREC**, and **ARTC**. There is a focus on these aspects since these seemed to be where these students needed the most guidance in order to make connections between a circle in Euclidean and Taxicab geometry. Of course, these activities are open for the reader to edit or adapt as he or she sees fit for use in a classroom, but these have the over-arching goal to facilitate the connections between representations for the concept of **Circle**.

First, activities to help with understanding the connections between the geometric and algebraic representation of distance in Euclidean and Taxicab geometry are presented. It was determined that that many students struggled to understand the distance formula in Euclidean geometry, which may have led to an issue with transferring particular knowledge from Euclidean to Taxicab geometry within their *circle schema* (e.g. - **Distance** as it relates to **Radius)**. I believe this was one of the main obstacles students faced with both drawing and writing the equation for the circle in Taxicab geometry on the questionnaire. As a note, these activities are not designed to necessarily be given sequentially in the same period of time, and students should be briefly introduced to the Taxicab metric prior to completing Activities 2, 3, and 5, as well as a brief introduction to Taxicab circles before completing activities 3 and 5. Once a student has achieved a trans-trans level of schema interaction, Activity 6 is provided as a supplement, which is intended to help in the thematization of the *circle schema* so that they can transfer this understanding to a new metric space. Within these activities in Appendix B, the task to be given to students along with the pedagogical goals are provided, along with ideal solutions for some of these tasks.

Lastly, another teaching implication for this study is the effect that constructions within atypical contexts have on students' understandings of definitions. In particular, having students explore concepts without explicitly stating the concept they are exploring significantly helped students in this study to generalize their understandings. For example, in The Red Line Investigation presented in Section 4.3.2 shows students the construction of a perpendicular bisector in Taxicab geometry with the goal that students could determine the relationship between the circles used in this construction and the constructed line and transfer this relationship back to Euclidean geometry to identify what was being constructed, based on the definition of that object (perpendicular bisector). In other words, by not stating "perpendicular bisector" within the problem, students were able to dissociate any misconceptions or preconceived understanding of properties that are involved with their concept images of **Perpendicular bisector** and approach the problem with limited associations. By doing this, it was the goal to avoid the conflict described by Çetịn (2009), in that "portions of concept image can be in conflict each other but also conflict might happen between concept image and its formal definition," (p. 24). These results also affirm that when attempting to apply mathematical definitions, many students have incomplete concept images from which they reason, resulting in them rejecting given definitions to use their imprecise concept image (Dickerson & Pitman, 2012).

Overall, these results imply exploration in a "simpler" non-Euclidean geometry led students to make more sense out of the definition of various concepts in Euclidean geometry. In addition these results are consistent with that of Smith (2013) and Berger (2016) in that many of these participants were able to generalize their understanding of why the shapes of figures can change as a result of the underlying structure of a definition. Güner and Gülten (2016) explain that geometry has three dimensions: definitions, images that represent these definitions, and their properties. I believe participants in this study deepened their understanding of the relationships between these three dimensions.

5.4 Limitations of the study

There are several limitations to this study. One limitation is that there were only 18 students enrolled in the course, and a greater number of participants could reveal more examples of various stages of conception for the concepts that emerged in the minds of these participants. Also, the enrollment in this course was cross-listed so there were both undergraduate and graduate students, as well as mathematics majors and students in an education program. These results did not explicitly take these characteristics into account other than noting what level and program they were in. These various differences in backgrounds may have had an effect on the interactions that took place within the class room and within the context of group work.

Further, it should be noted that I can only gain as much insight to these students' understanding of various concepts as they are willing to provide. In many cases, the conception a student is exhibiting may not be his or her overall conception that student has, but rather the "highest" conception the student finds necessary in order to answer a particular question or complete a particular task. Within the context of analysis, another limitation of this study was having only utilized the frameworks of APOS Theory and concept image and concept definition. As Baker et al. (2000) state, there are other theories or models that could have been used to analyze this data that may have led to other conclusions.

