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Essays on Risk Pricing in Insurance

BY

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A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree

Of

Doctor of Philosophy

In the Robinson College of Business

Of

Georgia State University

GEORGIA STATE UNIVERSITY

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ACCEPTANCE

This dissertation was prepared under the direction of the Qiheng Guo's Dissertation Committee. It has been approved and accepted by all members of that committee, and it has been accepted in partial fulfillment of the requirements for the degree of Doctoral of Philosophy in Business Administration in the J. Mack Robinson College of Business of Georgia State University.

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ABSTRACT

Essays on Risk Pricing in Insurance

BY

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Pricing risks in the insurance business is an essential task for actuaries. Implementing the appropriate pricing techniques to improve risk management and optimize its financial gain requires a thorough understanding of underlying risks and their interactions. In this dissertation, I address risk pricing in the context of insurance company by reviewing methods applied in practice, proposing new models, and also exploring different aspects of insurance risks.

This dissertation consists of three chapters. The first chapter provides a survey of existing capital allocation methods, including common approaches based on the gradients of risk measures and “economic” allocation arising from counterparty risk aversion. All methods are implemented in two example settings: binomial losses and using loss realizations from a catastrophe reinsurer. The stability of allocations is assessed based on sensitivity analysis with regards to losses. The results show that capital allocations appear to be intrinsically (geometrically) related, although the stability varies considerably. Stark differences exist between common and “economic” capital allocations.

The second chapter develops a dynamic profit maximization model for a financial institution with liabilities of varying maturity, and uses it for determining the term structure of capital costs.

As a key contribution, the theoretical, numerical, and empirical results show that liabilities with different terms are assessed differently, depending on the company's financial situation. In particular, for a financially constrained firm, value-adjustments due to financial frictions for liabilities in the far future are less pronounced than for short-term obligations, resulting in a strongly downward sloping term structure. The findings provide guidance for performance measurement in financial institutions.

The third chapter estimates a flexible affine stochastic mortality model based on a set of US term life insurance prices using a generalized method of moments approach to infer forward-looking, market-based mortality trends. The results show that neither mortality shocks nor stochasticity in the aggregate trend seem to affect the prices. In contrast, allowing for heterogeneity in the mortality rates across carriers is crucial. The major conclusion is that for life insurance, rather than aggregate mortality risk, the key risks emanate from the composition of the portfolio of policyholders. These findings have consequences for mortality risk management and emphasize important directions of mortality-related actuarial research.

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Chapter 1

Capital Allocation Techniques: Review and Comparison¹

1.1 Introduction

The question of how to allocate risk capital to different units or lines of business has generated considerable attention in the actuarial literature. This chapter reviews the form and the intellectual foundation of a variety of methods, and then compares their results in the context of a theoretical loss model and a specific real-world example. The goal is to offer insights on the similarities, differences, and stability of the different methods.

We find considerable variation in the results from different methods, although all of the allocations appear to be geometrically related. More precisely, depending on the example, the allocations seem to constitute a lower-dimensional manifold relative to the dimension of the allocation problem. Expected value allocations lie at one end of the frontier while extreme tail allocations are at the other. Furthermore, we also find significant differences in stability. Small changes in the underlying dataset can have dramatic effects on the allocation results associated with certain methods. In particular, allocations generated by Value-at-Risk (VaR) as well as those generated by extreme

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tail risk measures prove to be unstable due to their focus on a small number (in the case of VaR, just one) sample outcomes.

There are various surveys on capital allocation techniques and methods (Burkett et al., 2001; Albrecht, 2004; Venter, 2004; Bauer and Zanjani, 2013). It is important to stress that, in this chapter, we are not attempting to endorse or favor any particular method. While we have written elsewhere on the origins of “economic allocations” from a *theoretical* perspective (Bauer and Zanjani, 2016, 2018), the goal here is to gain perspective on *practical* differences. To elaborate, theoretical differences between two methods are less concerning if, in practice, they produce similar answers. Moreover, a theoretically appealing method is of little use if, in practice, it is unstable. Thus, we view the comparison of different allocation methods as a worthwhile endeavor to inform practice.

The rest of this essay is organized as follows. We start in Section 1.2 by reviewing the foundations of the capital allocation problem. In particular, we address the question of *why*—or, rather, under which conditions—the capital allocation problem is relevant, and *what* precisely we mean by a capital allocation. Section 1.3 then provide details on *how* capital is allocated. We commence by discussing the most popular conventional allocation approach, the so-called *Euler* or *Gradient* allocation principle, including its economic underpinnings. However, we also review alternative approaches, including so-called “distance minimizing” approaches, allocation by percentile layers, and “economic allocations” originating from counterparty risk aversion. We discuss relationships between the methods. We present a pedagogical allocation exercise illustrating the connections between “economic” methods and conventional methods in Section 1.5. In Section 1.6, we then implement all allocation methods in the context of data provided from a catastrophe reinsurer. We first describe the data and the specifics of the approaches. We then discuss and compare the resulting allocations and test their stability. Finally, Section 1.7 concludes.

1.2 The Foundations of Capital Allocation

1.2.1 Why Allocate Capital?

We must first establish *why* we allocate capital. The simple answer from the practitioner side is that allocation is a necessity for pricing and performance measurement. When setting benchmarks for lines of business within a multi-line firm, one must ensure that the benchmarks put in place are consistent with the firm's financial targets, specifically the target return on equity.

This seems logical at first glance, yet some of the academic literature has been skeptical. Phillips et al. (1998) noted that a “financial” approach to pricing insurance in a multi-line firm rendered capital allocation unnecessary, a point reiterated by Sherris (2006). The “financial” approach relies on applying the usual arbitrage-free pricing techniques in a complete market setting without frictional costs. In such a setting, one simply pulls out a market consistent valuation measure to calculate the fair value of insurance liabilities. Capital affects this calculation in the sense that the amount of capital influences the extent to which insurance claims are actually paid in certain states of the world, but, so long as the actuary is correctly evaluating the extent of claimant recoveries in various states of the world (including those where the insurer is defaulting), there is no need to apportion the capital across the various lines of insurance.

Once frictional costs of capital are introduced, the situation changes, as seen in Froot and Stein (1998), Zanjani (2002, 2010), and Bauer and Zanjani (2013, 2016, 2018). Frictions open up a gap between the expected profits produced by “financial” insurance prices and the targeted level of profits for the firm. In such a case, the “gap” becomes a cost that must be distributed back to business lines, like overhead or any other common cost whose distribution to business lines is not immediately obvious.

As a practical example, consider catastrophe reinsurers. Natural catastrophe risk is often argued to be “zero beta” in the sense of being essentially uncorrelated with broader financial markets. If we accept this assessment, basic financial theory such as the CAPM would then imply that a market rate of return on capital exposed to such risk would be the risk-free rate. Yet, target ROEs at these firms are surely well in excess of the risk-free rate. The catastrophe reinsurer thus has the problem of allocating responsibility for hitting the target ROE back to its various business lines

without any guidance from the standard arbitrage-free pricing models.

Viewed in this light, “capital allocation” is really shorthand for “capital cost allocation.” Capital itself, absent the segmentation of business lines into separate subsidiaries, is available for all lines to consume. A portion allocated to a specific line is not in any way segregated for that line’s exclusive use. Hence, the real consequence of allocation lies in the assignment of responsibility for capital cost: A line allocated more capital will have higher target prices.

An important point, to which we shall return later, is the economic meaning of the allocation. Merton and Perold (1993) debunk the notion that allocations could be used to guide business decisions involving inframarginal or supramarginal changes to a risk portfolio (e.g., entering or exiting a business line). The more common argument is that allocation is a marginal concept—offering accurate guidance on small, infinitesimal changes to a portfolio. As we will see, many methods do indeed have a marginal interpretation, but the link to marginal *cost* is not always a strong one.

1.2.2 Capital Allocation Defined

We first start with notation and by defining capital allocation.² Consider a one period model with N business lines with loss realizations $L^{(i)}$, $1 \leq i \leq N$, modeled as square-integrable random-variables in an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. At the beginning of the period, the insurer decides on a quantity of exposure in each business line and receives a corresponding premium $p^{(i)}$, $1 \leq i \leq N$, in return. The exposure is an indemnity parameter $q^{(i)}$, so that the actual exposure to loss $i \in \{1, 2, \dots, N\}$ is:

$$I^{(i)} = I^{(i)}(L^{(i)}, q^{(i)}).$$

We assume that an increase in exposure shifts the distribution of the claim random variable so that the resulting distribution has first order stochastic dominance over the former:

$$\mathbb{P}(I^{(i)}(L^{(i)}, \hat{q}^{(i)}) \geq z) \geq \mathbb{P}(I^{(i)}(L^{(i)}, q^{(i)}) \geq z) \quad \forall z \geq 0, \hat{q}_i \geq q_i.$$

²This subsection and the next borrows notation and approaches from Bauer and Zanjani (2013).

For simplicity, we typically consider $q^{(i)}$ representing an insurance company's quota share of a customer i 's loss:

$$I^{(i)} = L^{(i)} \times q^{(i)}.$$

Other specifications could be considered, but the specification above implies that the claim distribution is homogeneous with respect to the choice variable $q^{(i)}$. This simplifies capital allocation, although it should be noted that insurance claim distributions are not always homogeneous (Mildenhall, 2004), and the “adding up” property associated with a number of methods depends on homogeneity. Extensions to more general (non-linear) contracts are possible when generalizing the setting (Frees, 2017; Mildenhall, 2017).

We denote company assets as a and *capital* as k , where to fix ideas we adopt a common specification of the difference between the fair value of assets and the present value of claims. We denote by I the aggregate claims for the company, with the sum of the random claims over the sources adding up to the total claim:

$$\sum_{i=1}^N I^{(i)} = I.$$

However, actual payments made only amount to $\min\{I, a\}$ because of the possibility of default. We can also decompose actual payments, where the typical assumption in the literature is of equal priority in bankruptcy, so that the payment to loss i is:

$$\min\left\{I^{(i)}, \frac{a}{I}I^{(i)}\right\} \Rightarrow \sum_{i=1}^N \min\left\{I^{(i)}, \frac{a}{I}I^{(i)}\right\} = \min\{I, a\}.$$

Allocation is simply a division of the company's capital or assets across the N sources of risk, with $k^{(i)}$ representing the capital per unit of exposure assigned to the i -th source (and $a^{(i)}$ representing a similar quantity for assets). Of course, a full allocation requires that the individual amounts assigned to each of the lines “add up” to the total amount for the company:

$$\sum_{i=1}^N q^{(i)} k^{(i)} = k \text{ and } \sum_{i=1}^N q^{(i)} a^{(i)} = a.$$

It is worth noting that the question of *what* to allocate is not necessarily straightforward. Are

we to allocate the book value of equity? The market value of equity? Assets? In general, the answer to this question is going to be guided by the nature of costs faced by the firm. Even then, the costs may be difficult to define, as the decomposition of capital costs offered by Mango (2005) suggests.

1.3 Capital Allocation Techniques

Assuming we have answered the question of *what* to allocate, the remaining question is how to do it. Unfortunately, the answer is not straightforward: There is a bewildering variety of peddlers in the capital allocation market. Mathematicians bearing axioms urge us to adhere to their methods—failure to do so will result in some immutable law of nature being violated. Economists assure us that only their methods are “optimal.” Game theorists insist that only their solution concepts can be trusted. Practitioners wave off all of the foregoing as the raving of ivory tower lunatics, all the while assuring us that only their methods are adapted to the “real world” problems faced by insurance companies. Everyone has a “pet method,” perhaps one that has some intuitive appeal, or one that is perfectly adapted to some particular set of circumstances.

Given such variety, it is not surprising that allocation methods defy easy categorization. Many do end up in essentially the same place—the so-called Euler or Gradient Principle—a convergence noted by Urban et al. (2004) and Albrecht (2004). But others do not. In the following, we attempt to give an overview on the primary approaches. We keep the focus on concepts and examples. At the end of the section, we present in Table 1.1 a summary of the allocation examples and Figure 1.1 showing their relationship.

1.3.1 The Euler Method and Some Different Ways to Get There

Consider setting capitalization based on a differentiable risk measure $\rho(I) = k$ and further imagine allocating capital to line i based on:

$$k^{(i)} = \frac{\partial \rho(I)}{\partial q^{(i)}}. \quad (1.1)$$

This allocation is commonly referred to as *gradient* or *Euler* allocation, the latter being a ref-

erence to *Euler’s homogeneous function theorem*. This theorem states that for every positive homogeneous function of degree one $(q^{(1)}, \dots, q^{(N)}) \mapsto \rho(q^{(1)}, \dots, q^{(N)})$ —which is equivalent to requiring that the risk measure $\rho(I) = \rho(\sum_i q^{(i)} L^{(i)})$ be homogeneous—we automatically obtain the “adding up” property: $\rho(I) = \sum_{i=1}^N q^{(i)} \frac{\partial \rho(I)}{\partial q^{(i)}}$. The basic Euler approach can be found in Schmock and Straumann (1999) and Tasche (2004), among others.

The Euler or gradient allocation can also be implemented without requiring that $\rho(I) = k$ by normalizing:

$$\frac{k^{(i)}}{k} = \frac{\frac{\partial \rho(I)}{\partial q^{(i)}}}{\rho(I)}. \quad (1.2)$$

One of the major advantages of the Euler allocation is that it is possible to directly calculate (approximative) allocations given that one has an Economic Capital framework available that allows to derive $\rho(I)$ and k .³ More specifically, we can approximate the derivative occurring in the allocation rule by simple finite differences (although more advanced approaches may be used), that is:

$$\frac{\partial \rho(I)}{\partial q^{(i)}} \approx \frac{\rho(I + \Delta L^{(i)}) - \rho(I)}{\Delta}, \quad (1.3)$$

where $\Delta > 0$ is “small.”

A number of different paths lead to the Euler allocation. Denault (2001) proposes a set of axioms that define a *coherent* capital allocation principle when $\rho(I) = k$. His axioms required:

1. Adding up – The sum of allocations must be k .
2. No undercut – Any sub-portfolio would require more capital on a stand-alone basis.
3. Symmetry – If risk A and risk B yield the same contribution to capital when added to any disjoint subportfolio, their allocations must coincide.
4. Riskless allocation – A deterministic risk receives zero allocation in excess of its mean (see also Panjer (2002)).

³This is not at all to say that this task is simple. In fact, the computational complexity associated with evaluating economic capital presents a serious problem for financial institutions and frequently leads them to adopt second-best calculation techniques (Bauer et al., 2012). However, the availability of a suitable model for the different risk within a company’s portfolio and their interplay clearly is a necessity for the derivation for any coherent allocation of capital.

Denault shows that the risk measure must necessarily be linear in order for a coherent allocation to exist. This result essentially echoes the findings of Merton and Perold (1993), but shows that allocation based on a linear risk measure constitutes an exception to their indictment of using allocations to evaluate inframarginal or supramarginal changes to a portfolio. Linear risk measures are obviously of limited application, but Denault (2001) finds more useful results when analyzing *marginal* changes in the portfolio. In particular, he uses five axioms to define a “fuzzy” coherent allocation principle that exists for any given coherent, differentiable risk measure—and this allocation is given by the Euler principle applied to the supplied risk measure.

Kalkbrener (2005) uses a different set of axioms:

1. Linear aggregation – Which combines axioms 1 and 4 of Denault.
2. Diversification – Which corresponds to axiom 2 of Denault.
3. Continuity – Small changes to the portfolio should only have a small effect on the capital allocated to a subportfolio.

He finds that the unique allocation under these axioms is given by the Gâteaux derivative in the direction of the subportfolio, which again collapses to the Euler allocation:

$$k^{(i)} = \lim_{\varepsilon \rightarrow 0} \frac{\rho(I + \varepsilon L^{(i)}) - \rho(I)}{\varepsilon} = \frac{\partial \rho(I)}{\partial q^{(i)}}.$$

Some common homogeneous risk measures used in this axiomatic context are:

- Standard Deviation – derived from the so-called *standard deviation premium principle* (Deprez and Gerber, 1985);
- Value-at-Risk (VaR);
- Expected Shortfall (ES)/Tail Value at Risk (TVaR);
- Risk-Adjusted TVaR (RTVaR) – Furman and Landsman (2006) and under a different name in Venter (2010);
- Exponential risk measure – Venter et al. (2006);

- Distortion risk measures (Denneberg, 1990; Wang, 1996)
 - Proportional hazards transform (Wang, 1995, 1998);
 - Wang transform (Wang, 2000).

An alternative approach to the capital allocation problem is from the perspective of game theory. Lemaire (1984) and Mango (1998) both note the potential use of the Shapley Value, which rests on a different set of axioms, in solving allocation problems in insurance. The Shapley Value (Shapley, 1953) is a solution concept for cooperative games that assigns each player a unique share of the cost. Denault (2001) formally applies this idea to the capital allocation problem, in particular by relying on the theory of fuzzy cooperative games introduced by Aubin (1981). The key idea here is that the cost functional c of a cooperative game is defined via the risk measure ρ :

$$c(q^{(1)}, q^{(2)}, \dots, q^{(N)}) = \rho(q^{(1)}, q^{(2)}, \dots, q^{(N)}).$$

The problem is then to allocate shares of this “cost” to the players, with the set of valid solutions being defined as (see also Tsanakas and Barnett (2003)):

$$C = \left\{ (k^{(1)}, k^{(2)}, \dots, k^{(N)}) \mid c(q^{(1)}, q^{(2)}, \dots, q^{(N)}) = \sum k^{(i)} q^{(i)} \right. \\ \left. \& \ c(u) \geq \sum k^{(i)} u_i, \ u \in [0, q^{(1)}] \times \dots \times [0, q^{(N)}] \right\}.$$

Thus, for allocations in this set, any (fractional) subportfolio will feature an increase in aggregated per-unit costs, which connects to the usual solution concept in cooperative games requiring any solution to be robust to defections by subgroups of the players. The Aumann-Shapley solution is:

$$k^{(i)} = \frac{\partial}{\partial u_i} \int_0^1 c(\gamma u) d\gamma \Big|_{u_j = q^{(j)} \ \forall j}.$$

If the risk measure is subadditive, positively homogeneous, and differentiable, the solution boils down to the Euler method when loss distributions are homogeneous.⁴

⁴Aumann-Shapley values can also be used to cope with the problem of inhomogeneous loss distributions. In this case, Powers (2007) demonstrates that although the Euler principle will not apply, the Aumann-Shapley value can be used for the risk-allocation problem. Similarly, it may offer a solution if the underlying risk measure does not satisfy the homogeneity condition. For instance, Tsanakas (2009) shows how to allocate capital with convex risk measures, although the absence of homogeneity is shown to

The Euler method is also recovered in a number of “economic” approaches to capital allocation, where the risk measure is either embedded as a constraint in a profit maximization problem (e.g., Meyers (2003) or Stoughton and Zechner (2007)) or embedded in the preferences of policyholders (Zanjani, 2002). In either case, the marginal cost of risk ends up being defined in part by the gradient of the risk measure. To illustrate, consider the optimization problem adapted from Bauer and Zanjani (2016):

$$\max_{k, q^{(1)}, q^{(2)}, \dots, q^{(N)}} \underbrace{\left\{ \sum_{i=1}^N p^{(i)}(q^{(i)}) - V(\min\{I, a\}) - C \right\}}_{=\Pi} \quad (1.4)$$

subject to

$$\rho(q^{(1)}, q^{(2)}, \dots, q^{(N)}) \leq k.$$

From the optimality conditions associated with this problem, assuming a non-explosive solution exists, one can obtain:

$$\frac{\partial \Pi}{\partial q^{(i)}} = \left(-\frac{\partial \Pi}{\partial k} \right) \times \frac{\partial \rho}{\partial q^{(i)}} \quad (1.5)$$

at the optimal exposures and capital level. Hence, for the optimal portfolio, the risk adjusted marginal return $\frac{\frac{\partial \Pi}{\partial q^{(i)}}}{\frac{\partial \rho}{\partial q^{(i)}}}$ for each exposure i is the same and equals the cost of a marginal unit of capital $-\frac{\partial \Pi}{\partial k}$. More to the point, the right hand side of (1.5) allocates a portion of the marginal cost of capital to the i -th risk, an allocation that is obviously equivalent to the Euler allocation. In this sense, the Euler allocation is indeed “economic,” but it is important to stress that any economic content flows from the imposition of a risk measure constraint.

1.3.2 Distance-Minimizing Allocations

Not all approaches lead to the Euler principle. Laeven and Goovaerts (2004), whose work was later extended by Dhaene et al. (2003) and Dhaene et al. (2012), derive allocations based on minimizing a measure of the deviations of losses from allocated capital. Specifically, Laeven and Goovaerts

potentially produce an incentive for infinite fragmentation of portfolios. The intuition for this rather undesirable feature are risk aggregation penalties within inhomogeneous convex risk measures.

propose solving:

$$\begin{cases} \min_{k^{(1)}, k^{(2)}, \dots, k^{(N)}} \rho \left(\sum_{i=1}^N (I^{(i)}(L^{(i)}, q^{(i)}) - q^{(i)} k^{(i)})^+ \right), \\ \text{s.th. } \sum_{i=1}^N q^{(i)} k^{(i)} = k, \end{cases}$$

to identify an allocation, whereas Dhaene et al. (2012) consider:

$$\begin{cases} \min_{k^{(1)}, k^{(2)}, \dots, k^{(N)}} \sum_{i=1}^N q^{(i)} \mathbb{E} \left[\theta^{(i)} D \left(\frac{I^{(i)}(L^{(i)}, q^{(i)})}{q^{(i)}} - k^{(i)} \right) \right], \\ \text{s.th. } \sum_{i=1}^N q^{(i)} k^{(i)} = k, \end{cases}$$

where D is a (distance) measure and $\theta^{(i)}$ are weighting random variables with $\mathbb{E}[\theta^{(i)}] = 1$.

In the approach by Dhaene et al. (2012), certain choices for D and $\theta^{(i)}$ can reproduce various allocation methods. For instance, for $D(x) = x^2$ and $k = \sum \mathbb{E}[\theta^{(i)} I^{(i)}]$, they arrive at so-called *weighted risk capital allocations* $k^{(i)} = \mathbb{E}[\theta^{(i)} L^{(i)}]$ studied in detail by Furman and Zitikis (2008). Examples include the allocation based on the Esscher transform and the premium principle by Kamps (1998). Other choices lead to other allocation principles, including several that can be derived from the application of the Euler principle such as weighted TVaR (WTVaR).

1.3.3 Allocations by Co-Measures and the RMK Algorithm

Euler methods require calculation of gradient of risk measures, which sometimes can present a numerical challenge. An alternative approach is the Ruhm-Mango-Kreps (RMK) algorithm (Ruhm et al., 2003; Kreps, 2005), a popular approach of capital allocation in practice, partly due to its ease of implementation. According to Kreps (2005), it commences by defining $k = \sum_i q^{(i)} k^{(i)} = \mathbb{E}[I] + R$ as the *total capital* to support the company's aggregate loss I , where $\mathbb{E}[I]$ is the mean (reserve) and R is the risk load. Then the capital allocations $q^{(i)} k^{(i)}$ for risks i emanating from the asset or the liability side are defined as:

$$\begin{aligned} q^{(i)} k^{(i)} &= \mathbb{E}[I^{(i)}] + R_i \\ &= \mathbb{E}[I^{(i)}] + \mathbb{E}[(I^{(i)} - \mathbb{E}[I^{(i)}]) \phi(I)], \end{aligned} \tag{1.6}$$

where ϕ is the *riskiness leverage*, and “all” that one needs to do is to find the appropriate form of ϕ . This allocation method adds up by definition, it scales with a currency change if $\phi(\lambda x) = \phi(x)$ for a positive constant λ .

Different interpretations are possible, but key advantage ease of implementation since it solely relies on taking “weighted averages” (Ruhm, 2013):

Algorithm 1. *RMK Algorithm*

- *Simulate possible outcomes by component and total.*
- *Calculate expected values $\mathbb{E}[I^{(i)}]$ by taking simple averages.*
- *Select a risk measure on total company outcomes and express the risk measure as leverage factors.*
- *Calculate risk-adjusted expected values $\mathbb{E}[I^{(i)} \phi(I)]$ by taking “weighted averages”.*
- *Allocate capital in proportion to risk, by:⁵*

$$\frac{q^{(i)} k^{(i)}}{k} = \frac{\mathbb{E}[I^{(i)} \phi(I)] + \mathbb{E}[I^{(i)}](1 - E[\phi(I)])}{\mathbb{E}[I \phi(I)] + \mathbb{E}[I](1 - E[\phi(I)])}.$$

Of course, the RMK algorithm only presents the general framework. The crux lies in the determination of the riskiness leverage ϕ . Various examples are presented in Kreps (2005), some of which result in familiar allocation principles that can be alternatively derived by the gradient principles.

More generally, Venter (2004) and Venter et al. (2006) introduce so-called *co-measures*. Specifically, consider the risk measure:⁶

$$\rho(I) = \mathbb{E} \left[\sum_j h_j(I) \phi_j(I) \middle| \text{Condition on } I \right],$$

⁵We adjust Ruhm’s formula here to be in line with the allocation above.

⁶The definition in Venter (2010) allows for different conditions for the different j .

where the h are linear functions. They then define the *co-measure* as:

$$r(I^{(i)}) = \mathbb{E} \left[\sum_{j=1}^J h_j(I^{(i)}) \phi_j(I) \middle| \text{Condition on } I \right],$$

which satisfies $\sum_{i=1}^n r(I^{(i)}) = \rho(I)$ and thus serves as an allocation.

As Venter (2010) points out, even for one risk measure there may be different co-measures, i.e. the representation is not unique. Some of them yield representations that are equivalent to the gradient allocation, but this is not necessarily the case. In Table 1.1, the last column lists the riskiness leverage / co-measures of some common allocation methods. We introduce two allocation approaches by directly relying on their implementation via the RMK algorithm and co-measures.

Myers-Read Approach

Myers and Read (2001) argue that, given complete markets, default risk can be measured by the default value, i.e. the premium the insurer would have to pay for guaranteeing its losses in the case of a default. They then propose that “sensible” regulation will require companies to maintain the same default value per dollar of liabilities and effectively choose this latter ratio as their risk measure.

More precisely, following Mildenhall (2004), the default value can be written as:⁷

$$\begin{aligned} D(q^{(1)}, q^{(2)}, \dots, q^{(N)}) \\ &= \mathbb{E} [I_{\{I \geq a\}} (I - a)] \\ &= \mathbb{E} [I_{\{I \geq \mathbb{E}[I] + k^{(1)} q^{(1)} + \dots + k^{(N)} q^{(N)}\}} (I - [\mathbb{E}[I] + k^{(1)} q^{(1)} + \dots + k^{(N)} q^{(1)}])] , \end{aligned}$$

and the company’s default-to-liability ratio is:

$$c = \frac{D}{\mathbb{E}[I]} = \frac{\mathbb{E}[I_{\{I \geq a\}} (I - a)]}{\mathbb{E}[I]}.$$

Myers and Read (2001) verify the “adding up” property for D —which again shows a relationship

⁷In contrast to Myers and Read (2001), we ignore the asset side and possible adjustments in calculating the “option value”.

to the Euler principle. They continue to demonstrate that in order for the default value to remain the same as an exposure is expanded, it is necessary that:

$$c = \frac{\frac{\partial D}{\partial q^{(i)}}}{\mathbb{E}[L^{(i)}]},$$

which in turn yields:

$$\begin{aligned} c \mathbb{E}[L^{(i)}] &= \mathbb{E} \left[\left(L^{(i)} - (\mathbb{E}[L^{(i)}] + k^{(i)}) \right) 1_{\{I \geq a\}} \right] \\ \Rightarrow k^{(i)} &= \mathbb{E} \left[\left(L^{(i)} - \mathbb{E}[L^{(i)}] \right) | I \geq a \right] - c \frac{\mathbb{E}[L^{(i)}]}{\mathbb{P}(I \geq a)}. \end{aligned}$$

This is similar to the allocation found by Venter et al. (2006), although they allocate assets rather than capital so $\mathbb{E}[L^{(i)}]$ does not occur in the first term. As indicated in their paper, it is possible to represent this allocation as a co-measure using $J = 2$, $h_1(I) = I - \mathbb{E}[I]$, $\phi_1(I) = 1_{\{I \geq a\}}$, $h_2(I) = I$, and $\phi_2(I) = -\frac{c}{\mathbb{P}[I \geq a]}$.

D’Arcy (2011) Allocation

D’Arcy (2011) considers allocations by the RMK algorithm but identifies the flexibility in choosing the riskiness leverage as its “greatest flaw.” To uniquely identify the “right” function, he proposes to use capital market concepts, particularly cost-of-capital to “reflect the actual cost of recapitalizing the firm.” Specifically, he allows the riskiness leverage to depend both on the size of the loss realization as well as on the type of shock leading to the loss (idiosyncratic, industry-wide, or systemic). The riskiness leverage factor is the ratio of the cost of capital divided by the normal cost of capital, where the “realized” cost of capital, in addition to systemic factors, depends additively on the ratio of aggregate losses to the firm’s actual capital:

$$\phi(I) = \frac{1_{\{I \geq C\}}(\text{CoC}_{\text{market}} + \frac{I-a}{a})}{\text{CoC}_{\text{normal}}}. \quad (1.7)$$

It is important to note that D’Arcy (2011) only proposes the RMK algorithm for the “consumptive” aspect of capital allocation, whereas he also includes a “non-consumptive” allocation in the spirit of Mango (2005) (see Section 1.3.4).

1.3.4 Consumptive vs. Non-consumptive Capital

Mango (2005) argues that capital costs consist of two parts: On the one hand, an insurer's capital stock can be depleted if a loss realization exceeds the reserves for a certain segment or line, or when reserves are increased. He refers to this as a *consumptive* use of capital since in this case, funds are transferred from the (shared) capital account to the (segment-specific) reserve account. The second, *non-consumptive* component arises from a “capacity occupation cost” that compensates the firm for preclusion of other opportunities. It is thought to originate from rating agency requirements in the sense that taking on a certain liability depletes the underwriting capacity.

The importance of this distinction for our purposes is that it complicates practice in cases where the two sources of costs require different approaches to allocation. For example, D’Arcy (2011) follows Mango’s suggestion by first allocating consumptive capital via the RMK algorithm, where the riskiness leverage or *capital call cost factor* ϕ is associated with the cost of capital (see also Bear (2006) and D’Arcy (2011)). He then allocates capital according to regulatory rules, and the final allocation ends up as an average of the two allocations. Thus, the two different motivations for holding capital are reflected in a hybridization of allocation methods.

1.3.5 Capital Allocation by Percentile Layer

Bodoff (2009) argues that allocations according to Value at Risk or according to tail risk measures do not consider loss realizations at smaller percentiles, even though the firm’s capital obviously supports these loss levels as well. Thus, in order to allocate, he advocates considering *all* loss layers *up to* the considered confidence level. His approach considers allocating capital to loss events, but since we are interested in allocating capital to lines we follow the description from Venter (2010).

Assume the capital k is determined by some given risk measure. For instance, VaR is used in Bodoff (2009). Then the allocation for the layer of capital $[z, z + dz]$ is:

$$\mathbb{E} \left[\frac{I^{(i)}}{I} \mid I \geq z \right] \times dz.$$

Going over all layers of capital, we obtain the allocation:

$$q^{(i)} k^{(i)} = \int_0^k \mathbb{E} \left[\frac{I^{(i)}}{I} | I \geq z \right] dz,$$

where obviously:

$$\sum_{i=1}^N q^{(i)} k^{(i)} = \int_0^k \mathbb{E} \left[\frac{I}{I} | I \geq z \right] dz = \int_0^k dz = k.$$

As Venter (2010) points out, even if k is set equal to a risk measure and allowed to change with the volume of the writings, the resulting allocation does not collapse to the gradient allocation in any known cases.

When implementing the approach based on a sample of size N , obviously it is necessary to approximate the integral formulation above. When we base it on VaR_α and use the simple empirical quantile for its estimation, we can set:

$$q^{(i)} k^{(i)} = \sum_{j=1}^{\alpha N} \mathbb{E} \left[\frac{I^{(i)}}{I} | I \geq I_{(j)} \right] [I_{(j)} - I_{(j-1)}],$$

where we set $I_{(0)} = 0$. Since the conditional expectations within the sum have to be also approximated by taking averages, the implementation in a spreadsheet may be cumbersome (or even infeasible) for large samples.

