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# Comparison theorems and asymptotic behavior of solutions of discrete fractional equations

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# Comparison theorems and asymptotic behavior of solutions of discrete fractional equations

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**Abstract.** Consider the following $\nu$-th order nabla and delta fractional difference equations

$$\begin{aligned} \nabla^{\nu}_{\rho(a)} x(t) &= c(t)x(t), \qquad t \in \mathbb{N}_{a+1}, \\ x(a) &> 0. \end{aligned} \tag{$*$}$$

and

$$\begin{aligned} \Delta^{\nu}_{a+\nu-1} x(t) &= c(t)x(t+\nu-1), \qquad t \in \mathbb{N}_{a}, \\ x(a+\nu-1) &> 0. \end{aligned} \tag{$**$}$$

We establish comparison theorems by which we compare the solutions $x(t)$ of ($*$) and ($**$) with the solutions of the equations $\nabla^{\nu}_{\rho(a)} x(t) = bx(t)$ and $\Delta^{\nu}_{a+\nu-1} x(t) = bx(t+\nu-1)$, respectively, where $b$ is a constant. We obtain four asymptotic results, one of them extends the recent result [F. M. Atici, P. W. Eloe, *Rocky Mountain J. Math.* **41**(2011), 353–370].
These results show that the solutions of two fractional difference equations $\nabla^{\nu}_{\rho(a)} x(t) = cx(t)$, $0 < \nu < 1$, and $\Delta^{\nu}_{a+\nu-1} x(t) = cx(t+\nu-1)$, $0 < \nu < 1$, have similar asymptotic behavior with the solutions of the first order difference equations $\nabla x(t) = cx(t)$, $|c| < 1$ and $\Delta x(t) = cx(t)$, $|c| < 1$, respectively.

**Keywords:** nabla and delta fractional difference, discrete Mittag-Leffler function, rising and falling function.

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## 1 Introduction

Discrete fractional calculus has generated much interest in recent years. Some of the work has employed the fractional forward and delta difference operators. We refer the readers to [1,4], for example, and more recently [6,8]. Probably more work has been developed for the

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backward or nabla difference operator and we refer the readers to [5,7]. There has been some work to develop relations between the forward and backward fractional operators, $\Delta_a^\nu$ and $\nabla_a^\nu$ (see [2]) and fractional calculus on time scales (see [4]).

This work is motivated by F. Atici and P. Eloe [3] who obtained asymptotic results for the fractional difference equation $\nabla_{\rho(a)}^\nu x(t) = bx(t)$, $0.5 \leq \nu \leq 1$, $t \in \mathbb{N}_a$ with $0 < b < 1$, $x(a) > 0$. We shall consider the following $\nu$-th order nabla and delta fractional difference equations

$$\begin{aligned} \nabla_{\rho(a)}^\nu x(t) &= c(t)x(t), \qquad t \in \mathbb{N}_{a+1}, \\ x(a) &> 0. \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \Delta_{a+\nu-1}^\nu x(t) &= c(t)x(t+\nu-1), \qquad t \in \mathbb{N}_a, \\ x(a+\nu-1) &> 0. \end{aligned} \tag{1.2}$$

We establish comparison theorems by which we compare the solutions $x(t)$ of (1.1) and (1.2) with the solutions of the equations $\nabla_{\rho(a)}^\nu x(t) = bx(t)$ and $\Delta_{a+\nu-1}^\nu x(t) = bx(t+\nu-1)$, respectively, where $b$ is a constant. We obtain the following asymptotic results in which Theorem A extends the recent result of Atici and Eloe [3].

**Theorem A.** *Assume $0 < \nu < 1$ and there exists a constant $b$ such that $0 < b \leq c(t) < 1$. Then the solutions of the equation (1.1) satisfy*

$$\lim_{t\to\infty} x(t) = \infty.$$

**Theorem B.** *Assume $0 < \nu < 1$ and $c(t) \leq 0$. Then the solutions of the equation (1.1) satisfy*

$$\lim_{t\to\infty} x(t) = 0.$$

**Theorem C.** *Assume $0 < \nu < 1$ and there exists a constant $b$ such that $c(t) \geq b > 0$. Then the solutions of the equation (1.2) satisfy*

$$\lim_{t\to\infty} x(t) = \infty.$$

**Theorem D.** *Assume $0 < \nu < 1$ and $-\nu \leq c(t) < 0$. Then the solutions of the equation (1.2) satisfy*

$$\lim_{t\to\infty} x(t) = 0.$$

This shows that the solutions of two fractional difference equations $\nabla_{\rho(a)}^\nu x(t) = cx(t)$, $0 < \nu < 1$, and $\Delta_{a+\nu-1}^\nu x(t) = cx(t+\nu-1)$, $0 < \nu < 1$, have similar asymptotic behavior with the solutions of the first order difference equations $\nabla x(t) = cx(t)$, $|c| < 1$ and $\Delta x(t) = cx(t)$, $|c| < 1$, respectively.

