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Sign Patterns of J-orthogonal Matrices

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SIGN PATTERNS OF J-ORTHOGONAL MATRICES

by

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Under the Direction of Frank J. Hall, PhD and Zhongshan Li, PhD

ABSTRACT

This thesis builds upon the results in "G-matrices, J-orthogonal matrices, and their sign patterns", Czechoslovak Math. J. 66 (2016), 653-670, by Hall and Rozložník. Some general results about the sign patterns of J-orthogonal matrices are proved, including about block diagonal matrices. It is shown that every full 4×4 sign pattern allows J-orthogonality and as a result that, for $n \leq 4$, all $n \times n$ full sign patterns allow a J-orthogonal matrix as well as a G-matrix. The 3×3 sign patterns of the J-orthogonal matrices which have zero entires are also characterized.

INDEX WORDS: Qualitative matrix theory, J-orthogonal matrix, Sign pattern matrix, Exchange operator

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DEDICATION

This thesis is dedicated to all those who made it possible: to my family for allowing me to pursue the things I love, to my partner Makenzie for supporting me throughout, to my classmates for their help and friendship, and to all the professors at Georgia State who encouraged me to take this path.

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Contents

1 Introduction

The goal of this thesis is to study and characterize J-orthogonal matrices of certain orders and classes and their sign patterns. This introduction will familiarize the reader with the basic concepts of J-orthogonality and sign patterns. J-orthogonal matrices have been studied in many contexts such as group theory and generalized eigenvalue problems. In the recent decades and particularly in numerical mathematics, a class of problems appeared where the scalar products were indefinite and involved J-orthogonal matrices. Although J-orthogonality has many numerical connections, this thesis has more of a combinatorial matrix theory point of view, building on the results in [10]. The topic of sign pattern matrices originated with economist Paul Samuelson [12] in response to the need to solve problems given only the signs of entries in a given matrix. Since that time, sign pattern matrices have found new applications in areas such as communication complexity, neural networks, and chemistry [8].

Following [6], we say that a real matrix A is a *G-matrix* if A is nonsingular and there exist nonsingular diagonal matrices D_1 and D_2 such that

$$A^{-T} = D_1 A D_2$$

where A^{-T} denotes the transpose of the inverse of A. Now, denote by J a diagonal (signature) matrix whose diagonal entries are ± 1. Then a real nonsingular matrix Q is called *J-orthogonal* if

$$Q^{-T} = JQJ,$$

or equivalently if

$$Q^T JQ = J.$$

Thus we can see the close relationship that G-matrices and J-orthogonal matrices have, since every J-orthogonal matrix is a G-matrix with $D_1 = D_2 = J$. On the other hand, as shown in [10], every G-matrix can be transformed into a J-orthogonal matrix.

Definition 1.1. We say that two real matrices A and B are *positive-diagonally equivalent* if there exist diagonal matrices D_1 and D_2 with all positive diagonal entries such that $B = D_1AD_2$.

Theorem 1.2. *[10, Theorem 2.6] A matrix A is a G-matrix if and only if A is positive-diagonally equivalent to a column permutation of a J-orthogonal matrix.*

Hence, much discussion of J-orthogonal matrices also relies on properties of G-matrices.

A *sign pattern matrix* (or *sign pattern*) is a matrix with entries in the set $\{+, -, 0\}$. A sign pattern is called *full* if it has no 0 entries. For a real matrix B, the sign pattern $\mathrm{sgn}(B)$ is the matrix obtained by replacing each positive (respectively, negative, zero) entry of B with $+$ (respectively, $-$, 0). For example, given

$$B = \begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathrm{sgn}(B) = \begin{bmatrix} - & 0 \\ + & + \end{bmatrix}.$$

Given an $m \times n$ sign pattern A, the *sign pattern class* or *qualitative class* of A is

$$Q(A) = \{B \in M_{m,n}(\mathbb{R})\colon \mathrm{sgn}(B) = A\}.$$

For convenience of notation, we can say for a real matrix C that $Q(\mathrm{sgn}(C)) = Q(C)$. The set of all $n \times n$ sign pattern matrices is denoted Q_n. A sign pattern matrix P is called a *permutation pattern (generalized permutation pattern)* if exactly one entry in each row and column is equal to $+$ ($+$ or $-$). Just as for real matrices, we say P^TAP is *permutationally similar* to A if $A \in Q_n$ and P is a permutation pattern of order n. A *signature pattern* is a diagonal sign pattern matrix, each of whose diagonal entries is $+$ or $-$. If S_1 and S_2 are $n \times n$ signature patterns and $A, B \in Q_n$, then we say that B is *signature equivalent* to A if $B = S_1AS_2$.

The next theorem provides some easily proved properties of J-orthogonal matrices.

Theorem 1.3. *(i) For a fixed signature matrix J, the set of all J-orthogonal matrices is a multiplicative group, which is also closed under transposition and signature equivalence.*
(ii) The direct sum of square diagonal blocks $A_{11}, \ldots, A_{kk}$ is a J-orthogonal matrix if and only if each diagonal block A_{ii} is a J_i-orthogonal matrix, where J_i is the corresponding diagonal block of J.
(iii) The Kronecker product of J_i-orthogonal matrices is a J-orthogonal matrix with J equal to the Kronecker product of the J_i's.
(iv) If Q is J-orthogonal and P is a permutation of the same order, then $P^T QP$ is J_1-orthogonal with $J_1 = P^T JP$.

If P is a property referring to a real matrix, we say that a sign pattern A *requires* P if every matrix in $Q(A)$ has property P, and we say that A *allows* P if some matrix in $Q(A)$ has property P.

One property of real matrices of concern to this thesis is that of singularity. If every matrix $B \in \mathrm{sgn}(A)$ is singular, then A is said to be *sign singular.* If $A = [a_{ij}]$ is an $n \times n$ sign pattern matrix, then a *(simple) cycle* of *length* k (or a k-*cycle*) in A is a formal product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}$, where each of the elements is nonzero and the indices $i_1, i_2, \ldots, i_k$ are distinct. A *composite cycle* γ in A is a product of simple cycles, say $\gamma = \gamma_1 \gamma_2 \ldots \gamma_m$, where the index sets of the γ_i's are mutually disjoint. If the length of γ_i is l_i, then the length of γ is $\sum_{i=1}^{m} l_i$. It is known that an $n \times n$ sign pattern matrix A is sign singular if and only if A has no composite cycle of length n.

Of course, any orthogonal matrix is also J-orthogonal with $J = I$. One question that has been explored in the past is what sign patterns allow orthogonality. We denote by $\mathcal{PO}_n$ the set of all $n \times n$ sign patterns that allow orthogonality. A slightly more general question is which sign patterns allow J-orthogonality. Of particular interest are sign patterns that allow J-orthogonality but to not allow orthogonality. We let $\mathcal{J}_n$ denote the set of all sign patterns of the $n \times n$ J-orthogonal matrices (for various possible J). We also denote $\mathcal{G}_n$ to be the set of all $n \times n$ sign patterns that allow a G-matrix. Notice that if $A \in \mathcal{J}_n$, then A

cannot be sign singular.