1.3.6 “Economic” Counterparty Allocation

The previous Sections 1.3.1 to 1.3.5 outline so-called “conventional” capital allocation methods. While all conventional methods vary in techniques, they consider capital allocation as a technical problem but do not contemplate the motivation for holding capital in the first place. In contrast, Bauer and Zanjani (2016) argue that the demand side’s risk preferences would have an impact on the optimal capital and allocation decision. More precisely, Bauer and Zanjani introduce a theoretical framework of capital allocation that is derived from the insurer’s profit maximization subject to counterparty risk aversion. The concept is fundamentally different from problem (1.4), where the premium and risk measure are exogenously determined. Here, the premium is a choice variable for the insurer, subject to a participation constraint for the counterparty. In particular,

there is no risk measure to be imposed in the first place, but an endogenous expression for the risk measure can be derived from the allocation rule. In the remainder of the essay, we will refer to this allocation as Bauer-Zanjani allocation.

To elaborate on the setup, it is assumed that we have a group of N consumers/policyholders. Each consumer has wealth w_i and is susceptible to a loss L_i , which is random. Each consumer is risk averse by default, and has utility function U_i and expected utility v_i . Each consumer can purchase insurance to recover a portion $q_i \in [0, 1]$ of the loss at premium rate p_i . Both p_i and q_i are chosen by the insurer. The insurer will collect the premium upfront and deliver a total of $I = \sum q_i L_i$ for indemnity payments. The expected payback to each consumer is $e_i = \mathbb{E}[R_i] = \min(q_i L_i, q_i L_i a / I)$, with the latter amount triggered by insurer's default, or when its asset a are less than total indemnity payment I . The insurer then solves the following one-period optimization problem:

$$\max_{a, \{q_i\}, \{p_i\}} \sum_{i=1}^N p_i - \sum_{i=1}^N e_i - \tau a \quad (1.8)$$

subject to participation constraints for each consumer:

$$v_i = \mathbb{E}[U_i(w_i - p_i - L_i + R_i)] \geq \mathbb{E}[U_i(w_i - L_i)] \quad \forall i.$$

The solution suggests that an allocation weight for each consumer/policyholder is:

$$\frac{k_i}{k} = \frac{\mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U'_k}{v'_k} \frac{I_k}{I^2} \frac{\partial I_i}{\partial q_i} \right]}{\mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U'_k}{v'_k} \frac{I_k}{I} \right]}. \quad (1.9)$$

The supporting risk measure takes the form:

$$\rho(I) = \exp \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} [\log(I)] \right\},$$

where the measure $\tilde{\mathbb{P}}$ is given by its likelihood ratio:

$$\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}} = \frac{\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U'_k}{v'_k} \frac{I_k}{I}}{\mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U'_k}{v'_k} \frac{I_k}{I} \right]}.$$

It turns out that this risk measure does not satisfy the common axioms of coherence and convexity. It is important to note that within this framework, the allocation results are determined through optimization. In particular, the exposure parameters q_i are not fixed but determined in the optimization procedure – unlike in the “conventional” approaches.

1.3.7 Some Connections between the Allocations

Table 1.1 presents implementation of all allocation methods mentioned in the previous sections. As a side note, there are several connections between the various allocation methods beyond what has been pointed out so far in this section. We list them here:

- For elliptical distributions, the Euler allocation yields to the same relative amounts of capital allocated to each line, irrespective of which (homogeneous) risk measure we use (McNeil et al., 2015, Corollary 6.27).
- Asimit et al. (2012) show that risk capital allocation based on TVaR is asymptotically proportional to the corresponding Value-at-Risk (VaR) risk measure as the confidence level goes to 1.

Moreover, Figure 1.1 illustrates graphically the relationship between various methods discussed in this section. In the next three sections, we compare the methods based on two example settings.

1.4 Comparison of Capital Allocation Methods

Next, we analyze the allocation problems and methods discussed in the previous sections in the context of (i) a Binomial loss model and (ii) real-world catastrophe insurance losses. Specifically for comparison (ii), we gained access to (scaled) simulated loss data for a global catastrophe reinsurance company. We believe this data offers a degree of realism missing from previous contributions where proposed allocation methods are only studied in the context of stylized examples or based on Normal distributions (which is particularly limiting as discussed in Section 1.3.7).

We start by outlining the allocation approaches in Section 1.4.1. In the following two Sections 1.5 and 1.6, we present the implementation in the two settings. In Section 1.6, we also present a

Allocation	Risk Measure/ Capital to Hold $\rho(I)$	Allocation to Line i : $\partial\rho(I)/\partial q^{(i)}$	RMK Riskiness Leverage / Co-Measures
CoVar	$\mathbb{E}[I] + \beta \text{StDev}[I]$	$\mathbb{E}[I^{(i)}] + \beta \frac{\text{Cov}(I^{(i)}, I)}{\sqrt{\text{Var}[I]}}$	$\phi(I) = \beta \frac{I - \mathbb{E}[I]}{\text{StDev}(I)}$
VaR	$\text{VaR}_\alpha(I) = \inf\{x : F_I(x) \geq \alpha\}$	$\mathbb{E}[L^{(i)} I = \text{VaR}_\alpha(I)]$	$\phi(I) = \frac{\text{dirac}.\delta(I - \text{VaR}_\alpha(I))}{f_I(\text{VaR}_\alpha(I))}$
TVaR	$\text{TVaR}_\alpha(I) = \mathbb{E}[I I \geq \text{VaR}_\alpha(I)]$	$\mathbb{E}[L^{(i)} I \geq \text{VaR}_\alpha(I)]$	$\phi(I) = \frac{I_{\{I \geq \text{VaR}_\alpha(I)\}}}{1 - \alpha}$
RTVaR	$\mathbb{E}[I I \geq \text{VaR}_\alpha(I)]$ $+ \beta \text{StDev}[I I \geq \text{VaR}_\alpha(I)]$	$\mathbb{E}[L^{(i)} I \geq \text{VaR}_\alpha(I)]$ $+ \beta \frac{\text{Cov}(I_i, I I \geq \text{VaR}_\alpha(I))}{\text{StDev}(I I \geq \text{VaR}_\alpha(I))}$	$h_1(x) = h_2(x) = x, \phi_1 \equiv 1,$ $\phi_2 = \beta \frac{I - \mathbb{E}[I]}{\text{StDev}(I I \geq \text{VaR}_\alpha(I))}$
Exponential	$\mathbb{E}\left[I \exp\left\{\frac{cI}{\mathbb{E}[I]}\right\}\right]$	$\mathbb{E}\left[L^{(i)} \exp\left\{\frac{cI}{\mathbb{E}[I]}\right\}\right] + c \frac{\mathbb{E}[L^{(j)}]}{\mathbb{E}[I]}$ $\times \mathbb{E}\left[I \exp\left\{\frac{cI}{\mathbb{E}[I]}\right\} \times \left(\frac{L^{(i)}}{\mathbb{E}[L^{(i)}]} - \frac{I}{\mathbb{E}[I]}\right)\right]$	$h_1(I) = I, \phi_1(I) = e^{\frac{cI}{\mathbb{E}[I]}} + c \frac{I e^{\frac{cI}{\mathbb{E}[I]}}}{\frac{cI}{\mathbb{E}[I]}}$ $h_2(I) = -\mathbb{E}[I], \phi_2(I) = \frac{cI^2 e^{\frac{cI}{\mathbb{E}[I]}}}{\mathbb{E}[I^2]}$
Distortion	$\mathbb{E}[I g'(S_I(I))]$ *Proportional Hazard $g(p) = p^a$ * Wang Transformation $g(p) = 1 - \Phi(\Phi^{-1}(1-p) - \lambda)$	$\mathbb{E}[L^{(i)} g'(S_I(I))]$	—
Myers-Read	$\mathbb{E}[I_{\{I \geq a\}}(I - a)]$	$\mathbb{E}[(L^{(i)} - \mathbb{E}[L^{(i)}]) I \geq a]$ $-\frac{\mathbb{E}[I_{\{I \geq a\}}(I - a)]}{\mathbb{E}[I]} \frac{\mathbb{E}[L^{(i)}]}{\mathbb{P}(I \geq a)}$	$h_1(I) = I - \mathbb{E}[I], \phi_1(I) = I_{\{I \geq a\}},$ $h_2(I) = I, \phi_2(I) = -\frac{c}{\text{Pr}[I \geq a]}.$
Esscher	$\frac{\mathbb{E}[I e^{tI}]}{E[e^{tI}]}$	$\frac{\mathbb{E}[L^{(i)} e^{tI}]}{E[e^{tI}]}$	—
Kamps	$\frac{\mathbb{E}[I(1 - e^{-tI})]}{E[(1 - e^{-tI})]}$	$\frac{\mathbb{E}[L^{(i)}(1 - e^{-tI})]}{E[(1 - e^{-tI})]}$	—
D'Arcy	—	—	$\phi(I) = \frac{I_{\{I \geq C\}}(\text{CoC}_{\text{market}} + \frac{I - a}{a})}{\text{CoC}_{\text{normal}}}$
Bodoff	$\text{VaR}_\alpha(I)$	$\sum_{j=1}^{\alpha N} \mathbb{E}\left[\frac{L^{(i)}}{I} I \geq I_{(j)}\right] [I_{(j)} - I_{(j-1)}]$	—
Bauer-Zanjani	$\exp\left\{\mathbb{E}^{\tilde{\mathbb{P}}}[\log(I)]\right\}$	$\rho(I) \cdot \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{I_i}{I}\right]$	—

Table 1.1: Implementation of allocation methods

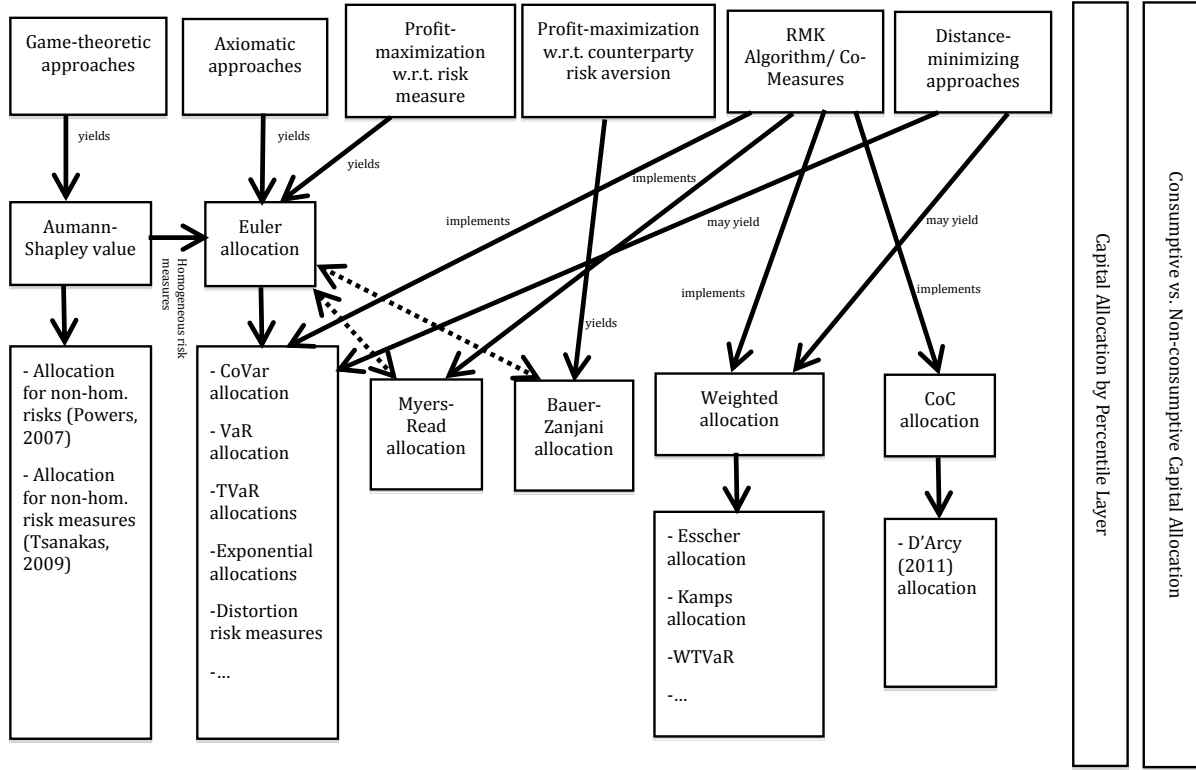


Figure 1.1: Overview of capital allocation methods

sensitivity analysis to examine the stability of allocations.

1.4.1 Allocation Approaches

For allocation techniques in both examples, we consider the following approaches:

- Allocation by expected values.
- A covariance allocation. Here we choose the parameter $\beta = 2$ due to the similarities of the supporting risk measure to a quantile for a Normal distribution, where 2 (or rather 1.96 for a two-sided confidence interval of 95%) is a common choice.
- TVaR (Expected Shortfall) allocations for confidence levels $\alpha = 75\%, 90\%, 95\%$, and 99% .

- VaR based allocations for confidence levels $\alpha = 95\%$ and 99% . In example (ii), in addition to estimating the allocations based on splitting up the corresponding empirical quantile in its loss components (labeled “simple”), we consider an estimation that takes into account the surrounding realizations by imposing a bell curve centered at the quantile with a standard deviation of three (labeled “bell”).
- Exponential allocations for parameters $c = 0.1, 0.25$, and 1 .
- Allocations based on a distortion risk measure—in particular proportional hazard transform and Wang transform. For the proportional hazard transform parameters, we use $a = 0.6$, $a = 0.8$, and $a = 0.95$, where we follow Wang (1998) indicating that a typical transform parameter ranges from 0.5 to 1 , depending on the ambiguity regarding the best-estimate loss distribution. For the Wang transform parameters, we use $\lambda = 0.25$, $\lambda = 0.5$, and $\lambda = 0.75$, where we follow Wang (2012) indicating that a typical transform parameters in the reinsurance domain range between 0.5 and 0.77 , whereas 0.25 is a typical assumption for long-termed Sharpe ratios in the financial market.
- Myers-Read allocations for different capital levels (Section 1.3.3). In particular, we choose the capital equal to the 99.94% quantile, which roughly depends on capital levels to support an AM Best AA+ rating; three times the premium which is roughly consistent with NAIC aggregate levels; and the 99% VaR just for comparison purposes.
- Weighted/transform-based allocations based on the Esscher and Kamps transform. Here we choose transform parameters such that an evaluation is possible (non-explosive) but sufficiently different from the expected value allocation (which results for $t = 0$).
- The D’Arcy (2011) implementation of the RMK algorithm (Section 1.3.3), where we rely two on the same (first) two capital levels as for the Myers-Read allocation.
- Allocations based on the percentile layer (Bodoff in Section 1.3.5), where we allocate the 90% , 95% , and 99% VaR.
- RTVaR allocations with $\alpha = 75\%$, 90% , and 95% and $\beta = 2$ as for the covariance allocation.

- Allocation on the (simple) average of the four considered TVaRs.
- Bauer-Zanjani allocations. Since their approach relies on the solution of an optimization problem, we need to specify the necessary ingredients. We provide details on the solution within both settings in Appendix Section 1.8.

1.5 The Case of Heterogeneous Bernoulli Losses

We consider individuals that face Bernoulli-distributed losses belonging to three groups. More precisely, we assume:

- $m = 3$ groups of identical consumers, with $N_1 = N_2 = N_3 = 5$, with the same probability of loss $\pi_1 = \pi_2 = \pi_3 = 0.1$;
- The size of the losses differ among groups, with $l_1 = 1$, $l_2 = 2$, and $l_3 = 3$;

For conventional allocation, we consider allocation of capital three groups of consumer and assume full exposure in all groups. Therefore, the total loss indemnity in each group is $I_i = k_i l_i$, where $k_i \sim \text{Binomial}(N_i, \pi_i)$ and the total indemnity of the company is $I = k_1 l_1 + k_2 l_2 + k_3 l_3$. The computation of expectation, standard deviation, VaR, TVaR, and other moments of I are trivial and those statistics are used to calculate allocation weights. The results are listed in Table 1.2.

For Bauer-Zanjani allocation, we assume that all groups have the same CARA preferences and absolute risk aversion parameters $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and solve the allocation problem using α from 0.1 to 2. It is well known that with CARA preferences, initial wealth is irrelevant. We impose a frictional cost of $\tau = 0.01$. We optimize objective function (1.8), obtain the parameters, and use equation (1.10) in Appendix 1.8.1 to obtain the allocation weights. The optimization and allocation results are shown in Table 1.3.

Figure 1.2 provides a direct comparison of the different allocation methods (except for Exp c=1). More precisely, since we have three different groups and allocation percentages k_i/k add up to one, we can depict allocations by two numbers. We choose allocation percentages to groups 1 and 3, where of course the allocation to Group 2 can be calculated as the difference of their sum and one.

Surprisingly, all methods lie along a line that shows a trade off between allocating more to Group 3, which has the biggest loss size, and allocating more to Group 1, which has the smallest loss size. We observe that the tail-focused allocations such as Myers-Read lie on one extreme end and allocate the most to Group 3, while expected value allocation lies on the other extreme end and allocates the least to Group 3. Methods with distortion risk measures and weighted distribution transformation result in allocations closer to expected value method. VaR, TVaR and Risk-adjusted TVaR methods are in between, with RTVaR (more tailed focused) allocating more to Group 3, followed by TVaR and VaR.

The Bauer-Zanjani allocations also adhere to the same relationship, where a higher risk aversion parameter α pushes towards the tail risk measures – e.g. BZ-2.0 (the Bauer-Zanjani allocation with $\alpha = 2$) resembles the TVaR 95% allocation (seen in Figure 1.2(a)). For smaller risk aversion, the allocation is closer to the allocations focusing on the whole distribution – e.g. BZ-0.1 (the Bauer-Zanjani allocation with $\alpha = 0.1$) resembles the Bodoff, Wang transformation with large λ , proportional hazard with small a , and the Kamps allocations (seen in Figure 1.2(b)).

Overall, we find that despite the variety in capital allocation methods proposed, it appears that they produce rather similar results with differences being explained by a single parameter that roughly corresponds to how much tail scenarios are emphasized.

1.6 The Case of Catastrophe Reinsurance Losses

For this application, we begin by describing in detail the data and the approach to aggregation in Section 1.6.1. In particular, for our analyses, we limit the presentation to an aggregation to four lines only in order to facilitate interpretation of the results. Here we follow Bauer and Zanjani (2018) where the same data and aggregations are used. We then compare allocation methods in Section 1.6.2. Finally, we consider their stability in Section 1.6.3.

1.6.1 Description of the Data

We are given 50,000 joint loss realizations for 24 distinct lines differing by peril and geographical region. Figure 1.3 provides a histogram of the aggregate loss distribution, and Table 1.4 lists the

Allocation	Group 1	Group 2	Group 3	Sum	RiskMeas
Exp Val	0.5000	1.0000	1.5000	3.0000	3.0000
	16.67%	33.33%	50.00%	100.00%	
CovWBeta	0.8586	2.4343	4.7271	8.0200	8.0200
beta = 2	10.71%	30.35%	58.94%	100.00%	
VaR 95%	0.6611	2.4447	4.8942	8.0000	8.0000
	8.26%	30.56%	61.18%	100.00%	
VaR 99%	0.8780	2.9425	6.1795	10.0000	10.0000
	8.78%	29.42%	61.80%	100.00%	
TVaR 75%	0.6656	2.0093	3.7754	6.4502	6.4502
	10.32%	31.15%	58.53%	100.00%	
TVaR 90%	0.7582	2.0146	4.5103	7.2832	7.2832
	10.41%	27.66%	61.93%	100.00%	
TVaR 95%	0.7810	2.5699	5.6869	9.0378	9.0378
	8.64%	28.44%	62.92%	100.00%	
TVaR 99%	0.8953	3.0652	6.9330	10.8935	10.8935
	8.22%	28.14%	63.64%	100.00%	
Exponential	0.5445	1.1633	1.8607	3.5684	3.5684
c = 0.1	15.26%	32.60%	52.14%	100.00%	
Exponential	0.6026	1.4657	2.6257	4.6939	4.6939
c = 0.25	12.84%	31.22%	55.94%	100.00%	
Exponential	-1.6958	4.5706	22.1425	25.0172	25.0172
c = 1	-6.78%	18.27%	88.51%	100.00%	
Proportional Hazard	0.6464	1.5479	2.8057	5.0000	5.0000
a = 0.6	12.93%	30.96%	56.11%	100.00%	
Proportional Hazard	0.5599	1.2152	1.9970	3.7722	3.7722
a = 0.8	14.84%	32.22%	52.94%	100.00%	
Proportional Hazard	0.5133	1.0467	1.6054	3.1653	3.1653
a = 0.95	16.22%	33.07%	50.72%	100.00%	
Wang Trans.	0.5713	1.2297	1.9857	3.7868	3.7868
$\lambda = 0.25$	15.09%	32.47%	52.44%	100.00%	
Wang Trans.	0.6428	1.4808	2.5548	4.6784	4.6784
$\lambda = 0.5$	13.74%	31.65%	54.61%	100.00%	
Wang Trans.	0.7148	1.7523	3.2058	5.6729	5.6729
$\lambda = 0.75$	12.60%	30.89%	56.51%	100.00%	
Myers-Read	0.2463	1.7674	4.9863	7.0000	7.0000
a = 10	3.52%	25.25%	71.23%	100.00%	
Myers-Read	0.1080	1.2239	3.6681	5.0000	5.0000
a = 8	2.16%	24.48%	73.36%	100.00%	
Esscher	0.5468	1.1949	1.9563	3.6981	3.6981
t = 0.1	14.79%	32.31%	52.90%	100.00%	
Kamps	0.6391	1.5347	2.6560	4.8299	4.8299
t = 0.1	13.23%	31.78%	54.99%	100.00%	
Esscher	0.5045	1.0181	1.5410	3.0637	3.0637
t = 0.01	16.47%	33.23%	50.30%	100.00%	
Kamps	0.6487	1.5926	2.8280	5.0694	5.0694
t = 0.01	12.80%	31.42%	55.79%	100.00%	
Esscher	0.5005	1.0018	1.5041	3.0063	3.0063
t = 0.001	16.65%	33.32%	50.03%	100.00%	
Kamps	0.6499	1.5993	2.8478	5.0969	5.0969
t = 0.001	12.75%	31.38%	55.87%	100.00%	
D'Arcy	0.5073	1.0380	1.6067	3.1519	3.1519
a = 10	16.09%	32.93%	50.98%	100.00%	
D'Arcy	0.5312	1.1442	1.9093	3.5847	3.5847
a = 8	14.82%	31.92%	53.26%	100.00%	
Bodoff 90%	0.9016	1.9442	3.1542	6.0000	6.0000
	15.03%	32.40%	52.57%	100.00%	
Bodoff 95%	1.0894	2.5262	4.3844	8.0000	8.0000
	13.62%	31.58%	54.80%	100.00%	
Bodoff 99%	1.2622	3.0769	5.6610	10.0000	10.0000
	12.62%	30.77%	56.61%	100.00%	
RTVaR 95%	0.9740	3.1442	7.6200	11.7382	11.7382
	8.30%	26.79%	64.92%	100.00%	
RTVaR 99%	0.9909	3.5364	8.7626	13.2899	13.2899
	7.46%	26.61%	65.93%	100.00%	

Table 1.2: Conventional allocation results—heterogeneous Bernoulli case

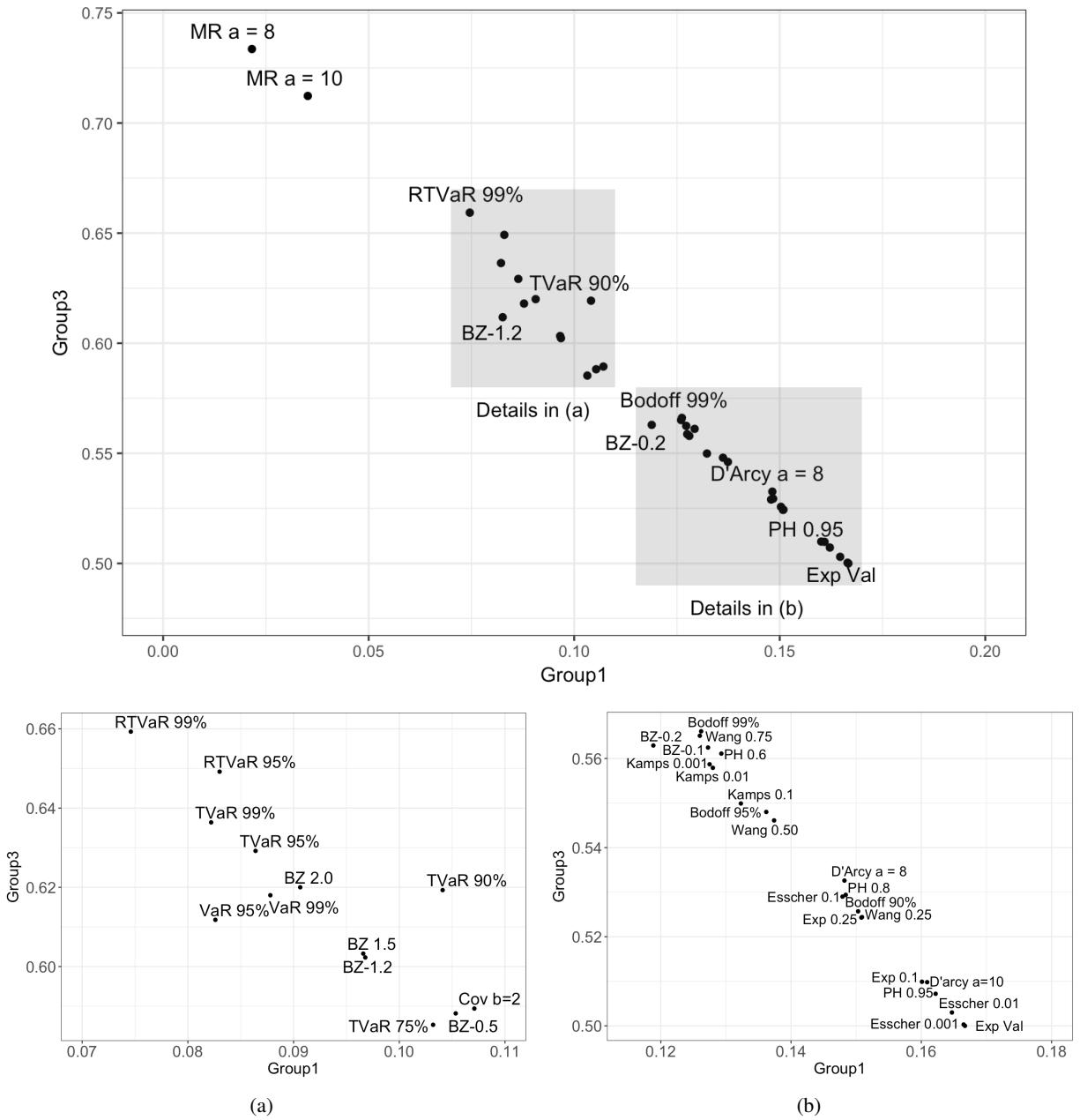


Figure 1.2: Comparison of allocations–heterogeneous Bernoulli losses

α	Optimization Results							Allocation Results		
	a	p_1	p_2	p_3	q_1	q_2	q_3	Group 1	Group 2	Group 3
0.1	5.2608	0.0913	0.1961	0.3023	0.9232	0.9747	1.0000	12.73%	31.03%	56.25%
0.2	6.3701	0.1023	0.2270	0.3719	0.9584	0.9846	1.0000	11.89%	31.82%	56.29%
0.3	6.9498	0.1085	0.2542	0.4375	0.9589	0.9889	1.0000	10.17%	31.13%	58.70%
0.4	7.4627	0.1162	0.2820	0.5087	0.9773	0.9912	1.0000	10.53%	30.72%	58.75%
0.5	7.7758	0.1226	0.3114	0.5868	0.9810	0.9927	1.0000	10.53%	30.65%	58.82%
0.6	8.0124	0.1290	0.3429	0.6720	0.9835	0.9936	1.0000	10.59%	29.23%	60.18%
0.7	8.2992	0.1356	0.3766	0.7639	0.9855	0.9943	1.0000	10.58%	29.21%	60.22%
0.8	8.5196	0.1425	0.4126	0.8607	0.9871	0.9948	1.0000	10.57%	29.19%	60.24%
0.9	8.6959	0.1495	0.4507	0.9606	0.9884	0.9952	1.0000	10.56%	29.17%	60.27%
1.0	8.8415	0.1569	0.4909	1.0617	0.9894	0.9955	1.0000	10.55%	29.16%	60.29%
1.1	8.9645	0.1646	0.5327	1.1617	0.9903	0.9957	1.0000	9.71%	28.19%	62.10%
1.2	9.1344	0.1725	0.5759	1.2592	0.9909	0.9960	1.0000	9.68%	30.09%	60.23%
1.3	9.2885	0.1807	0.6200	1.3527	0.9915	0.9962	1.0000	9.67%	30.06%	60.27%
1.4	9.4235	0.1892	0.6647	1.4412	0.9920	0.9964	1.0000	9.66%	30.04%	60.30%
1.5	9.5434	0.1980	0.7096	1.5242	0.9924	0.9965	1.0000	9.66%	30.01%	60.33%
1.6	9.6511	0.2071	0.7541	1.6015	0.9928	0.9967	1.0000	9.65%	29.99%	60.35%
1.7	9.7489	0.2165	0.7981	1.6731	0.9931	0.9968	1.0000	9.65%	29.97%	60.38%
1.8	9.8384	0.2261	0.8411	1.7392	0.9934	0.9969	1.0000	9.64%	29.96%	60.40%
1.9	9.9211	0.2360	0.8829	1.8001	0.9937	0.9970	1.0000	9.64%	29.94%	60.42%
2.0	10.0048	0.2461	0.9233	1.8563	0.9939	0.9971	1.0000	9.06%	28.94%	62.00%

Table 1.3: Optimization and allocation results: Bauer-Zanjani allocation heterogeneous Bernoulli losses

lines and provides some descriptive statistics about each line. The largest lines for our reinsurer (by premiums and expected losses) are “US Hurricane,” “N American EQ West” (North American Earthquake West), and “ExTropical Cyclone” (Extratropical Cyclone). The expected aggregate loss is \$187,819,998 with a standard deviation of \$162,901,154, and the aggregate premium income is \$346,137,808.

We consider an aggregation to four lines, with line numbers listed in Column “Agg.” Here, where we lump together all lines by perils. In particular, we can think of Line 1 as “earthquake,” Line 2 as “storm and flood,” Line 3 as “fire & crop,” and Line 4 as “terror & casualty.” In order to keep the results comprehensible, we limit the exposition to this four-line aggregation level. Figure 1.4 shows histograms for each of these four lines.

We notice that the “earthquake” distribution is concentrated at low loss levels with only relatively few realizations exceeding \$50,000,000 (the 99% VaR only slightly exceeds \$300,000,000). However, the distribution depicts relatively fat tails with a maximum loss realization of only slightly under one billion. The (aggregated) premium for this line is \$46,336,664 with an expected loss of \$23,345,695.