## 2 Asymptotic behavior, nabla case, $0 < b \leq c(t) < 1$

Let $\Gamma(x)$ denote the gamma function. Then we define the rising function (see [10]) by

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)},$$

for those values of $t$ and $r$ such that the right-hand side of this equation is well defined. We also use the standard extensions of their domains to define these functions to be zero when the numerator is well defined, but the denominator is not defined. We will be interested in functions defined on sets of the form

$$\mathbb{N}_a := \{a, a+1, a+2, \dots\},$$

where $a \in \mathbb{R}$. The delta and the nabla integral of a function $f : \mathbb{N}_a \to \mathbb{R}$ are defined by the following

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \qquad \int_a^b f(t)\nabla t = \sum_{t=a+1}^{b} f(t),$$

where $b \in \mathbb{N}_a$. We will use elementary properties of these integrals throughout this paper (see Goodrich and Peterson [8] for these properties). The nabla fractional Taylor monomial of degree $\nu$ based at $\rho(a) := a-1$ (see [8]) is defined by

$$H_\nu(t, \rho(a)) := \frac{(t-a+1)^{\overline{\nu}}}{\Gamma(\nu+1)}.$$

The following definition of the discrete Mittag-Leffler function is given in Atici and Eloe [3] (see also [8]).

**Definition 2.1.** For $|p| < 1$, $0 < \alpha < 1$, we define the discrete Mittag-Leffler function by

$$E_{p,\alpha,\alpha-1}(t, \rho(a)) := \sum_{k=0}^{\infty} p^k H_{\alpha k+\alpha-1}(t, \rho(a)), \qquad t \in \mathbb{N}_a.$$

To study the asymptotic behavior of the solutions of (2.3) for the case $0.5 \le \nu \le 1$, the authors in [3] used the Laplace transformation, the convolution theorem and the properties of a hypergeometric function. They proved that the solutions of the fractional difference equation $\nabla^\nu_{\rho(a)} x(t) = bx(t)$, $0.5 \le \nu \le 1$, $t \in \mathbb{N}_a$ where $0 < b < 1$ tend to $\infty$ as $t \to \infty$.

A natural question arises: if $0 < \nu < 0.5$ and $|b| < 1$, then how about the asymptotic behavior of the solutions of equation (2.3)? In this paper we will answer this question and related questions. First we will establish a useful comparison theorem. We will use the following lemma which appears in [8].

**Lemma 2.2.** *Assume that* $f : \mathbb{N}_{a-1} \to \mathbb{R}$, $\nu > 0$, $\nu \notin \mathbb{N}_1$, *and choose* $N \in \mathbb{N}_1$ *such that* $N-1 < \nu < N$. *Then*

$$\nabla^\nu_{\rho(a)} f(t) = \int_{\rho(a)}^{t} H_{-\nu-1}(t, \rho(\tau)) f(\tau)\nabla\tau,$$

*for* $t \in \mathbb{N}_a$.

**Lemma 2.3.** *Assume that* $0 < \nu < 1$, $|b| < 1$. *Then*

$$\nabla^\nu_{\rho(a)} E_{b,\nu,\nu-1}(t, \rho(a)) = E_{b,\nu,-1}(t, \rho(a))$$

*for* $t \in \mathbb{N}_a$.

*Proof.* From Lemma 2.2, we have

$$\begin{aligned}\nabla^\nu_{\rho(a)} E_{b,\nu,\nu-1}(t, \rho(a)) &= \int_{\rho(a)}^{t} H_{-\nu-1}(t, \rho(s)) E_{b,\nu,\nu-1}(s, \rho(a))\nabla s \\ &= \int_{\rho(a)}^{t} H_{-\nu-1}(t, \rho(s)) \sum_{k=0}^{\infty} b^k H_{\nu k+\nu-1}(s, \rho(a))\nabla s.\end{aligned} \tag{2.1}$$

In the following, we first prove that the infinite series

$$H_{-\nu-1}(t,\rho(s))\sum_{k=0}^{\infty} b^k H_{\alpha k+\alpha-1}(s,\rho(a)) \tag{2.2}$$

for each fixed $t$ is uniformly convergent for $s \in [\rho(a), t]$.