The following result gives some straightforward properties of the set of J-orthogonal matrices.

Theorem 1.4. *[10, Lemma 6.3] The set $\mathcal{J}_n$ is closed under the following operations:*

(i) negation;

(ii) transposition;

(iii) permutation similarity;

(iv) signature equivalence.

The use of these operations yields "equivalent" sign patterns, which will be referenced throughout the thesis.

With the above notations, we have an immediate consequence of Thorem 1.2.

Theorem 1.5. *[10, Theorem 4.3] The sign patterns in $\mathcal{G}_n$ are exactly the column permutations of the sign patterns in $\mathcal{J}_n$.*

Theorem 1.5 can be paraphrased as follows: $\mathcal{G}_n = \mathcal{J}_n\mathcal{P}_n$, where $\mathcal{P}_n$ is the set of all $n \times n$ permutation patterns. Observe that $\mathcal{G}_n^T = \mathcal{G}_n$, $\mathcal{J}_n^T = \mathcal{J}_n$, and $\mathcal{P}_n^T = \mathcal{P}_n$. Hence, by taking the transpose on both sides in the equation $\mathcal{G}_n = \mathcal{J}_n\mathcal{P}_n$, we obtain $\mathcal{G}_n = \mathcal{P}_n\mathcal{J}_n$, which is the content of the next theorem.

Theorem 1.6. *The set of all $n \times n$ sign patterns that allow a G-matrix is the same as the set of all row permutations of the $n \times n$ sign patterns allowing J-orthogonality.*

In fact, we can generalize this result as follows:

Theorem 1.7. *The set of all $n \times n$ sign patterns that allow a G-matrix is the same as the set of all permutation equivalences of the $n \times n$ sign patterns allowing J-orthogonality.*

Proof. From Theorem 1.5, we have $\mathcal{G}_n = \mathcal{J}_n\mathcal{P}_n$. Thus, it suffices to show that $\mathcal{J}_n\mathcal{P}_n = \mathcal{P}_n\mathcal{J}_n\mathcal{P}_n$. Since the identity sign pattern is in $\mathcal{P}_n$, obviously $\mathcal{J}_n\mathcal{P}_n \subseteq \mathcal{P}_n\mathcal{J}_n\mathcal{P}_n$. To show the reverse inclusion, let $P_1QP_2 \in \mathcal{P}_n\mathcal{J}_n\mathcal{P}_n$, where P_1 and P_2 are permutation patterns and Q allows J-orthogonality. By Theorem 1.4 we know $P_1QP_1^T$ allows J-orthogonality, and hence $P_1QP_2 = (P_1QP_1^T)(P_1P_2) \in \mathcal{J}_n\mathcal{P}_n$. □

Let A be an $n \times n$ sign pattern matrix. From [10], the very important fundamental *sign potentially J-orthogonal* (SPJO) conditions are that there exists a $(+,-)$ signature pattern J such that

$$A^TJA \overset{c}{\longleftrightarrow} J \tag{1}$$

and

$$AJA^T \overset{c}{\longleftrightarrow} J, \tag{2}$$

where $\overset{c}{\longleftrightarrow}$ denotes (generalized) sign pattern compatibility.

These are necessary conditions for $A \in \mathcal{J}_n$. If these conditions do not hold, then $A \notin \mathcal{J}_n$. When $J = I$, we get the normal SPO conditions for orthogonal matrices, see for example [4]. The SPJO conditions are not sufficient for an $n \times n$ sign pattern matrix to allow J-orthogonality, as illustrated in [10].

Observe that A^TJA and AJA^T are symmetric generalized sign pattern matrices. So, to verify the SPJO conditions we need only to find a J which fulfills the upper-triangular part of the compatible conditions. Let $J = \operatorname{diag}(\omega_1, \ldots, \omega_n)$. Note that (1) and (2) may be restated as

$$\sum_{k=1}^{n} \omega_k a_{ki} a_{kj} \overset{c}{\longleftrightarrow} \delta_{ij}\omega_j \quad \text{for all } i, j \tag{3}$$

and

$$\sum_{k=1}^{n} \omega_k a_{ik} a_{jk} \overset{c}{\longleftrightarrow} \delta_{ij}\omega_j \quad \text{for all } i, j. \tag{4}$$

(With an $n \times n$ $(+,-)$ sign pattern A, for $i = j$, (3) and (4) automatically hold for any J.)

In [10], the following important result was proved.

Theorem 1.8. *[10, Theorem 6.11] For all $n \geq 1$, each $n \times n$ full sign pattern A satisfies the SPJO conditions.*

If we allow zero entries, then Theorem 1.8 may fail. For example, an $n \times n$ sign pattern A with a zero column does not satisfy $A^T JA \stackrel{c}{\longleftrightarrow} J$ and an $n \times n$ sign pattern A with a zero row does not satisfy $AJA^T \stackrel{c}{\longleftrightarrow} J$, for any signature pattern J.

In Section 2, a number of other general results on sign patterns are proved and used in subsequent sections. The 3×3 sign patterns of the J-orthogonal matrices which have zero entries are characterized in Section 3. In Section 4 it is shown that all 4×4 full sign patterns allow J-orthogonality. Important tools in this analysis are Theorem 2.2 on the exchange operator and Theorem 3.2 on the characterization of J-orthogonal matrices in the paper [11] by Nick Higham.

2 Block Upper Triangular Matrices and Sign Patterns

Definition 2.1. An $n \times n$ matrix A that contains an $s \times (n-s)$ zero submatrix for some integer $1 \leq s \leq n-1$ is said to be *partly decomposable.* If no such submatrix exists, then A is *fully indecomposable.*

The following structural result of G-matrices was established in [10].

Theorem 2.2. *[10, Theorem 2.1] Let A be a nonsingular real matrix in block upper triangular form*

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ & \ddots & \vdots \\ 0 & & A_{mm} \end{bmatrix},$$

where all the diagonal blocks are square. Then A is a G-matrix if and only if each A_{ii} $(i = 1, \dots, m)$ is a G-matrix and all the strictly upper triangular blocks A_{ij} are equal to 0. Furthermore, if A is a G-matrix that has a row (or a column) with no 0 entry, then A is fully indecomposable.

Example 2.3. Let

$$A = \begin{bmatrix} 0 & + & + & 0 \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & 0 \end{bmatrix}.$$

Notice that A is permutationally equivalent to the sign pattern

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix},$$

which by Theorem 2.2 does not allow a G-matrix. Hence, A does not allow a J-orthogonal matrix. The same holds for any similar $n \times n$ sign pattern.