<i>Line</i>	Premiums	Expected Loss	Standard Deviation	Agg
<i>N American EQ East</i>	6,824,790.67	4,175,221.76	26,321,685.65	1
<i>N American EQ West</i>	31,222,440.54	13,927,357.33	47,198,747.52	1
<i>S American EQ</i>	471,810.50	215,642.22	915,540.16	1
<i>Australia EQ</i>	1,861,157.54	1,712,765.11	13,637,692.79	1
<i>Europe EQ</i>	2,198,888.30	1,729,224.02	5,947,164.14	1
<i>Israel EQ</i>	642,476.65	270,557.81	3,234,795.57	1
<i>NZ EQ</i>	2,901,010.54	1,111,430.78	9,860,005.28	1
<i>Turkey EQ</i>	214,089.04	203,495.77	1,505,019.84	1
<i>N Amer. Severe Storm</i>	16,988,195.98	13,879,861.84	15,742,997.51	2
<i>US Hurricane</i>	186,124,742.31	94,652,100.36	131,791,737.41	2
<i>US Winterstorm</i>	2,144,034.55	1,967,700.56	2,611,669.54	2
<i>Australia Storm</i>	124,632.81	88,108.80	622,194.10	2
<i>Europe Flood</i>	536,507.77	598,660.08	2,092,739.85	2
<i>ExTropical Cyclone</i>	37,033,667.38	23,602,490.43	65,121,405.35	2
<i>UK Flood</i>	377,922.95	252,833.64	2,221,965.76	2
<i>US Brushfire</i>	12,526,132.95	8,772,497.86	24,016,196.20	3
<i>Australian Terror</i>	2,945,767.58	1,729,874.98	11,829,262.37	4
<i>CBNR Only</i>	1,995,606.55	891,617.77	2,453,327.70	4
<i>Cert. Terrorism xCBNR</i>	3,961,059.67	2,099,602.62	2,975,452.18	4
<i>Domestic Macro TR</i>	648,938.81	374,808.73	1,316,650.55	4
<i>Europe Terror</i>	4,512,221.99	2,431,694.65	8,859,402.41	4
<i>Non Certified Terror</i>	2,669,239.84	624,652.88	1,138,937.44	4
<i>Casualty</i>	5,745,278.75	2,622,161.64	1,651,774.25	4
<i>N American Crop</i>	21,467,194.16	9,885,636.27	18,869,901.33	3

Table 1.4: Descriptive Statistics

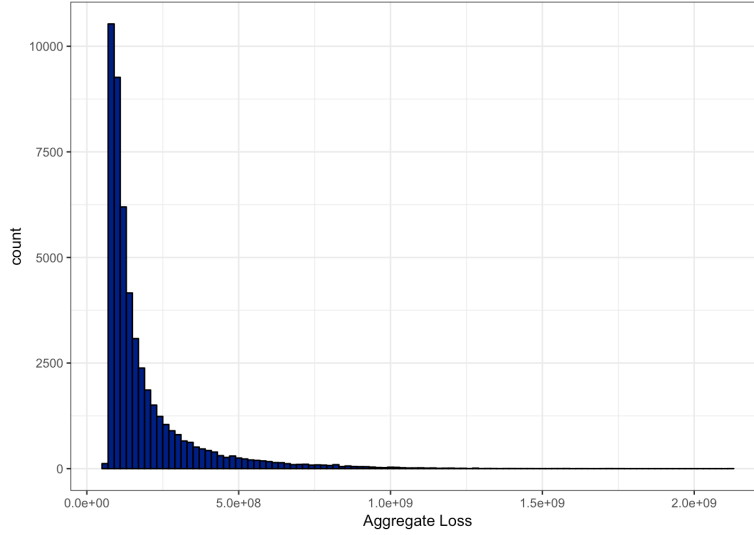


Figure 1.3: Histograms for aggregate loss

“Storm & flood” is by far the largest line, both in terms of premiums (\$243,329,704) and expected losses (\$135,041,756). The distribution is concentrated around loss realizations between 25 and 500 million, though the maximum loss in our 50,000 realizations is almost four times that size. The 99% VaR is approximately 700 million USD.

In comparison, the “fire & crop” and “terror & casualty” lines are small with an (aggregated) premiums (expected loss) of about 34 (19) million and 22.5 (11) million, respectively. The maximal realizations are around 500 million for “fire & crop” (99% VaR = 163,922,557) and around 190 million for “terror & casualty” (99% VaR = 103,308,358).

We consider the same allocation approaches outlined in Section 1.4.1. For the Bauer-Zanjani allocations, we again consider CARA preferences with different absolute risk aversion levels. The results of the optimization procedure are provided in Table 1.6. For details on the implementation procedure, we refer to Appendix 1.8.2.

1.6.2 Comparisons for the Unmodified Portfolio

Table 1.5 presents conventional allocation results for the (unmodified) portfolio of the company. Here for each allocation method, we list the capital levels for each line, their sum, as well as the risk measure evaluated for the aggregate loss distributions. Obviously, the last two numbers should

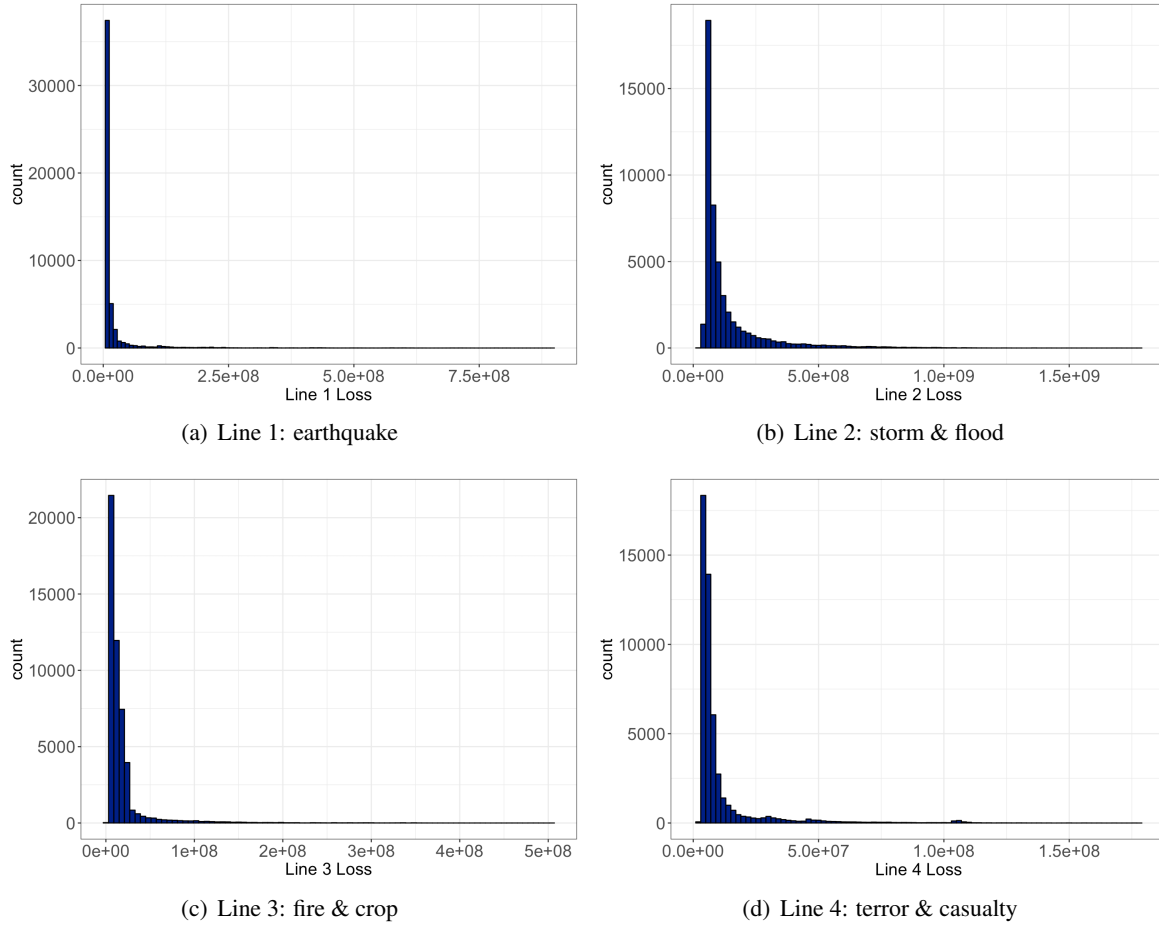


Figure 1.4: Histograms for four aggregated lines

coincide—which can serve as a simple check for the calculations.

These aggregate risk measures vary tremendously, and thus so do the by-line allocations. For instance, it is trivial that the 99% quantile (VaR) is far greater than the 95% quantile (VaR). Thus, in the second line for each method, Table 1.5 again lists the allocations as a percentage of the aggregate risk measure. These are the percentages on which we will base our comparisons. This is not only because it facilitates comparisons, but also because this is in line with practice where the actual capital of a company may not be given in terms of a risk measure at all, or even if it is this may not be the measure used for allocation.

The first observation when comparing the allocations is the realizations that many of them look quite similar, which resonates with observations in other studies. For instance, in the context of assumptions used for the CAS DFA modeling challenge (“Bohra-Weist DFAIC distributions”),

Vaughn (2007) points out that a variety of methods, including allocations based on “covariance, Myers/Read, RMK with Variance, Mango Capital Consumption, and XTVaR99 are all remarkably similar.” We find similar results in the context of an example from life insurance (Bauer and Zanjani, 2013). There are a few outliers, however, most notably the Exponential Allocation with $c = 1$. The reason is that here there is an extreme weight on the extreme tail—that in turn is driven by very extreme realizations of line 2. Indeed there are various realizations in the aggregate tail where the line realizations for lines 1, 3, and 4 are under the expected loss, which explains the resulting negative allocations to these lines.

For comparing the remaining allocations, in analogy to the comparison in Section 1.5, we note that each allocation in our four-line context is characterized by *three*—not four—real numbers, since the fourth follows by subtracting the sum of the others from 100%. Hence, similarly to Figure 1.2 we can compare allocations as points in a three-dimensional space, and moreover we can evaluate the “distance” between two allocations by identifying it with the distance between the two points in terms of its Euclidean norm.

Figure 1.5(a) plots all of our allocations except for the aforementioned exponential allocation with $c = 1$. From Panel 1.5(a), we see that there are a few other outliers in the sense that the distance to other allocation methods is quite significant: Three value at risk allocations, namely the “simple” calculation (VaR1S, VaR2S) for both confidence levels and the bell-curve based calculation for the higher confidence level (VaR2B); and the Esscher allocation for the (high) parameter of $1E-7$ (Essch1). The intuition for the latter is, again, the exponential weight pushing all relevance to the extreme tail where line 2 dominates the others. Hence, both the exponential allocation and the Esscher allocations are extremely sensitive to the choice of the parameter (although this sensitivity does not appear to apply to the Kamps allocation). For VaR, on the other hand, it is well-known that estimation based on Monte Carlo simulation is erratic (Kalkbrener, 2005)—so it may be numerical errors driving these outliers (at least for VaR1S).

Interestingly, aside from the “outlier” allocations mentioned above and two Myers-Read allocations, the points all appear to lie on a parabola-shaped curve in three-dimensional space that is suggestive of a systematic pattern. Hence, similarly to the findings in Section 1.5, the differences among allocation seems to be explained by a single parameter. In order to zoom in on the

Allocation	Line 1	Line 2	Line 3	Line 4	Sum	RiskMeas
Exp Val	23,345,695 12.43%	135,041,756 71.90%	18,658,134 9.93%	10,774,413 5.74%	187,819,998 100%	187,819,998
CovWBeta	62,292,648 12.13%	398,599,625 77.61%	38,938,608 7.58%	13,791,425 2.69%	513,622,306 100%	513,622,306
$\beta = 2$	51,212,126 12.76%	301,490,365 75.14%	34,960,054 8.71%	13,571,979 3.38%	401,234,524 100%	401,234,524
TVaR _{75%}	72,975,560 12.36%	462,489,152 78.36%	42,432,578 7.19%	12,350,749 2.09%	590,248,038 100%	590,248,038
TVaR _{90%}	85,607,782 11.59%	596,451,028 80.76%	44,523,108 6.03%	11,979,729 1.62%	738,561,647 100%	738,561,647
TVaR _{95%}	106,293,324 10.17%	869,605,928 83.19%	58,168,596 5.56%	11,294,985 1.08%	1,045,362,832 100%	1,045,362,832
TVaR _{99%}	9,776,274 1.85%	330,811,906 62.57%	173,984,700 32.91%	14,118,944 2.67%	528,691,824 100%	528,691,824
VaR _{95%} (simple)	62,367,597 11.80%	404,259,189 76.47%	51,869,109 9.81%	10,157,414 1.92%	528,653,309 100%	528,653,309
VaR _{95%} (bell)	7,170,815 0.83%	816,870,497 94.18%	39,256,266 4.53%	4,060,778 0.47%	867,358,356 100%	867,358,356
VaR _{99%} (simple)	40,819,623 4.70%	780,226,792 89.92%	29,835,131 3.44%	16,807,649 1.94%	867,689,194 100%	867,689,194
VaR _{99%} (bell)	27,850,426 12.30%	166,607,986 73.61%	20,890,454 9.23%	10,989,763 4.86%	226,338,629 100%	226,338,629
Exponential	37,216,272 11.80%	245,941,709 77.99%	23,451,637 7.44%	8,755,958 2.78%	315,365,576 100%	315,365,576
Exponential	-3,671,904,055 -30.81%	23,720,252,516 199.06%	-4,880,193,171 -40.95%	-3,252,109,822 -27.29%	11,916,045,468 100%	11,916,045,468
Proportional Hazard	36,080,499 12.05%	226,703,991 75.69%	25,110,583 8.38%	11,640,570 3.89%	299,535,643 100%	299,535,643
$a = 0.6$	28,192,014 12.32%	168,301,360 73.52%	21,223,234 9.27%	11,199,323 4.89%	228,915,932 100%	228,915,932
Proportional Hazard	24,363,235 12.41%	141,822,074 72.26%	19,212,540 7.99%	10,880,196 5.54%	196,278,045 100%	196,278,045
$a = 0.95$	27,893,622 12.40%	164,362,273 73.04%	21,313,685 9.47%	11,452,085 5.09%	225,021,665 100%	225,021,665
Wang	33,402,026 12.32%	201,407,441 74.29%	24,294,793 8.96%	12,004,713 4.43%	271,108,972 100%	271,108,972
$\lambda = 0.5$	39,867,730 12.20%	247,043,116 75.57%	27,573,810 8.44%	12,410,105 3.80%	326,894,761 100%	326,894,761
Wang	120,879,204 10.33%	1,006,221,885 86.01%	50,599,598 4.33%	-7,876,720 -0.67%	1,169,823,967 100%	1,169,823,967
MyersRead,	64,285,407 7.56%	756,847,100 88.98%	37,746,242 4.44%	-8,285,324 -0.97%	850,593,426 100%	850,593,426
$a = 1,357,643,965$	60,821,986 8.95%	606,579,733 89.26%	21,827,406 3.21%	-9,690,768 -1.43%	679,538,358 100%	679,538,358
MyersRead,	8,226,240 0.39%	1,987,777,539 93.65%	54,262,714 2.56%	72,291,051 3.41%	2,122,557,544 100%	2,122,557,544
$a = 867,358,356$	27,359,301 12.31%	163,201,071 73.42%	20,690,228 9.31%	11,031,052 4.96%	222,281,653 100%	222,281,653
Esscher, $t =$	26,654,710 12.47%	155,020,509 72.51%	20,725,105 9.69%	11,406,228 5.33%	213,806,553 100%	213,806,553
1.E-07	40,195,291 12.23%	249,028,113 75.75%	27,434,525 8.35%	12,082,021 3.68%	328,739,950 100%	328,739,950
Esscher, $t =$	55,387,701 11.20%	408,352,811 82.57%	25,296,290 5.12%	5,487,442 1.11%	494,524,244 100%	494,524,244
1.E-08	104,209,232 9.89%	875,467,350 83.10%	62,492,051 5.93%	11,388,390 1.08%	1,053,557,023 100%	1,053,557,023
1.E-09	47,772,088 12.70%	273,086,001 72.63%	36,572,490 9.73%	18,591,385 4.94%	376,021,964 100%	376,021,964
Kamps, $t =$	66,892,452 12.65%	393,236,293 74.38%	46,970,415 8.88%	21,592,664 4.08%	528,691,824 100%	528,691,824
1.E-08	104,581,664 12.06%	670,427,703 77.30%	66,183,982 7.63%	26,165,006 3.02%	867,358,356 100%	867,358,356
Kamps, $t =$	93,544,093 11.55%	654,561,402 80.85%	50,055,237 6.18%	11,458,019 1.42%	809,618,751 100%	809,618,751
1.E-11	107,255,495 10.76%	824,861,293 82.72%	53,516,378 5.37%	11,599,036 1.16%	997,232,202 100%	997,232,202
D'Arcy, $\alpha =$	114,157,530 10.17%	935,144,574 83.30%	61,454,489 5.47%	11,804,261 1.05%	1,122,560,854 100%	1,122,560,854
1,357,643,965	79,022,198 11.39%	557,509,118 80.35%	45,021,084 6.49%	12,299,361 1.77%	693,851,760 100%	693,851,760
D'Arcy, $\alpha =$						
1,038,413,423						
Bodoff,						
VaR _{90%}						
Bodoff,						
VaR _{95%}						
Bodoff,						
VaR _{99%}						
RTVaR						
$\alpha = 75\%, \beta = 2$						
RTVaR						
$\alpha = 90\%, \beta = 2$						
RTVaR						
$\alpha = 95\%, \beta = 2$						
AvgTVaR						

Table 1.5: Conventional allocation results—catastrophe reinsurance case

α	Optimization Results					Allocation Results			
	a	$q^{(1)}$	$q^{(2)}$	$q^{(3)}$	$q^{(4)}$	Line 1	Line 2	Line 3	Line 4
5.00E-10	2.89E+08	0.8290	0.6538	0.5235	0.4896	16.06%	77.86%	4.70%	1.38%
7.00E-10	3.98E+08	0.9076	0.7990	0.6436	0.6268	13.70%	80.54%	4.47%	1.30%
1.00E-09	5.02E+08	0.9467	0.8863	0.7300	0.7284	11.33%	83.07%	4.38%	1.23%
2.00E-09	6.94E+08	0.9761	0.9616	0.8359	0.8457	7.72%	86.91%	4.32%	1.05%
3.00E-09	8.13E+08	0.9832	0.9799	0.8766	0.8752	5.86%	88.78%	4.35%	1.02%
4.00E-09	9.18E+08	0.9874	0.9874	0.9007	0.8758	4.32%	90.28%	4.27%	1.13%
5.00E-09	1.02E+09	0.9902	0.9913	0.9195	0.8615	3.54%	91.10%	3.99%	1.38%
6.00E-09	1.13E+09	0.9925	0.9934	0.9348	0.8431	2.93%	91.76%	3.59%	1.72%
7.00E-09	1.22E+09	0.9942	0.9949	0.9457	0.8299	2.25%	92.64%	3.08%	2.03%
8.00E-09	1.31E+09	0.9954	0.9958	0.9526	0.8237	1.66%	93.20%	2.85%	2.28%
9.00E-09	1.39E+09	0.9963	0.9965	0.9574	0.8235	1.39%	93.48%	2.67%	2.47%
1.00E-08	1.45E+09	0.9970	0.9971	0.9610	0.8272	1.15%	93.68%	2.57%	2.60%

Table 1.6: Optimization and allocation results: Bauer-Zanjani allocation catastrophe reinsurance losses

remaining allocations, Figure 1.5(b) re-plots the same points, but this time we exclude outlying allocations as well as the two outer Myers-Read allocations. Again, the allocations seem related and we find that the expected value allocation (EV) plays an “extreme role.” This may not come as a surprise since suitable allocation methods should penalize risk “more than linearly” (Venter, 2010).

A number of allocation methods are very close to the expected value allocation: The Kamps allocations, the Wang allocations, the Bodoff allocations, and the Covariance allocation are all within 0.06 of the EV allocation. In contrast, all but one TVaR/RTVaR/AvgTVaR allocations, the D’Arcy allocations, and the Myers-Read allocations all bunched together between 0.07 and 0.19—and all roughly along the parabola-shaped curve, where the order appears to be driven by the parameters. The former methods are all driven by the entire distribution, whereas the focus of the latter allocation methods is on the tails (though the Myers-Read allocation does depend on the entire distribution).

Figure 1.5(a) shows that Bauer-Zanjani allocations ($BZ\alpha$) roughly lie along the parabola when the risk aversion parameters are large, so that the allocations roughly coincide with many tail-based allocation when the counterparty is more risk-averse. However, when the counterparty is approaching risk neutrality, or when α is small, Bauer-Zanjani allocations produce results that deviate from other allocations. A key reason for this finding lies in the underpinning optimization problem that delivers an optimal portfolio on which the allocation is based. As is evident from Table 1.6, espe-

cially for smaller choices of α , the portfolio weights vary in their values. In particular, the weight for line 3, $q^{(3)}$, is relatively low explaining why less capital is allocated to it.

All in all, two key observations emerge. First, we observe a dissonance between tail-based allocations and allocations that are based on the entire distribution. But which one is more appropriate? Should we, or should we not, focus on tails? Venter (2010) argues that from an economic stance, risk-taking is not risk free—any modification to risk taking should carry some charge, so that a focus on the tails is misguided. He supports using marginal (i.e., Euler-based) methods that are based on the entire distribution such as the Wang transform, since they are most “the most commensurate with pricing theory.” However, D’Arcy (2011) and Myers and Read (2001) also present approaches with an economic motivations. Second, “conventional” allocations behave qualitatively different than the “economic” Bauer-Zanjani allocations.

1.6.3 Stability of the Methods

In this section, we study the stability of the allocation methods. In particular, we recalculate the allocations from the previous subsection for two distorted portfolios:

- *Sensitivity 1:* We eliminate 1,000 arbitrary sample realizations leaving us with 49,000 realizations.
- *Sensitivity 2:* We replace the five worst case (aggregate) scenarios with the sixth worst aggregate scenario (so that our sample now contains six identical scenarios).

The intuition behind the first stability test is clear: An allocation should be robust to unsystematic changes in the sample. When adding, changing, or subtracting from the sample in an unsystematic way, we would hope to see the allocation staying more or less the same. And since we cannot add to or change the sample because we do not know the data-generating process, we subtract.

The second test is motivated by ideas from Kou et al. (2013), who discuss robustness properties of risk measures and—based on the observation that *coherent* risk measures are not always robust—define so-called *natural* risk statistics. It is important to note that our angle is different in that we consider allocations and not risk measures, even though the underlying issues are the same.

Specifically, extreme tail scenarios are very hard to assess—for instance, even with 5,000 observations one can not distinguish between the Laplace distribution and the t-distributions (Heyde and Kou, 2004). Therefore, modifications in the extreme tail should not have a tremendous impact on the allocation.

As indicated in the previous subsection, we can identify allocations for our four business lines with points in three dimensional space, and we can identify the “difference” between two allocations with the (Euclidean) distance between the corresponding points. For a yardstick when assessing allocations, note that the difference between the 90% TVaR and the 99% TVaR is 0.056, which is thus a sizable difference. The difference between the 95% and the 99% TVaR is 0.026, which is still considerable.

Figure 1.6 plots the distance between the allocations for the original portfolio and the modified portfolio for both sensitivity portfolios and all considered allocations methods. Again, we find that VaR-based allocations and the Exp3 allocation stand out as extreme outliers, though on different tests. More specifically, VaR allocations respond particularly poorly to unsystematic changes in the portfolio, whereas the exponential allocation is particularly sensitive to changes in the tail. This contrasts with Kou et al. (2013), who argue that VaR has good robustness properties for risk measurement. We eliminate (all) VaR-based and exponential allocations and plot the differences for the remaining methods for both tests separately. Figure 1.7 displays the results.

Figure 1.7(a) shows the results for the (unsystematic) modification via eliminating 1,000 samples. We find that when ignoring VaR-based allocations, all methods are relatively stable. The maximal difference now is about 0.0025 which is not too sizable for the Myers-Read 2 allocation: the corresponding allocation vectors are (7.56%;88.98%;4.44%;-0.97%) and (7.78%;88.85%;4.38%;-1.01%), respectively.

In contrast, when eliminating tail scenarios, the impact can be considerable. Figure 1.7(b) shows that in some cases it can amount to more than 0.04. The most sensitive methods are the Myers-Read allocations, the D’Arcy allocations, the Esscher, and the Bauer-Zanjani allocation for high α —all of which are “tail-focused.” However, we do not find the same for TVaR based allocations, which again is contrary to the findings from Kou et al. (2013) for risk measurement. Also noteworthy is the stability of the proportional hazard, the Wang, the Kamps, and the Bodoff

allocations, so it appears that stability is less critical for non-tail-focused methods.

1.7 Conclusion

The actuarial literature entails numerous contributions on capital allocation. While the theoretical questions are not settled and deserve continued attention, questions concerning implementation issues have received much less attention in the literature but are of great importance to practitioners. This chapter attempts to contribute by exploring differences and commonalities between various methods that have been proposed.

We find substantial differences across the universe of methods, although we find that all allocations appear systematically related in the context of our example. Stability issues, predictably, arise in methods where allocations are keyed to one outcome or to a small set of outcomes—as is the case with VaR-based allocations and allocations based on tail risk measures. While the analysis here is based on specific data, we find the systematic relationship surprising and also encouraging in view of the companies’ problem of choosing the “correct” method. More research is obviously needed to verify whether these findings carry over to other situations.

1.8 Appendix A: Bauer-Zanjani Allocation Implementation

1.8.1 The Case of Heterogeneous Bernoulli Losses

We consider consumers that face Bernoulli distributed losses. We allow for heterogeneity in consumer preferences as well as in the losses. More specifically, we assume that there are m groups of consumers, where group i contains N_i identical consumers with wealth level w_i and utility function $U_i(\cdot)$ that face independent losses l_i occurring with a probability $\pi_i, i = 1, \dots, m$. The participation constraint again is given by their autarky levels:

$$\gamma_i = \mathbb{E}[U_i(w_i - l_i)] = \pi_i U_i(w_i - l_i) + (1 - \pi_i) U_i(w_i).$$

The optimization problem in the one period model without a regulatory constraint can then be conveniently set up by observing that the number of losses in the different groups follow independent

Binomial(N_i, π_i) distributions.

For counterparty-based allocation, we obtain for each group i :

$$\begin{aligned}
q_i \tilde{\phi}_i = & \tilde{c} \sum_{k_1=0}^{N_1} \cdots \sum_{k_i=1}^{N_i} \cdots \sum_{k_m=0}^{N_m} \binom{N_1}{k_1} \cdots \binom{N_i-1}{k_i-1} \cdots \binom{N_m}{k_m} \\
& \times \pi_1^{k_1} \cdots \pi_i^{k_i} \cdots \pi_m^{k_m} (1 - \pi_1)^{N_1-k_1} \cdots (1 - \pi_i)^{N_i-k_i} \cdots (1 - \pi_m)^{N_m-k_m} \\
& \times \left[\mathbf{1}_{\{\sum_{s=1}^m k_s q_s l_s \geq a\}} \left\{ \sum_{j=1}^m \frac{k_j U'_j \left(w_j - p_j - l_j + q_j l_j \frac{a}{\sum_{s=1}^m k_s q_s l_s} \right) q_j l_j}{v'_j \times \sum_{s=1}^m k_s q_s l_s} \right\} \right] \\
& \times \frac{q_i l_i}{\sum_{s=1}^m k_s q_s l_s}, \tag{1.10}
\end{aligned}$$

where \tilde{c} is a constant such that $\sum_i N_i q_i \tilde{\phi}_i = 1$.

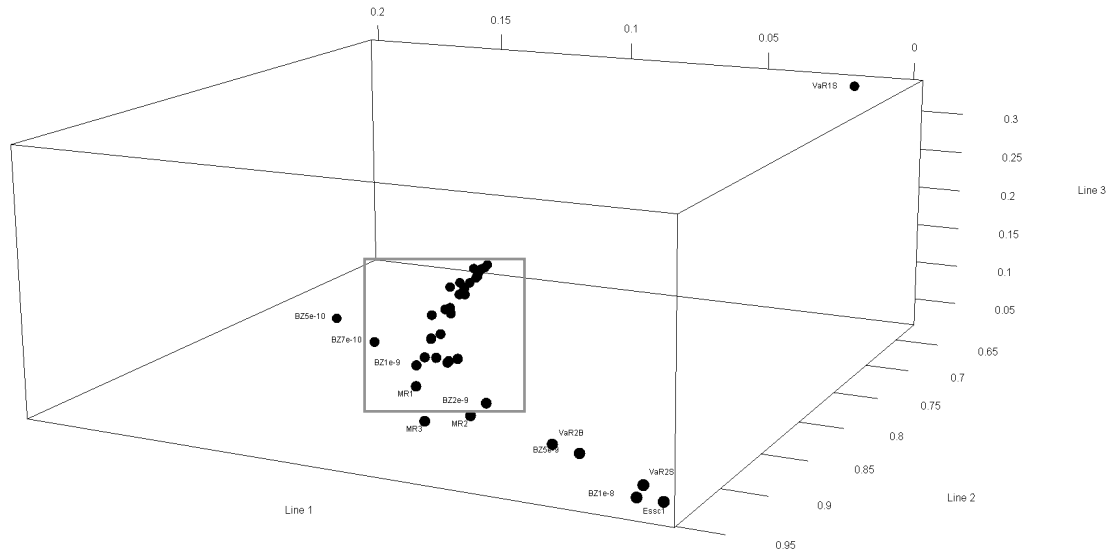
1.8.2 The Case of Catastrophe Reinsurance Losses

We assume each line represents a counterparty with CARA preference. The indemnity of each line follows the simulated distribution from the data. We consider the profit maximization problem (1.8) and we can show that under CARA preference, the solution does not depend on initial wealth and

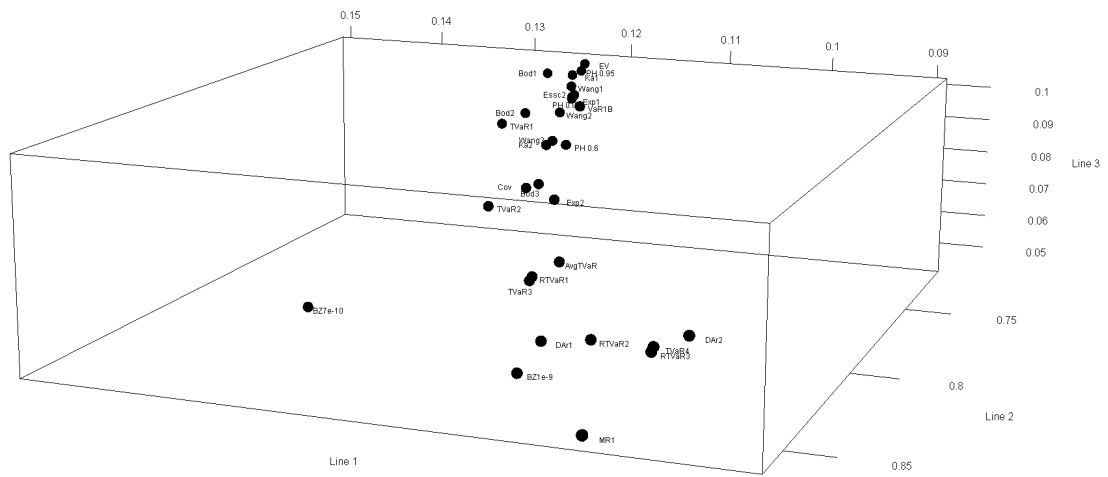
$$p_i = \frac{1}{\alpha_i} \log \frac{\mathbb{E}[e^{\alpha_i L^{(i)}}]}{\mathbb{E}[e^{\alpha_i L^{(i)}(1-q^{(i)})} \mathbf{1}_{\{I \leq a\}} + e^{\alpha_i L^{(i)}(1-q^{(i)} \frac{a}{I})} \mathbf{1}_{\{I > a\}}]}, \quad i = 1, 2, 3, 4.$$

Therefore, we can simplify the optimization problem (1.8) to five choice variables $a, q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}$. The allocation are calculated based on optimized variables and equation (1.9).