5.5 Future research

Moore-Russo (2008) points out that there is not as much literature on attributes of geometric figures and the relationships between them than there is on the construction of definitions of geometric figures, which was a major motivation for this study. Future research could benefit from investigating how differently designed activities affect student understanding of definitions and mathematical terms. In particular, do students better understand mathematical concepts from constructing definitions, using their personal definitions to observe properties and incorporate these with prior knowledge, or using given definitions and applying them in a situation to determine what mathematical concept is under consideration? Further investigations into this can help to develop more instructional material that will help facilitate an understanding of mathematical definitions, their roles, and their uses. Moore (1994) discusses how mathematical language or definitions are a large influence on students' inability to write proofs. Thus, more research on how students understand mathematical definitions and how they apply them in their geometric reasoning could result in bettering student understanding of proofs.

As an extension of this direction in approach to instruction, this report provided examples of activities and interactions in which students generalized definitions using technology and group work. These results lead to further research questions such as (1) In what other ways can technology help students to interiorize their action conceptions or encapsulate their process conceptions of definitions of various concepts in mathematics? (2) How else can we utilize APOS Theory to analyze student interactions with technology in mathematics? (3) What other frameworks can be used to model this interaction?

There is a lack of research that uses of the stages of the triad in APOS Theory (for examples, see Clark et al. (1997) and McDonald et al. (2000)), but even fewer studies that use the idea of the "double triad" or schema interaction. A model of schema interaction was described in detail, using the original model by Baker et al. (2000) as a guide (see Trigueros (2004) for a second model pertaining to differential equations). It was determined that analyzing this data using APOS Theory without the triad or schema development hindered the researcher from understanding the thought process of many of the participants. By involving these facets of APOS Theory into the analysis I was able to better describe the structure of these participants' *circle schema* and their overall understanding of the definition of various concepts in geometry. This leads to future research questions such as (1) What pedagogical activities could help in the development of a student's *circle schema* with this schema interaction in mind? (2) Within the *circle schema*, what other schemata and their interaction would be beneficial for a student to help with the development of its underlying structure? (3) What does the thematization of other schemata look like? (4) What other mathematical situations could be better understood through analysis using schema interaction? I suggest that future research investigates these issues.

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APPENDICES

Appendix A INTERVIEW QUESTIONNAIRE AND PROTOCOL

We note that red text indicates questions part of the protocol given to the interviewer to ask each student, orange text indicates sample solutions students could provide for each portion of the questionnaire, and black text indicates questions that students responded to on the questionnaire prior to each interview.

Preliminary Questions:

- What are your majors/programs?
- What are some of the highest-level mathematics courses you have completed prior to this course?
- Do you have a degree in a different area of study? If so, what area and where did you earn your degree from?
- How do you feel about the course? *[Elaborate on any interesting responses]*

  [Ask students to elaborate on any of the Likert-Scale questions they feel strongly about one way or another]

Interview Questionnaire

1. Define and draw an image (or images) that represents each of the following terms, however you see fit.
   a. Circle - $\{P: d(P,C) = r, where\ r > 0\ and\ C\ is\ fixed\}$, or the locus of points equidistant from a fixed point.
   b. Distance – a function that gives measurement to how far apart two objects are
   c. Congruent Triangles – all corresponding angles and sides are congruent

d. Tangent line – a line that intersects another figure at exactly one point; line that is perpendicular to a radius at a given point (point of tangency)

e. Perpendicular bisector of a segment – the locus of points equidistant from endpoints of segment; or the line that is perpendicular to the segment that crosses through the midpoint of the segment

f. Ellipse - $\{P: d(P, F_1) + d(P, F_2) = d, where\ d > 0\ and\ F_1\ and\ F_2\ are\ fixed\}$, or the locus of points such that the sum of distances from any point of the locus of points and two other distinct fixed points is constant

(a) What was your thought process in coming up with these definitions?

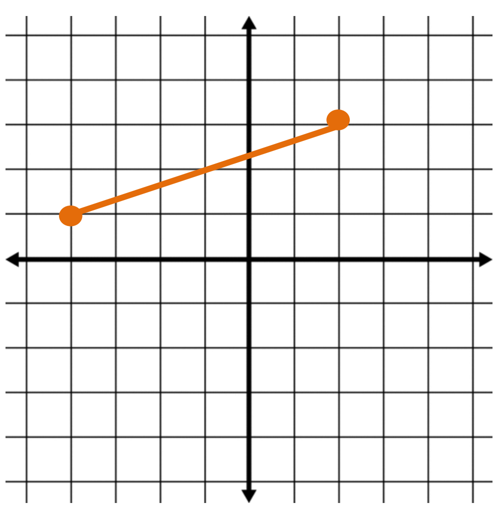
(b) Does your definition work in both Euclidean and Taxicab geometry?