Using the result given in Theorem 1.7, we give the following generalization of Threom 2.2.

Theorem 2.4. *Let A be an $n \times n$ sign pattern matrix, and P and Q be permutation patterns such that PAQ has the block upper triangular form*

$$PAQ = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ & \ddots & \vdots \\ 0 & & A_{mm} \end{bmatrix},$$

where all the diagonal blocks are square. If $A \in \mathcal{J}_n$, then $PAQ \in \mathcal{G}_n$, each A_{ii} $(i = 1, \dots, m)$ allows a G-matrix, and all the strictly upper triangular blocks A_{ij} are equal to 0. If $PAQ \notin \mathcal{G}_n$, then $A \notin \mathcal{J}_n$.

We note that when the sign pattern A in Theorem 2.4 is not sign singular, such a PAQ block upper triangular form where specifically the square diagonal blocks are fully indecomposable, is always possible [2, Theorem 4.2.6].

Of specific interest is the following.

Theorem 2.5. *If A is an $n \times n$ sign pattern matrix with exactly $n+1$ nonzero entries, then $A \notin \mathcal{J}_n$.*

Proof. If A has no composite cycle of length n then of course $A \notin \mathcal{J}_n$ because A is sign singular. If A does have a composite cycle of length n, then for some permutation sign pattern P, the permutation equivalence AP has no zero diagonal entries and exactly one nonzero off-diagonal entry. By Theorem 2.2, $AP \notin \mathcal{G}_n$. Hence, by Theorem 2.4, we have $A \notin \mathcal{J}_n$. □

Notice that the results from Theorem 1.5 can also be applied to sign patterns.

Theorem 2.6. *Let the $n \times n$ sign pattern matrix be the direct sum*

$$A = \begin{bmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{mm} \end{bmatrix},$$

where all the diagonal blocks are square. Then A allows a J-orthogonal matrix if and only if each A_{ii} $(i = 1, \ldots, m)$ allows a J-orthogonal matrix.

Remark 2.7. The Kronecker product of sign patterns which allow a J-orthogonal pattern also allows a J-orthogonal matrix. For a fixed sign pattern matrix J, a product of J-orthogonal matrices can produce a different sign pattern allowing a J-orthogonal matrix for the same J.

Observe that any generalized permutation pattern allows orthogonality (and indeed J-orthogonality with $J = 1$), since if B is a generalized permutation matrix, then $B^T I B = B^T B = I$. Hence we have another result:

Theorem 2.8. *If A is an $n \times n$ generalized permutation sign pattern, then $A \in \mathcal{J}_n$.*

The following structural result can be of general use.

Theorem 2.9. *Suppose that B is an $n \times n$ real nonsingular matrix, and suppose that B is both J_1-orthogonal and J_2-orthogonal, where $J_1 = \operatorname{diag}(I_{p_1}, -I_{q_1})$, $J_2 = \operatorname{diag}(I_{p_2}, -I_{q_2})$, and $J_1 \neq J_2$. Then it follows that*

$$B = \begin{bmatrix} B_{11} & 0 & B_{13} \\ 0 & B_{22} & 0 \\ B_{31} & 0 & B_{33} \end{bmatrix},$$

where the partitioning of B results from the partitioning of the matrix $J_2 J_1$.

Proof. Since the matrix B is J_1-orthogonal, from $B^T J_1 B = J_1$ we have that $J_1 B = B^{-T} J_1$. Similarly, we can obtain $BJ_2 = J_2 B_{-T}$. These two identities give $(J_2 J_1)B = B(J_2 J_1)$. Notice that $J_2 J_1 = \operatorname{diag}(I_{\min(p_1,p_2)}, -I_{\max(p_1,p_2)-\min(p_1,p_2)}, I_{\min(q_1,q_2)})$. We partition B using the same dimensions as $J_2 J_1$ to get

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}.$$

Equating the blocks of $(J_2 J_1)B = B(J_2 J_1)$ forces $B_{12} = B_{21} = B_{23} = B_{32} = 0$. $\square$

Remark 2.10. Let P be the permutation matrix such that $\tilde{J} = P(J_2 J_1)P^T = \operatorname{diag}(I_p, -I_q)$, where $p = \min(p_1, p_2)$ and $q = \max(p_1, p_2) - \min(p_1, p_2)$. Using this permutation matrix P, the matrix B is permutationally similar to the block diagonal matrix

$$\tilde{B} = \begin{bmatrix} B_{11} & B_{13} & 0 \\ B_{31} & B_{33} & 0 \\ 0 & 0 & B_{22} \end{bmatrix}$$

that is $\tilde{J}$-orthogonal, satisfying $\tilde{B}^T \tilde{J} \tilde{B} = \tilde{J}$.

Corollary 2.11. *If A is an $n \times n$ full sign pattern matrix, then there does not exist $B \in Q(A)$ that is both J_1-orthogonal and J_2-orthogonal, where $J_1 \neq J_2$.*

Example 2.12. Given the sign pattern

$$A = \begin{bmatrix} + & - & - \\ + & + & - \\ + & - & - \end{bmatrix},$$

there are two possible choices for J that satisfy the SPJO conditions, namely $J_1 = \mathrm{diag}(1, 1, -1)$ and $J_2 = \mathrm{diag}(1, -1, -1)$.

For $d > \frac{1}{\sqrt{2}}$, the real matrix

$$\begin{bmatrix} 1 & -1 & -1 \\ d & \frac{1}{2d} & -\frac{2d^2-1}{2d} \\ d & -\frac{1}{2d} & -\frac{2d^2+1}{2d} \end{bmatrix} \in Q(A)$$

is J_1-orthogonal. For example, if $d = 1$, then

$$B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} \in Q(A)$$

satisfies $B^T J_1 B = J_1$.

On the other hand, for $j > \frac{1}{\sqrt{2}}$, the real matrix

$$\begin{bmatrix} \frac{2j^2+1}{2j} & -\frac{2j^2-1}{2j} & -1 \\ \frac{1}{2j} & \frac{1}{2j} & -1 \\ j & -j & -1 \end{bmatrix} \in Q(A)$$

is J_2-orthogonal. For example, if $j = 1$, then

$$B = \begin{bmatrix} 3/2 & -1/2 & -1 \\ 1/2 & 1/2 & -1 \\ 1 & -1 & -1 \end{bmatrix} \in Q(A)$$

satisfies $B^T J_2 B = J_2$.

In the above example, the two signature matrices used are equivalent (though not all the resulting J-orthogonal matrices are equivalent). In the next example, we exhibit the property with non-equivalent signature matrices in the case of two sign patterns.