1.9 Appendix B: Additional Tables and Figures



(a) All methods (w/o Exp3)



(b) Restricted methods (squared area in (a))

Figure 1.5: Comparison of allocations—catastrophe insurance losses

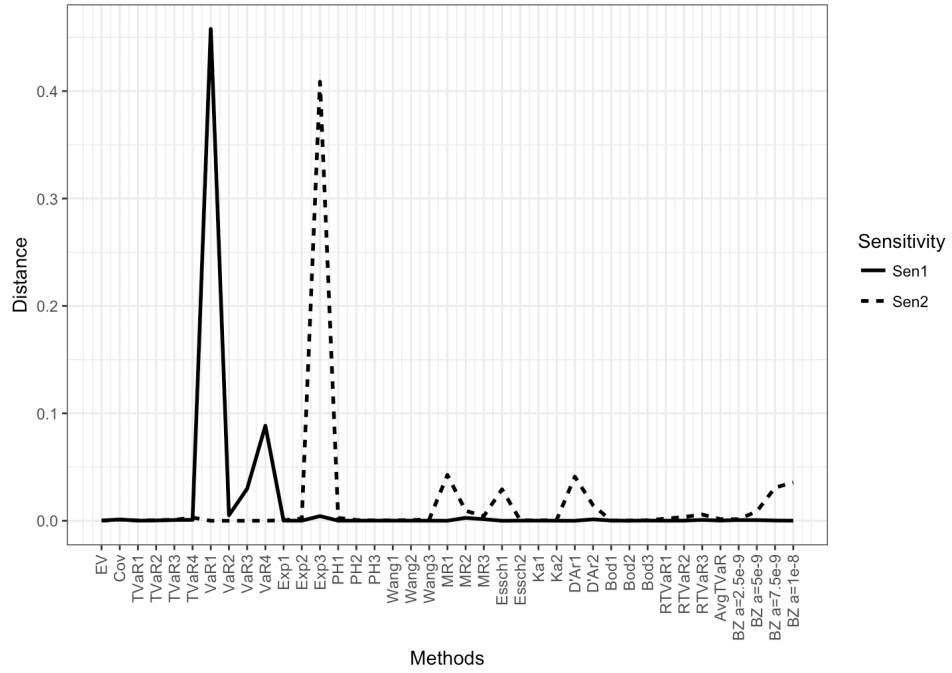


Figure 1.6: Stability of allocations: distance between allocations on basic and modified portfolios for all methods

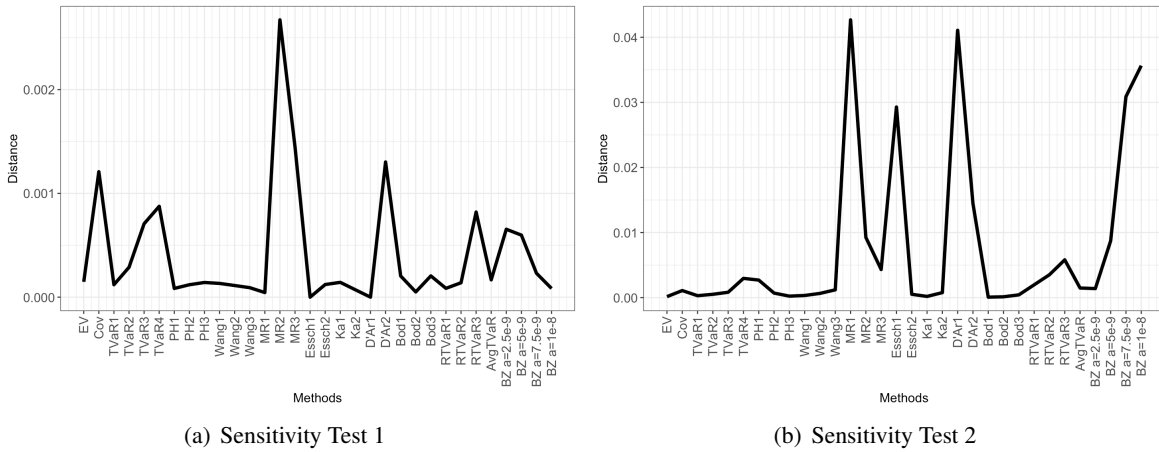


Figure 1.7: Stability of allocations: distance between allocations on basic and modified portfolios for allocation methods (except Exp and VaR)

Chapter 2

The Term Structure of Capital Costs¹

2.1 Introduction

How do financial institutions discount liability cash flows in the near and distant future? It has long been established that capital costs are an important valuation component for financial institutions facing risk, particularly in the insurance sector (Cummins and Phillips, 2005), and market-consistent valuation frameworks such as IFRS 17 and Solvency II include corresponding “risk margins” for non-replicable risks (Albrecher et al., 2018).² Such firm-specific risk penalties arise in financial models with financing frictions (Froot and Stein, 1998; Zanjani, 2002; Bauer and Zanjani, 2016). However, thus far little is known about the effect of the markups on liabilities materializing in the near and far future, that is, about the *term structure* of these capital costs. This essay closes this gap in literature by analyzing how markups affect liabilities with different maturities, both theoretically and empirically.

Relying on an extension of such risk management models with financial frictions, we devise

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²Solvency II is a directive within the European Union that codifies and harmonizes insurance regulation. International Financial Reporting Standard (IFRS) 17 is the new international accounting standard for insurance contracts, and particularly their valuation. Both emphasize market-consistent valuation principles.

a theoretical model for the firm-specific term structure of capital costs. We develop our theory in the context of a property and casualty (P&C) insurer, which typically carry business lines that vary in the length of time it takes for claims to be reported and to settle—referred to as *short* and *long* tailed business lines.³ Hence, this industry provides an ideal laboratory setting for our ideas, also since unique data are available due to regulatory reporting requirements. We estimate discount curves (net of discounting at risk-free rates) that depend on firm characteristics. Our key theoretical and empirical finding is that firms that face financial constraints include hefty markups for liabilities in the near future, with a term structure that is rapidly declining. In contrast, well capitalized firms include a relatively modest markup that is less steep over time—and can even be negative and increasing for companies with extremely high capital levels. The key intuition is that due to the generally profitable though risky business, capital costs—that are high for meagerly capitalized firms and modest for well capitalized firms—have a *mean-reverting* character at the firm level.

Within the theoretical model, we integrate a general P&C loss structure given via so-called *loss triangles* into a dynamic profit maximization model for an insurer that economizes on different capitalization options similar to that from Bauer and Zanjani (2018). The model is set in an economic environment with financing frictions (Duffie, 2010), and includes both internal and external capital that can be raised at different costs (Brunnermeier et al., 2012). We derive our key equation of the marginal cost of risk from the company’s optimality conditions, along with a rule for the economic allocation of capital to the different lines. In line with Bauer and Zanjani (2018), we find that while the marginal cost takes the conventional form of the value of future liabilities *plus* allocated capital costs, the company *evaluates* uncertain liabilities under adjusted probabilities that reflect company *effective* risk aversion (Froot and Stein, 1998). However, we demonstrate that the adjustment differs for payments due in the next year versus payments in future development years, since the associated probability weights depend on the company’s (expected) financial situation. This difference in treatment for liabilities with different durations implies differences in the assessment of shorter versus longer tailed business lines. In particular, both differences in the

³Indeed, this is one of the primary aspects addressed in the vast actuarial literature on loss reserving in non-life insurance. We refer to the textbooks by Wüthrich and Merz (2008), Taylor (2012), and Radtke et al. (2016) for details.

loss distribution as well as expected settlement times between business lines will interact with the financial situation of the company to determine their valuations.

We explore this relationship by solving our model numerically in a setting with two business lines and two development periods for the long-tailed line (2L2DY). More precisely, we consider a business selling a workers' compensation insurance as the long-tailed line, where we assume that the losses develop according to a Chain-Ladder model with jointly normal innovations (Mack, 1993), and selling commercial automobile insurance as the short-tailed line. We implement the firm's profit maximization problem by dynamic programming on a discretized state space. In line with Bauer and Zanjani (2018), we find that the value of the P&C insurer is concave with an optimal point that results from balancing profit expectations and capital costs. However, our differentiation between the long- and short-tailed business lines allows for analyzing the impact of firm capitalization on the optimal line mix. We find that exposure in the long-tailed line—where payments occur further in the future—is relatively higher for financially constrained firms, whereas the opposite is true for the short-tailed line. This is due to the former facing high capital costs in the short-term and lower capital costs in the long term, so that "delaying" indemnity payments by increasing exposure to the long-tailed line is optimal.

To obtain firm-specific markups empirically, we aggregate the marginal cost equation across lines so that we express aggregate company premiums in terms of expected aggregate discounted liability payments and capital costs—where the expected value features risk adjustment terms due to future capital costs. We can then identify a firm-specific term structure of risk adjustments since different companies have a different mix of business lines, so that the risk adjustment terms will have a different impact on the firm's right-hand side of marginal cost equation. We rely on a simple version of the term structure specification by Nelson and Siegel (1987), where the parameters depend on firm capitalization as measured by the surplus to asset ratio or the leverage ratio. Our estimation delivers current period industry cost-of-capital, and parameters governing the company-specific term structure as a function of firm characteristics.

Our industry cost of capital figures vary between 7.5% and 13%, depending on the considered year. The term structure of firm-specific markups differs markedly in the financial situation of the firm. More precisely, for a firm with an average capitalization level (equity to asset ratio \simeq

0.5), its valuation of near-future (due in 1-3 years) liabilities is 20-80% higher than their present values. However, for liabilities that are due much later (in 4-10 years), the markups are around 10-20% or even less. This pattern is similar for poorly capitalized firms, although here short-term markups are even higher and the term structure is even more steeply downward sloping. For firms with high capitalization level, in contrast, we observe much lower markup levels overall—and they may even display markdowns with an upward sloping term structure. That is, extremely well capitalized firms may evaluate near-term liabilities below their expected discounted value, though this markdown subsides for liabilities in the far future. The findings are in line with our theory, and they are robust to the inclusion of additional firm characteristics.

Related Literature and Organization of the Essay

This essay relates to several strands of literature. First, we bring together economic approaches for risk pricing in financial institutions with actuarial loss forecasting methods, for the purpose of deriving the term structure of capital costs in P&C insurance. A seminal contribution with regards to the former literature is Froot and Stein (1998), who present economic foundations for risk pricing in a setting with costly capital. We directly build on the dynamic extension of Froot and Stein's work by Bauer and Zanjani (2018), where different modes of capitalization are included. The actuarial literature on loss forecasting and claims reserving methods is extensive (Taylor and Ashe, 1983; Wüthrich and Merz, 2008; Radtke et al., 2016, e.g.). We rely on the common Chain-Ladder forecasting approach (Mack, 1993), although generalizations are possible.

The empirical section borrows from the literature on yield curve specification and estimation, particularly from Nelson and Siegel (1987). The drivers for the choice between long- and short-tailed lines and associated costs relate to the finance literature on debt maturity (Custódio et al., 2013; Mian and Santos, 2011; Xu, 2016). Finally, we contribute to the literature on cost of capital estimation in the insurance sector (Cox and Griepentrog, 1988; Cummins and Lamm-Tennant, 1994; Lee and Cummins, 1998; Cummins and Phillips, 2005). In particular, while we rely on a completely different approach, it is comforting that resulting figures fall in the same region as previous estimates.

The remainder of the chapter is organized as follows: Section 2.2 presents the term structure

equation derived from a general model of multi-period profit maximization with a general loss structure in a P&C company, an implementation of the model and numerical results; Section 2.3 presents details of the empirical study on estimating the term structure of capital costs; Section 2.4 concludes.

2.2 Multi-Period Profit Maximization with Loss History

2.2.1 Loss Structure for a P&C Company

Setting up a profit maximization framework for a P&C company requires modeling the asset and the liability sides. For simplicity, we assume the company's assets bear no risk and that all the uncertainty originates from the liability side, modeled via claim payment amounts.

A P&C company writes new insurance contracts in each of its business lines at the beginning of every year (*accident year*), during which accidents occur and losses are reported. However, some of the losses are not reported until the next year or even years after the origination of the contract. Furthermore, only a portion of the payments is settled in the accident year, whereas the remainder of the (unrealized) payments will take several years to settle. The lags in reporting and paying losses are accounted for by considering so-called *loss development years*. Such a loss structure is typically represented via so-called *loss triangles*, with one triangle recording incurred (reported) losses, and another triangle recording paid losses. To illustrate, in Figure 2.1 we consider a P&C insurance company with N business lines with corresponding (paid) loss random variables $L_{i,j}^{(n,i+j-1)}$, with line identifier $n = 1, 2, \dots, N$, accident year (AY) $i = 1, 2, \dots, t - d_n, \dots, t, \dots$, development year (DY) $j = 1, 2, \dots, d_n$, and $i + j - 1$ being the calendar year (period). For each variable, we only need to identify the development and calendar year and thus drop the accident year subscripts for simplicity.

In the paid loss triangle, for example, $L_1^{(n,1)}$ to $L_{d_n}^{(n,d_n)}$ denote amounts paid (if positive, or amount received if negative) for insurance sold at the beginning of year 1 in line n . Thus, every year, there are payments for losses incurred in the current year, as well as for losses developed from previous years. Specifically, payments in the same calendar year consist of the diagonal entries in the paid loss triangle. For example, payments in calendar year t correspond to

AY \ DY	1	2	d_n
1	$L_1^{(n,1)}$	$L_2^{(n,2)}$...	$L_{d_n}^{(n,d_n)}$
2	$L_1^{(n,2)}$
...
$t - d_n$	$L_1^{(n,t-d_n)}$	$L_{d_n}^{(n,t-1)}$
$t - d_n + 1$	$L_1^{(n,t-d_n+1)}$...	$L_{d_n-1}^{(n,t-1)}$	$L_{d_n}^{(n,t)}$
...	
$t - 1$	$L_1^{(n,t-1)}$	$L_2^{(n,t)}$		
t	$L_1^{(n,t)}$			

Figure 2.1: Loss triangle for a P&C insurer in business line n with t accident years and d_n development years

$(L_1^{(n,t)}, L_2^{(n,t)}, \dots, L_{d_n}^{(n,t)})$, which are double-boxed inside Figure 2.1. $L_1^{(n,t)}$ represents losses from the contract sold in period t . Other losses $(L_2^{(n,t)}, \dots, L_{d_n}^{(n,t)})$ are developed from previous years' losses, which are in oval boxes and themselves make up a triangle in Figure 2.1. We denote this “historical” loss triangle at time $t - 1$ as $\Delta^{(n,t-1)} = \{L_j^{(n,i)}, t - d_n + 1 \leq i \leq t - 1\}$, which contains (partial) loss information from $t - d_n + 1$ to $t - 1$ and is the only source of uncertainty. Denote $\mathbf{L}^{(n,t)} = (L_2^{(n,t)}, \dots, L_{d_n}^{(n,t)})$ as losses paid in year t that developed from $\Delta^{(n,t-1)}$. To account for the loss development in each accident year, it is common to assume that the paid losses triangles have a Markov structure:

$$\mathbb{P}(\Delta^{(n,t)} | \Delta^{(n,t-1)}, \Delta^{(n,t-2)}, \dots, \Delta^{(n,1)}) = \mathbb{P}(\Delta^{(n,t)} | \Delta^{(n,t-1)}).$$

Also, as is common, we assume independence across accident years. A Markov structure and the independence assumptions together fit most of the loss reserving methods in the P&C industry. It is possible to relax the independence assumption and allow cross-sectional correlations between accident years, at the cost of more complex derivations.

Under independence and Markov assumptions, loss random variables in each accident year are

related as follows:

$$\mathbb{P}\left(L_j^{(n,t)} \mid h(L_{1:j-1}^{(n,t-j+1:t-1)}), \dots, h(L_1^{(n,t-j+1)})\right) = \mathbb{P}\left(L_j^{(n,t)} \mid h(L_{1:j-1}^{(n,t-j+1:t-1)})\right).$$

The losses in the j^{th} development year only depend on the information of the same accident year and on a function h of loss information on the previous development years. For example, in the most popular stochastic loss reserving method, the so-called Chain-Ladder approach (Mack, 1993), h is the cumulative summation operation:

$$\mathbb{P}\left(L_j^{(n,t)} \mid h(L_{1:j-1}^{(n,t-j+1:t-1)}), \dots, h(L_1^{(n,t-j+1)})\right) = \mathbb{P}\left(L_j^{(n,t)} \mid \sum_{k=1}^{j-1} L_k^{(n,t-k)}\right).$$

2.2.2 A Multi-Line Multi-Period Profit Maximization Model

To fully describe the dynamic liabilities that the P&C company faces, we assume the following underwriting process: At the beginning of every period t , the insurer chooses to underwrite certain amounts in each line of business and charges premium $p^{(n,t)}$ in return. The underwriting decision corresponds to choosing an exposure parameter $q^{(n,t)}$. The losses will be realized over the development years, but the payments are always contingent on the exposure parameter and paid loss random variables. Also note that in each period, the total indemnity payment includes losses incurred and paid in the current year, as well as losses developed from the past years and to be paid in the current calendar year. Thus, for business line n in period t , the indemnity payment can be presented via the following function $\mathcal{I}_t^{(n)}(\cdot)$:

$$I^{(n,t)} = \mathcal{I}^{(n,t)}\left(\underbrace{\{q^{(n,t)}, L_1^{(n,t)}\}}_{\text{current}}, \underbrace{\{\mathbf{Q}^{(n,t-1)}, \mathbf{L}^{(n,t)}\}}_{\text{history}}\right),$$

where we assume $\mathcal{I}^{(n,t)}(\{q^{(n,t)}, 0\}, \{\mathbf{Q}^{(n,t-1)}, \mathbf{0}\}) = 0$. $\mathbf{Q}^{(n,t-1)}$ is the vector of exposure parameters associated with triangle $\Delta^{(n,t-1)}$ and losses $\mathbf{L}^{(n,t)}$. In what follows, we will assume that indemnity payments are proportional to the exposure parameters:

$$I^{(n,t)} = q^{(n,t)} \times L_1^{(n,t)} + \mathbf{Q}^{(n,t-1)} \times \mathbf{L}^{(n,t)},$$

but generalizations are possible at the expense of a more cumbersome analysis (Frees, 2017; Mildenhall, 2017). We denote the aggregate period indemnity across business lines by $I^{(t)} = \sum_{n=1}^N I^{(n,t)}$.⁴

The company collects the full premium $p^{(n,t)}$ at the beginning of each period t on each line. The aggregate period premium is $p^{(t)} = \sum_{n=1}^N p^{(n,t)}$. The company can raise capital $B^{(t)} \geq 0$ (or shed capital $B^{(t)} < 0$). The cost of raising capital $B^{(t)}$ is $c(B^{(t)})$ if $B^{(t)} \geq 0$. There is no cost of shedding capital, i.e. $c(B^{(t)}) = 0$ if $B^{(t)} < 0$. The company carries over capital $a^{(t-1)}(1 - \tau)$ from the last period, with τ denoting the unit frictional cost of internal capital. Raising external capital is always marginally more expensive than keeping internal capital, so we always have $c'(\cdot) > \tau > 0$.

Thus, the company's assets at the beginning of period t are

$$a^{(t-1)}(1 - \tau) + B^{(t)} - c(B^{(t)}) + p^{(t)}.$$

During period t , the assets are invested at a fixed annual interest rate r . At the end of period t , the company pays the aggregate indemnity $I^{(t)}$ from its insurance policies sold in the current period and previous periods. The surplus of assets over aggregate indemnity, denoted by $a^{(t)}$, can then be carried over to period $t + 1$. Thus, we have the following law of motion for the company's capital:

$$a^{(t)} = (a^{(t-1)}(1 - \tau) + B^{(t)} - c_1(B^{(t)}) + p^{(t)}) (1 + r) - I^{(t)}, \quad (2.1)$$

assuming $a^{(t)} \geq 0$. If the company defaults, it pays out all remaining assets to policyholders. The company cannot shed more capital than it has available. Hence, for $a^{(t-1)} \geq 0$, we require that:

$$B^{(t)} \geq -a^{(t-1)}(1 - \tau). \quad (2.2)$$

The objective function for each period can be derived using the revenue (premium collected), minus the costs (indemnity, frictional costs on carrying capital, and financing costs). For each

⁴In the rest of the essay, we use $X^{(:,t)}$ as the sum across the lines $\sum_{n=1}^N X^{(n,t)}$. $X^{(:,t)}$ is used to represent the line-by-line collection (vector) $(X^{(1,t)}, \dots, X^{(N,t)})$, and its subset $X^{(m:n,t)} = (X^{(m,t)}, \dots, X^{(n,t)})$. $X^{(n,t:t+s)}$ represents a collection of random variables over discrete time $(X^{(n,t)}, X^{(n,t+1)}, \dots, X^{(n,t+s)})$.

period, the expected aggregate indemnity takes the following form:

$$e^{(t)} = \mathbb{E} \left[I^{(t)} \mathbf{1}_{\{a^{(1)} \geq 0, \dots, a^{(t)} \geq 0\}} + (a^{(t)} + I^{(t)}) \mathbf{1}_{\{a^{(1)} \geq 0, \dots, a^{(t)} < 0\}} \mid \Delta^{(:,t-1)} \right].$$

Note that here we write the remaining assets in case of default as $a^{(t)} + I^{(t)} < I^{(t)}$.

Hence, the company's period profit function f is:

$$\begin{aligned} f(s_t = \{a^{(t-1)}, \mathbf{Q}^{(:,t-1)}, \Delta^{(:,t-1)}\}, c_t = \{q^{(:,t)}, p^{(:,t)}, B^{(t)}\}) \\ = (1+r)p^{(t)} - e^{(t)} - (1+r)(\tau a^{(t-1)} + c(B^{(t)})) \\ = \mathbb{E} \left[\mathbf{1}_{\{a^{(1)} \geq 0, \dots, a^{(t)} \geq 0\}} \{ (1+r)p^{(t)} - I^{(t)} - (1+r)(\tau a^{(t-1)} + c(B^{(t)})) \} \right. \\ \left. - \mathbf{1}_{\{a^{(1)} \geq 0, \dots, a^{(t)} < 0\}} (1+r)(a^{(t-1)} + B^{(t)}) \mid s_t \right]. \end{aligned} \quad (2.3)$$

The state “ s_t ” contains all the variables that determine the state of the company at the beginning of period t . The control “ c_t ” contains all the variables that the company chooses in maximizing the objective function, also at the beginning of period t . In particular, both s_t and c_t are predictable with the information from the loss triangle $\Delta^{(:,t-1)}$. The insurance company's ultimate objective is to maximize future expected discounted cash flows, which corresponds to the following infinite horizon optimization problem:

$$\max_{c_t} \sum_{t=1}^{\infty} \mathbb{E}[\beta^t f(s_t, c_t)], \quad (2.4)$$

where $\beta = (1+r)^{-1}$ is the discount factor. The objective function can be equivalently represented as present value of future dividends as follows:

$$\max_{c_t} \mathbb{E} \left[\sum_{t \leq t^*} -\beta^{t-1} B^{(t)} - a^{(0)} \right], \quad (2.5)$$

where t^* is the time such that $a^{(1)} \geq 0, a^{(2)} \geq 0, \dots, a^{(t^*-1)} \geq 0, a^{(t^*)} < 0$ (see Appendix 2.5.1 for the proof).

We solve the optimization with constraints (2.1), (2.2), a premium function for each line n , and a regulatory constraint if needed. For the premium function, we follow Bauer and Zanjani (2018) and assume that the premium charged for one line is the expected present (actuarial) value of future

losses multiplied by a markup function. The present value of future losses for each line at the end of period t can be represented as

$$R^{(n,t)} = \sum_{j=1}^{d_n} \beta^{j-1} q^{(n,t)} L_j^{(n,t+j-1)}.$$

The markup function is a (decreasing) function of company risk ϕ and size $\theta = \mathbb{E} [R^{(t)} \mid \Delta^{(:,t-1)}]$, defined as

$$\pi^{(n)} = \pi^{(n)}(\phi, \theta),$$

with the assumption on partial derivatives:

$$\pi_1^{(n)} = \frac{\partial \pi^{(n)}(\phi, \theta)}{\partial \phi} < 0, \text{ and } \pi_2^{(n)} = \frac{\partial \pi^{(n)}(\phi, \theta)}{\partial \theta} < 0,$$

so a company with greater risk and larger size charges a smaller markup over actuarial value.

ϕ is a risk metric that measures the risk of a company given its total indemnities and assets. For measuring “risk,” we assume that the policyholders are concerned with the company’s period solvency, so that the risk depends on total indemnities paid $I^{(t)}$ and total end-of-periods assets $S^{(t)}$:

$$\phi = \phi(I^{(t)}, S^{(t)}),$$

where $S^{(t)} = (a^{(t-1)}(1 - \tau) + B^{(t)} - c(B^{(t)}) + p^{(t)}) (1 + r)$. Here, similarly to Bauer and Zanjani (2018), in addition to obvious monotonicity assumptions ($\phi(I, x) \leq \phi(I, y), x \geq y$, and $\phi(X, x) \leq \phi(Y, x), X \leq Y$), we assume scale invariance of the risk metric, i.e. $\phi(aI, ax) = \phi(I, x), a > 0$. The key example that we will rely on in our numerical applications is the conditional default probability:

$$\phi(I^{(t)}, S^{(t)}) = \mathbb{P}(I^{(t)} > S^{(t)} \mid \Delta^{(:,t-1)}).$$

We note that this specification assumes consumers are myopic in that they are only concerned with the coming period—and not necessary the performance of their contract. This may be justified with the assumption that consumers rely on company ratings that obviously do not depend on the term of the obligation.

Altogether, we have the following premium function for line n :

$$p^{(n,t)} = \mathbb{E} [\beta R^{(n,t)} \mid \Delta^{(:,t-1)}] \times \pi^{(n)}(\phi, \theta). \quad (2.6)$$

According to Bertsekas (1995), the optimization problem (2.4) is an infinite-horizon discrete-time stochastic optimal control problem, resulting in the following Bellman equation:

Proposition 1. (*Bellman Equation*). *The Bellman equation for problem (2.4) reads:*

$$\begin{aligned} & V(a^{(t-1)}, \mathbf{Q}^{(:,t-1)}, \Delta^{(:,t-1)}) \\ &= \max_{q^{(:,t)}, p^{(:,t)}, B^{(t)}} \mathbb{E} \left[I_{\{I^{(t)} \leq S^{(t)}\}} \left(p^{(t)} - \beta I^{(t)} - \tau a^{(t-1)} - c(B^{(t)}) + \beta V(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \right) \right. \\ & \quad \left. - I_{\{I^{(t)} > S^{(t)}\}} (a^{(t-1)} + B^{(t)}) \mid \Delta^{(:,t-1)} \right], \end{aligned}$$

subject to (2.1), (2.2), and (2.6)

Here the default threshold for the company is $S^{(t)}$. Once the aggregate indemnity is greater than $S^{(t)}$, the company defaults. We do not consider the option of raising emergency capital to save the company as in Bauer and Zanjani (2018), since the focus of this essay is on how loss history, i.e. past exposures $\mathbf{Q}^{(:,t-1)}$ and losses $\Delta^{(:,t-1)}$, affect the optimal exposure, raising, and allocation decisions. However, incorporating emergency capital is theoretically straightforward. In particular, when incorporating emergency raising capital, we note that the model in Bauer and Zanjani (2018) will be a special case of the general model here with one development year in all business lines, thus effectively reducing the value function to one dimension with $a^{(t-1)}$. In our general setting, with the company having N lines and each line n having d_n development years, there are a total of $1 + \frac{1}{2} \sum_{n=1}^N (d_n^2 + d_n - 2)$ state variables.

2.2.3 Term Structure of Capital Costs

We rely on the first order conditions of the Bellman equation in Proposition (1) to derive the marginal cost of risk (the proof is provided in Appendix 2.5.1).

Proposition 2. (*Term Structure Equation*). *We have the marginal cost of risks presented in the form*

of term structure of capital costs:

$$\begin{aligned}
& \mathbb{E} \left[\sum_{j=1}^{d_n} \beta^j L_j^{(n,t+j-1)} \mid \Delta^{(:,t-1)} \right] * \pi^{(n)} \left(1 + \sum_{i=1}^N \frac{\pi_2^{(i)}}{\pi^{(n)}} \mathbb{E} \left[R^{(i,t)} \mid \Delta^{(:,t-1)} \right] \right) \\
&= \sum_{s=0}^{d_n-1} (1 - c'(B^{(t)})) \cdot \left(\mathbb{E} \left[I_{\{I^{(t)} \leq S^{(t)}, \dots, I^{*(t+s)} \leq S^{*(t+s)}\}} \beta^{s+1} L_{s+1}^{(n,t+s)} w_{t+s} \mid \Delta^{(:,t-1)} \right] \right. \\
&\quad \left. + \beta \frac{\partial \rho(I^{(t)})}{\partial q^{(n,t)}} \mathbb{E} \left[I_{\{I^{(t)} > S^{(t)}\}} w_t \mid \Delta^{(:,t-1)} \right] \right), \tag{2.7}
\end{aligned}$$

$$\text{where } w_t = \begin{cases} 1 + V_1(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) & I^{(t)} \leq S^{(t)} \\ \sum_{i=1}^N \frac{1}{1 - c'(B^{(t)})} \mathbb{E} \left[R^{(i,t)} \mid \Delta^{(:,t-1)} \right] * \pi_1^{(i)} \frac{\partial \phi}{\partial S^{(t)}} & I^{(t)} > S^{(t)}, \end{cases}$$

ρ is the risk measure associated with the risk metric ϕ , and $V_1 = \partial V / \partial a$.

w_t is a weighting function similar to the function in Bauer and Zanjani (2018), with

$$\mathbb{E} \left[(1 - c'(B^{(t)})) w_t \right] = 1.$$

The weighting function $w_{t+s} = 1 + V_1(a^{*(t+s)}, \mathbf{Q}^{*(:,t+s)}, \Delta^{*(:,t+s)})$ on $I^{(t)} \leq S^{(t)}, \dots, I^{*(t+s)} \leq S^{*(t+s)}$ from time t to $t+d_n-1$, where $a^{*(t+s)}, \mathbf{Q}^{*(:,t+s)}$ etc. are the (stochastic) future state variables under the optimal policy.

The left-hand side of the equation represents the marginal premium income, adjusted down by company size. The first line of the right-hand side of the equation consists of evaluation of future losses for accident year t , namely, losses $(L_1^{(n,t)}, L_2^{(n,t+1)}, \dots, L_{d_n}^{(n,t+d_n-1)})$, which are dependent through a stochastic loss reserving model. The last line of the right-hand side of the equation are about the capital allocation $\frac{\partial \rho(I^{*(t)})}{\partial q^{(n,t)}}$ multiplied by associated capital costs.

The interpretation of the term structure equation is that valuation of liability and capital allocation should be done by line and development year in each line, in contrast to the previous literature, where only by-line allocation is considered. Not only does the equation highlight the importance of loss reserving in pricing P&C insurance, but it also redefines the capital allocation and risk pricing goal in a P&C insurer. We are interested in the term structure of capital costs, namely $V_1(a^{*(t+s)}, \mathbf{Q}^{*(:,t+s)}, \Delta^{*(:,t+s)})$ and how it varies with capitalization level. In what follows,

we solve a basic version of our theoretical model numerically, we explore insurer's decision at optimality, and we analyze how V_1 behaves in a dynamic setting.

2.2.4 Implementation – Two Lines and Two Development Years

In this section, we provide an implementation of our theory in the previous section in the context of a P&C insurer. Specifically, the P&C insurer has two business lines and two development years on the long-tailed line (2L2DY). We then calibrate and solve for the model using numerical methods.

In 2L2DY, Line 1 is the long-tailed line with development year loss, the paid loss triangle is a 2x1 triangle, as illustrated in Figure 2.2. Line 2 is assumed to be the short-tailed line with no development years beyond the accident year. The time period equals to $AY + DY - 1$. Therefore, at the end of current period t , the insurer faces losses $L_1^{(1,t)}$ and $L_2^{(1,t)}$ from its long-tailed line 1, and $L_1^{(2,t)}$ from its short-tailed line 2. The loss random variables above the solid lines in Figure 2.2 are realized before t . The grayed-out $L_2^{(1,t+1)}$ is not a part of the loss triangle and not realized until the end of the next period $t + 1$, but it is relevant to the premium written for the accident year t and therefore related to the insurer's problem.

Line 1			Line 2	
AY \ DY	DY		AY \ DY	DY
	1	2		1
1	$L_1^{(1,1)}$	$L_2^{(1,2)}$	1	$L_1^{(2,1)}$
...
t-2	$L_1^{(1,t-2)}$	$L_2^{(1,t-1)}$	t-2	$L_1^{(2,t-2)}$
t-1	$L_1^{(1,t-1)}$	$L_2^{(1,t)}$	t-1	$L_1^{(2,t-1)}$
t	$L_1^{(1,t)}$	$L_2^{(1,t+1)}$	t	$L_1^{(2,t)}$

Figure 2.2: Losses for a company under 2L2DY

Line 1			Line 2	
AY \ DY	DY		AY \ DY	DY
	1	2		1
t-1	$L_1^{(1)}$	$L_2^{(1)}$	t-1	$L_1^{(2)}$
t	$L_1'^{(1)}$	$L_2'^{(1)}$	t	$L_1'^{(2)}$

Figure 2.3: Losses relevant to the bellman equation for a company under 2L2DY

We simplify notations $L_j^{(n,t-2+j)}$ to $L_j^{(n)}$ and $L_j^{(n,t-1+j)}$ to $L_j'^{(n)}$, shown in Figure 2.3, as the

prime “ t ” denotes state variables in the next period. We put three assumptions on loss triangles: (i) Chain-Ladder in loss development; (ii) conditional normality of loss distribution; (iii) linear correlation between lines. These assumptions make the model tractable and more efficient to calculate the moments of loss random variables in the Bellman equation. Details of three distributional assumptions are presented in Appendix 2.5.2 and useful in developing numerical solutions.