2. For any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

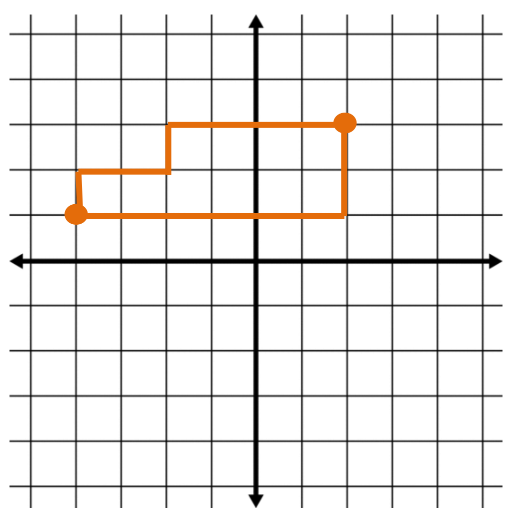
(i) Euclidean distance is given by $d_E(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

(ii) Taxi distance is given by $d_T(P, Q) = |x_2 - x_1| + |y_2 - y_1|$

a) Using the grids below, illustrate each of these two distances. Be as detailed as possible in labeling each of them.



Euclidean distance



Taxicab distance

(a) How do each of your drawings represent each distance?

(b) Did you use the definition of each type of metric when you made your illustrations?

(c) If I gave you a different metric or way to measure distance, do you think you could illustrate that easily?

b) Is it possible for these two distances to be same, i.e. $d_E(P,Q) = d_T(P,Q)$? If yes, explain.

Yes, it is possible for them to be equal if the points are vertical or horizontal from one another.

(a) Elaborate on your response to this question.

(b) Is it possible for the Euclidean distance be more than the Taxicab distance between two points?

(c) Is it possible for the Taxicab distance be more than the Euclidean distance between two points?

3. Is the following definition true in both geometries? Explain.

"The circle (Euclidean or Taxi) is a set of points in the plane equally distant from a fixed point."

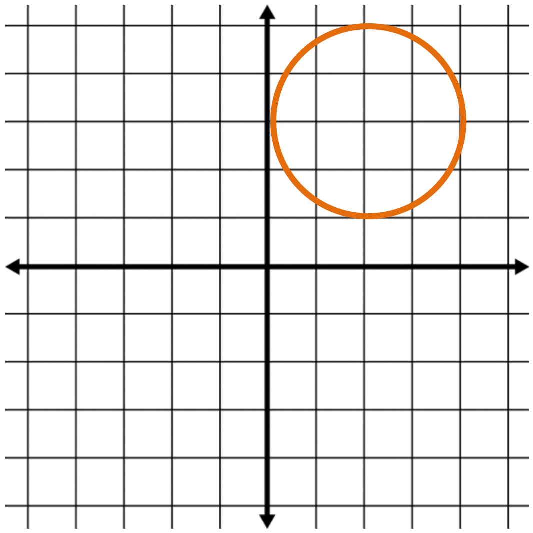
Yes, it is true. Definitions are constant between geometries, although they produce different looking objects.

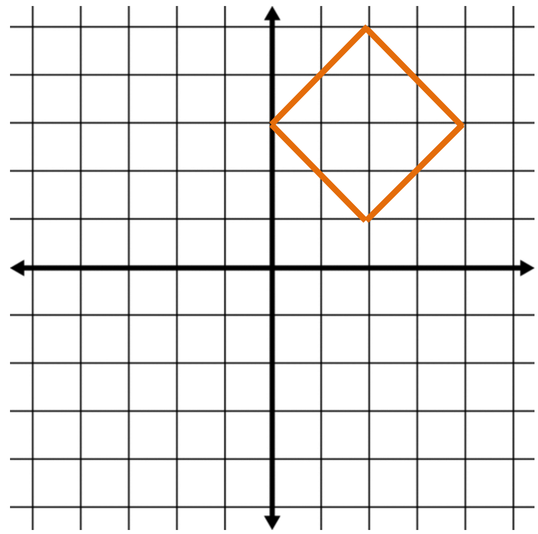
(a) Does this definition differ from your definition in the Preliminary questions?

a. If so, what is the difference? Are they still equivalent?

b. If not, what makes them the same?