We will first show that

$$|H_{-\nu-1}(t,\rho(s))| = \left|\frac{\Gamma(-\nu+t-s)}{\Gamma(t-s+1)\Gamma(-\nu)}\right| \leq 1$$

for $\rho(a) \leq s \leq t$. For $s = t$ we have that

$$|H_{-\nu-1}(t,\rho(s))| = 1.$$

Now assume that $\rho(a) \leq s < t$, then

$$\begin{aligned}\left|\frac{\Gamma(-\nu+t-s)}{\Gamma(t-s+1)\Gamma(-\nu)}\right| &= \left|\frac{(t-s-\nu-1)(t-s-\nu-2)\cdots(-\nu)}{(t-s)!}\right| \\ &= \left|\frac{t-s-(\nu+1)}{t-s}\right|\left|\frac{t-s-1-(\nu+1)}{t-s-1}\right|\cdots\left|\frac{-\nu}{1}\right| \\ &\leq 1.\end{aligned}$$

Also consider

$$\begin{aligned}H_{\nu k+\nu-1}(s,\rho(a)) &= \frac{\Gamma(\nu k+\nu+s-a)}{\Gamma(s-a+1)\Gamma(\nu k+\nu)} \\ &= \frac{(\nu k+\nu+s-a-1)\cdots(\nu k+\nu)}{(s-a)!}.\end{aligned}$$

Note that for large $k$ it follows that

$$\begin{aligned}H_{\nu k+\nu-1}(s,\rho(a)) &\leq (\nu k+\nu+s-a-1)^{s-a} \\ &\leq (\nu k+\nu+t-a-1)^{t-a}\end{aligned}$$

for $\rho(a) \leq s \leq t$. Since

$$\lim_{k\to\infty} \sqrt[k]{|b|^k(\nu k+\nu+t-a-1)^{t-a}} = |b| < 1,$$

we get by the Root Test that for each fixed $t$ the infinite series (2.2) is uniformly convergent for $s \in [\rho(a), t]$. So from (2.1), integrating term by term, we get, (using $\nabla^{\nu}_{\rho(a)} H_{\nu k+\nu-1}(s,\rho(a))) = H_{\nu k-1}(s,\rho(a))$),

$$\begin{aligned}\nabla^{\nu}_{\rho(a)} E_{b,\nu,\nu-1}(t,\rho(a)) &= \sum_{k=0}^{\infty} b^k \int_{\rho(a)}^{t} H_{-\nu-1}(t,\rho(s))H_{\nu k+\nu-1}(s,\rho(a))\nabla s \\ &= \sum_{k=0}^{\infty} b^k \nabla^{\nu}_{\rho(a)} H_{\nu k+\nu-1}(t,\rho(a)) \\ &= \sum_{k=0}^{\infty} b^k H_{\nu k-1}(t,\rho(a)) \\ &= E_{b,\nu,-1}(t,\rho(a)).\end{aligned}$$

This completes the proof. □

Atici and Eloe [3] gave a formal proof of the following result using Laplace transforms. With the aid of Lemma 2.3 we now give a rigorous proof of this result.

**Lemma 2.4.** *Assume that* $0 < \nu < 1$, $|b| < 1$. *Then* $E_{b,\nu,\nu-1}(t, \rho(a))$ *is the unique solution of the initial value problem*

$$\begin{aligned} \nabla^{\nu}_{\rho(a)} x(t) &= bx(t), \qquad t \in \mathbb{N}_{a+1} \\ x(a) &= \frac{1}{1-b} > 0. \end{aligned} \tag{2.3}$$

*Proof.* If $b = 0$, then

$$E_{0,\nu,\nu-1}(t, \rho(a)) = H_{\nu-1}(t, \rho(a)).$$

So from [8, Chapter 3], we have

$$\nabla^{\nu}_{\rho(a)} H_{\nu-1}(t, \rho(a)) = H_{-1}(t, \rho(a)) = 0,$$

using out convention $H_{-1}(t, \rho(a)) = 0$. Now assume $b \neq 0$. From Lemma 2.3, we have (using $H_{-1}(t, \rho(a)) = 0$)

$$\begin{aligned} \nabla^{\nu}_{\rho(a)} E_{b,\nu,\nu-1}(t, \rho(a)) &= E_{b,\nu,-1}(t, \rho(a)) \\ &= \sum_{k=0}^{\infty} b^k H_{\nu k-1}(t, \rho(a)) \\ &= b \sum_{k=1}^{\infty} b^{k-1} H_{\nu k-1}(t, \rho(a)) \\ &= b \sum_{j=0}^{\infty} b^j H_{\nu j+\nu-1}(t, \rho(a)) \\ &= b E_{b,\nu,\nu-1}(t, \rho(a)). \end{aligned}$$

This completes the proof. □

The following comparison theorem plays an important role in proving our main results.