All in all, we solve the following simplified Bellman equation:

$$\begin{aligned} & V(a, q^{(1)}, L_1^{(1)}) \\ &= \max_{q^{(1)}, q^{(2)}, p^{(1)}, p^{(2)}, B} \beta \mathbb{E} \left[\mathbf{1}_{\{I \leq S\}} (S - I) + \mathbf{1}_{\{I > S\}} V(a', q^{(1)}, L_1^{(1)}) \right] - a - B, \end{aligned} \quad (2.8)$$

where:

$$S = (a(1 - \tau) + B - c_1(B) + p^{(1)} + p^{(2)})e^r \quad a' = S - I.$$

For the premium functions, akin to Bauer and Zanjani (2018), we assume the following specification:

$$p_n = \mathbb{E} \left[\beta R^{(n)} \right] \times \exp \left\{ \alpha_n - \delta_n \mathbb{P}(I > S) - \gamma_n \mathbb{E}[R] \right\}, n = 1, 2$$

where $R^{(1)} = q^{(1)} L_1^{(1)} + \beta q^{(1)} L_2^{(1)}$, $R^{(2)} = q^{(2)} L_1^{(2)}$, and $R = R^{(1)} + R^{(2)}$ representing the aggregate risk in the premium $p_1 + p_2$. Note that the aggregate risk R does not equal to the aggregate indemnity I , because of the long-tailed line. I reflects losses to be paid out in the current time period, while R entails risks exposed in one accident year across two periods. $\mathbb{E}[R]$, instead of $\mathbb{E}[I]$ used in the models without development year, reflects the aggregate scale of the insurance business. The motivation for this specification is that policyholders assess company quality via ratings that reflect the default probability, and increasing the scale of insurance business decreases profit margins. We then can specify the premium function as the product of expected present value of future exposed losses and a corresponding markup function.

In the numerical implementation, we calibrate the 2L2DY model with premium parameters, loss triangle parameters and company level parameters listed in Table 2.1. We choose the same premium parameters for both lines. Although a generalization to two distinct sets of premium parameters is possible, it complicates the model solution and may yield results that are difficult

	Long-Tailed Line	Short-Tailed Line
Premium		
α	0.1	0.1
β	-10.0	-10.0
γ	-5.0E-10	-5.0E-10
Triangle		
f	2.0	N/A
σ^2	2.5E7	N/A
$\mu_1^{(n)}$	1.0E8	1.0E8
$(\sigma_1^{(n)})^2$	6.25E14	6.25E14
ρ	0.5	
Company		
τ		0.03
$c_1^{(1)}$		0.075
$c_1^{(2)}$		1E-10

Table 2.1: Model parameters

to interpret. We leave corresponding extensions for future research. We set the capital costs as $\tau = 0.03$, $c_1^{(1)} = 0.075$, $c_1^{(2)} = 1.0E-10$, and the risk-free interest rate is $r = 0.03$, as in the “base case” scenario in Bauer and Zanjani (2018). We then use value iteration method to solve the Bellman equation (2.8) numerically on discretized grids of $a \in [0, 2.0E9]$, $q^{(1)} \in [0, 2.0]$, and $L^{(1)} \in [5.0E7, 1.5E8]$. A detailed solution to the 2L2DY model and corresponding numerical techniques are detailed in Appendix 2.5.3.

2.2.5 Results – Two Lines and Two Development Years

Since the value function and optimal policies are functions of three state variables, it is impossible to capture the results in a single graph. Therefore, we graph functions of capital a and previous exposure on the long-tailed line $q^{(1)}$ on the x and y axis, given two extreme levels of the previous shock $L_1^{(1)}$ (large and small). Figure 2.4 shows the value function and optimal policies under an extremely small previous shock of two standard deviations below the mean at $L_1^{(1)} = 5.0E7$. Figure 2.8 shows the solution under an extremely large previous shock of two standard deviations above the mean at $L_1^{(1)} = 1.5E8$. We choose extreme levels to documents the effects.

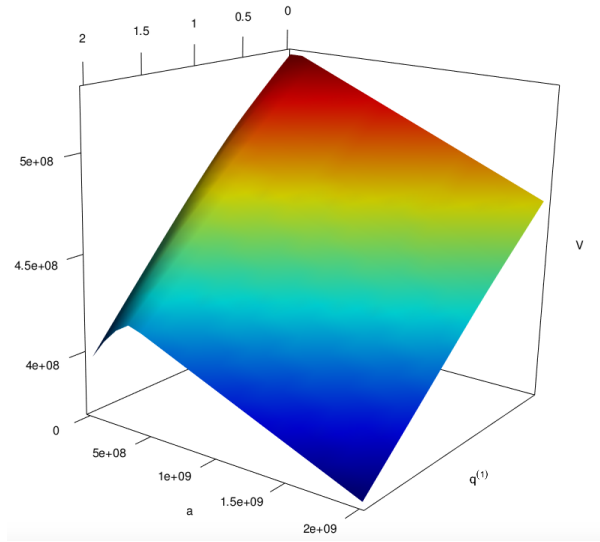
The value function, optimal raising of external capital, and exposure to the short-tailed line all

match the corresponding characteristics in Bauer and Zanjani (2018). More precisely, the value function is concave with an optimal capitalization level that economizes on costly external financing, internal capital costs, and an optimal company size. The firm raises capital if it is severely underfunded or sheds capital (pays dividends) if it is severely overfunded, but remains inactive for capitalization levels around the optimal point. The optimal exposure to the short-tailed line is concave and increasing in the capital level, up to a saturation point where costs associated with scale do not warrant further expansion.

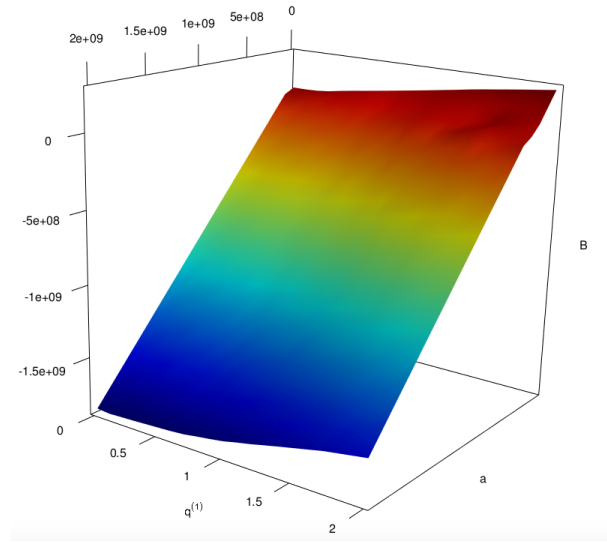
However, the solution here additionally provides insights on how previous exposure in the long-tailed line affect the value function and optimal policies. The ridge in Figures 2.4(a) and 2.8(a) depict the “optimal capital line”, which connects a ’s that maximize V under $q^{(1)} \in [0, 2.0]$. To the left of the line, firm value decreases with capital, reflecting the cost of raising external financing; to the right of the line, firm value decreases as the capital level increases, reflecting the cost of carrying internal capital. The optimal capitalization point increases in both the previous exposure $q^{(1)}$ and the loss realization $L_1^{(1)}$. The pattern is also shown in two-dimensional Figures 2.5(a) and 2.9(a).

The optimal capital raising decision is slightly more subtle as seen from Figures 2.4(b) and 2.8(b). While with a higher previous exposure on long-tailed line, the company will keep more capital and will start raising capital earlier, with lower previous exposure the company will raise more aggressively for low capital levels. This reflects the increased value with limited loss legacy.

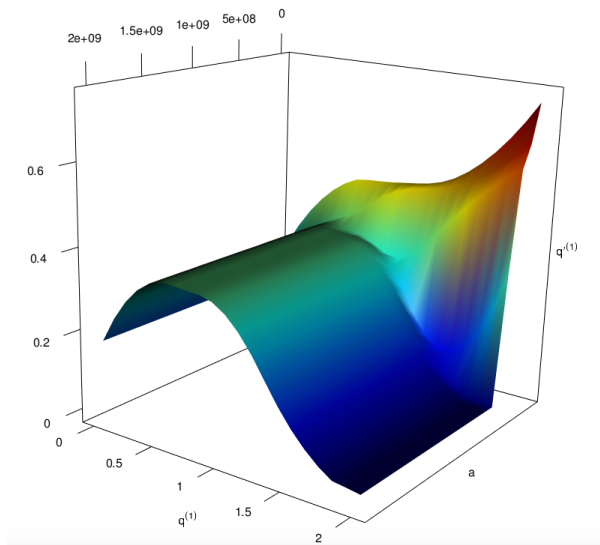
The optimal long-tailed line exposure is depicted in Figures 2.4(c) and 2.8(c). Interestingly, the optimal exposure is strictly increasing with previous exposure when the capital level is low—which may be counterintuitive at first sight. As the insurer starts out with low capital, it would sell more insurance on the long-tailed line, less on short-tailed line and raise external capital. The reason is that the insurer will be paying 100% of the loss incurred in the short-tailed line, but only a fraction of the loss incurred in the short-tailed line, while it earns full premium on both lines. As a result, the long-tailed line offers a relatively attractive source of financing for a firm in need of funds. In other words, the long-tailed line can serve to gain short-term financing, at lower cost than raising external capital. This effect is more pronounced for a high previous exposure, since more short-term financing is needed.



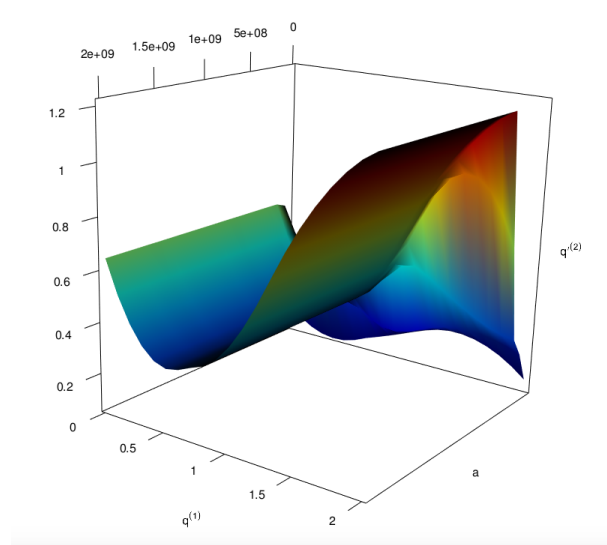
(a) Value function



(b) External capital

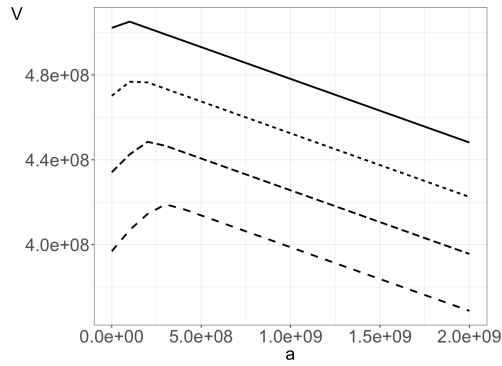


(c) Line 1 (long-tailed) exposure

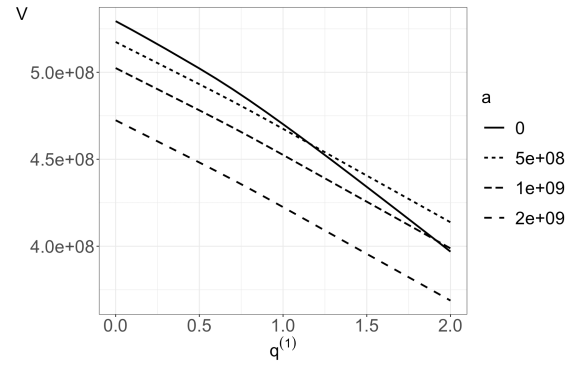


(d) Line 2 (short-tailed) exposure

Figure 2.4: Value function, optimal external capital raising and exposure decision under small previous shock

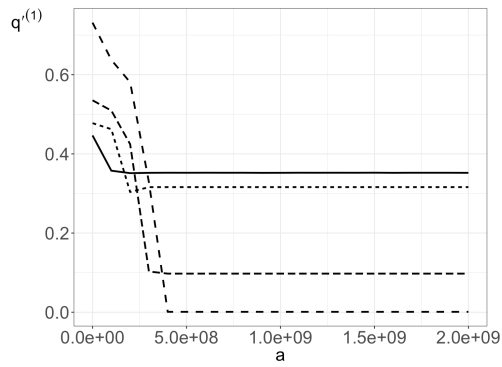


(a) Function of a

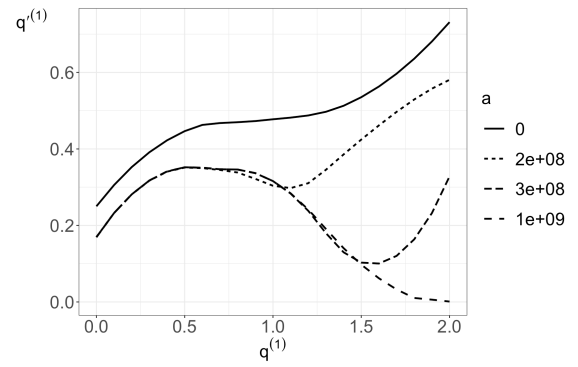


(b) Function of $q^{(1)}$

Figure 2.5: 2-dimensional representations of value function (small previous shock)

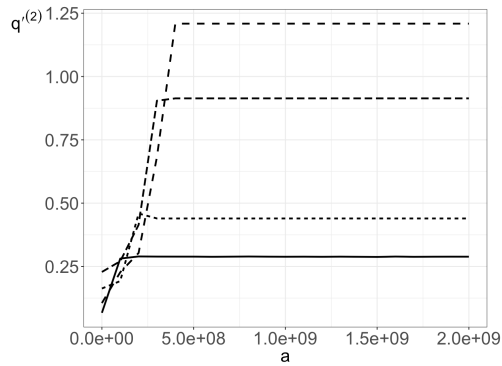


(a) Function of a

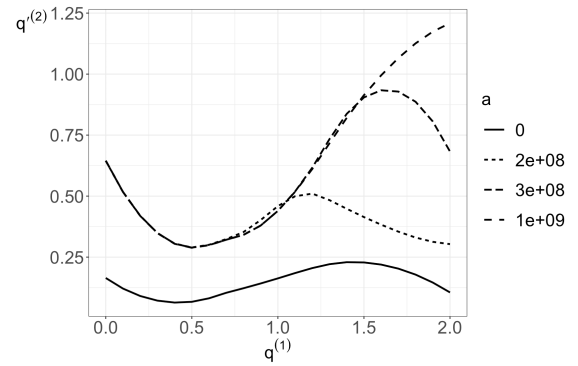


(b) Function of $q^{(1)}$

Figure 2.6: 2-dimensional representations of long-tailed line exposure (small previous shock)

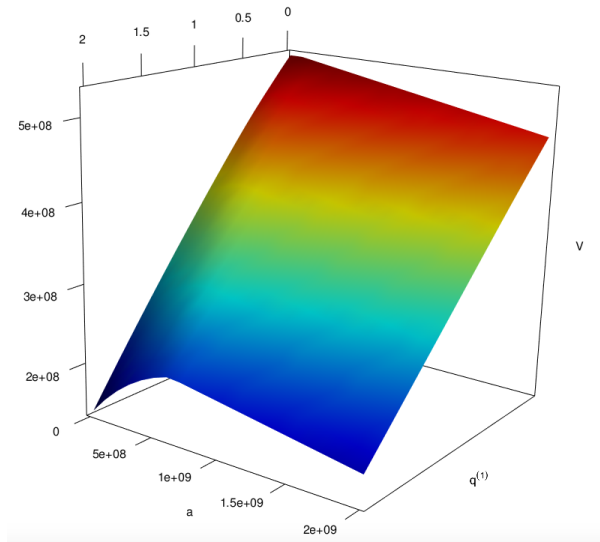


(a) Function of a

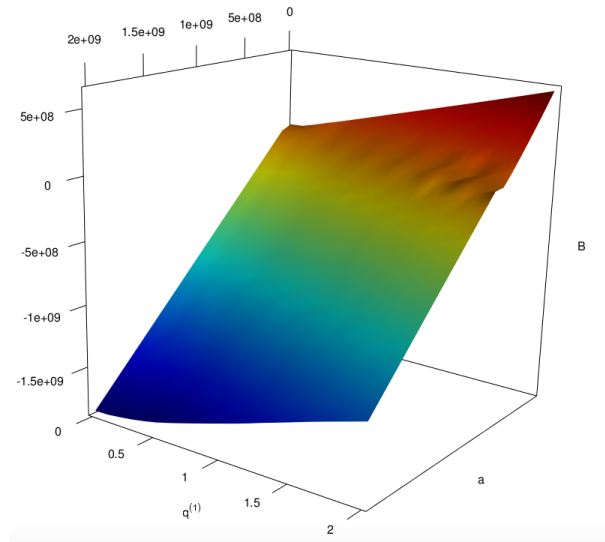


(b) Function of $q^{(1)}$

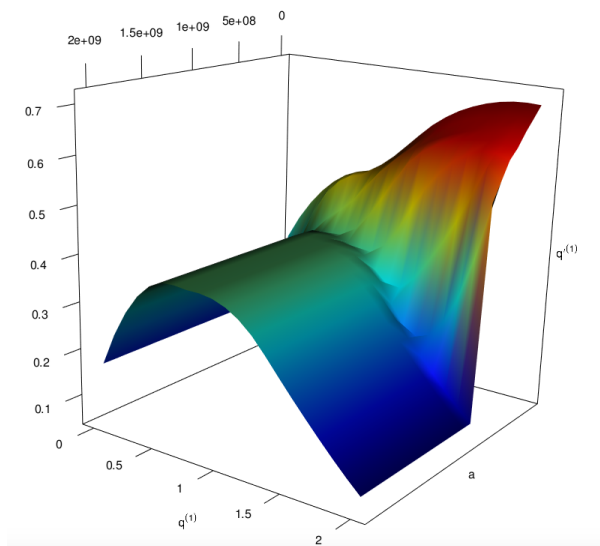
Figure 2.7: 2-dimensional representations of short-tailed line exposure (small previous shock)



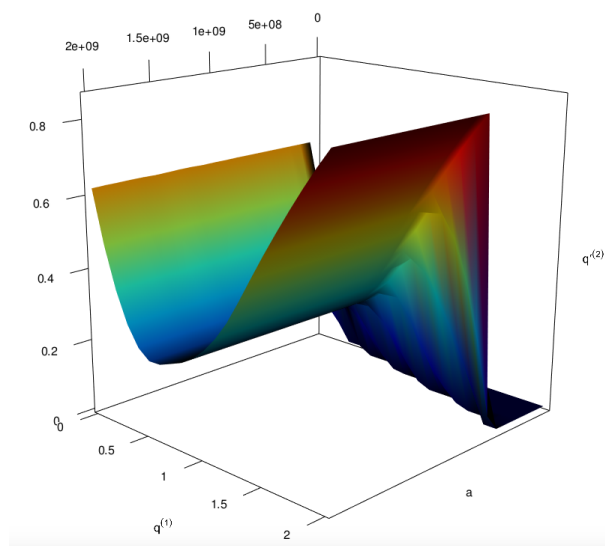
(a) Value function



(b) External capital

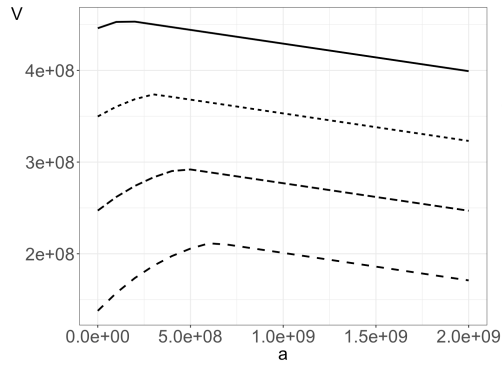


(c) Line 1 (long-tailed) exposure

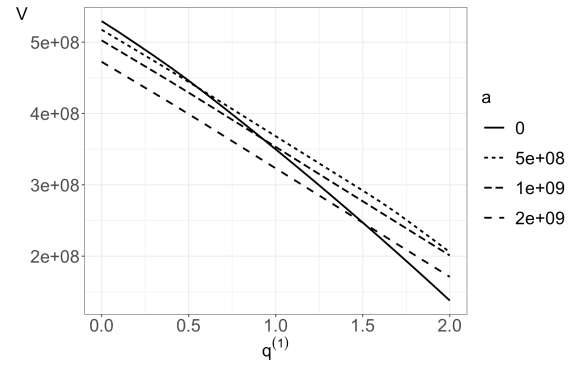


(d) Line 2 (short-tailed) exposure

Figure 2.8: Value function, optimal external capital raising and exposure decision under large previous shock

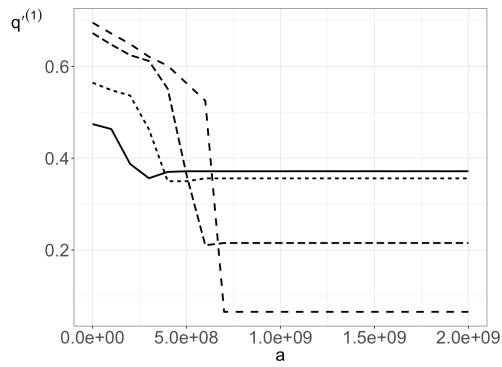


(a) Function of a

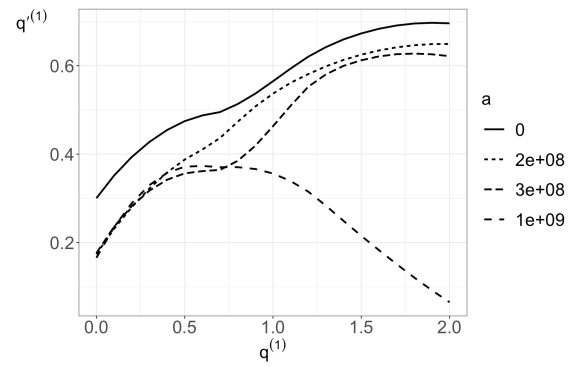


(b) Function of $q^{(1)}$

Figure 2.9: 2-dimensional representations of value function (large previous shock)

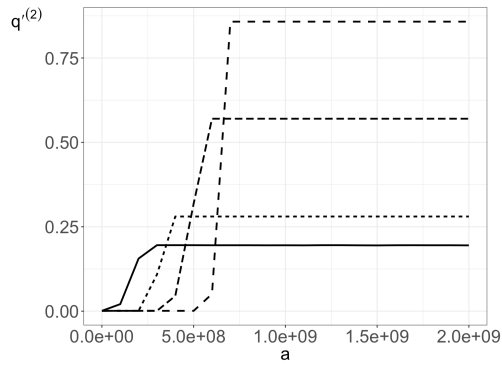


(a) Function of a

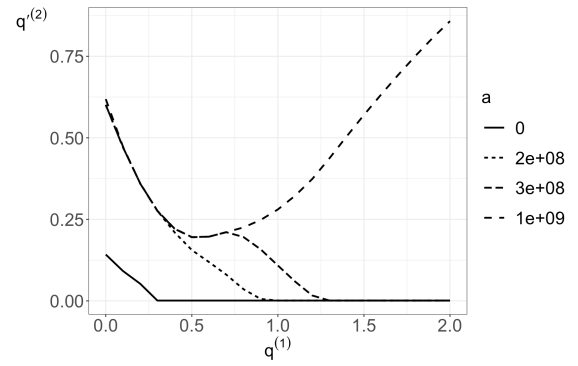


(b) Function of $q^{(1)}$

Figure 2.10: 2-dimensional representations of long-tailed line exposure (large previous shock)



(a) Function of a



(b) Function of $q^{(1)}$

Figure 2.11: 2-dimensional representations of short-tailed line exposure (large previous shock)

When the capital is over optimal capital line, the optimal exposure to the long-tailed line increases first with previous long-tailed line exposure and then decreases. The need for financing declines and the insurer's objective is to balance the current long-tailed line exposure and previous one. Thus, when the previous exposure is too high, the insurer chooses to decrease the current exposure to long-tailed line business. In particular, Figure 2.6(b) shows that while optimal exposure to line 1 is increasing in previous exposure for low a , the relationship inverts for large capital level a . Similarly, for high capital levels, optimal exposure to the long-tailed line becomes flat as shown in Figure 2.6(a).

The optimal exposure to the short-tailed line complements with the optimal exposure to the long-tailed line as evident from Figures 2.4(d) and 2.8(d). As a company has more exposure on its long-tailed line in the last period, it will be responsible for a greater amount of indemnity developed from the last period. As a result, the insurer will keep more capital and reduce exposure on its short-tailed line business. When the capital is high, the insurer would increase its short-tailed line exposure as it complements with decreasing long-tailed line exposure, as we observe at $a = 1E09$ in Figure pairs 2.6(b) & 2.7(b), and 2.10(b) & 2.11(b). In terms of exposure with respect to capital, we can see in Figures 2.7(a) and 2.11(a) that the optimal short-tailed exposure increases with capital but becomes flat when the capital is over optimal capital line, similar to the findings in Bauer and Zanjani (2018). Most evidently in Figure 2.11(a) when $q^{(1)} = 2$ and capital is low, it is optimal for the insurer to almost completely shut down the short-tailed line. On the other hand, when the capital is high, it is optimal for the insurer to almost completely shut down the long-tailed line, as seen in Figure 2.10(a).

The value function also provides insight to the term structure of capital costs. According to the term structure equation in Section 2.2.3, the gradient of value function with respect to capital V_1 forms the term structure of capital costs. Because of the concavity of the value function with respect to capital as seen in Figures 2.5(a) and 2.9(a), when the company is undercapitalized or its capital is less than the optimal capitalization level, then $V_1 > 0$ and the company effectively apply markup to losses in premium pricing. On the other hand, when the company is overcapitalized or its capital is more than the optimal capitalization level, then $V_1 < 0$ and the company marks down on losses. Specifically in Figure 2.9(a), we observe that the value function displays greater

At the beginning of period t	At the end of period t	At the end of period $t + 1$
Financially constrained	0.0254	-0.0170
Well capitalized	-0.0303	-0.0196

Table 2.2: Numerical calculation of V_1

concavity when $q^{(1)}$ is high, thus larger V_1 , than when $q^{(1)}$ is low. As a result, the term structure of capital costs is higher for firms that are financially constrained, lower for firms that are better capitalized, and even negative for firms that are extremely well capitalized.

The natural next steps would be finding the term structure numerically by calculating V_1 for every development year. However, for our 2L2DY implementation, we can only obtain numerical calculation of V_1 for two periods. A sample calculation is shown in Table 2.2. When the company begins the period with financial constraint, i.e. with low capitalization and high past exposure, V_1 at the end of the period is positive. For a company that starts with high capital and low past exposure, V_1 at the end of the period is positive. Both V_1 's turn closer to zero at the end of period $t + 1$. From the sample numerical results, we can see a downward sloping V_1 for a financially constrained company, who puts more adjustment to the expected losses in the current period than the next period. On the other hand, we can see an upward sloping V_1 for the well capitalized company. Alas, the numerical calculation provides very limited information on the term structure, which ideally spans for ten years for a P&C company. If we set out to numerically solve for a 10-year term structure, it requires implementation of full 10×10 loss triangle. As discussed in Section 2.2.2, we will face a model with more than 100 state variables and such model is impossible to be solved using dynamic programming approach. Instead, in the next section, we propose an estimation of the term structure of capital costs in an empirical setting. Our empirical findings are in line with our theoretical results.

2.3 Empirical Study

An insurer relies on premium income, and also capital to support its expected future losses. Capitals are costly to hold, but necessary to obtain whenever the insurance businesses are less profitable and/or the insurer does not do well financially. In calculating how much premium to charge for

additional exposure in future losses, a company considers valuation other than simply pricing at actuarial value of expected future losses. The reason is that companies are risk averse about loss shocks that may happen in the future. If such shocks occur, they consume capital quickly and will put the companies at great risk of default and/or out of competition. For most companies, the losses to be paid out in the near future (e.g. 1-3 years) are of the most concern and companies choose to price them higher than their actuarial value, while losses to be paid out in the far future (e.g. 4-10 years) are priced closer to their actuarial value. We are interested in how a P&C company weighs on their valuation of expected future losses at different point of time, and how the valuation changes among companies with different capitalization levels.

Our theoretical finding, i.e. term structure equation (2.7), shows that the valuation of loss reserve in each line of business involves two discount factors. The first one is a yield curve, in our theory being flat at r . The second is a term structure of future losses, described in our theory by $(1 + V_1(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}))$ from t to $t + d_n - 1$. Since V_1 is $\partial V / \partial a$ and does not vary by business lines, the term structure summarizes the sensitivity of premium (aggregate/by line) to future losses (aggregate/by line) from development year 1 to d_n . Although it makes sense to generalize term structures of future losses by line, in this essay we remain focus on finding a term structure at company level for different years. Our goal is to estimate the sensitivity of premium to the expected future losses and capital in a P&C company in an empirical study.

Now, we start from identifying the variables of interest and back out an equation for estimation. From our theoretical results (2.7), multiplying exposure $q^{(n,t)}$ to both sides of the equation and summing up by line, we obtain the following equation:

$$\begin{aligned}
& \underbrace{\sum_{n=1}^N \mathbb{E} \left[\sum_{j=1}^{d_n} \beta^j q^{(n,t)} L_j^{(n,t+j-1)} \mid \Delta^{(:,t-1)} \right]}_{\text{Total Premium}} * \pi^{(n)} \left(1 + \sum_{i=1}^N \frac{\pi_2^{(i)}}{\pi^{(n)}} \mathbb{E} \left[R^{(i,t)} \mid \Delta^{(:,t-1)} \right] \right) \\
&= \underbrace{\sum_{n=1}^N \sum_{s=0}^{d_n-1} (1 - c'(B^{(t)})) \cdot \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}, \dots, I^{*(t,s)} \leq S^{*(t+s)}\}} \beta^{s+1} q^{(n,t)} L_{s+1}^{(n,t+s)} (1 + V_1^{(t+s)}) \mid \Delta^{(:,t-1)} \right]}_{\text{Valuation of Total Loss Reserve}} \\
&+ \underbrace{\beta \rho(I^{(t)}) \left(1 - \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} (1 + V_1^{(t)}) (1 - c'(B^{(t)})) \mid \Delta^{(:,t-1)} \right] \right)}_{\text{Capital Costs}} \quad (2.9)
\end{aligned}$$

We obtain total premium for all lines plus a fraction on the left-hand-side and valuation of loss reserve plus capital costs on the right-hand-side. The left-hand side of Equation 2.9 can be identified using net premium written for all lines in company i in a given year, denoted by $P_i = \sum_{n=1}^N P_{n,i}$, where $P_{n,i}$ is the net premium written for line n . The capital costs component on the right-hand side is identified using capital/surplus of a company multiplied by unit cost of capital or return-on-capital, which is a parameter to be estimated. We denote surplus for company i as $SURP_i$ and unit cost as c .

Next, we wish to identify both the expected future losses component and term structure component $V_1^{(t+s)}$. The expected paid loss can be identified/constructed using chain-ladder approach. Specifically, we use the following relationships introduced in the numerical implementation section:

$$\mathbb{E} \left[L_1^{(n,t)} \right] = p^{(n,t)} \cdot f_1$$

$$\mathbb{E} \left[\sum_{m=1}^s L_m^{(n,t+m-1)} \mid \sum_{m=1}^{s-1} L_m^{(n,t+m-1)} \right] = \sum_{m=1}^{s-1} L_m^{(n,t+m-1)} \cdot f_{m+1}, \quad m = 1, 2, \dots, d_n - 1,$$

where f_s 's are the chain-ladder factor estimated using loss triangles in the last 10 years. Thus, in addition to the chain-ladder factors, we need the premium for each line and we use net premium written $P_{n,i}$. The interest rate discount factor β , though being flat in our theoretical model, can be identified using FRED yield curve of a given year, with m -year yield denoted by r_m . We denote expected future loss (adjusted for interest rate discount and without adjustment for term structure of reserve) for company i as EL_i with definition:

$$EL_i = \sum_{n=1}^N \left((1 + r_1)^{-1} P_{n,i} \cdot f_1 + \sum_{m=2}^{d_n} (1 + r_m)^{-m} P_{n,i} \cdot f_1 \cdot \dots \cdot f_{m-1} \cdot (f_m - 1) \right).$$

We use the following function from Nelson and Siegel (1987) to start identifying the term structure component.

$$g(m; \beta_0, \beta_1, \beta_2, \tau) = \left(\beta_0 + \beta_1 \left(\frac{1 - e^{-\frac{m}{\tau}}}{\frac{m}{\tau}} \right) + \beta_2 \left(\frac{1 - (m + \tau)e^{-\frac{m}{\tau}}}{m} \right) \right), \quad m \in (0, \infty).$$

Even though more complex modelings of yield curve have developed, the above function pro-

vides a flexible fit for any yield curve while adds minimum complexity to the overall modeling. β_0 describes the long-term value of f (as m goes to infinity). $\beta_0 + \beta_1$ describes the short-term value of f (as m goes to zero). β_2 gives f function a hump or S shape. τ can be understood as a tuning parameter that determines how quickly the curve decays from the short-term value to its long-term value.