4. Sketch the following circle in each of the geometries: Circle with center at $C(3,3)$ and radius $r = 2$.





Euclidean circle Taxicab circle

(a) How did you go about drawing each of your circles?

(b) How did you use the definition of a circle in each situation?

(c) Could you write the equations for each of these circles? Describe what they would look like.

(d) If I gave you any center and any radius, do you think you could draw the Taxicab circle associated with it?

(e) If I gave you a different metric or way to measure distance, do you think you would be able to sketch a circle using that metric? How would you go about sketching a circle?

Appendix B SUGGESTED ACTVITIES

Activity 1. ***(Euclidean distance)***

(a) On the first grid provided, plot points at (1,2) and (4,6). Illustrate the distance between these two points in Euclidean geometry.

(b) Calculate the distance between the x-coordinates and the y-coordinates of these two points and draw/label these distances on your graph. What figure is formed from your original illustration of the distance between these two points and the distances you just drew?

(c) Using your answer to the question in part (b), can you use the distances you found for part (b) to find the distance between these two points? What theorem did you use to do this?

(d) Repeat parts (a)-(c) on the second grid, but with points arbitrary points (x_1, y_1) and (x_2, y_2).

(e) Reflection: The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in Euclidean geometry can be calculated by $d_E = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Write a few sentences to explain how this formula is illustrated in your graph.

Pedagogical goal for Activity 1.

For an ideal solution, the students should realize they have formed a right triangle in parts (a) and (b), and that they can find the length of the hypotenuse, which is the distance between these two points, by using the Pythagorean theorem (part (c)). Then, by generalizing this to arbitrary points (part (d)) and having them reflect on what similarities they identify between the provided formula for distance in Euclidean geometry and their graph from part (d), I hope that students will come to understand the derivation of this formula better both geometrically and algebraically. Ideal graphical solutions for parts (a)-(d) can be found in Figures B.1 and B.2.

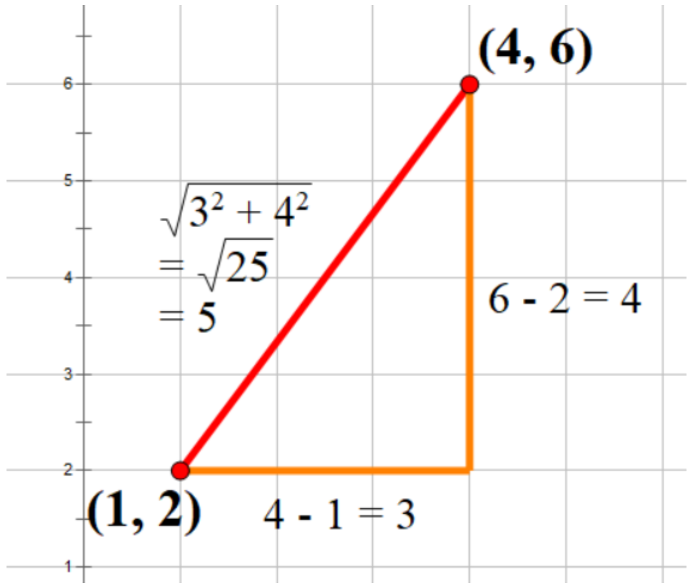


Figure B.1 An ideal graphical solution for parts (a)-(c) of Activity 1.