**Theorem 2.5.** *Assume* $c_2(t) \leq c_1(t) < 1$, $0 < \nu < 1$. *Then if* $x(t), y(t)$ *are the solutions of the equations*

$$\nabla^{\nu}_{\rho(a)} x(t) = c_1(t) x(t), \tag{2.4}$$

*and*

$$\nabla^{\nu}_{\rho(a)} y(t) = c_2(t) y(t), \tag{2.5}$$

*respectively, for* $t \in \mathbb{N}_{a+1}$ *satisfying* $x(a) \geq y(a) > 0$, *then*

$$x(t) \geq y(t),$$

*for* $t \in \mathbb{N}_a$.

*Proof.* For simplicity, we let $a = 0$. From Lemma 2.2, we have for $t = k$

$$\begin{aligned}\nabla_{\rho(0)}^{\nu} x(t) &= \int_{\rho(0)}^{t} H_{-\nu-1}(t, \rho(s)) x(s) \nabla s \\ &= \sum_{s=0}^{k} H_{-\nu-1}(k, s-1) x(s) \\ &= x(k) - \nu x(k-1) - \frac{\nu(-\nu+1)}{2} x(k-2) \\ &\quad - \cdots - \frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!} x(0).\end{aligned}$$

Using (2.4) and (2.5), we have that

$$\begin{aligned}(1 - c_1(k)) x(k) &= \nu x(k-1) + \frac{\nu(-\nu+1)}{2} x(k-2) \\ &\quad + \cdots + \frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!} x(0).\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}(1 - c_2(k)) y(k) &= \nu y(k-1) + \frac{\nu(-\nu+1)}{2} y(k-2) \\ &\quad + \cdots + \frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!} y(0).\end{aligned} \tag{2.7}$$

We will prove $x(k) \geq y(k) \geq 0$ for $k \in \mathbb{N}_0$ by using the principle of strong induction. When $i = 0$, from the assumption, the result holds. Suppose that $x(i) \geq y(i) \geq 0$, for $i = 0, 1, \ldots, k-1$. Since

$$\frac{\nu(-\nu+1)\cdots(-\nu+i-1)}{i!} > 0$$

for $i = 2, 3, \ldots, k-1$, from (2.6), (2.7) we have

$$(1 - c_1(k)) x(k) \geq (1 - c_2(k)) y(k) \geq 0.$$

Using $c_2(t) \leq c_1(t) \leq 1$, we get

$$x(k) \geq \frac{1 - c_2(k)}{1 - c_1(k)} y(k) \geq y(k) \geq 0.$$

This completes the proof. □

**Theorem 2.6.** *Assume $0 < b \leq c(t) < 1$, $0 < \nu < 1$. Then for any solution $x(t)$ of*

$$\nabla_{\rho(a)}^{\nu} x(t) = c(t) x(t), \qquad t \in \mathbb{N}_{a+1} \tag{2.8}$$

*satisfying $x(a) > 0$ we have that*

$$x(t) \geq \frac{(1-b) x(a)}{2} E_{b,\nu,\nu-1}(t, \rho(a)), \qquad t \in \mathbb{N}_a.$$

*Proof.* From Lemma 2.4, we have

$$\nabla_{\rho(a)}^{\nu} E_{b,\nu,\nu-1}(t, \rho(a)) = b E_{b,\nu,\nu-1}(t, \rho(a))$$

and $E_{b,\nu,\nu-1}(a,\rho(a)) = \frac{1}{1-b}$. Let $c_2(t) = b$, then $x(t)$ satisfies

$$\nabla^{\nu}_{\rho(a)} x(t) = c(t)x(t), \qquad t \in \mathbb{N}_{a+1} \tag{2.9}$$

and

$$y(t) := \frac{(1-b)x(a)}{2} E_{b,\nu,\nu-1}(t,\rho(a))$$

satisfies

$$\nabla^{\nu}_{\rho(a)} y(t) = by(t), \qquad t \in \mathbb{N}_{a+1} \tag{2.10}$$

and

$$x(a) > \frac{(1-b)x(a)}{2} E_{b,\nu,\nu-1}(a,\rho(a)) = y(a).$$

From the comparison theorem (Theorem 2.5), we get that

$$x(t) \geq \frac{(1-b)x(a)}{2} E_{b,\nu,\nu-1}(t,\rho(a)), \qquad t \in \mathbb{N}_a.$$

This completes the proof. □

The following lemma is from [11, page 4].