To help with our estimation goal, we make three assumptions on the term structure. First, the term structure is monotonic, i.e. $\beta_2 = 0$. In this case, a positive β_1 results in a decreasing yield curve and a negative β_1 results in an increasing one. Second, the term structure approaches zero in infinite future, or $\lim_{m \rightarrow \infty} g(m; \beta_0, \beta_1, \beta_2, \tau) = \beta_0 = 0$. The intuition is that companies are effectively risk neutral about the losses in infinite future and thus evaluate them at actuarial value. Also intuitively from our theoretical results, a firms capitalization level would approach optimality as firm always choose optimal level of exposure and external capital raising in the infinite future, thus V_1 will approach zero when time goes to infinity. Third, β_1 depends on how well the company is capitalized. More precisely, $\beta_1 = b_0 + b_1 \cdot X_i$, where X_i is a variable that assesses the capitalization of a company. Along with our assumptions, we can write down the term structure function as follows:

$$g_i(m) = (b_0 + b_1 X_i) \left(\frac{1 - e^{-\frac{m}{\tau}}}{\frac{m}{\tau}} \right),$$

and together with expected paid loss, we have the identification for valuation of total loss reserve:

$$\begin{aligned} EL_i + & b_0 \sum_{n=1}^N \sum_{m=1}^{d_n} \left((1+r_1)^{-1} \left(\frac{1 - e^{-\frac{1}{\tau}}}{\frac{1}{\tau}} \right) \cdot P_{n,i} \cdot f_1 + \sum_{m=2}^{d_n} (1+r_m)^{-m} \left(\frac{1 - e^{-\frac{m}{\tau}}}{\frac{m}{\tau}} \right) \cdot P_{n,i} \cdot f_1 \cdots f_{m-1} \cdot (f_m - 1) \right) \\ & + b_1 X_i \underbrace{\sum_{n=1}^N \sum_{m=1}^{d_n} \left((1+r_1)^{-1} \left(\frac{1 - e^{-\frac{1}{\tau}}}{\frac{1}{\tau}} \right) \cdot P_{n,i} \cdot f_1 + \sum_{m=2}^{d_n} (1+r_m)^{-m} \left(\frac{1 - e^{-\frac{m}{\tau}}}{\frac{m}{\tau}} \right) \cdot P_{n,i} \cdot f_1 \cdots f_{m-1} \cdot (f_m - 1) \right)}_{VEL_i} \end{aligned}$$

VEL_i means the valuation of loss reserve for company i . Now Equation 2.9 looks like:

$$P_i - \alpha_i = EL_i + b_0 \cdot VEL_i + b_1 \cdot X_i \times VEL_i + c \cdot SURP_i,$$

where α_i correspond to the addition amount over/under total premium on the left-hand side of

	2006		2011		2017	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
P_i	454,120	2,448,032	421,235	2,454,344	577,997	3,340,674
EL_i	205,130	1,180,483	184,314	1,177,542	250,948	1,532,206
EP_i	248,990	1,291,260	236,921	1,301,678	327,049	1,842,551
$SURP_i$	473,793	2,866,170	580,728	4,112,805	758,166	5,790,749
SA_i	0.4450	0.1988	0.4638	0.1958	0.4763	0.1890
No. Obs.	895		955		900	

Table 2.3: Summary statistics

Equation 2.9. We write $\alpha_i = \alpha + \epsilon_i$, with $\mathbb{E}[\epsilon_i] = 0$. We define $EP_i = P_i - EL_i$ as excess premium over expected losses. Rearrange the equation above, we propose the following model for estimation:

$$EP_i = \alpha + b_0 \cdot VEL_i + b_1 \cdot X_i \times VEL_i + c \cdot SURP_i + \epsilon_i \quad (2.10)$$

The estimation of the model takes two steps. We notice that there is a hidden parameter τ . We follow the empirical approach in Nelson and Siegel (1987) and first find a grid of τ values. Conditional on τ , the model essentially becomes a linear model and can be estimated using OLS approach. We analyze the model for each τ on a grid and find the “best” fit with the least residual standard deviation.

We use data obtained from combined NAIC⁵ annual statements of P&C insurance companies at group level in the U.S. and across multiple years. From each company, we can obtain net premium written and paid loss triangles (Schedule P part 3) for each business line. We have two candidates for X_i : surplus to asset ratio SA_i and logged leverage ratio $LR_i = \log\left(\frac{1 - SA_i}{SA_i}\right)$. We obtain $SURP_i$, SA_i and LR_i for year 2006, 2011 and 2017. We use 10 years of triangle to compute EL_i and VEL_i . For example, we obtain triangles from year 1996–2005 to estimate chain-ladder factors for expected losses in year 2006.

The estimation results are listed in Table 2.4. We use τ values from 0.1 to 2 with 0.01 increment and τ^* is one that results in the least residual standard error. For both candidates of X_i , the estimated unit capital cost c is about the same, with about 8% in 2006, 13% in 2011, and 7.5% in

⁵The National Association of Insurance Commissioners (NAIC) is the U.S. standard-setting and regulatory support organization created and governed by the chief insurance regulators from the 50 states, the District of Columbia and five U.S. territories.

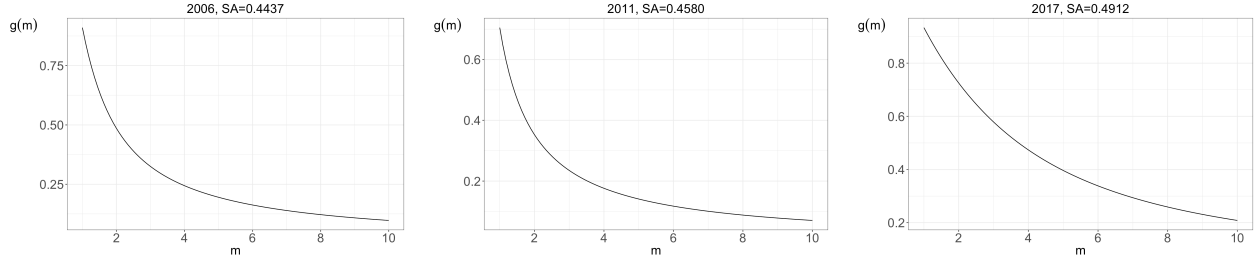


Figure 2.12: Term structure of loss reserves for representative company with normal capitalization

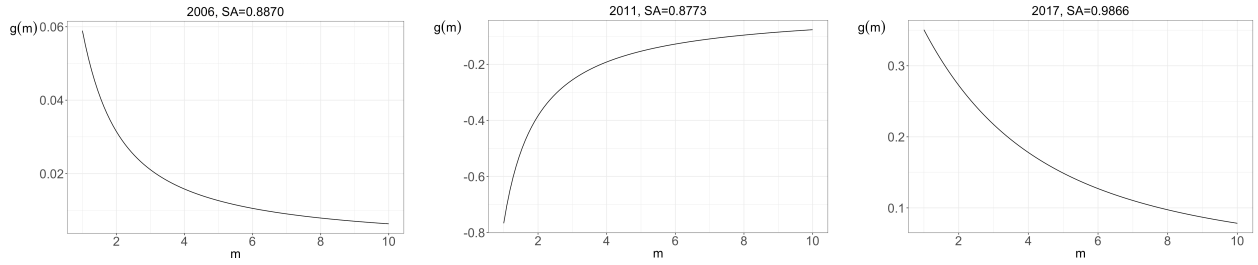


Figure 2.13: Term structure of loss reserves for representative company with high capitalization

2017. The estimate for b_0 and b_1 are all significant. Our estimation is robust to additional regressors as shown in Table 2.5.

We can use the estimates to back out term structure function $g_i(m)$. Figure 2.12 and 2.13 shows the term structure in three years for two different companies. The first company has lower capitalization level than the second one. In year 2006 and 2017, both companies has downward sloping term structure, but Company 1 price future expected losses at a much higher level. In year 2011, Company 2 has a upward sloping curve, which is not usually seen and only happens when the company has a very high level of capitalization. Company 2 price the future losses below actuarial value and the premium mostly recoup the high capital costs. Again, the empirical findings are in line with our theoretical results presented in Section 2.2.5.

2.4 Conclusion

In this chapter, we set out to explore how a financial institution evaluates its future cash flows as a term structure of capital costs in a P&C insurance company setting. We find both theoretical and empirical implication that a well-capitalized insurer discount its future claim payments into

	2006		2011		2017	
$X_i =$	SA_i	LR_i	SA_i	LR_i	SA_i	LR_i
(Intercept)	32558.96 (5897.93)	32382.95 (5969.82)	17498.88 (4499.02)	17783.77 (4524.51)	25307.95 (5271.12)	25612.60 (5295.69)
VEL_i	5.1050 (0.1010)	1.8723 (0.0345)	23.1275 (0.3170)	5.7211 (0.1792)	2.0026 (0.0345)	1.1889 (0.0219)
$X_i VEL_i$	-5.5629 (0.2373)	0.8747 (0.0399)	-35.1023 (0.6866)	7.9418 (0.1566)	-1.5588 (0.0850)	0.3395 (0.0192)
$SURP_i$	0.0829 (0.0045)	0.0783 (0.0045)	0.1297 (0.0065)	0.1297 (0.0065)	0.0758 (0.0026)	0.0753 (0.0026)
τ^*	0.37	0.52	0.10	0.10	1.69	1.86
R^2	0.9753	0.9747	0.9839	0.9837	0.9799	0.9797
Residual std. err.	172871	174963	134988	135766	153920	154662
No. Observation	895	895	955	955	900	900

Table 2.4: Regression results

	2006		2011		2017	
	Original	Robustness	Original	Robustness	Original	Robustness
(Intercept)	32558.96 (5897.93)	-150131.8 (28491.26)	17498.88 (4499.02)	-114796.80 (24427.16)	25307.95 (5271.12)	-167300.30 (28274.30)
VEL_i	5.1050 (0.1010)	4.8677 (0.1063)	23.1275 (0.3170)	23.2212 (0.3779)	2.0026 (0.0345)	1.9356 (0.0412)
$SA_i VEL_i$	-5.5629 (0.2373)	-4.7650 (0.3019)	-35.1023 (0.6866)	-34.9597 (1.0240)	-1.5588 (0.0850)	-1.4460 (0.1321)
$SURP_i$	0.0829 (0.0045)	0.0303 (0.0151)	0.1297 (0.0065)	0.1286 (0.0142)	0.0758 (0.0026)	0.0804 (0.0074)
$\log(ASSET)$	-	16593.96 (2601.42)	-	12172.29 (2226.08)	-	17289.7800 (2496.99)
Policyholder Dividend	-	3.1151 (0.3111)	-	0.1935 (0.3654)	-	-0.1940 (0.2959)
Investment Income	-	0.3120 (0.0958)	-	-0.1276 (0.0139)	-	0.0910 (0.0923)
R^2	0.9753	0.9792	0.9839	0.9844	0.9799	0.9809
Residual std. err.	172871	158900	134988	133000	153920	150400
No. Observation	895	895	955	955	900	900

Table 2.5: Regression results with robustness

premium and has a upward sloping term structure, while other insurers with lower capitalization level mark up their future claim payments and have a downward sloping term structure.

The model presented in this chapter takes into account the loss structure of a P&C insurance company, which is a novel feature relative to the previous literature. The general model is very flexible and can be applied to insurance companies that have both short-tailed and long-tailed business lines. The implementation of the model with two lines and two development years shows previous loss exposure and loss realization significantly affect the company's optimal policies. We find that long-tailed lines are employed as short-term sources of financing, an insight that considerably changes the characteristics and optimal policies relative to short-tailed lines.

Various extensions are possible. First, for tractability, we adopt a chain-ladder method with normal distributions. In the actuarial literature, there are more advanced models for estimating and forecasting loss triangles that may be considered. Second, we begin with the assumption that all the assets are invested at a fixed interest rate. Adding securities markets to the general model as well as other financing options or reinsurance would further bridge the gap between model and reality.

2.5 Appendix A: Technical Appendix

2.5.1 Proofs of the Lemmas and Propositions

Lemma 2.1 *The optimization problem (2.4) can be equivalently represented as a maximization of the present value of future dividends:*

$$\max_{c_t} \mathbb{E} \left[\sum_{t \leq t^*} -\beta^{t-1} B^{(t)} - a^{(0)} \right],$$

where t^* is the time such that $a^{(1)} \geq 0, a^{(2)} \geq 0, \dots, a^{(t^*-1)} \geq 0, a^{(t^*)} < 0$.

Proof. The capital motion equation (2.1) can be rewritten into:

$$\beta^t a^{(t)} - \beta^{t-1} a^{(t-1)} - \beta^{t-1} B^{(t)} = \beta^t [(1+r)p^{(t)} - I^{(t)} - (1+r)(\tau a^{(t-1)} + c(B^{(t)}))]$$

We can rewrite the objective function in (2.4) as the following:

$$\begin{aligned}
& \sum_{t=1}^{\infty} \mathbb{E} \left[\mathbf{1}_{\{a^{(1)} \geq 0, \dots, a^{(t)} \geq 0\}} \beta^t \{ (1+r)p^{(t)} - I^{(t)} - \beta^t (1+r)(\tau a^{(t-1)} + c(B^{(t)})) \} \right. \\
& \quad \left. - \mathbf{1}_{\{a^{(1)} \geq 0, \dots, a^{(t)} < 0\}} (1+r)(a^{(t-1)} + B^{(t)}) \right] . \\
& = \mathbb{E} \left[\sum_{t < t^*} \beta^t a^{(t)} - \beta^{t-1} a^{(t-1)} - \beta^{t-1} B^{(t)} - \beta^{t^*-1} (a^{(t^*-1)} + B^{(t^*)}) \right] \\
& = \mathbb{E} \left[\sum_{t \leq t^*} -\beta^{t-1} B^{(t)} + \beta^{t^*-1} a^{(t^*-1)} - a^{(0)} - \beta^{t^*-1} a^{(t^*-1)} \right] \\
& = \sum_{t \leq t^*} \mathbb{E} \left[-\beta^{t-1} B^{(t)} - a^{(0)} \right]
\end{aligned}$$

■

Proposition 2.1 (Bellman Equation). *The Bellman equation for problem (2.4) reads:*

$$\begin{aligned}
& V(a^{(t-1)}, \mathbf{Q}^{(:,t-1)}, \Delta^{(:,t-1)}) \\
& = \max_{q^{(:,t)}, p^{(:,t)}, B^{(t)}} \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} \left(p^{(t)} - \beta I^{(t)} - \tau a^{(t-1)} - c(B^{(t)}) + \beta V(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \right) \right. \\
& \quad \left. - \mathbf{1}_{\{I^{(t)} > S^{(t)}\}} (a^{(t-1)} + B^{(t)}) \mid \Delta^{(:,t-1)} \right],
\end{aligned}$$

subject to (2.1), (2.2), (2.6)

Proof. Since our objective function in (2.4) is bounded from above, following Bertsekas (1995), the infinite horizon optimization problem (2.4) subject to (2.1) is exactly resulting in the Bellman equation (1). ■

Proposition 2.2 (Term Structure Equation). *We have the marginal cost of risks presented in the form of term structure of reserve and capital costs:*

$$\begin{aligned}
& \mathbb{E} \left[\sum_{j=1}^{d_n} \beta^j L_j^{(n,t+j-1)} \mid \Delta^{(:,t-1)} \right] * \pi^{(n)} \left(1 + \sum_{i=1}^N \frac{\pi_2^{(i)}}{\pi^{(n)}} \mathbb{E} \left[R^{(i,t)} \mid \Delta^{(:,t-1)} \right] \right) \\
& = \sum_{s=0}^{d_n} \beta^{s+1} \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}, \dots, I^{*(t+s)} \leq S^{*(t+s)}\}} L_{s+1}^{(n,t+s)} \left(1 + V_1(a^{*(t+s)}, \mathbf{Q}^{*(t+s)}, \Delta^{(:,t+s)}) \right) (1 - c'(B^{(t)})) \mid \Delta^{(:,t-1:t+s-1)} \right] \\
& \quad + \beta \frac{\partial \rho(I^{(t)})}{\partial q^{(n,t)}} \left(\sum_{i=1}^N \mathbb{E} \left[R^{(i,t)} \mid \Delta^{(:,t-1)} \right] * \pi_1^{(i)} \frac{\partial \phi}{\partial S^{(t)}} \right)
\end{aligned} \tag{2.11}$$

Proof. The Bellman equation reads:

$$\begin{aligned} & V(a^{(t-1)}, \mathbf{Q}^{(:,t-1)}, \Delta^{(:,t-1)}) \\ &= \max_{q^{(:,t)}, p^{(:,t)}, B^{(t)}} \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} \left(p^{(:,t)} - \beta I^{(t)} - \tau a^{(t-1)} - c(B^{(t)}) + \beta V(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \right) \right. \\ & \quad \left. - \mathbf{1}_{\{I^{(t)} > S^{(t)}\}} (a^{(t-1)} + B^{(t)}) \mid \Delta^{(:,t-1)} \right], \end{aligned}$$

where,

$$\begin{aligned} I^{(:,t)} &= \sum_{n=1}^N (q^{(n,t)} * L_1^{(n,t)} + \mathbf{Q}^{(n,t-1)} * \mathbf{L}^{(n,t)}) \\ S^{(t)} &= \left(a^{(t-1)}(1 - \tau) + B^{(t)} - c(B^{(t)}) + p^{(:,t)} \right) (1 + r) \\ R^{(n,t)} &= \sum_{j=1}^{d_n} \beta^{j-1} q^{(n,t)} L_j^{(n,t+j-1)} \end{aligned}$$

subject to:

$$\begin{aligned} a^{(t)} &= S^{(t)} - I^{(:,t)} \\ p^{(n,t)} &= \mathbb{E} \left[\beta R^{(n,t)} \mid \Delta^{(:,t-1)} \right] * \pi^{(n)}(\phi, \theta), \quad n = 1, 2, \dots, N \end{aligned}$$

The premium function is the product of conditional expected present value of $R^{(n,t)}$, or future losses of the accident year t , and a markup function $\pi^{(n)}$. The markup function consists of two arguments: a risk metric ϕ and company size. We assume the risk metric is scale invariant, or $\phi(wI^{(t)}, wS^{(t)}) = \phi(I^{(t)}, S^{(t)})$, $w > 0$. Denote $\frac{\partial \pi^{(n)}(x,y)}{\partial x} = \pi_1^{(n)}$ and $\frac{\partial \pi^{(n)}(x,y)}{\partial y} = \pi_2^{(n)}$.

In addition, we denote the gradients of the value function

$$V_1(a, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) = \lim_{\delta \rightarrow 0} \frac{V(a + \delta, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) - V(a, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)})}{\delta}$$

For $s = 1, 2, \dots, d_n - 1$

$$V_{2,s}^{(n)}(a, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) = \lim_{\delta \rightarrow 0} \frac{V(a, \mathbf{Q}^{(1:n-1)}, \overbrace{\dots, q^{(n)} + \delta, \dots}^{s^{th} \text{ element in } \mathbf{Q}^{(n)}}, \mathbf{Q}^{(n+1:N)}, \Delta^{(:,t)}) - V(a, \mathbf{Q}^{(1:n-1)}, \overbrace{\dots, q^{(n)}, \dots}^{s^{th} \text{ element in } \mathbf{Q}^{(n)}}, \mathbf{Q}^{(n+1:N)}, \Delta^{(:,t)})}{\delta}$$

The Lagrangian writes:

$$\begin{aligned} \mathcal{L}^{(t)} &= \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} \left(p^{(:,t)} - \beta I^{(t)} - \tau a^{(t-1)} - c(B^{(t)}) + \beta V(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \right) - \mathbf{1}_{\{I^{(t)} > S^{(t)}\}} (a^{(t-1)} + B^{(t)}) \mid \Delta^{(:,t-1)} \right] \\ & \quad - \sum_{i=1}^N \lambda^{(i,t)} \left(p^{(i,t)} - \mathbb{E} \left[\beta R^{(i,t)} \mid \Delta^{(:,t-1)} \right] * \pi^{(i)}(\phi, \theta) \right) \end{aligned}$$

Take first order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}^{(t)}}{\partial q^{(n,t)}} &= \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} \left(-\beta L_1^{(n,t)} + \beta \left(-L_1^{(n,t)} V_1(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) + V_{2,1}^{(n)}(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \right) \right) \mid \Delta^{(:,t-1)} \right] \\
&\quad + \lambda^{(n,t)} \left(\mathbb{E} \left[\sum_{j=1}^{d_n} \beta^j L_j^{(n,t+j-1)} \mid \Delta^{(:,t-1)} \right] * \pi^{(n)} \left(\phi(I^{(t)}, S^{(t)}), \mathbb{E}_{t-1} [R^{(t)} \mid \Delta^{(:,t-1)}] \right) \right) \\
&\quad + \sum_{i=1}^N \lambda^{(i,t)} \mathbb{E} \left[\beta R^{(i,t)} \mid \Delta^{(:,t-1)} \right] * \left(\pi_1^{(i)} \frac{\partial \phi}{\partial q^{(n,t)}} + \pi_2^{(i)} \mathbb{E} \left[\sum_{j=1}^{d_n} \beta^j L_j^{(n,t+j-1)} \mid \Delta^{(:,t-1)} \right] \right) = 0 \quad (2.12)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}^{(t)}}{\partial p^{(n,t)}} &= \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} \left(1 + V_1(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \right) \mid \Delta^{(:,t-1)} \right] \\
&\quad - \lambda^{(n,t)} + \sum_{i=1}^N \lambda^{(i,t)} \mathbb{E} \left[\beta R^{(i,t)} \mid \Delta^{(:,t-1)} \right] * \pi_1^{(i)} \frac{\partial \phi}{\partial p^{(n,t)}} = 0 \quad (2.13)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}^{(t)}}{\partial B^{(t)}} &= \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} \left(-c'(B^{(t)}) + (1 - c'(B^{(t)})) V_1(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \right) - \mathbf{1}_{\{I^{(t)} > S^{(t)}\}} \mid \Delta^{(:,t-1)} \right] \\
&\quad + \sum_{i=1}^N \lambda^{(i,t)} \mathbb{E} \left[\beta R^{(i,t)} \mid \Delta^{(:,t-1)} \right] * \pi_1^{(i)} \frac{\partial \phi}{\partial B^{(t)}} = 0 \quad (2.14)
\end{aligned}$$

The envelope theorem suggests that

$$\begin{aligned}
&V_{2,1}^{(n)}(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \\
&= -\beta \mathbb{E} \left[\mathbf{1}_{\{I^{*(t+1)} \leq S^{*(t+1)}\}} L_2^{(n,t+1)} \left(1 + V_1(a^{*(t+1)}, \mathbf{Q}^{*(:,t+1)}, \Delta^{(:,t+1)}) \right) \right. \\
&\quad \left. + V_{2,2}^{(n)}(S^{*(t+1)} - I^{*(t+1)}, \underbrace{q^{*(:,t+1)}, q^{(:,t)}, \dots, q^{(:,t-d_n+2)}}_{\mathbf{Q}^{*(:,t+1)}}, \Delta^{(:,t+1)}) \mid \Delta^{(:,t)} \right] \\
&V_{2,2}^{(n)}(a^{(t+1)}, \mathbf{Q}^{(:,t+1)}, \Delta^{(:,t+1)}) \\
&= -\beta \mathbb{E} \left[\mathbf{1}_{\{I^{*(t+2)} \leq S^{*(t+2)}\}} L_3^{(n,t+2)} \left(1 + V_1(a^{*(t+2)}, \mathbf{Q}^{*(:,t+2)}, \Delta^{(:,t+2)}) \right) \right. \\
&\quad \left. + V_{2,3}^{(n)}(S^{*(t+2)} - I^{*(t+2)}, q^{*(:,t+2)}, q^{(:,t+1)}, \dots, q^{(:,t-d_n+3)}, \Delta^{(:,t+2)}) \mid \Delta^{(:,t+1)} \right] \\
&\cdot \\
&\cdot \\
&\cdot \\
&V_{2,d_n-1}^{(n)}(a^{(t+d_n-2)}, \mathbf{Q}^{(:,t+d_n-2)}, \Delta^{(:,t+d_n-2)}) \\
&= -\beta \mathbb{E} \left[\mathbf{1}_{\{I^{*(t+d_n-1)} \leq S^{*(t+d_n-1)}\}} L_{d_n}^{(n,t+d_n-1)} \left(1 + V_1(a^{*(t+d_n-1)}, \mathbf{Q}^{*(:,t+d_n-1)}, \Delta^{(:,t+d_n-1)}) \right) \right. \\
&\quad \left. + \underbrace{V_{2,d_n}^{(n)}(S^{*(t+d_n-1)} - I^{*(t+d_n-1)}, q^{*(:,t+d_n-1)}, q^{(:,t+d_n-2)}, \dots, q^{(:,t)}, \Delta^{(:,t+d_n-1)})}_{=0} \mid \Delta^{(:,t+d_n-2)} \right] \quad (2.15)
\end{aligned}$$

In the end, we have

$$\begin{aligned}
& V_{2,1}^{(n)}(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)}) \\
= & -\beta \mathbb{E} \left[\mathbf{1}_{\{I^*(:,t+1) \leq S^*(t+1)\}} L_2^{(n,t+1)} \left(1 + V_1(a^{*(t+1)}, \mathbf{Q}^{(:,t+1)}, \Delta^{(:,t+1)}) \right) \mid \Delta^{(:,t)} \right] \\
& -\beta^2 \mathbb{E} \left[\mathbf{1}_{\{I^*(:,t+1) \leq S^*(t+1) \text{ \& } I^*(:,t+2) \leq S^*(t+2)\}} L_3^{(n,t+2)} \left(1 + V_1(a^{*(t+2)}, \mathbf{Q}^{(:,t+2)}, \Delta^{(:,t+2)}) \right) \mid \Delta^{(:,t)}, \Delta^{(:,t+1)} \right] \\
& \dots \\
& -\beta^{d_n-1} \mathbb{E} \left[\mathbf{1}_{\{I^*(:,t+1) \leq S^*(t+1), \dots, I^*(:,t+d_n-1) \leq S^*(t+d_n-1)\}} L_{d_n}^{(n,t+d_n-1)} \right. \\
& \quad \times \left. \left(1 + V_1(a^{*(t+d_n-1)}, \mathbf{Q}^{(:,t+d_n-1)}, \Delta^{(:,t+d_n-1)}) \right) \mid \Delta^{(:,t+d_n-2)}, \dots, \Delta^{(:,t)} \right] \\
= & -\sum_{s=1}^{d_n-1} \beta^s \mathbb{E} \left[\mathbf{1}_{\{I^*(:,t+1) \leq S^*(t+1), \dots, I^*(:,t+s) \leq S^*(t+s)\}} L_{s+1}^{(n,t+s)} \left(1 + V_1(a^{*(t+s)}, \mathbf{Q}^{(:,t+s)}, \Delta^{(:,t+s)}) \right) \mid \Delta^{(:,t:t+s-1)} \right]
\end{aligned} \tag{2.16}$$

Now, since

$$\frac{\partial \phi}{\partial B^{(t)}} = \frac{\partial \phi}{\partial S^{(t)}} \frac{\partial S^{(t)}}{\partial B^{(t)}} = (1+r)(1-c'(B^{(t)})) \frac{\partial \phi}{\partial S^{(t)}} = (1-c'(B^{(t)})) \frac{\partial \phi}{\partial S^{(t)}} \frac{\partial S^{(t)}}{\partial p^{(n,t)}} = (1-c'(B^{(t)})) \frac{\partial \phi}{\partial p^{(n,t)}}$$

From equations (2.13) and (2.14), we have

$$\lambda^{(n,t)} = \frac{1}{1-c'(B^{(t)})}, \quad \forall n = 1, 2, \dots, N, \quad t = 1, 2, \dots$$

and

$$\sum_{i=1}^N \frac{1}{1-c'(B^{(t)})} \mathbb{E} \left[R^{(i,t)} \mid \Delta^{(:,t-1)} \right] * \pi_1^{(i)} \frac{\partial \phi}{\partial S^{(t)}} = \frac{1}{1-c'(B^{(t)})} - \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} (1 + V_1(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)})) \mid \Delta^{(:,t-1)} \right] \tag{2.17}$$

The scale invariance property of ϕ yields the following (cf. Bauer and Zanjani, 2018):

$$\begin{aligned}
0 &= \frac{\partial}{\partial w} \phi(wI^{(t)}, wS^{(t)}) = S^{(t)} \frac{\partial}{\partial S^{(t)}} \phi(wI^{(t)}, wS^{(t)}) + I^{(t)} \frac{\partial}{\partial I^{(t)}} \phi(wI^{(t)}, wS^{(t)}) \\
\Rightarrow S^{(t)} &= \sum_{i=1}^N \sum_{j=0}^{d_n-1} q^{(i,t-j)} \frac{\frac{\partial}{\partial q^{(i,t-j)}} \phi(I^{(t)}, S^{(t)})}{-\frac{\partial}{\partial S^{(t)}} \phi(I^{(t)}, S^{(t)})}
\end{aligned} \tag{2.18}$$

Define ρ as the risk measure associated with the risk metric ϕ , with adding-up property:

$$\sum_{i=1}^N \sum_{j=0}^{d_n-1} q^{(i,t-j)} \frac{\partial \rho(I^{(t)})}{\partial q^{(i,t-j)}} = \sum_{i=1}^N \sum_{j=0}^{d_n-1} q^{(i,t-j)} \frac{\frac{\partial}{\partial q^{(i,t-j)}} \phi(I^{(t)}, S^{(t)})}{-\frac{\partial}{\partial S^{(t)}} \phi(I^{(t)}, S^{(t)})} = S^{(t)}$$

Hence for every i, j and t , we have:

$$\frac{\partial \phi}{\partial q^{(i,t-j)}} = -\frac{\partial \phi}{\partial S^{(t)}} * \frac{\partial \rho(I^{(t)})}{\partial q^{(i,t-j)}} \quad (2.19)$$

With equations (2.16), (2.17) and (2.19), (2.12) becomes:

$$\begin{aligned} \frac{\partial \mathcal{L}^{(t)}}{\partial q^{(n,t)}} &= -\beta \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}\}} \left(L_1^{(n,t)} (1 + V_1(a^{(t)}, \mathbf{Q}^{(:,t)}, \Delta^{(:,t)})) \right) \mid \Delta^{(:,t-1)} \right] \\ &\quad - \sum_{s=1}^{d_n-1} \beta^{s+1} \mathbb{E} \left[\mathbf{1}_{\{I^{(t)} \leq S^{(t)}, \dots, I^{*(t+s)} \leq S^{*(t+s)}\}} L_{s+1}^{(n,t+s)} \left(1 + V_1(a^{*(t+s)}, \mathbf{Q}^{(:,t+s)}, \Delta^{(:,t+s)}) \right) \mid \Delta^{(:,t-1)} \right] \\ &\quad + \frac{1}{(1 - c'(B^{(t)}))} \mathbb{E} \left[\sum_{j=1}^{d_n} \beta^j L_j^{(n,t+j-1)} \mid \Delta^{(:,t-1)} \right] * \left(\pi^{(n)} + \sum_{i=1}^N \pi_2^{(i)} \mathbb{E} \left[R^{(i,t)} \mid \Delta^{(:,t-1)} \right] \right) \\ &\quad - \beta \frac{\partial \rho(I^{(t)})}{\partial q^{(n,t)}} \left(\sum_{i=1}^N \frac{1}{1 - c'(B^{(t)})} \mathbb{E} \left[R^{(i,t)} \mid \Delta^{(:,t-1)} \right] * \pi_1^{(i)} \frac{\partial \phi}{\partial S^{(t)}} \right) = 0 \end{aligned}$$

Rearrange and obtain the term structure equation (2.7). ■

2.5.2 2L2DY Loss Distribution Assumptions

In loss triangle depicted by Figure 2.3, $L_1^{(n)}$ are losses paid in the previous accident year $t-1$. $L_1^{(1)}$ is a realization included in the Markov structure, and therefore is a state variable in the optimal control problem, Line 2 has no development year, so $L_1^{(2)}$ is irrelevant in the optimal control problem. $L_1'^{(1)}$, $L_1'^{(2)}$, and $L_2^{(1)}$ are the losses to be paid in the current period t , with $L_1'^{(1)}$ and $L_1'^{(2)}$ being paid losses for accident year t and $L_2^{(1)}$ being the second development year paid loss for the previous accident year $t-1$ of line 1. Therefore, $L_1'^{(1)}$, $L_1'^{(2)}$, and $L_2^{(1)}$ are the stochastic disturbances and have distributions subject to probability measures $p(dL_1'^{(1)} | L_1^{(1)})$, $p(dL_1'^{(2)} | L_1^{(2)})$ and $p(dL_2^{(1)} | L_1^{(1)})$. Meanwhile, $L_2'^{(1)}$ is the paid loss in the next period $t+1$, developed from $L_1'^{(1)}$ and therefore related to the current period's premium in Line 1, $L_2'^{(1)}$ is also the stochastic disturbance subject to probability measures $p(dL_2'^{(1)} | L_1'^{(1)})$. We make the following assumptions for the properties of the probability measures:

Following **Chain-Ladder**, we assume:

$$\begin{aligned} \mathbb{E}(L_2^{(1)} | L_1^{(1)}) &= (f-1)L_1^{(1)}, & \mathbb{E}(L_2'^{(1)} | L_1'^{(1)}) &= (f-1)L_1'^{(1)}, \\ \mathbb{V}(L_2^{(1)} | L_1^{(1)}) &= \sigma^2 L_1^{(1)}, & \mathbb{V}(L_2'^{(1)} | L_1'^{(1)}) &= \sigma^2 L_1'^{(1)}. \end{aligned}$$

We assume **conditional normality**:

$$\begin{aligned} L_1'^{(n)} | L_1^{(n)} = L_1'^{(n)} &\sim \mathcal{N}(\mu_1^{(n)}, (\sigma_1^{(n)})^2) \quad n = 1, 2, \\ L_2^{(1)} | L_1^{(1)} &\sim \mathcal{N}((f-1)L_1^{(1)}, \sigma^2 L_1^{(1)}), \\ L_2'^{(1)} | L_1'^{(1)} &\sim \mathcal{N}((f-1)L_1'^{(1)}, \sigma^2 L_1'^{(1)}). \end{aligned}$$

We assume a **linear correlation** between lines:

$$\text{corr}(L_1^{(1)}, L_1^{(2)}) = \text{corr}(L_1'^{(1)}, L_1'^{(2)}) = \rho.$$

The chain-ladder property follows Mack (1993), whose model assumes $M_{t,j-1}^{(n)} = C_{t,j-1}^{(n)}$. That is, the cumulative paid loss in each accident year is a Markov chain, with the ultimate development year being the time horizon of the chain. f and σ^2 are respectively the chain-ladder factor and its variance factor.