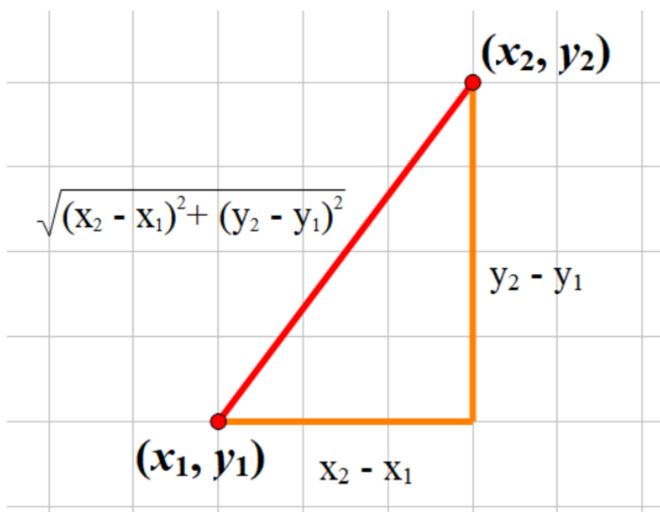


Figure B.2 An ideal graphical solution for part (d) of Activity 1.

Activity 2. *(**Taxicab distance**/cEg)*

(a) On the first grid provided, plot points at (1,2) and (4,6). Illustrate the distance between these two points in Taxicab geometry such that it forms the legs of a right triangle.

(b) Calculate the length of each “leg” of this distance using the x-coordinates and the y-coordinates of these two points. Label these distances on your graph.

(c) Using part (b), calculate the total distance in Taxicab geometry between these two points. Is it possible to draw any other "route" between these two points that is the same distance in Taxicab geometry? Shorter? Longer?

(d) Repeat parts (a)-(b) on the second grid, but with points arbitrary points (x_1, y_1) and (x_2, y_2). Use this information to calculate the distance in Taxicab geometry between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(e) The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in Taxicab geometry can be calculated by $d_T = |x_2 - x_1| + |y_2 - y_1|$. In 2-3 sentences, summarize how this formula is illustrated in your graph. Why is it necessary to include the absolute value signs?

(f) Reflection: Compare and contrast how the differences in the x and y coordinates of two points ($x_2 - x_1$ and $y_2 - y_1$) are used in the formulas for distance in Euclidean and Taxicab geometry in relation to your graphs.

Pedagogical goal for Activity 2.

For this ideal solution, the students should realize in parts (a)-(c) they can use the difference in x coordinates and y coordinates to find the distance between two points in Taxicab geometry. In addition, by asking them about other routes of shortest Taxicab distance, I hope they will abstract that, although there is more than one way to illustrate Taxicab distance, this method results in calculating the length of all shortest paths. Then, by generalizing a form of this calculation to arbitrary points (part (d)) and having them reflect on what similarities they identify between the provided formula for distance and their graph in part (e), I hope that students will come to understand the derivation of this formula better both geometrically and algebraically. Lastly, for part (f), I hope that students will begin to make better connections between distance in these two geometries, and how they are represented geometrically and algebraically. Ideal graphical solutions for parts (a)-(d) are below in Figures B.3 and B.4.

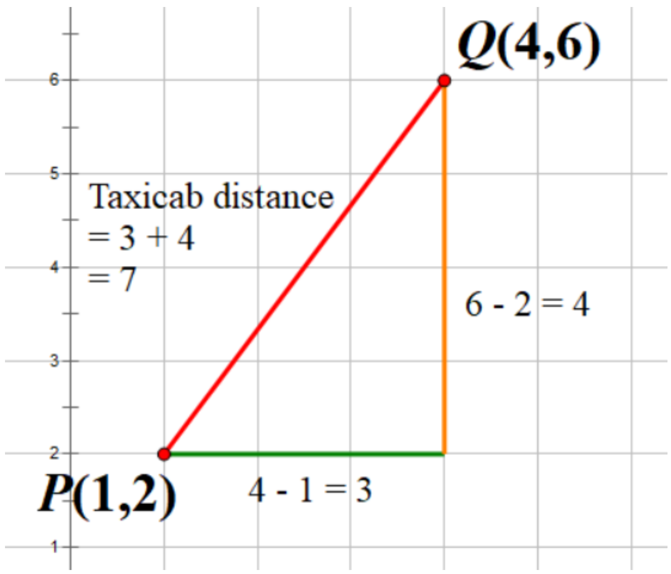


Figure B.3 An ideal graphical solution for parts (a)-(c) of Activity 2.