Example 2.13. Consider the 4×4 all $+$ sign pattern matrix. This pattern is J-orthogonal with the two non-equivalent signature matrices $J_1 = \mathrm{diag}(1, 1, 1, -1)$ and $J_2 = \mathrm{diag}(1, 1, -1, -1)$:

$$A_1 = \frac{1}{3}\begin{bmatrix} 4 & 1 & 1 & 3 \\ 1 & 4 & 1 & 3 \\ 1 & 1 & 4 & 3 \\ 3 & 3 & 3 & 6 \end{bmatrix} \text{ is } J_1\text{-orthogonal,}$$

$$A_2 = \frac{1}{3}\begin{bmatrix} 4 & 1 & 2 & 2 \\ 1 & 4 & 2 & 2 \\ 2 & 2 & 4 & 1 \\ 2 & 2 & 1 & 4 \end{bmatrix} \text{ is } J_2\text{-orthogonal.}$$

Also consider the sign pattern

$$A = \begin{bmatrix} + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \\ + & + & - & + & + \\ + & + & + & + & + \end{bmatrix}.$$

Let $J_1 = \mathrm{diag}(1, 1, 1-1, -1)$ and $J_2 = \mathrm{diag}(1, 1, 1, 1-1)$. Then $A_1, A_2 \in Q(A)$ and

$$A_1 = \frac{1}{106}\begin{bmatrix} 203 & 97 & 158 & 85 & 239 \\ 97 & 203 & 158 & 85 & 239 \\ 202 & 202 & 46 & 118 & 242 \\ 35 & 35 & -2 & 117 & 1 \\ 281 & 281 & 202 & 161 & 429 \end{bmatrix} \text{ is } J_1\text{-orthogonal}$$

$$A_2 = \frac{1}{81}\begin{bmatrix} 8 & 89 & 103 & 64 & 127 \\ 89 & 8 & 103 & 64 & 127 \\ 149 & 149 & 106 & 58 & 229 \\ 20 & 20 & -26 & 79 & 34 \\ 155 & 155 & 163 & 106 & 304 \end{bmatrix} \text{ is } J_2\text{-orthogonal.}$$

An interesting question is the following: Is it true that whenever a square full sign pattern A and a signature pattern J satisfy the SPJO conditions, then A allows J-orthogonality (for this particular J)? That the answer is no is seen in [4, Example 3.8] where the 6×6 pattern is SPO but is not in $\mathcal{PO}_6$. However, specifying the $*$ entries as $+$, this pattern is in $\mathcal{J}_6$, as shown as follows.

Example 2.14. Let

$$A = \begin{bmatrix} + & + & + & + & + & + \\ + & + & + & + & + & - \\ + & + & + & + & - & + \\ + & + & + & + & - & - \\ + & + & - & - & + & + \\ + & - & + & - & + & + \end{bmatrix}$$

and let $J = \text{diag}(1, -1, -1, -1, -1, -1)$. Then we produced the following decimal approximation of a matrix $B \in Q(A)$ such that $B^T JB = J$ to within four decimal places:

$$B = \begin{bmatrix} 1.8457 & 0.1748 & 1.2301 & 0.5382 & 0.0023 & 0.7572 \\ 0.4467 & 0.4877 & 0.5807 & 0.5934 & 0.3467 & -0.3900 \\ 1.2188 & 0.1332 & 0.7813 & 0.7961 & -0.0450 & 1.1053 \\ 0.1207 & 0.4068 & 0.1700 & 0.0456 & -0.8983 & -0.1055 \\ 0.0121 & 0.7684 & -0.0923 & -0.4680 & 0.2659 & 0.3339 \\ 0.8408 & -0.1379 & 1.2361 & -0.2876 & 0.0113 & 0.2776 \end{bmatrix}$$

Thus, the question of whether a full sign pattern satisfying the SPJO conditions for some J implies J-orthogonality is still open.

3 Characterization of sign patterns in $\mathcal{J}_3$ with 0 entries

We want to identify all those 3×3 sign patterns with 0 entries which allow J-orthogonality. To organize our argument, we consider sign patterns with varying numbers of zero entries.

Note that all 3×3 full sign patterns allow J-orthogonality [10].

Sign patterns with 9, 8, or 7 zero entires. Any 3×3 sign pattern with only 2, 1 or 0 nonzero entires cannot contain a composite cycle of length 3; thus, any such pattern is sign singular and hence cannot allow J-orthogonality, since if B is J-orthogonal, then B is nonsingular.

Sign patterns with 6 zero entries. Note that a 3×3 sign pattern with exactly 3 nonzero entries must not be sign singular in order to allow J-orthogonality, so we only consider such sign patterns which have a composite cycle of length 3, namely the 3×3 generalized permutation patterns. By Theorem 2.8, these patterns allow J-orthogonality. Thus, the sign patterns in $\mathcal{J}_3$ with exactly 6 zero entries are precisely the 3×3 generalized permutation patterns.

Sign patterns with 5 zero entries. That no 3×3 sign pattern with exactly five zero entries allows a J-orthogonal matrix simply follows from Theorem 2.5.

Sign patterns with four zero entries. In order to determine the sign patterns with four zero entries that allow J-orthogonality, we can systematically consider the number of zero entries on the main diagonal. Let $\star$ denote a $+$ or $-$ entry. Note that if we require all nonzero entries to occur on the main diagonal, then, up to equivalence, there are three patterns to consider. Two of these patterns

$$\begin{bmatrix} \star & \star & \star \\ 0 & \star & 0 \\ 0 & 0 & \star \end{bmatrix}, \begin{bmatrix} \star & \star & 0 \\ 0 & \star & \star \\ 0 & 0 & \star \end{bmatrix}$$

do not satisfy the SPJO conditions for any J, while it can be seen that

$$\begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{bmatrix}$$

does allow J-orthogonality.

Now suppose there is one zero entry on the main diagonal. Then we may permute it to the $(1,1)$ position. By systematic inspection it can be seen that no pattern of this form allows J-orthogonality.

Now, if there are two zero entries on the main diagonal, then up to equivalence there is one pattern of this form that allows J-orthogonality:

$$\begin{bmatrix} \star & 0 & \star \\ \star & 0 & \star \\ 0 & \star & 0 \end{bmatrix}.$$

Finally, with three zero entries on the main diagonal, there is no pattern that allows J-orthogonality.

Sign patterns with three or two zero entries. We can conduct a similar investigation of the sign patterns by systematically inspecting the possibilities. Once again the SPJO conditions come into play. In this way, we find that there is no 3×3 sign pattern with exactly three or two zero entries that allows J-orthogonality.

Sign patterns with one zero entry. In this case, we first eliminate from consideration all those sign patterns which are sign potentially orthogonal, since for $n = 3$, every SPO pattern allows orthogonality [1].