As is conventional in this context, the indemnity is assumed to be proportional to the current-period exposures $q'^{(n)}$ and last-period exposures $q^{(n)}$. Hence, the indemnity random variable is specified as $I = q'^{(1)}L_1'^{(1)} + q'^{(2)}L_1'^{(2)} + q^{(1)}L_2^{(1)}$. This linearity assumption entails that the marginal claim distribution is fixed, so that the loss distribution is homogeneous. In addition to linearity of indemnity, the conditional normality assumption ensures that I is also normal, with conditional mean and variance:

$$\begin{aligned} \mu_I = \mathbb{E}(I | L_1^{(1)}) &= q'^{(1)}\mu_1^{(1)} + q'^{(2)}\mu_1^{(2)} + (f-1)q^{(1)}L_1^{(1)}, \\ \sigma_I^2 = \mathbb{V}(I | L_1^{(1)}) &= (q'^{(1)})^2(\sigma_1^{(1)})^2 + (q'^{(2)})^2(\sigma_1^{(2)})^2 + (q^{(1)})^2\sigma^2 L_1^{(1)} + 2q'^{(1)}q'^{(2)}\rho\sigma_1^{(1)}\sigma_1^{(2)}. \end{aligned}$$

Again, this is in line with typical assumptions, and generalizations are possible.

2.5.3 Numerical Solution of 2L2DY Model

To solve the Bellman equation, we rely on numerical methods. By our premium function assumption, $q'^{(1)}$, $q'^{(2)}$, and B endogenously determine the sum of premiums P , and therefore these are the only choice variables. Note that the expectations of Equation (2.8) entail functions of $L_1'^{(1)}$

and I , which renders the problem two-dimensional. To solve it, we use the numerical integration method from Tanskanen and Lukkarinen (2003).

First, let $X = L_1^{(1)}$ and $Y = I|L_1^{(1)}$. Note that X and Y , both univariate normal, can be represented as a bivariate normal distribution:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} \mu_1^{(1)} \\ \mu_I \end{pmatrix}, \begin{pmatrix} (\sigma_1^{(1)})^2 & \rho_{x,y} \sigma_1^{(1)} \sigma_I \\ \rho_{x,y} \sigma_1^{(1)} \sigma_I & \sigma_I^2 \end{pmatrix} \right),$$

where $\rho_{x,y} = \frac{q^{(1)} \sigma_1^{(1)} + q^{(2)} \rho \sigma_1^{(2)}}{\sigma_I}$. Hence, the conditional distribution of Y given X is also univariate normal with mean and variance

$$\begin{aligned} \mu_{y|x} &= \mu_I + \rho_{x,y} \sigma_I \frac{x - \mu_1^{(1)}}{\sigma_1^{(1)}}, \\ \sigma_{y|x}^2 &= \sigma_I^2 (1 - \rho_{x,y}^2), \end{aligned} \tag{2.20}$$

respectively.

Let $f(x, y)$ be the density of the bivariate normal of X and Y , $f(y|x)$ be the conditional density of Y given X , and $f(x)$ be the marginal density of X . Because of the nature of the value function is unknown, we need to use the value iteration method to solve the Bellman equation (2.8) on a discretized state-space. For 2L2DY model, there are three state variables: capital a , last-period exposures on long-tailed line 1 $q^{(1)}$, and last-period loss realizations on long-tailed line 1 $L_1^{(1)}$.

Here are the steps of solving the Bellman equation using value iteration.

(1) Pick grids for $a = (a_1, a_2, \dots, a_s)$, $q^{(1)} = (q_1, q_2, \dots, q_n)$, and $L_1^{(1)} = (x_1, x_2, \dots, x_p)$. Set $V_0 = v_0(a, q^{(1)}, L_1^{(1)})$, where v_0 is an arbitrary function.

(2) Solve the optimization problem on the right hand side of the Bellman equation and get optimized state variables c^* and yield policy function $c = u_1((a, q^{(1)}, L_1^{(1)}); c^*)$. Then obtain the next value function $V_1((a, q^{(1)}, L_1^{(1)}); u_1) \dots$ until V_j converges.

We can obtain a simplified Bellman equation for implementation:

$$\begin{aligned}
\beta \mathbb{E} \left\{ \mathbf{1}_{\{I \leq S\}} S - I \right\} &= \int_{-\infty}^S \beta(S - y) f(y) \, dy \\
&= \beta S \Phi_I(S) - \beta \int_{-\infty}^S y \frac{1}{\sqrt{2\pi}\sigma_I} e^{-\frac{(y-\mu_I)^2}{2\sigma_I^2}} \, dy \\
&= \beta ((S - \mu_I) \Phi_I(S) + \sigma_I \phi_I(S))
\end{aligned}$$

And

$$\begin{aligned}
\mathbb{E} \left\{ \mathbf{1}_{\{I \leq S\}} V(a', L_1'^{(1)}) \right\} &= \int_{-\infty}^{\infty} \int_{-\infty}^S V(a', q'^{(1)}, x) f(x, y) \, dy \, dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^S V(a', q'^{(1)}, x) f(y|x) \, dy \right) f(x) \, dx,
\end{aligned} \tag{2.21}$$

where Φ and ϕ , respectively, are the CDF and PDF of a standard normal distribution.

To solve the inner integral on a grid, we apply the Tanskanen and Lulkarinen (2003) method. First, we interpolate on a . We pick $(l+1)$ -point grids for $I \geq 0$, say (y_0, y_1, \dots, y_l) , with $0 = y_0 < y_1 < \dots < y_l = S$, let $\varphi_i = V(a'(y_i), q'^{(1)}, L_1'^{(1)})$.

For $a'(y_i) \in (a_k, a_{k+1})$, we approximately have by linear interpolation:

$$\varphi_i = \frac{a_{k+1} - a'(y_i)}{a_{k+1} - a_k} V(a_k, q'^{(1)}, L_1'^{(1)}) + \frac{a'(y_i) - a_k}{a_{k+1} - a_k} V(a_{k+1}, q'^{(1)}, L_1'^{(1)})$$

If $a'(y_i) > a_l$, we can extrapolate:

$$\varphi_i = \frac{a'(y_i) - a_{l-1}}{a_l - a_{l-1}} V(a_l, q'^{(1)}, L_1'^{(1)}) + \frac{a_l - a'(y_i)}{a_l - a_{l-1}} V(a_{l-1}, q'^{(1)}, L_1'^{(1)})$$

The linear interpolation w.r.t. Y is

$$V(a', q'^{(1)}, L_1'^{(1)}) = \sum_{k=0}^{l-1} \left(\varphi_k + \frac{y - y_k}{y_{k+1} - y_k} (\varphi_{k+1} - \varphi_k) \right) \mathbf{1}_{[y_k, y_{k+1})}(y)$$

We then break down the integral into sums:

$$\begin{aligned}
& \int_{-\infty}^S V(a', q'^{(1)}, x) f(y|x) dy \\
&= \sum_{k=0}^{l-1} \left[\left(\varphi_k - \frac{y_k(\varphi_{k+1} - \varphi_k)}{y_{k+1} - y_k} \right) \int_{y_k}^{y_{k+1}} f(y|x) dy + \left(\frac{\varphi_{k+1} - \varphi_k}{y_{k+1} - y_k} \right) \int_{y_k}^{y_{k+1}} y f(y|x) dy \right] \\
&= \sum_{k=0}^{l-1} \left\{ \left(\varphi_k - \frac{y_k(\varphi_{k+1} - \varphi_k)}{y_{k+1} - y_k} \right) \left[\Phi \left(\frac{y_{k+1} - \mu_{y|x}}{\sigma_{y|x}} \right) - \Phi \left(\frac{y_k - \mu_{y|x}}{\sigma_{y|x}} \right) \right] \right. \\
&\quad + \left(\frac{\varphi_{k+1} - \varphi_k}{y_{k+1} - y_k} \right) \left[\mu_{y|x} \left(\Phi \left(\frac{y_{k+1} - \mu_{y|x}}{\sigma_{y|x}} \right) - \Phi \left(\frac{y_k - \mu_{y|x}}{\sigma_{y|x}} \right) \right) \right. \\
&\quad \left. \left. - \sigma_{y|x} \left(\phi \left(\frac{y_{k+1} - \mu_{y|x}}{\sigma_{y|x}} \right) - \phi \left(\frac{y_k - \mu_{y|x}}{\sigma_{y|x}} \right) \right) \right] \right\} \\
&= h(x)
\end{aligned} \tag{2.22}$$

Therefore the right-hand side of our Bellman equation can be written as

$$\beta \left((S - \mu_I) \Phi \left(\frac{S - \mu_I}{\sigma_I} \right) + \sigma_I \phi \left(\frac{S - \mu_I}{\sigma_I} \right) \right) + \int_{-\infty}^{\infty} \beta h(x) f(x) dx - a - B, \quad (**)$$

which can be solved using a discretized grid of $L_1'^{(1)}$.

For our interpolation, we choose a $m+1$ -point equally spaced grid on $[\mu_1^{(1)} - 5\sigma_1^{(1)}, \mu_1^{(1)} + 5\sigma_1^{(1)}]$, say (x_0, x_1, \dots, x_m) . In our case the grid size is 26. Use the trapezoidal rule to break the integral down into sums, say $F(x) = \beta(g(x) + h(x))f(x)$, then the integral becomes:

$$\int_{-\infty}^{\infty} F(x) dx = \frac{x_m - x_0}{2m} (F(x_0) + 2F(x_1) + \dots + 2F(x_{m-1}) + F(x_m))$$

Hence we successfully convert double integrals into sums and therefore significantly reduce the computation time without compromising the accuracy. The value iteration is implemented and run in Julia. The optimization is executed using Julia's NLOpt package. The value function is defined on a 21 x 21 x 3 discretized grid (with 21 grid points on a and $q'^{(1)}$). We ran the program for 80 iterations and the value function converges for both value function and choice variables.

The total runtime was about 100,000s, on a Intel I7 dual-core CPU. In this specific numerical task, Julia is six times faster than the popular high-level language such as R and MatLab. In

particular, Julia is much faster with loops, which is heavily found in the iterations and numerical integrals. According to Julia language developers, Julia is a high-performance language suitable for dynamic programming and its syntax is easily adapted from R or Matlab. Julia's high efficiency helps shorten the runtime from one week that would have taken on R, to just under one day.

Compared to previous models without considering DY, which only has one state variable capital, the 2L2DY model suffers from “the curse of dimensionality”. As the general $nLjDY$ model has hundreds, million, or even trillion times more states, each iteration of value function would take proportional more time to complete, resulting the value iteration to finish in months or even years. Solving this high-dimensional problem seems infeasible even five years ago, but thanks to the power of modern day computing, it is feasible under proper assumptions. We start with solving 2L2DY, which has the least dimension in the general $nLjDY$ model. In the future, we will continue to refine the algorithm and implement parallel computing to further shorten the running time.

2.6 Appendix B: Additional Figures

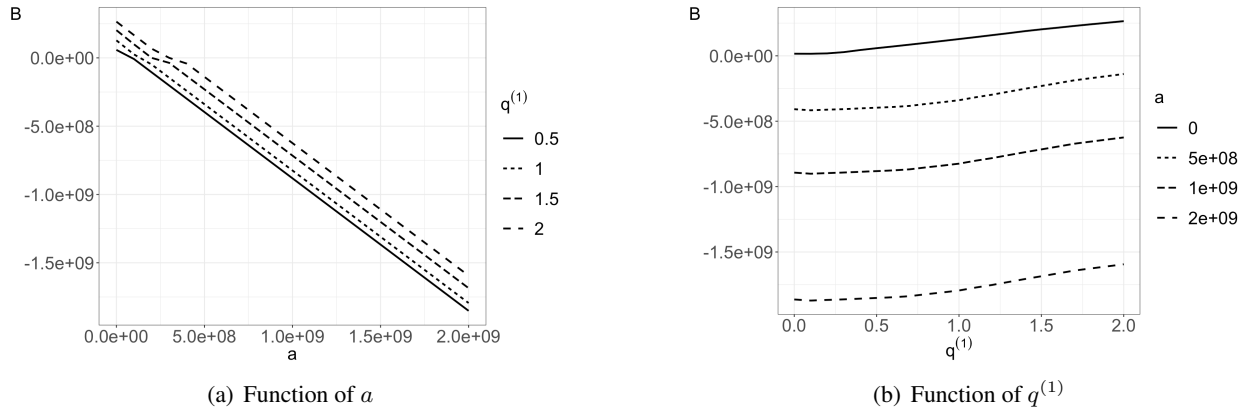
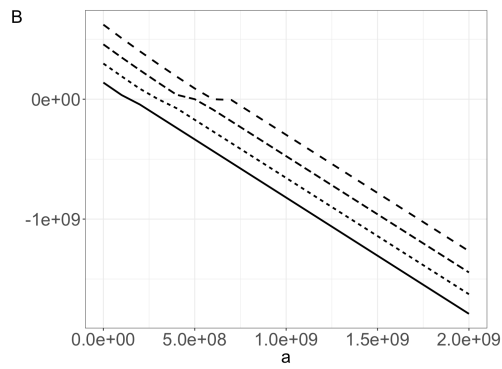
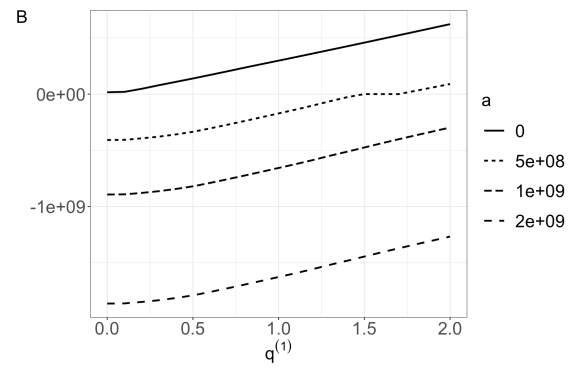


Figure 2.14: 2-dimensional representations of external capital raising (small previous shock)



(a) Function of a



(b) Function of $q^{(1)}$

Figure 2.15: 2-dimensional representations of external capital raising (large previous shock)

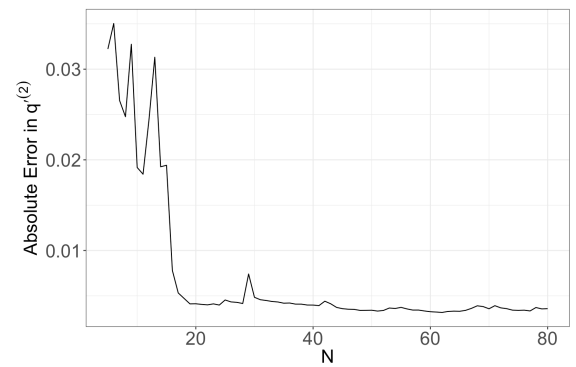
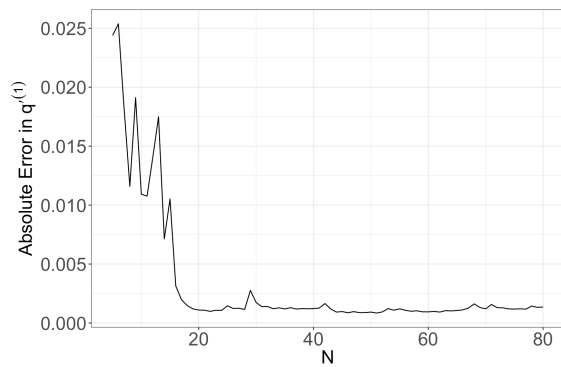
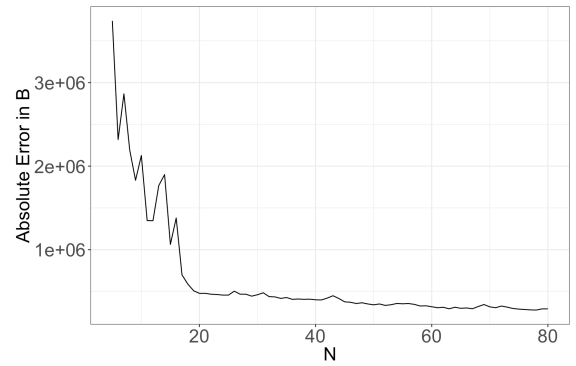
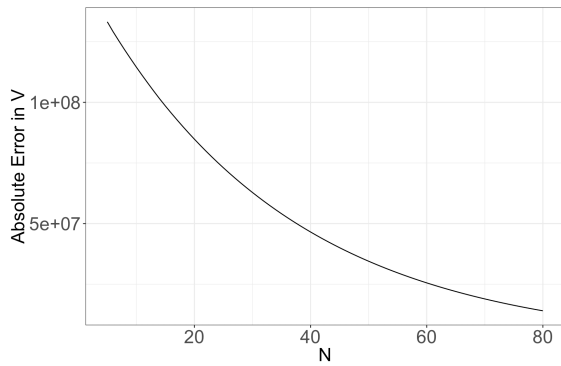


Figure 2.16: Convergence of value function and choice variables

Chapter 3

Different Shades of Risk: Mortality Trends Implied by Term Insurance Prices¹

3.1 Introduction

Term life insurance policies are typically considered to be fairly homogeneous products. Aside from conversion options and financial strength ratings—the relevance of which is mitigated to some extent by guaranty funds protection—the ensuing cash flows are typically relative congenerous across issuers. The key risk factors are investment/interest and mortality risks, where in view of the former investment opportunities and strategies again do not vary much across companies. Thus, given competitive forces in this large and undifferentiated market, it seems proximate to infer forward-looking, market-based estimates of future mortality dynamics from insurance prices (Mullin and Philipson, 1997).

Building on this logic, we estimate the stochastic mortality model from Bauer and Kramer (2016) with a set of US (term) life insurance prices using a generalized method of moments (GMM) approach, where we allow for selection/underwriting effects, surrenders, pertinent expenses, etc.

¹This essay is co-authored with Daniel Bauer and is forthcoming at the *North American Actuarial Journal*. A previous version was presented at the Twelfth International Longevity Risk and Capital Markets Solutions Conference (Longevity 12) under the title “Mortality Trends Implied by Term Insurance Prices”, and parts are taken from the earlier working paper “The Risk in Catastrophe Mortality Securitization Transactions” by the second author. We are grateful for helpful comments from an anonymous referee, Jin-Chuan Duan, Yue Kuen Kwok, Johnny Li, and other participants of the Insurance Risk and Finance Research Centre (IRFRC) 2017 Annual Conference, the Longevity 12 conference, Perspectives on Actuarial Risks in Talks of Young Researchers, and for financial support under the Society of Actuaries CAE grant “New Trends in Longevity”.

The model includes a catastrophe component to pick up mortality shocks associated with pandemics or natural disasters, a fairly flexible mortality trend component, and a diffusion term to capture stochastic variations in the mortality trend. Our results are striking: Neither the catastrophe component nor the diffusion term is significant, and a model comparison favors a simple deterministic mortality model. In contrast, allowing for heterogeneity among the carriers is of utmost importance: A model that does not incorporate differences in mortality trends between the different companies is strongly rejected. The company effects are large in magnitude and point towards three possible sub-markets.

Our interpretation is that for pricing and managing term life insurance products—potentially in contrast to annuity and pension products (cf. Sec. 3.3.4)—the key risks emanate from the composition of the pool of policyholders, rather than the uncertainty in aggregate mortality trends. Differences in underwriting criteria, primary distribution channels and distribution area, and other factors jointly determine the composition of the insurer’s portfolio of policies in each rating class. The relevant mortality rates for a given company will be based on the demographic evolution of this subgroup, and their behavior with regards to lapsing their contracts. Given heterogeneity in mortality trends for different population subgroups and different causes of death, the cross-sectional risk dimension dwarfs uncertainties in the aggregate trend. In other words, *basis risk*, i.e. the deviation of the experience in the particular pool relative to the aggregate population, seems to dominate *systematic mortality risk*, i.e. uncertainties in the aggregate trend component. And, indeed, an active management of the composition of the pool, by accepting certain risks and rejecting others, may be a way to compete in the marketplace.

Our findings have consequences for mortality risk management and for the corresponding actuarial literature. On the one side, they do not support assertions that aggregate mortality risks are important for life insurance products. For instance, papers presenting “natural hedging” between life insurance and annuity lines as an internal way to manage a company’s mortality exposure implicitly assume that the corresponding populations of policyholders are subject to the same variations. Thus, our results cast doubt on the effectiveness of such strategies, pointing to the necessity of market-based solutions for managing longevity. On the other side, our conclusion that mortality trends for different partitions of the population and different conditions determine the mortality

profile of the relevant portfolio of policyholders, and that it may be possible to actively manage this profile, emphasizes the importance of studying mortality at a more granular level.

Related Literature and Organization of the Essay

A closely related paper to ours is Mullin and Philipson (1997), who argue that under the assumption that insurance companies are close to risk-neutral with respect to their mortality exposure, it is possible to derive market-based estimates from zero expected profit (moment) conditions. In contrast to Mullin and Philipson, however, we include additional institutional factors affecting life insurance prices such as expenses, selection/underwriting effects, and policy surrenders/lapses. Furthermore, we consider variations in mortality in the time and the cross-sectional (across companies) direction. The latter aspect, in particular, allows us to draw our primary conclusions. Several papers in the actuarial literature also rely on prices of insurance products to obtain parameters in mortality models. In particular, Lin and Cox (2005) and Bauer et al. (2010) use annuity quotes to estimate risk premiums for longevity risk.

In the estimation process, we use the stochastic mortality model from Bauer and Kramer (2016). As discussed in their paper, the model is flexible enough to fit a relatively long time series of US mortality data, it includes a mortality catastrophe component, and it is tractable. The latter property originates from it falling in the class of affine mortality models (Biffis, 2005; Dahl and Møller, 2006). In particular, this allows for an efficient computation of survival probabilities and, thus, basic life insurance prices, which is essential for our estimation approach.

Our results relate to recent results in the economic and medical literature on the heterogeneity of mortality trends across different subpopulations that show that there are large disparities in life expectancy between different racial, regional, and socio-economic groups (Case and Deaton, 2015; Chetty et al., 2016). Furthermore, they relate and endorse analyses of cause-specific mortality rates, and how these in aggregate affect the mortality of a certain population (Boumezoued et al., 2017; Arnold and Sherris, 2016, and references therein). An insurer's underwriting process, together with its regional presence, its advertising strategy, etc., determines the composition of the portfolio of policyholders, and it is the mortality dynamics of that group that are relevant for the insurer's future cash flows.

As indicated, our findings are relevant for so-called “natural hedging” approaches to managing longevity risk by trading off annuity and life insurance exposures (Cox and Lin, 2007; Li and Haberman, 2015, and references therein). In particular, our results suggest that beyond difficulties with natural hedging within the same population (Zhu and Bauer, 2014), basis risk may inhibit the effectiveness of such strategies.

The remainder of the chapter is structured as follows: Section 3.2 introduces the affine mortality model from Bauer and Kramer (2016) with some extensions to suit our setting. Section 3.3 introduces our GMM estimation method, presents results, their implications, and a discussion on their robustness. Finally, Section 3.4 concludes.

3.2 Model

In what follows, we introduce the mortality model from Bauer and Kramer (2016), which we rely on in the remainder of the text. As discussed in their paper, the model consists of three parts: 1) a catastrophe component, 2) an age-dependent affine mortality component, and 3) a temporary component. In contrast to their analysis, we use the third part to include selection/underwriting effects, rather than a temporary deterministic trend. Furthermore, we subsequently augment the model by company, risk class, and calendar year effects that account for heterogeneity in mortality trends.

As in Bauer and Kramer (2016), we introduce the stochastic force of mortality by relying on conventional concepts from credit risk modeling (Lando, 1998). More precisely, given a stochastic process $X = (X_t)_{0 \leq t \leq T}$ and a positive, continuous function $\mu(\cdot, \cdot)$, we define an individual’s time of death as the first jump time of a Cox-process with intensity $\mu(x_0 + t, X_t)$:

$$\tau_{x_0} = \inf \left\{ t : \int_0^t \mu(x_0 + s, X_s) ds \geq E \right\}, \quad (3.1)$$

where E is a $Exp(1)$ -distributed random variable and independent among individuals.

Considering only one single insured for now, let the filtrations $\mathbf{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$ and $\mathbf{H} = (\mathcal{H}_t)_{0 \leq t \leq T^*}$ be given as the augmentations of the filtrations generated by $(X_t)_{0 \leq t \leq T^*}$ and

$(\mathbf{1}_{\{\tau_{x_0} \leq t\}})_{0 \leq t \leq T^*}$, respectively, and set $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. From Equation (3.1), we can then derive the $(T - t)$ -year survival probability at time t for an $x_t = x_0 + t$ year old individual as

$${}_{T-t}p_{x_t}(t) := \mathbb{E} [\mathbf{1}_{\{\tau_{x_0} > T\}} | \mathcal{G}_t, \tau_{x_0} > t] = \mathbb{E} \left[\exp \left\{ - \int_t^T \mu(x_0 + s, X_s) ds \right\} \middle| \mathcal{G}_t, \tau_{x_0} > t \right], \quad (3.2)$$

and, from results of Lando (1998),

$$\mathbb{E} [\mathbf{1}_{\{\tau_{x_0} > T\}} | \mathcal{F}_t] = \mathbf{1}_{\{\tau_{x_0} > t\}} {}_{T-t}p_{x_0+t}(t)$$

Following Bauer and Kramer (2016), we use the following model for the baseline stochastic force of mortality:

$$\mu_t(x_0) = \mu(x_0 + t, Y_t, \Gamma_t) = e^{b(x_0+t)} Y_t + \Gamma_t + D_t(x_0), \quad Y_0 > 0, \Gamma_0 \geq 0. \quad (3.3)$$

The catastrophe component follows the dynamics:

$$d\Gamma_t = -\kappa \Gamma_t dt + dJ_t, \quad \Gamma_0 \geq 0,$$

where (J_t) is a compound Poisson process with intensity λ and $Exp(\zeta)$ -distributed jumps. And the baseline component follows the dynamics:

$$dY_t = \alpha \left(\underbrace{(Y_0 - \beta^{(2)})}_{\beta^{(3)}} e^{-\beta^{(1)} t} + \beta^{(2)} - Y_t \right) dt + \sigma \sqrt{Y_t} dW_t, \quad Y_0 > 0,$$

where $(W_t)_{0 \leq t \leq T^*}$ is a one-dimensional Brownian motion and α , $\beta^{(1)}$, $\beta^{(2)}$, and σ are positive constants with $\alpha \neq \beta^{(1)}$. Here $\beta^{(2)} + \beta^{(3)}$ describes the trend level at time 0. We refer to their paper for the motivation of the model in the context of demographic research.

We use the temporary component $D_t(x_0)$ to introduce selection effects acting to temporary reduce mortality. Here the *selection effect* does not refer to the potential impact of adverse selection on life insurance prices but the impact of underwriting during the early policy years.² In particular,

²In insurance practice, actuaries rely on so-called “select-and-ultimate” tables to account for this type of selection, where the “select” tables used in early policy years display lower mortality due to underwriting examinations.

this is important for the evaluation of life insurance prices as here mandatory health examinations lead to significant selection effects. Within our model, this can be captured via the temporary component D by a roughly proportional structure akin to a proportional hazards model:

$$D_t(x_0) = c - (c + e^{b(x_0+t)} \tau) \times \frac{\gamma}{\bar{T}} \times (\bar{T} - t)^+.$$

Note that the component is still deterministic and it is only roughly proportional since we use an approximation of the relevant force of mortality based on the initial values. Moreover, this specification of D only relies on the four additional parameters $(c, \tau, \gamma, \bar{T})$ minding the complexity of the estimation process. We obtain:

$$\begin{aligned} \bar{D}_{t,T}(x_0) &= \int_t^T D_s(x_0) ds \\ &= c(T-t) - \frac{c\gamma}{\bar{T}} \left[((\bar{T}-t)^+)^2 - ((\bar{T}-T)^+)^2 \right] \\ &\quad - \frac{\tau\gamma}{\bar{T}b} \left[(\bar{T}-T)^+ \exp\{b(x_0 + \min\{T, \bar{T}\})\} - (\bar{T}-t)^+ \exp\{b(x_0 + \min\{t, \bar{T}\})\} \right] \\ &\quad - \frac{\tau\gamma}{\bar{T}b^2} \left[\exp\{b(x_0 + \min\{T, \bar{T}\})\} - \exp\{b(x_0 + \min\{t, \bar{T}\})\} \right]. \end{aligned}$$

The (exponential-)affine structure enables us to write down an analytical representation of the survival function (cf. Prop. 1 in Duffie et al. (2000)):

$$\begin{aligned} {}_{T-t}p_{x_0+t}(t) &= \exp \left\{ u(T-t) + v(T-t) Y_t - \frac{\Gamma_t}{\kappa} (1 - e^{-\kappa(T-t)}) - \frac{\lambda(T-t)}{\zeta\kappa + 1} \right\} \\ &\quad \times \exp \left\{ \frac{\lambda\zeta}{\zeta\kappa + 1} \log \left[1 + \frac{1}{\zeta\kappa} (1 - e^{-\kappa(T-t)}) \right] - \bar{D}_{t,T}(x_0) \right\}, \end{aligned} \quad (3.4)$$

where u and v satisfy the following Riccati ODEs:

$$\begin{aligned} v'(s) &= -e^{b(x_0+T-s)} - \alpha v(s) + \frac{1}{2} \sigma^2 v^2(s), \quad v(0) = 0, \\ u'(s) &= v(s) \alpha \left(e^{-\beta^{(1)}(T-s)} Y_0 + (1 - e^{-\beta^{(1)}(T-s)}) \beta^{(2)} \right), \quad u(0) = 0. \end{aligned} \quad (3.5)$$

Here, $\mu_t(x_0)$ and ${}_{T-t}p_{x_0+t}(t)$ represent the baseline force of mortality and survival function, respectively.

We introduce heterogeneity in mortality trends by assuming that the aggregate force of mortality in each company's portfolio is proportional to the baseline. Thus, the force of mortality for company i can be written as:

$$\mu_t^i(x_0) = \mu_t(x_0)(1 + E_i^{co}), \quad \sum_i E_i^{co} = 0, \quad i = 1, 2, \dots, I,$$

where E_i^{co} is the company effect. We further add a risk class effect E_j^{rc} and a calendar year effect E_h^{year} to the model to account for variation within a company and across different calendar years:

$$\begin{aligned} \mu_t^{(i,j,h)}(x_0) &= \mu_t(x_0)(1 + E_i^{co} + E_j^{rc} + E_h^{year}), \\ \sum_i E_i^{co} &= \sum_j E_j^{rc} = \sum_h E_h^{year} = 0, \quad E_i^{co} + E_j^{rc} + E_h^{year} > -1 \quad \forall i, j, h \end{aligned}$$

with risk class j ranging from 1 (highest mortality risk, e.g. regular class) to J (lowest mortality risk, e.g. preferred plus class) and year h spanning all calendar years of insurance prices. The three effect components do not relate to the industry baseline force of mortality.