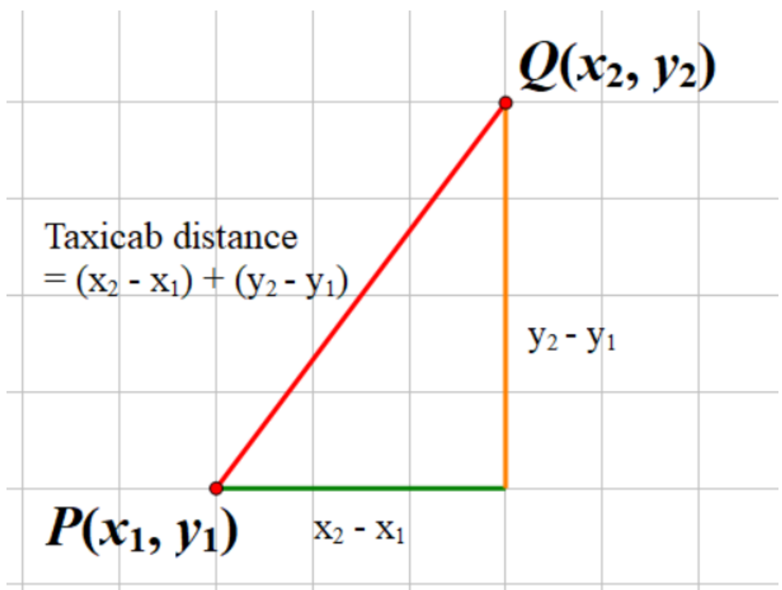


Figure B.4 An ideal graphical solution for part (d) of Activity 2.

Activity 3. *(**GRTC**/cEg)*

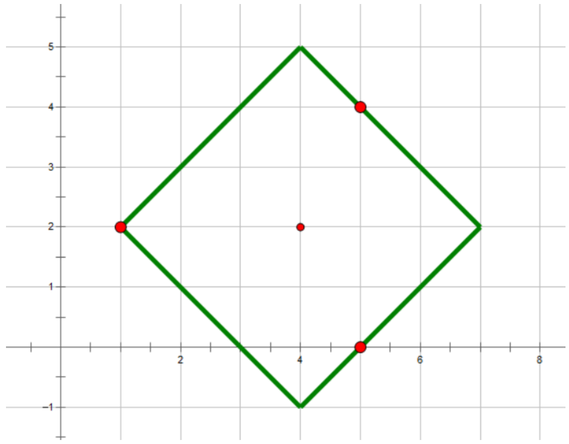


Figure B.5 Provided figure for Activity 3.

(a) Using the figure provided (Figure B.5), illustrate the Taxicab distance between the point in the center of this figure and each of the red points on the figure. Show or disprove that all of the red points on this figure have equal Taxicab distance from the center point.

(b) What is the definition of a circle? Does this figure provided satisfy this definition? How do you know?

(c) Draw/sketch a circle in Taxicab geometry centered at (2,1) with radius 4 by finding at least 6 different points that would be on this circle. Label the center of this circle and illustrate the 6 different radii that you have constructed.

(d) Reflection: Does the definition of a circle change from Euclidean to Taxicab geometry? What properties of a circle hold from Euclidean to Taxicab geometry? What properties do not?

Pedagogical goal for Activity 3.

From this activity it is aimed that students can begin to reflect on their understanding of the definition of a circle and abstract upon this to not feel the “figural constraints” Fischbein (1993) refers to. In particular, students can make geometrical connections between their concept of

Radius in both Euclidean and Taxicab geometry, geometrically and algebraically, since this was a common obstacle for the participants in this study.

Activity 4. ***(GREC/AREC)***

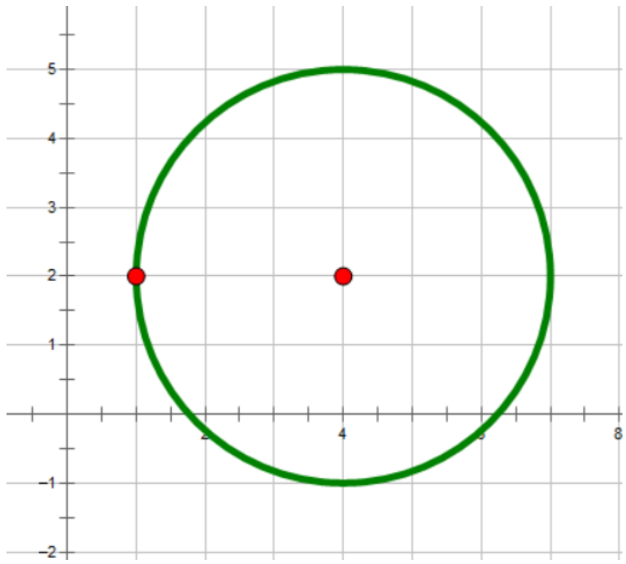


Figure B.6 Provided figure for Activity 4.