So suppose A is a 3×3 non-SPO pattern with exactly one zero entry. If the zero is on the main diagonal, we permute it to the $(3, 3)$ position. Suppose first that the inner product of the first two columns is not 0 or #. Since they are nonzero, these columns are either the same or negative of each other. So we can multiply on the left and right by suitable signature patterns so that all the entries in the first two columns are $+$. We can also multiply the third column by $-$ if necessary to obtain the form

$$A = \begin{bmatrix} + & + & + \\ + & + & \star \\ + & + & 0 \end{bmatrix},$$

leaving two possible patterns up to equivalence. Note that if $\star = -$, then A does not satisfy the SPJO conditions for any J. On the other hand, if $\star = +$, then we can obtain a J-orthogonal matrix of this form; for example

$$\begin{bmatrix} 2 & 1 & \sqrt{2} \\ \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \sqrt{3} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 \end{bmatrix}$$

allows J-orthogonality with $J = \mathrm{diag}(1, -1, -1)$.

Similarly, if the first and third columns are not SPO, then by signature equivalence we can obtain the form

$$\begin{bmatrix} + & + & + \\ + & \star & + \\ + & \star & 0 \end{bmatrix},$$

while if the second and third columns are not SPO, we obtain

$$\begin{bmatrix} + & + & + \\ \star & + & + \\ \star & + & 0 \end{bmatrix}.$$

Upon inspection we find that no matrix of the above forms (except for all the $\star$ equal to $+$, as described above) allows J-orthogonality.

Now suppose that the zero entry is off the main diagonal, and without loss of generality, permute the zero to the $(2, 3)$ position. Then similar to the above discussion, we obtain three possible forms:

$$\begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & \star \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & \star & 0 \\ + & \star & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ \star & + & 0 \\ \star & + & + \end{bmatrix}.$$

Of these possible patterns, four allow J-orthogonality. They are listed below along with examples of J-orthogonal matrices with those sign patterns:

$$A = \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & + \end{bmatrix};$$

$$B = \begin{bmatrix} \frac{4}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \sqrt{3} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 \\ 2 & 1 & 2 \end{bmatrix} \in Q(A) \text{ is } J\text{-orthogonal with } J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & - \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{\sqrt{3}}{2} \end{bmatrix} \in Q(A) \text{ is } J\text{-orthogonal with } J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} + & + & + \\ + & - & 0 \\ + & + & + \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & 2 & 2 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} & \sqrt{5} \end{bmatrix} \in Q(A) \text{ is } J\text{-orthogonal with } J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} + & + & + \\ - & + & 0 \\ + & + & + \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \in Q(A) \text{ is } J\text{-orthogonal with } J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

These are all of the non-SPO 3×3 sign patterns, up to equivalence, with exactly one zero entry which allow J-orthogonality.

We have thus proved the following result.

Theorem 3.1. *Up to equivalence, the sign patterns in $\mathcal{J}_3$ with at least one zero entry are*

$$\begin{bmatrix} + & + & 0 \\ + & \star & 0 \\ 0 & 0 & + \end{bmatrix}, \begin{bmatrix} + & 0 & + \\ + & 0 & \star \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & + \end{bmatrix},$$

$$\begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & - \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & - & 0 \\ + & + & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ - & + & 0 \\ + & + & + \end{bmatrix},$$

as well as the 3×3 generalized permutation sign patterns and the 3×3 SPO sign patterns with one zero entry, where $\star$ denotes a $+$ or $-$ entry.

4 The 4×4 full sign pattern case

An initial investigation of the question of whether the full $n \times n$ sign patterns always allow J-orthogonality was begun in [10], and for $n \leq 3$ it was shown to be true.

Remark 4.1. It was observed in [4] that for $n \leq 4$, the SPO patterns are the same as the sign patterns in $\mathcal{PO}_n$, and that this is also the case for full sign patterns of order 5 (see [1] and [15]). So, regarding the above question with $n \leq 5$, we need only consider non-SPO patterns.

In this section, we establish that every 4×4 full sign pattern allows J-orthogonality. As observed above, for $n \leq 4$, the SPO patterns are the same as the patterns in $\mathcal{PO}_n$. Therefore, since every orthogonal matrix is also J-orthogonal, we need only consider those patterns which are not sign potentially orthogonal. Since $\mathcal{J}_4$ is closed under transposition, we consider those patterns which are not sign potentially column orthogonal.

Any given full sign pattern can be multiplied on the left and right by signature patterns so that it has the form

$$\begin{bmatrix} + & + & + & + \\ + & & & \\ + & & & \\ + & & & \end{bmatrix}.$$

Moreover, since we are considering sign patterns which are not sign potentially column orthogonal and which have no zero entries, this means that two columns must be the same. Thus we can use permutation similarity and signature equivalence to reduce to the case

$$\begin{bmatrix} + & + & + & + \\ + & + & & \\ + & + & & \\ + & + & & \end{bmatrix}, \tag{5}$$

which leaves us with $2^6 = 64$ distinct sign patterns to consider. We can reduce this number of cases by noting that

$$\begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & & \\ + & + & & \end{bmatrix} \text{ and } \begin{bmatrix} + & + & + & + \\ + & + & - & + \\ + & + & & \\ + & + & & \end{bmatrix}$$

are equivalent, since we can switch the third and fourth columns, and simultaneously switch the third and fourth row. Now, using (5) as our template, there are three possibilities to consider, depending on the sign of the $(2,3)$ and $(2,4)$ entries. We inspect each case individually.

4.1. If the $(2,3)$ *and* $(2,4)$ *entries are both* $+$

There are 16 possible ways to fill the remaining entries from the set $\{+,-\}$. By inspection, we find that four of these are symmetric staircase patterns and therefore in $\mathcal{J}_4$ [10, Theorem 6.2]. A further 2 patterns are permutationally similar to symmetric staircase patterns, so these too are in $\mathcal{J}_4$.

Now consider the non-symmetric staircase pattern

$$A = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & - & - \end{bmatrix}.$$

If we construct J_1 and J_2 as in [10, Remark 5.7], then $J_2 = PJ_1P^T$ where $P = [e_1, e_2, e_4, e_3]$.

Since $AP = A$, we have $A \in \mathcal{J}_4$. The transpose of A is also in $\mathcal{J}_4$. Additionally,

$$B = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & - \\ + & + & + & + \end{bmatrix}$$

is permutationally similar to A, so B, $B^T \in \mathcal{J}_4$.

Now consider another non-symmetric staircase pattern

$$A = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & + & - \end{bmatrix}.$$

Similarly to the above discussion, A and A^T allow J-orthogonality with $J = \mathrm{diag}(+, +, -, +)$ and $P = [e_1, e_3, e_2, e_4]$, since $AP = A$. Let $S = \mathrm{diag}(+, +, +, -)$ and $Q = [e_3, e_2, e_1, e_4]$. Then

$$B = Q^T ASQ = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & - \\ + & + & + & + \end{bmatrix} \in \mathcal{J}_4; \;\; B^T \in \mathcal{J}_4.$$

There are 3 remaining patterns, up to equivalence, which we still must show are in $\mathcal{J}_4$. We address these patterns further down in Subsection 4.4.