Therefore, the survival probabilities for company i , risk class j in calendar year h take the form:

$${}_{T-t}p_{x_0+t}^{(i,j,h)}(t) = {}_{T-t}p_{x_0+t}(t)^{(1+E_i^{co}+E_j^{rc}+E_h^{year})}.$$

Given risk class, calendar year, and company effect parameters, it remains to calculate the survival probabilities (3.4) and actuarial present values by simply solving the ODEs from Equation (3.5). The full analytical representation of the survival probabilities facilitates the estimation process.

3.3 Estimation with Insurance Price Data

3.3.1 Insurance Price GMM Estimator

We are given annual term insurance premiums $P_{x,n}^{(i,j,h)}$ for age x , term n , risk class $j \in 1, 2, \dots, J$, calendar year $h \in 1, 2, \dots, H$, and a fixed benefit B payable upon death at the end of the year from I companies, i.e. $i \in \{1, 2, \dots, I\}$, which we take to be i.i.d. Denote by \mathcal{C} the collection of all available age-term combinations (x, n) and we have $N_{x,n}$ such combinations. As is common in

actuarial modeling and in contrast to Mullin and Philipson (1997), we consider various types of expenses to be reflected in our insurance price data. In particular, we include initial expenses both as a percentage of the first premium $c_{IP}^{(1)}$ and as a fixed amount depending on the death benefit $c_{IP}^{(2)}$, as well as a fixed maintenance expense c_M (Society of Actuaries, 2004).

We assume policyholders surrender at a fixed proportion $q_u^{(l)}$ in policy year u , $u \geq 1$, immediately before premiums become due. Thus, policies remain in force for k years with a probability of:

$${}_k p_{x_0}^{(\tau)(i,j,h)}(0) = {}_k p_{x_0}^{(i,j,h)}(0) \underbrace{\prod_{1 \leq u \leq k} (1 - q_u^{(l)})}_{{}_k p_{x_0}^{(l)}}, \quad k \geq 1.$$

Upon surrendering at time k , policyholders may be entitled to so-called *cash surrender values* (CSVs) ${}_k C_{x,n}^{(i,j,h)}$, which are to be calculated according to the *National Association of Insurance Commissioners'* (NAIC) *Standard Nonforfeiture Law for Life Insurance*. This regulation essentially entails the calculation of guaranteed reserve levels according to given interest rates, a given mortality table, and given expense levels. For simplicity, we assume the cost parameters and surrender probabilities are the same across all companies.³

We start by writing down an insurance company's loss function, or the negative profit function:

$$\begin{aligned} f(\theta, P_{x,n}^{(i,j,h)}) = & \sum_{k=0}^{n-1} p(0, k+1) \left[{}_k p_x^{(l)} \left({}_k p_x^{(i,j,h)}(0) - {}_{k+1} p_x^{(i,j,h)}(0) \right) B + {}_k p_x^{(\tau)(i,j,h)}(0) p_{x+k}^{(i,j,h)}(0) q_{k+1}^{(l)} {}_{k+1} C_{x,n}^{(i,j,h)} \right] \\ & + c_{IP}^{(2)} + c_M \sum_{k=0}^{n-1} p(0, k) {}_k p_x^{(\tau)(i,j,h)}(0) - P_{x,n}^{(i,j,h)} \left(\sum_{k=0}^{n-1} p(0, k) {}_k p_x^{(\tau)(i,j,h)}(0) - c_{IP}^{(1)} \right), \end{aligned} \quad (3.6)$$

where $p(t, \tau)$ denotes the time t price of a zero coupon bond with maturity $t + \tau$. Of course, equation (3.6) depends on the (risk-neutral) parameters of the mortality model as well as on the parameters governing expenses, selection, and surrenders, which we stack in the parameter vector θ . The key assumption is now that the so-called *equivalence principle* holds, i.e. that the expected present value of future benefits (including expenses) equals the expected present value of future premiums for all of the company's policies for one mortality profile, that is, for one risk class in one calendar year. Thus, the equivalence principle must hold for each i , j , and h combination.

³This assumption, again, is motivated by competitive pressures in the market. Large differences in costs should be competed away unless they are linked to aspects relating to underlying heterogeneity.

That is to say, for each company and each risk class in one calendar year, the expected profit (loss) from selling insurance contracts is zero. Therefore, we can write down our moment conditions for the GMM estimation as follows:

$$\mathbb{E}_{x,n} [f(\theta, P_{x,n}^{(i,j,h)})] = 0, (x, n) \in \mathcal{C}.$$

The *efficient* GMM estimator can be obtained using the so-called “two-step feasible GMM” method, which yields efficient and consistent estimators. First, choose a weighting matrix W , where W is an $I \times J \times H$ dimensional square matrix. W can be any such matrix in the first step. We choose W such that

$$W^{-1} = \text{diag} \{ \sigma_{i,j,h}^2, (i, j, h) \},$$

where $\sigma_{i,j,h}^2 = \text{Var}_{x,n} [P_{x,n}^{(i,j,h)}]$ with corresponding sample version $\hat{\sigma}_{i,j,h}^2$. We obtain a GMM estimate $\hat{\theta}_1$ by minimizing the following function of θ :

$$\hat{\theta}_1 = \arg \min_{\theta} \left[\frac{1}{N_{x,n}} \sum_{x,n} f(\theta, P_{x,n}^{(i,j,h)}) \right]'_{(i,j,h)} W \left[\frac{1}{N_{x,n}} \sum_{x,n} f(\theta, P_{x,n}^{(i,j,h)}) \right]_{(i,j,h)}. \quad (3.7)$$

Then, we update the weighting matrix using $\hat{\theta}_1$:

$$\hat{W}(\hat{\theta}_1)^{-1} = \text{diag} \left\{ \frac{1}{N_{x,n}} \sum_{j=1}^{N_{x,n}} \left(f(\hat{\theta}_1, P_{x,n}^{(i,j,h)}) \right)^2, (i, j, h) \right\} \quad (3.8)$$

and minimize function (3.7) using $W = \hat{W}(\hat{\theta}_1)$ to obtain our estimate $\hat{\theta}$. We carry out both minimizations numerically.

3.3.2 Data and Estimation

We use quotes for annual life insurance premiums from the *CompuLife* price quotation system (historical data) for April 2012, 2013, 2014, and 2015. More precisely, we focus on contracts with a face amount of \$500,000 for male non-smokers under two underwriting categories: regular (Rg, residual standard), and preferred plus (Pf+, super preferred). From 31 companies, we retrieved

data for 33 age-term combinations, with terms of 10, 15, 20, and 30 years, and ages from 25 to 75, where for numerical convenience we use ages in five year intervals only. We only allow for a single quote per company per age/term-combination. Since the records include several instances of multiple quotes from the same company for different states—though prices usually coincide—we average over these quotes. All in all, we rely on 7,688 different quotes in our estimation process.

Figures 3.1(a) and 3.1(b) plot the quotes by term in April 2012, for age 25 and 40, respectively. For each risk class, the variance of insurance prices for each term is about the same and the coefficient of variation decreases as the term increases. This goes against the intuition of stochastic mortality because it would imply an increasing variation in mortality (and likely insurance prices) as the term increases. Similar characteristics of the variance of the insurance prices are observed in the other calendar years and ages.

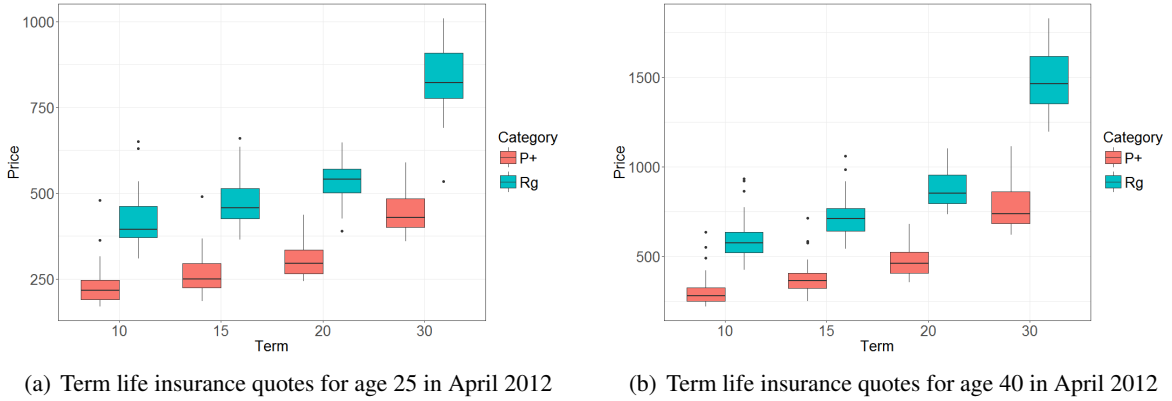


Figure 3.1: Illustration of the insurance price data

In addition to the parameters of the mortality model, the GMM estimator depends on various *business-related* parameters. The CSVs ${}_k C_{x,n}^{(i,j,h)}$, $k \geq 0$, are calculated according to the NAIC *Standard Nonforfeiture Law for Life Insurance*. The interest rates derive from Moody’s *Corporate Bond Yield Averages Index* (we refer to Towers Watson (2015) for details and an illustration of the relevant rates), and the relevant mortality rates are taken from the *Commissioners Standard Ordinary* (CSO) 2017 mortality table (mandatory in 48 U.S. states and optional in all 50; see American Academy of Actuaries (2008)). We do not include expense, surrender and other business-related parameters in the minimization, but rather fix them according to Table 3.1, whose column 2 and 3

list their values and the source, respectively.⁴

Parameter	Value	Source
$c_{IP}^{(1)}$	60%	Avg. values from the 2005
$c_{IP}^{(2)}$	\$882.5	“Generally Recognized Expense Table”
c_M	\$45	(see e.g. Society of Actuaries (2004))
\bar{T}	18 years	Avg. value from Society of Actuaries (2007)
γ	60%	Value matched to CSO 2017 mort. table
q_i	15%, $i \leq 3$, 5%, $i > 3$	Roughly matches pattern (non-renewable) according to Purushotham (2006)

Table 3.1: Parameters relevant to the insurance contracts.

For the numerical optimization in (3.7), we rely on the Julia language and apply the COBYLA (constrained optimization by linear approximation) algorithm, suitable for a minimization task with a large number of equality and inequality constraints of the parameters. For the numerical solution of the ordinary differential equations arising in each time step, we rely on an implementation of the Runge-Kutta method with a variable time step as available within Julia (ode45).

We estimate five versions of the model. The first model version (Full model) includes catastrophe component parameters (κ , λ and ζ). In the second version (w/o CAT), we exclude the catastrophe component. In the third model (w/o Sigma), we assume that the stochastic diffusion parameter σ is zero. In the fourth model (w/o Trend), we set the trend parameters α , β_1 , and β_3 to zero so that Y_t is fixed at β_2 . The fifth model (w/o Co. effect) takes away company effects from the “w/o CAT” model. We record the estimates, standard errors, and also a likelihood-like \mathcal{J} -statistics, which is the basis for testing the overall specification and parametric restrictions as in Sargan (1958) and Hansen (1982). It is calculated as follows:

$$\mathcal{J} = N_{x,n} \left[\frac{1}{N_{x,n}} \sum_{x,n}^{N_{x,n}} f(\hat{\theta}, P_{x,n}^{(i,j,h)}) \right]'_{(i,j,h)} W(\hat{\theta}) \left[\frac{1}{N_{x,n}} \sum_{x,n}^{N_{x,n}} f(\hat{\theta}, P_{x,n}^{(i,j,h)}) \right]_{(i,j,h)}.$$

\mathcal{J} , a Wald statistic, converges in distribution to $\chi^2(L - K)$ under the null hypothesis that the model

⁴Including them in the estimation procedure leads to problems when including a complex mortality model, since the impact on insurance prices is similar to those originating from certain components. We obtain reasonable magnitudes not too different from the set values in the context of simple mortality models.

is valid, or well-specified, where L is the number of moment equations and K is the number of parameters. In terms of testing parametric restrictions, according to Newey and West (1987), the difference of two \mathcal{J} -statistics, the GMM counterpart to the likelihood ratio test statistic, converges in distribution to $\chi^2(K_1 - K_2)$, where $K_1 - K_2$ is the number of restricted parameters in model 2 compared to model 1. This “Likelihood-ratio-like” statistic can be used to test the parameter restrictions and offer guidance to model comparison, which is highlighted in the next part.

3.3.3 Results and Discussion

Table 3.2 provides the parameter estimates for all five model versions, with the first four having the company effect parameters. In the Full model, the catastrophe parameter estimates (κ , λ , ζ) have large standard errors and are not significant, so do the trend parameter β_1 and diffusion parameter σ . In contrast, the trend parameter β_2 and Gompertz parameter b are significant, pointing to a simple deterministic mortality model. Parameters related to selection/underwriting effect (c , τ , γ) are significant in the first four model estimations, highlighting such effects in life insurance underwriting.

The estimation results for company effects are shown in Figure 3.2(a). Note that estimations of company effects in all models yield similar results, so we illustrate the company effects using the estimates in “w/o Trend” model. Figure 3.2(c) summarizes the significance of the estimates compared to zero and we find around one third of companies having mortality rates significantly above the industry average, and around one third of companies having mortality rates significantly below the average. In the fifth model, without the company effects capturing the heterogeneity of mortality, parameter estimates turn out to have large standard errors and the overall specification statistic \mathcal{J} is much larger compared to the other four models, indicating that the “w/o Co. effect” model is unlikely to be favorable. The estimation results strongly favor the model with company effects, which overshadow the mortality catastrophe and diffusion parameters.

We conduct model comparisons using the difference of \mathcal{J} statistics as discussed in the previous subsection. Figure 3.3 shows the value of the statistics (referred to as “LR”) between models, with arrows pointing toward the more favorable model. Table 3.3 shows the hypothesis testing results at significance level 95%. The results suggest strong rejection of the “w/o Co. effect” model, again

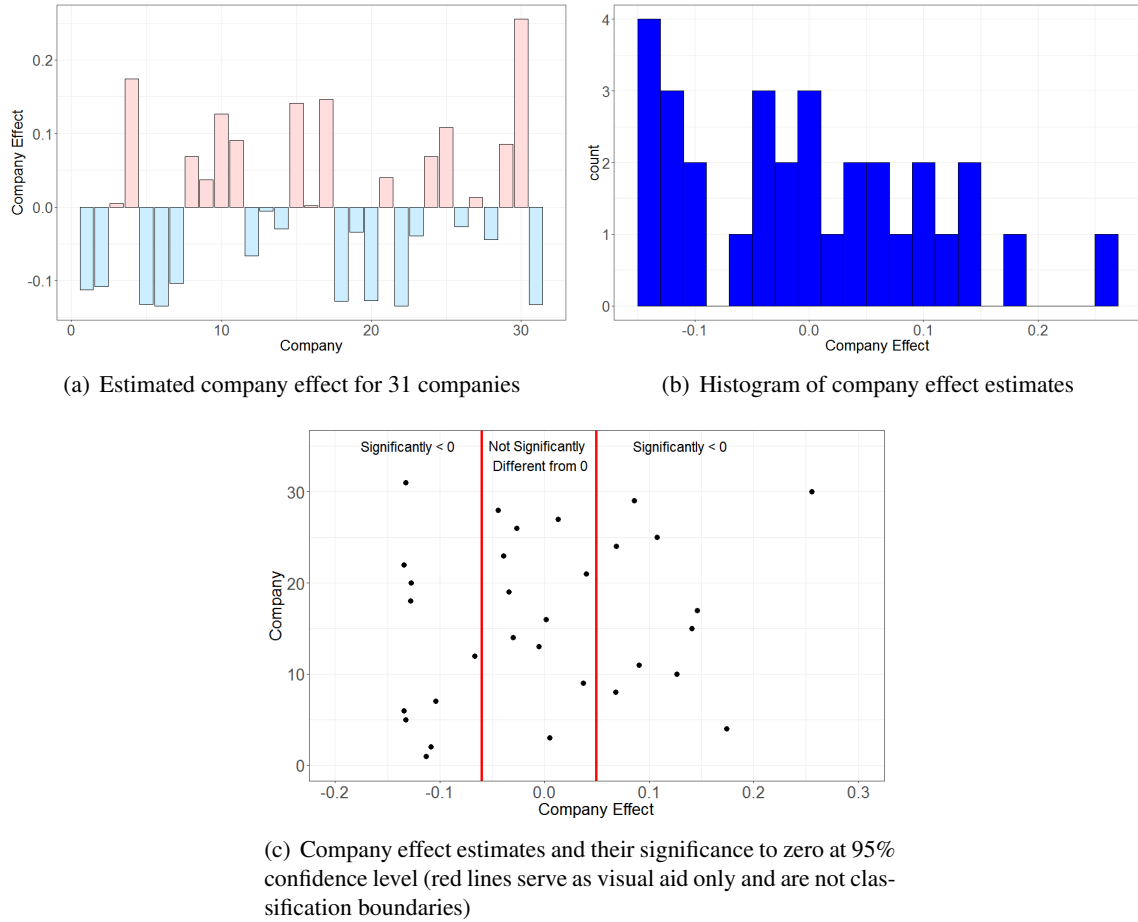


Figure 3.2: Company effect in three representations

highlighting the importance of including the company heterogeneity in the model. Among the four models with company effects, we always fail to reject a simpler model with fewer parameters. As a result, the preferred specification turns out to be the “w/o Trend” model, which has a simple deterministic Gompertz mortality component and selection effect. These results suggest that company effects matter while the catastrophe components and the stochastic diffusion are not important in explaining the prices.

It is important to note that the estimated mortality rates correspond to mortality rates used in pricing, and they may entail aggregated risk adjustments (as expectations under a risk-neutral measure). Hence, we do not propose these mortality rates adequately project population mortality, also because they correspond to a special subpopulation, or that mortality catastrophes and stochasticity are not relevant in forecasting population mortality. Rather, our estimation results and the model

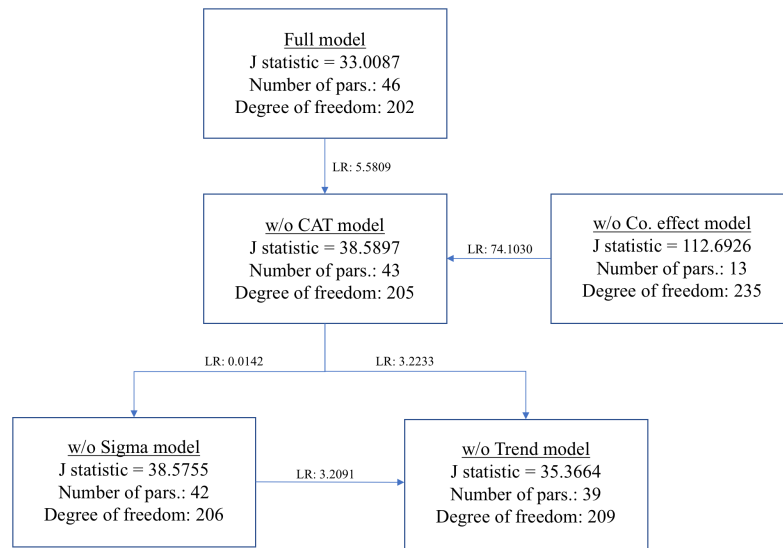


Figure 3.3: Comparison of the model specifications

comparisons point towards the relevance of company effects in life insurance pricing, whereas other aspects (mortality catastrophes, stochastic trends) do not seem to be of key relevance.

The distribution of the company effects takes a roughly tri-modal shape as shown in Figure 3.2(b), with several companies bunching at positive effects (worse mortality experience) of around 0.1, some companies bunching around 0 (average mortality experience), and some companies bunching at negative effects (better mortality experience) of around -0.1, also seen in Figure 3.2(c). The results suggests that there are roughly three segments in the market.

We retrieve information from A.M. Best to check whether these groups can be explained by company characteristics. More precisely, from A.M. Best Rating/Information Services, we obtain company ratings and company size (two companies' information are not available). Here, financial strength rating means A.M. Best's independent opinion of "an insurer's financial strength and ability to meet its ongoing insurance policy and contract obligations" and issuer credit rating refers to A.M. Best's independent opinion of "an entity's ability to meet its ongoing financial obligations." Table 3.4 provides the information together with the company effects from our estimation. More precisely, we code the companies according to the three segments, -,0,+, belonging to the group with negative, insignificant, and positive company effects, respectively.

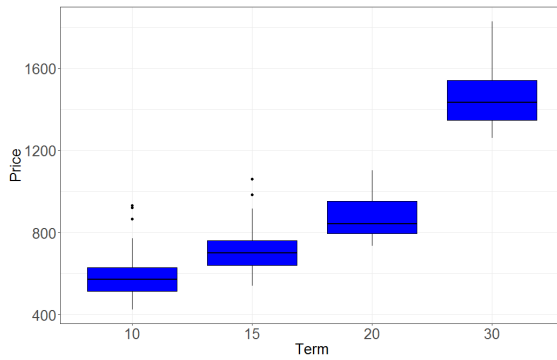
There are no obvious relationships to company size or company ratings. There are very highly rated companies and very large companies in each segment, and there are smaller and relatively low-rated companies in each segment. In other words, company characteristics do not seem to explain the market segmentation—or, more generally, the significant heterogeneity in prices. Thus, it appears that differences in the pools of policyholders between companies must lead to the segmentation.

These differences in the mortality profile of each company’s portfolio of policies may arise from different channels. First, underwriting criteria, and particularly the categorization of individuals with a certain health history to rating classes, differ among carriers. Since in the end the mortality experience is driven by morbidity rates and cause-specific mortality rates, and since these in aggregate shape the mortality profile of the relevant pool, heterogeneity may arise (Boumezoued et al., 2017; Arnold and Sherris, 2016, and references therein). Moreover, different companies penetrate different regions dissimilarly and they use different distribution channels, which in turn affect the demographic and socio-economic distribution of policyholders. Since there are large differences in mortality across different subpopulations (Case and Deaton, 2015; Chetty et al., 2016), again these aspects may generate heterogeneity in company mortality profiles.

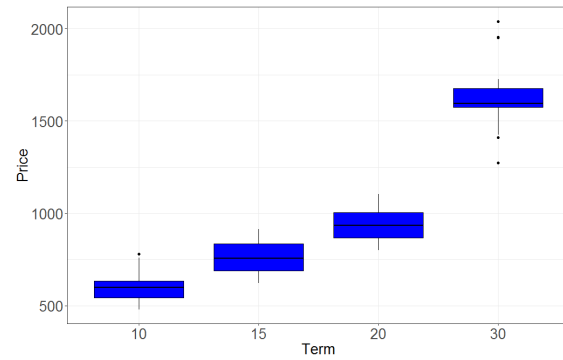
3.3.4 Robustness: Time Period and Market Segment

In the estimation process, we rely on insurance price data for different ages and different terms for the years 2012-2015. In particular, we use insurance quotes with terms of up to 30 years, so that the relevant mortality experience for ex-post companies profits spans the period 2012-2045 (when the 30-year term policies sold in 2015 mature). To evaluate whether our results are driven by idiosyncrasies of insurance prices in recent (post financial crisis) years, Figure 7 illustrates term life quotes issued in January 2006 (panel 3.4(b)) relative to the quotes from 2012 (panel 3.4(a)). We observe that the general pattern across companies is similar. In particular, although the 2006 quotes for term 30 are somewhat less dispersed, the insurance prices still exhibit a significant variance across companies, which is indicative of the relevance of company effects.

While the 2006 quotes by themselves do not allow for an estimation of company effects, estimating model versions without company effects results in similar parameter estimates as the “w/o



(a) Term life insurance quotes in April 2012



(b) Term life insurance quotes in April 2006

Figure 3.4: Illustration of the insurance price data for age 40, male, non-smoker, regular, \$500,000 face value

Co. effects” model for the 2012-2015 time period (last column in Table 3.2). In particular, the catastrophe, the stochastic diffusion, and the trend parameters are all insignificant, and the data strongly favors the simple model.

To illustrate that the results here are not likely to carry over to other life-contingent product classes such as annuities or pension products, Figure 3.5 taken from Bauer (2008) shows annuity quotes in the UK market. More precisely, the figure depicts the evolution of annual annuity rates for a purchase price £10,000 within open market option between January 2000 and December 2005, payable monthly in advance and without guarantee period, for leading companies in UK.

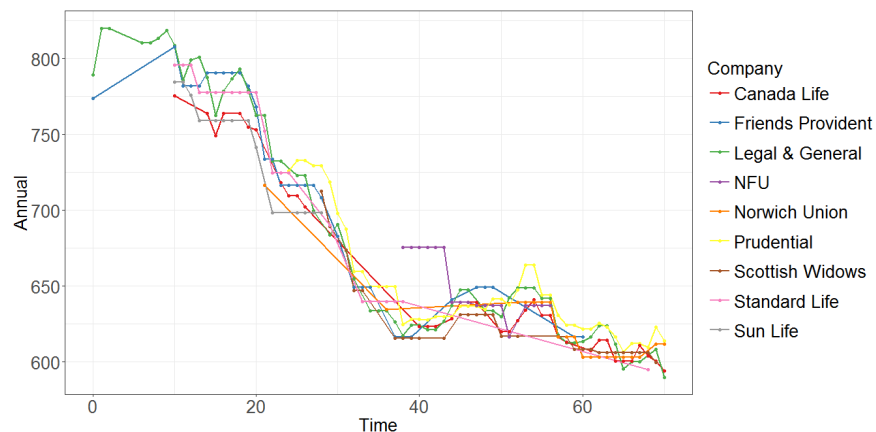


Figure 3.5: Monthly British annuity rates data from January 2000 (0) to December 2005 (70), open market option, male, age 60, purchase price £10,000

Here, the findings appear to be quite different. In contrast to our term insurance quotes, the annuity rates throughout the period show a relatively small variance at any point of time, and the

relative order between the companies changes throughout the observation period. This suggests that unlike our estimation results for term insurance prices, there are no persistent company effects in this market.

These differences may not come as a surprise given the differences between the market segments. In the annuities market, there is no underwriting process (Hoermann and Ruß, 2008) so that companies cannot actively control their portfolio. Life annuities cannot be surrendered, so that policy lapsation/surrender is not relevant. Moreover, the full price is paid upfront so that the contracts are fully and solely exposed to longevity risk. In particular, Lin and Cox (2005) and Bauer et al. (2010) rely on annuity for deriving a (usually positive) “market price of longevity risk” implied by life annuity prices.

3.4 Conclusion

Our study of term life insurance prices goes against the common perception that life insurance is fairly homogeneous and that aggregate mortality trends are highly relevant in this marketplace. Our estimates do not reflect significance of mortality catastrophe or stochastic mortality trends. The model comparison strongly rejects the model without company specific mortality effects, however, pointing to a significant heterogeneity in the mortality profiles among different life insurance companies. Thus, “basis risk” seems to dominate systematic mortality risk in life insurance. This questions the efficiency of so-called “natural hedging” of longevity risk using life insurance exposure. Furthermore, it points to the relevance of understanding granular mortality trends.

Aside from explanations on the supply side, it is possible that demand-side effects contribute to our findings. For instance, companies may be able to sell overpriced insurance to existing customers in other business lines, or carriers may pursue certain segments of the market by their pricing strategy. We leave exploring these angles for future research.

	Full	w/o CAT	w/o Sigma	w/o Trend	w/o Co. effects
α	1.8793 (6.1387)	1.8176 (1.2603)	1.8178 (1.2613)	- -	1.3609 (13.7273)
β_1	2.4470 (10.5424)	2.4440 (2.2557)	2.4446 (2.2593)	- -	2.6631 (50.4369)
β_2	1.20E-06 (5.58E-08)	1.20E-06 (5.99E-08)	1.20E-06 (6.00E-08)	1.09E-06 (5.90E-08)	1.01E-06 (2.64E-06)
σ	0.0002 (1.4153)	0.0004 (0.4272)	- -	- -	0.0002 (98.2278)
b	0.1015 (0.0008)	0.1018 (0.0008)	0.1017 (0.0008)	0.1027 (0.0010)	0.0439 (0.0383)
κ	2.2554 (76.0239)	- -	- -	- -	- -
λ	0.0007 (0.0043)	- -	- -	- -	- -
ζ	19.4661 (367.7243)	- -	- -	- -	- -
β_3	2.15E-06 (1.60E-06)	2.15E-06 (8.24E-07)	2.15E-06 (8.24E-07)	- -	3.87E-06 (2.33E-05)
c	0.0047 (0.0001)	0.0047 (0.0001)	0.0046 (0.0001)	0.0047 (6.00E-05)	0.0057 (0.0001)
γ	0.9628 (0.0071)	0.9606 (0.0070)	0.9603 (0.0070)	0.9372 (0.0073)	0.9603 (0.0073)
τ	3.00E-07 (1.06E-07)	3.00E-07 (1.13E-07)	3.00E-07 (1.13E-07)	3.10E-07 (1.14E-07)	3.53E-07 (4.52E-06)
Y_0	3.35E-06 (1.60E-06)	3.35E-06 (8.26E-07)	3.35E-06 (8.26E-07)	1.09E-06 (5.90E-08)	4.88E-06 (2.34E-05)
E_i^{co}	Yes	Yes	Yes	Yes	No
E_1^{rc}	0.3652 (0.0087)	0.3651 (0.0086)	0.3651 (0.0086)	0.3652 (0.0090)	0.3747 (0.0110)
E_{2012}^{year}	0.0131 (0.0122)	0.0130 (0.0121)	0.0130 (0.0121)	0.0129 (0.0127)	0.0148 (0.0155)
E_{2013}^{year}	-0.0043 (0.0121)	-0.0044 (0.0120)	-0.0044 (0.0120)	-0.0043 (0.0125)	-0.0037 (0.0153)
E_{2014}^{year}	0.0172 (0.0122)	0.0173 (0.0121)	0.0173 (0.0121)	0.0173 (0.0126)	0.0183 (0.0155)
J-stat	33.0087	38.5897	38.5755	35.3664	112.6926
df	202	205	206	209	235

Table 3.2: Estimated parameters (company effect estimates not shown) for four models based on GMM. Standard errors for each parameter are shown in parentheses.

H_0	H_1	Reject H_0 at 95% confidence?
w/o CAT	Full	No
w/o Sigma	w/o CAT	No
w/o Sigma	Full	No
w/o Trend	w/o Sigma	No
w/o Trend	w/o CAT	No
w/o Trend	Full	No
w/o Co. effect	w/o CAT	Yes
w/o Co. effect	Full	Yes

Table 3.3: Model comparison results

Company Effect	Financial Strength Rating	Long-Term Issuer Credit Rating	Financial Size Category
-	A- (Excellent)	a- (Excellent)	X (\$250 mil ~ \$500 mil)
-	A (Excellent)	a+ (Excellent)	XV (>\$2 billion)
o	A (Excellent)	aa- (Superior)	XIV (\$1.5 bil ~ \$2 bil)
+	A- (Excellent)	a- (Excellent)	IX (\$250 mil ~ \$500 mil)
-	A+ (Superior)	aa- (Superior)	IX (\$250 mil ~ \$500 mil)
-	-	-	-
-	A (Excellent)	a+ (Excellent)	VIII (\$100 mil ~ \$250 mil)
+	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
o	A- (Excellent)	a- (Excellent)	VIII (\$100 mil ~ \$250 mil)
+	A++ (Superior)	aa+ (Superior)	XV (>\$2 billion)
+	A- (Excellent)	a- (Excellent)	VIII (\$100 mil ~ \$250 mil)
-	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
o	A (Excellent)	a+ (Excellent)	XV (>\$2 billion)
o	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
+	A- (Excellent)	a- (Excellent)	VII (\$50 mil ~ \$100 mil)
o	-	-	-
+	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
-	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
o	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
-	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
o	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
-	A (Excellent)	a+ (Excellent)	VIII (\$100 mil ~ \$250 mil)
o	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
+	A- (Excellent)	a- (Excellent)	VIII (\$100 mil ~ \$250 mil)
+	A (Excellent)	a+ (Excellent)	XV (>\$2 billion)
o	A++ (Superior)	aaa (Exceptional)	XV (>\$2 billion)
o	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
o	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
+	A+ (Superior)	aa- (Superior)	XV (>\$2 billion)
+	A+ (Superior)	aa (Superior)	XV (>\$2 billion)
-	A+ (Superior)	aa- (Superior)	IX (\$250 mil ~ \$500 mil)

Table 3.4: Company effects vs. A.M. Best ratings and company size (-,o,+, belonging to the group with negative, insignificant, and positive company effects, respectively)

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