Using the figure provided (Figure B.6), the formula for distance in Euclidean geometry, and the format of the example below:

(a) Calculate and illustrate the distances between each of the following points and the center of the circle

Point (1,2)	$\sqrt{(1-4)^2+(2-2)^2}=3$
Point (4,-1)	______________________
Point (7,2)	______________________
Point (4,5)	______________________

(b) Do you notice any similarities/differences in all of your responses to part (a)? Explain how these similarities/differences relate to the graph of this circle.

(c) Reflection: Use your responses to parts (a)-(b) to write the equation for this Euclidean circle. In your own words, use 2-3 sentences to explain what this equation is saying in relation to your graph and to the definition of a circle.

Pedagogical goal for Activity 4.

We hope through this activity, that students can better understand where values in the equation of a circle in Euclidean geometry are located on a graph, and not only that the distance formula is used within this equation, but why it is used in relation to the definition of a circle, i.e. – that the distance between any point on the circle and center should be constant, and is equal to the radius of that circle.

Activity 5. ***(GRTC/ARTC)***

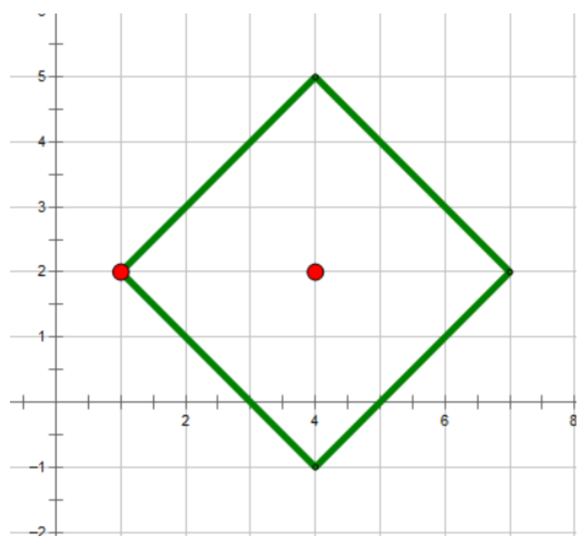


Figure B.7 Provided figure for Activity 5

Using the figure provided (Figure B.7), the formula for distance in Taxicab geometry formula, and the format of the example below:

(a) Calculate and illustrate the distances between each of the following points and the center of the circle

Point (1,2)	$\|1-4\|+\|2-2\|=3$
Point (6,1)	__________________
Point (6,3)	__________________
Point (3,4)	__________________

(b) Do you notice any similarities/differences in all of your responses to part (a)? Explain how these similarities/differences relate to the graph of this circle.

(c) Reflection: Use your responses to parts (a)-(b) to write the equation for this Taxicab circle. In your own words, use 2-3 sentences to explain what this equation is saying in relation to your graph and to the definition of a circle.

(d) Reflection: Does the definition of a circle change between Euclidean and Taxicab geometry? What geometric/algebraic properties of a circle in Euclidean geometry still hold in Taxicab geometry? What geometric/algebraic properties do not?

Pedagogical goal for Activity 5.

We hope through this activity, that students can better understand where values in the equation of a circle in Taxicab geometry are located on a graph, and not only that the distance formula is used within this equation, but why it is used in relation to the definition of a circle, i.e. – that the distance between any point on the circle and center should be constant, and is equal to the radius of that circle. By asking them to relate their findings to their understanding of a circle in Euclidean geometry, I hope students will begin to coordinate more processes across their *cEg* and *cTg schemata* in order to better the development of their *circle schema* in general.

Activity 6. (Thematization of the *circle schema*)

Define a new metric (or way to measure distance) between two points $P(x_1, y_1)$ and $Q\ (x_2, y_2)$ as $d_M = \max\{|x_2 - x_1|, |x_2 - x_1|\}$. This is sometimes referred to as Maxi-distance.