4.2. If the* $(2,3)$ *and* $(3,4)$ *entries are both $-$

In this case there is one staircase pattern

$$A = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & - \\ + & + & - & - \end{bmatrix}$$

for which, following [10, Remark 5.7], we see that $AP = A$. So $A \in \mathcal{J}_4$. Note from A we can also multiply the third and fourth columns by $-$ and permute the first and second lines to obtain

$$\begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \in \mathcal{J}_4. \tag{6}$$

More matrices in this case can be obtained as follows:

We begin with the staircase pattern

$$A = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & - & - \end{bmatrix}.$$

If we compute P as in [10, Remark 5.7], then we find that $AP \neq A$. But in fact,

$$AP = \begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & - & + & + \\ + & - & - & + \end{bmatrix}.$$

So $AP \in \mathcal{J}_4$. Now we can obtain the pattern

$$B = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & - \\ + & + & + & - \end{bmatrix}$$

from AP by permutation similarity, so $B \in \mathcal{J}_4$. If $Q = [e_1, e_2, e_4, e_3]$, then $Q^T BQ$ is another sign pattern in this subcase that allows J-orthogonality.

Similarly, if we begin with the staircase pattern $A = \left[\begin{smallmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & - \\ + & + & - & - \end{smallmatrix}\right]$, we find that

$$AP = \begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & - & - & + \\ + & - & - & + \end{bmatrix} \in \mathcal{J}_4 \tag{7}$$

and by permutation similarity, $B = \left[\begin{smallmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & - \\ + & + & + & - \end{smallmatrix}\right] \in \mathcal{J}_4$. If $Q = [e_1, e_2, e_4, e_3]$, then $Q^T BQ$ is another pattern in this subcase that allows J-orthogonality.

We can obtain another pattern in $\mathcal{J}_4$ by letting A be the pattern in equation (7), and letting $S = \text{diag}(+, -, -, +)$ and $P = [e_2, e_4, e_3, e_1]$. Then $B = P^T ASP = \left[\begin{smallmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & + \\ + & + & - & + \end{smallmatrix}\right] \in \mathcal{J}_4$, and permuting the third and fourth lines of B yields another pattern in this subcase.

There are 5 more patterns, up to equivalence, in this subcase still to be determined. They will be addressed in Subsection 4.4.

4.3. If the $(2,3)$ *entry is* $+$ *and the* $(2,4)$ *entry is* $-$

The staircase pattern $\begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & + & - \end{bmatrix}$ can be seen to be in $\mathcal{J}_4$ by the process in [10, Example 5.9]. If we take $S = \operatorname{diag}(+,+,+,-)$ and $P = [e_2, e_1, e_3, e_4]$, then $P^T ASP = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \in \mathcal{J}_4$.

We can obtain two more patterns in this subcase from (6) above by taking the transpose and performing permutation similarity to obtain the patterns

$$\begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & + & + \end{bmatrix}, \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & + & + & - \end{bmatrix}.$$

Up to equivalence, there are 5 unresolved patterns in this subcase, to be addressed in the next subsection.

4.4. Remaining cases

To this point, there remain 11 unresolved patterns, up to equivalence:

$$A_1 = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & + \\ + & + & + & + \end{bmatrix}, A_2 = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & + \\ + & + & + & + \end{bmatrix}, A_3 = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & + & - & + \end{bmatrix},$$

$$A_4 = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}, A_5 = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & - \\ + & + & - & + \end{bmatrix}, A_6 = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & + \\ + & + & - & - \end{bmatrix},$$

$$A_7 = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & - \\ + & + & + & + \end{bmatrix}, A_8 = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & + \\ + & + & + & - \end{bmatrix}, A_9 = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & - \\ + & + & - & - \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & + & - & - \end{bmatrix}, A_{11} = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & + \\ + & + & - & - \end{bmatrix}.$$

To settle most of these remaining sign patterns, we use the following result contained in [11]. As stated in [11], this decomposition was first derived in [7]; it is also mentioned in [11] that in a preliminary version of [14] (which was published later) the authors treat this decomposition in more depth.

Theorem 4.2. *[11, Theorem 3.2] We define*

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad p + q = n.$$

Assume also that $p \leq q$. Let

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

be J-orthogonal with $B_{11} \in \mathbb{R}^{p,p}$, $B_{12} \in \mathbb{R}^{p,q}$, $B_{21} \in \mathbb{R}^{q,p}$ and $B_{22} \in \mathbb{R}^{q,q}$. Then there are *orthogonal matrices $U_1, V_1 \in \mathbb{R}^{p,p}$ and $U_2, V_2 \in \mathbb{R}^{q,q}$ such that*

$$\begin{bmatrix} U_1^T & 0 \\ 0 & U_2^T \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = \begin{bmatrix} C & -S & 0 \\ -S & C & 0 \\ 0 & 0 & I_{q-p} \end{bmatrix}, \tag{8}$$

where $C = \mathrm{diag}(c_i)$, $S = \mathrm{diag}(s_i)$ and $C^2 - S^2 = I_p$ ($c_i > s_i \geq 0$). Any matrix B satisfying (8) is J-orthogonal.

Remark 4.3. In the case $n = 4$ and $J = \mathrm{diag}(1, 1, -1, -1)$, every J-orthogonal matrix B has a factorization of the form

$$\begin{aligned} B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} &= \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} C & -S \\ -S & C \end{bmatrix} \begin{bmatrix} V_1^T & 0 \\ 0 & V_2^T \end{bmatrix} \\ &= \begin{bmatrix} U_1 & 0 \\ 0 & -U_2 \end{bmatrix} \begin{bmatrix} C & S \\ S & C \end{bmatrix} \begin{bmatrix} V_1^T & 0 \\ 0 & -V_2^T \end{bmatrix}. \end{aligned}$$

For $J = \mathrm{diag}(1, 1, -1, -1)$, we can choose suitable 2×2 orthogonal matrices U_1, U_2 and V_1, V_2, we can generate some 4×4 J-orthogonal matrices. However, some sign patterns are quite difficult to achieve by a product of two 2×2 orthogonal matrices and a diagonal matrix. For a fixed pair V_1, V_2, the two block rows of the matrix B can be interpreted as two orthogonal transformations of four vectors in the plane. Thus, the sign pattern will allow a J-orthogonal matrix only if there exists an orthogonal transformation mapping the four vectors with the sign pattern of the first block row of the four vectors with the sign pattern of the second block row. This is clearly not always possible.

Remark 4.4. In the case $n = 4$ and $J = \mathrm{diag}(1, -1, -1, -1)$, every J-orthogonal matrix B

has a factorization of the form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} c_1 u_1 v_1 & -s_1 u_1 (V_2 e_1)^T \\ -s_1 v_1 U_2 e_1 & U_2 \begin{bmatrix} c_1 & 0 \\ 0 & I_2 \end{bmatrix} V_2^T \end{bmatrix}, \tag{9}$$

where $u_1, v_1 \in \mathbb{R}$, $U_2, V_2 \in \mathbb{R}^{3,3}$ are orthogonal and $e_1 = [1\ 0\ 0]^T \in \mathbb{R}^3$.

It was noted that a given 4×4 full sign pattern can be multiplied on the left and right by signature patterns so that it has the form

$$\begin{bmatrix} + & + & + & + \\ + & & & \\ + & & & \\ + & & & \end{bmatrix}. \tag{10}$$

For $J = \mathrm{diag}(1, -1, -1, -1)$ this sign pattern essentially leads to the condition $u_1 v_1 = 1$ due to $c_1 > 1$. Taking $u_1 = -1$ and $v_1 = -1$ we get to the conditions that both $U_2 e_1$ and $V_2 e_1$ should have the sign pattern equal to $[+\ +\ +]^T$. So, given the orthogonal matrices $U_2, V_2 \in \mathbb{R}^{3,3}$ such that $\mathrm{sgn}(U_2 e_1) = \mathrm{sgn}(V_2 e_1) = [+\ +\ +]^T$, then there exists a J-orthogonal matrix of the form (9) with the sign pattern (10). The sign pattern of the lower right diagonal block is given by the sign pattern of the matrix

$$U_2 \begin{bmatrix} c_1 & 0 \\ 0 & I_2 \end{bmatrix} V_2^T = U_2 V_2^T + (c_1 - 1) U_2 e_1 e_1^T V_2^T.$$

Note that the sign pattern of $U_2 e_1 e_1^T V_2^T$ is the 3×3 matrix of all $+$. In addition, for sufficiently small $c_1 - 1$, the sign pattern of $U_2 \begin{bmatrix} c_1 & 0 \\ 0 & I_2 \end{bmatrix} V_2^T$ becomes equal to the sign pattern of the 3×3 orthogonal matrix $U_2 V_2^T$. This is the way we can generate 3×3 J-orthogonal matrices with some prescribed sign patterns of the form (10).

We can now address some of the sign patterns remaining in our analysis.

We can handle A_3 by the approach given in Remark 4.4. Let

$$U_2 = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix} V_2; \quad V_2 = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ 3 & 2 & -6 \\ 2 & 6 & 3 \end{bmatrix}.$$

Then the matrix $U_2 V_2^T$ has the same sign pattern as the lower right block of the pattern A_3. It can also be verified that the first column of the matrix U_2 has all positive entries. Then, in Remark 4.4, the matrix

$$B = \begin{bmatrix} 2 & \frac{6}{7}\sqrt{3} & \frac{3}{7}\sqrt{3} & \frac{2}{7}\sqrt{3} \\ \frac{11}{21}\sqrt{3} & \frac{164}{147} & \frac{82}{147} & -\frac{76}{147} \\ \frac{16}{21}\sqrt{3} & \frac{145}{147} & \frac{146}{147} & \frac{130}{147} \\ \frac{8}{21}\sqrt{3} & \frac{146}{147} & -\frac{74}{147} & \frac{65}{147} \end{bmatrix}$$

is J-orthogonal with respect to $J = \operatorname{diag}(1, -1, -1, -1)$. Eight other patterns from the list of unresolved patterns can be handled by this approach.

Note that A_7 is equivalent to the sign pattern

$$\begin{bmatrix} - & + & - & + \\ + & + & + & + \\ + & + & + & + \\ + & - & + & - \end{bmatrix}.$$

This pattern can be handled by the approach given in Remark 4.3. We choose

$$U_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, -U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, V_1^T = -V_2^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, S_1 = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}.$$

Then, as can be checked, the matrix

$$B = \begin{bmatrix} -\sqrt{2} & \sqrt{2} & -\frac{\sqrt{3}}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 2 & 2 \\ 2 & 2 & \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{\sqrt{3}}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

is J-orthogonal with respect to $J = \mathrm{diag}(1, 1, -1, -1)$.

Now, A_9 is the only 4×4 full sign pattern that we have not shown to be in $\mathcal{J}_4$. To handle this final case, we need the notion of the exchange operator. Let p and n be positive integers with $p \le n$. Let B be an $n \times n$ matrix partitioned as $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ such that B_{11} is $p \times p$ and is nonsingular. The exchange operator applied to B with respect to the above partition yields

$$\mathrm{exc(B)} = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12} \\ B_{21}B_{11}^{-1} & B_{22} - B_{21}B_{11}^{-1}B_{12} \end{bmatrix}.$$

The following theorem found in [13, Theorem 2.1] and [11, Theorem 2.2] reveals the close connections between orthogonal matrices and J-orthogonal matrices via the exchange operator.

Theorem 4.5. *Let p and n be positive integers with $p \le n$. Let B be an $n \times n$ real matrix partitioned as $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ such that B_{11} is $p \times p$. Let $J = \mathrm{diag}(I_p, -I_{n-p})$. If B is J-orthogonal, then the leading $p \times p$ principal submatrix of B is nonsingular and exc(B) is orthogonal. Conversely, if B is orthogonal and B_{11} is nonsingular, then exc(B) is J-orthogonal.*

Hence, every J-orthogonal matrix can be constructed from a suitable orthogonal matrix using the exchange operator and permutation similarity. This approach can be used to show that a given full sign pattern allows J-orthogonality for a particular signature matrix J. This process can be done for hundreds of thousands of "random" rational orthogonal matrices using MATLAB.

For every full 4×4 sign pattern A that satisfies the SPJO conditions for a specific signature pattern J, we can generate a J-orthogonal matrix in $Q(A)$. In particular, note that A_9 satisfies the SPJO conditions with the signature pattern $J_1 = \mathrm{diag}(+,-,+,-)$. With the help of MATLAB running the procedure, for $J = \mathrm{diag}(1,1,-1,-1)$, we obtain the following J-orthogonal matrix

$$B = \frac{1}{12}\begin{bmatrix} 8 & 18 & 12 & 10 \\ 26 & -9 & 18 & -17 \\ 20 & 6 & 24 & -2 \\ 14 & -15 & 6 & -23 \end{bmatrix},$$

which satisfies $P^T B P = A_9$ for $P = [e_1, e_3, e_2, e_4]$. It follows that A_9 allows a J_1-orthogonal matrix with $J_1 = P^T J P = \mathrm{diag}(1,-1,1,-1)$, and hence $A_9 \in \mathcal{J}_4$.

We now reach the following conclusion.

Theorem 4.6. *Every 4×4 full sign pattern allows a J-orthogonal matrix.*

Combined with known results on full sign patterns of orders at most three, we get the following result.