(a) Using the first grid provided, plot the points (1,2) and (4,6). Use the provided distance formula to calculate the distance between these two points and illustrate this distance on your graph. In your own words, explain how to calculate Maxi-distance.

(b) Use the definition of a circle and the grid provided to draw/sketch the Maxi-circle centered at (4,2) with radius 3. (Hint: think about how you use a metric to construct the radii of a circle in Euclidean/Taxicab geometry, and try to use this idea here).

(c) Using the definition of a circle, try to write the equation of this circle. If necessary, use some points on your circle to verify that your equation is correct.

(d) Reflection: Write 2-3 sentences to compare and contrast your geometric representation and the equation of this circle to that of a circle in Euclidean and Taxicab geometry.

Pedagogical goal of Activity 6.

From this activity, I hope that students can take what they have abstracted about a circle from their exploration in Euclidean and Taxicab geometry (and comparing these) and be able to illustrate a new metric and a circle using this metric. It is also a goal that from the definition of a circle, a student will be able to determine the equation of that circle by identifying that that the Maxi-distance between any point on the circle and center should be constant, and is equal to the radius of that circle. Thus, the student will have reflected and abstracted enough on their understanding of a circle to have a coherent understanding of how to complete these tasks. After a student is able to complete this activity, if they were given a new metric and asked to sketch a particular circle within that space, that they could utilize their *circle schema* and geometrical reasoning skills to do this on their own, without prompting.

Activity 7. (**Midpoint, Perpendicular bisector, Circle**)

As a note, the original problem given to students simply asked them to graphically illustrate where their apartment could be located and to associate a mathematical term with this illustration. While the openness of that problem was a positive for this collected data and analysis, the following suggestion for altering the problem is to to probe for certain understandings of mathematical concepts. As such, this problem can be used as an assessment tool or as an in class activity.

> Assume Georgia State's campus and surrounding streets are designed explicitly in a grid pattern, i.e.- distance is measured by Taxi-distance. You are looking for an apartment near campus, but you want to make sure that from your apartment, the walking distance to Aderhold (located at (-2, -2)) is the same as the walking distance to the College of Education and Human Development (located at (4, 3)), since you have classes in both locations.
>
> a. Provide one possible location of your apartment, given that you want it to be half way between the two buildings. What mathematical term would describe the point you provided? Is there more than one possible point that you could have given?
> b. Is it possible to find a location of your apartment, given that you do not want to be half way between the buildings but still equidistant to them? Is there more than one possible point that you could have provided? What mathematical concept would you associate with your solution?

c. Draw a graphical representation of how a taxicab circle could be used to identify a location for your apartment, given that you don't care if you are halfway between the buildings or not. Is there more than one way to do this? Does this relate at all to your answers from (a) and (b)?

d. How would your responses to these questions change if we provided two new locations for the buildings?

For part (a), the goal is for students to identify a midpoint between the buildings and use their definition of midpoint to understand this is what they have drawn. In addition, I hope students can generalize this point to all points that are half way between the buildings and understand their point is not unique.

For part (b), the ideal solution would be that a student provides a point that is on the perpendicular bisector and uses their personal definition of perpendicular bisector to understand their point falls on this line. Also, the aim is for students to see that they could have chosen any point on the perpendicular bisector that is not in the midset, bringing awareness again to the fact that a solution is not unique.

For part (c), the goal is for students to visualize how a Taxicab circle could be used to find such a point. Hopefully, they arrive at the conclusion that the center of the circle would be a possible location for their apartment and see that this is not a unique place. The answers to this part could be analyzed in more depth with regard to the coordination of processes discussed in Figure 3.9 in Section 3.1.2.3.

For part (d), the goal is to probe students to see if they can observe what happens when the original two points change. In particular, if students are able to compare properties of a mathematical definition between Euclidean and Taxicab geometries, or able to compare properties of a mathematical definition as you change certain conditions within one geometry, they may be exhibiting an object conception of that mathematical concept. For example, if a student is able to describe how, in Taxicab geometry, the perpendicular bisector appears differently if you change the slope of the original segment, this is evidence of an object conception of **Perpendicular bisector.**