Corollary 4.7. *For $n \leq 4$, every $n \times n$ full sign pattern allows a J-orthogonal matrix.*

In view of Theorem 1.5, we also have

Corollary 4.8. *For $n \leq 4$, every $n \times n$ full sign pattern allows a G-matrix.*

Thus we have the following nice result.

Corollary 4.9. *For every $n \times n$ full sign pattern A with $n \leq 4$, $A \in \mathcal{G}_n$ iff $A \in \mathcal{J}_n$.*

Suppose a full $n \times n$ sign pattern A allows a J-orthogonal matrix $B \in Q(A)$. Without loss of generality, we may assume that all the positive entries of J occur at the leading diagonal entries. By Theorem 4.5, exc(B) is an orthogonal matrix. Observe that exc(exc(B))=B. Write exc(B) as a product of real Householder matrices $H_{v_1}, \ldots, H_{v_k}$ (where $k \leq n$). Replace each v_i with a rational approximation $\tilde{v}_i$. Since matrix multiplication and exchange operator are continuous, we see that when the rational approximations $\tilde{v}_i$ are sufficiently close to v_i, $\tilde{B} = \text{exc}(H_{\tilde{v}_1} \cdots H_{\tilde{v}_k})$ is a rational J-orthogonal matrix in $Q(A)$. Thus we have shown the following interesting result.

Theorem 4.10. *Let A be a full $n \times n$ sign pattern. If A allows a J-orthogonal matrix, then A allows a rational J-orthogonal matrix with the same signature matrix J. In particular, if A allows orthogonality, then A allows a rational orthogonal matrix.*

As a consequence, if the $n \times n$ full sign pattern A does not allow a rational J-orthogonal matrix for any signature matrix J, then A does not allow a real J-orthogonal matrix.

5 Concluding remarks

The question of whether every $n \times n$ full sign pattern allows a J-orthogonal matrix is still open. It seems to be a complicated and impressive problem. Even for $n = 5$ the number of cases is daunting. Some examples of rational 5×5 J-orthogonal matrices are given in Appendix A. Some other techniques will need to be developed to prove that every full sign pattern allows J-orthogonality.

This thesis forms the basis for the publication [9].

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Appendix A

This appendix gives some examples of 5×5 J-orthogonal matrices generated in MATLAB. The program used is based on the exchange operator. Some random Householder matrices are generated in order to create an orthogonal matrix to which the exchange operator can be applied and a signature matrix J is designated. Then the randomly generated J-orthogonal matrix is checked against some desired sign pattern to determine whether or not it will be output. This method can be used to generate a large number of J-orthogonal matrices of a given sign pattern. Thus far, every full 5×5 sign pattern that has been checked with this method has turned out to allow J-orthogonality. However, to check every case is a large task.

Examples:

$$\begin{bmatrix} \frac{28}{25} & \frac{6}{25} & \frac{1}{5} & \frac{1}{25} & \frac{13}{25} \\ \frac{6}{25} & \frac{37}{25} & \frac{2}{5} & \frac{2}{25} & \frac{26}{25} \\ \frac{11}{25} & \frac{22}{25} & \frac{2}{5} & \frac{-13}{25} & \frac{31}{25} \\ \frac{7}{25} & \frac{14}{25} & \frac{-1}{5} & \frac{19}{25} & \frac{22}{25} \\ \frac{1}{5} & \frac{2}{5} & 1 & \frac{2}{5} & \frac{1}{5} \end{bmatrix} \text{ is } J\text{-orthogonal with } J = \operatorname{diag}(1, 1, -1, -1, -1)$$

$$\begin{bmatrix} \frac{25}{18} & \frac{37}{126} & \frac{40}{63} & \frac{31}{63} & \frac{4}{9} \\ \frac{1}{18} & \frac{121}{126} & \frac{-2}{63} & \frac{-11}{63} & \frac{-2}{9} \\ \frac{1}{9} & \frac{13}{63} & \frac{-40}{63} & \frac{32}{63} & \frac{-2}{9} \\ \frac{5}{9} & \frac{11}{63} & \frac{34}{63} & \frac{61}{63} & \frac{-2}{9} \\ \frac{7}{9} & \frac{19}{63} & \frac{53}{63} & \frac{8}{63} & \frac{8}{9} \end{bmatrix} \text{ is } J\text{-orthogonal with } J = \operatorname{diag}(1, -1, -1, -1, -1)$$

$$\begin{bmatrix} \frac{88}{57} & \frac{67}{57} & \frac{1}{57} & \frac{1}{57} & \frac{2}{57} \\ \frac{43}{57} & \frac{58}{57} & \frac{-27}{57} & \frac{-27}{57} & \frac{-34}{57} \\ \frac{7}{19} & \frac{9}{19} & \frac{18}{19} & \frac{-1}{19} & \frac{-2}{19} \\ \frac{7}{19} & \frac{9}{19} & \frac{-1}{19} & \frac{18}{19} & \frac{-2}{19} \\ \frac{14}{19} & \frac{18}{19} & \frac{-2}{19} & \frac{-2}{19} & \frac{15}{19} \end{bmatrix} \text{ is } J\text{-orthogonal with } J = \operatorname{diag}(1, -1, -1, -1, -1)$$

$$\begin{bmatrix} \frac{27}{16} & \frac{1}{26} & \frac{1}{13} & \frac{5}{26} & \frac{5}{26} \\ \frac{1}{26} & \frac{27}{26} & \frac{1}{13} & \frac{5}{26} & \frac{5}{26} \\ \frac{1}{13} & \frac{1}{13} & \frac{-11}{13} & \frac{5}{13} & \frac{5}{13} \\ \frac{5}{26} & \frac{5}{26} & \frac{5}{13} & \frac{25}{26} & \frac{-1}{26} \\ \frac{5}{26} & \frac{5}{26} & \frac{5}{13} & \frac{-1}{26} & \frac{25}{26} \end{bmatrix} \text{is } J\text{-orthogonal with } J = \mathrm{diag}(1,1,-1,-1,-1)$$

$$\begin{bmatrix} \frac{19}{18} & \frac{1}{12} & \frac{5}{27} & \frac{23}{108} & \frac{11}{54} \\ \frac{1}{12} & \frac{9}{8} & \frac{5}{18} & \frac{23}{72} & \frac{11}{36} \\ \frac{1}{12} & \frac{1}{8} & \frac{17}{18} & \frac{-1}{72} & \frac{-13}{36} \\ \frac{1}{18} & \frac{1}{12} & \frac{-4}{27} & \frac{95}{108} & \frac{-25}{54} \\ \frac{1}{3} & \frac{1}{2} & \frac{4}{9} & \frac{11}{18} & \frac{8}{9} \end{bmatrix} \text{s } J\text{-orthogonal with } J = \mathrm{diag}(1,1,-1,-1,-1)$$