**Lemma 2.7.** *Assume* $\Re(z) > 0$. *Then*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

The following lemma gives an asymptotic property concerning the nabla fractional Taylor monomial.

**Lemma 2.8.** *Assume that* $0 < \nu < 1$. *Then we have*

$$\lim_{t\to\infty} H_{\nu k+\nu-1}(t,\rho(a)) = \infty, \quad \textit{for } k > \frac{1-\nu}{\nu},$$
$$\lim_{t\to\infty} H_{\nu k+\nu-1}(t,\rho(a)) = \frac{1}{\nu k+\nu}, \quad \textit{for } k = \frac{1-\nu}{\nu},$$
$$\lim_{t\to\infty} H_{\nu k+\nu-1}(t,\rho(a)) = 0, \quad \textit{for } k < \frac{1-\nu}{\nu}.$$

*Proof.* Taking $t = a+1+n$, $n \geq 0$, we have

$$\begin{aligned} &\lim_{t\to\infty} H_{\nu k+\nu-1}(t,\rho(a)) \\ &= \lim_{n\to\infty} \frac{(n+2)^{\overline{\nu k+\nu-1}}}{\Gamma(\nu k+\nu)} \\ &= \lim_{n\to\infty} \frac{\Gamma(\nu k+\nu+n+1)}{\Gamma(n+2)\Gamma(\nu k+\nu)} \\ &= \lim_{n\to\infty} \frac{(\nu k+\nu+n)(\nu k+\nu+n-1)\cdots(\nu k+\nu)}{n!n^{\nu k+\nu}} \cdot \frac{n^{\nu k+\nu}}{n+1}. \end{aligned} \tag{2.11}$$

Using Lemma 2.7, we have

$$\lim_{n\to\infty} \frac{(\nu k+\nu+n)(\nu k+\nu+n-1)\cdots(\nu k+\nu)}{n!n^{\nu k+1}} = \frac{1}{\Gamma(\nu k+\nu)},$$

and

$$\lim_{n\to\infty} \frac{n^{\nu k+\nu}}{n+1} = \infty, \quad \text{for } k > \frac{1-\nu}{\nu},$$
$$\lim_{n\to\infty} \frac{n^{\nu k+\nu}}{n+1} = 1, \quad \text{for } k = \frac{1-\nu}{\nu},$$
$$\lim_{n\to\infty} \frac{n^{\nu k+\nu}}{n+1} = 0, \quad \text{for } k < \frac{1-\nu}{\nu}.$$

Using (2.11), we complete the proof. □

Since there are only a finite number of $k$ which satisfy $k < \frac{1-\nu}{\nu}$, from Lemma 2.8 and the definition of $E_{b,\nu,\nu-1}(t,\rho(a))$, we obtain the following theorem.

**Theorem 2.9.** *For $0 < b < 1$, we have*

$$\lim_{t\to\infty} E_{b,\nu,\nu-1}(t,\rho(a)) = +\infty.$$

From Theorem 2.6 and Theorem 2.9, we have the following.

**Theorem A.** *Assume $0 < \nu < 1$ and there exists a constant $b$ such that $0 < b \le c(t) < 1$. Then the solutions of the equation (1.1) satisfy*

$$\lim_{t\to\infty} x(t) = \infty.$$

**Remark 2.10.** Theorem A can be regarded as an extension of the following result which appears in Atici and Eloe [3].

**Theorem 2.11.** *Let $0.5 \le \nu \le 1, -1 < c < 0$. Then the solution of $\nabla^{\nu}_{\rho(0)} x(t) + cx(t) = 0, x(0) > 0$ diverges to infinity as $t \to \infty$.*

## 3 Asymptotic behavior, nabla case, $c(t) \le 0$

**Lemma 3.1.** *For any $\nu > 0$ such that $N-1 < \nu < N$, where $N \in \mathbb{N}_1$, the following equality holds:*

$$\nabla_a^{-\nu}\nabla^N f(t) = \nabla^N \nabla_{\rho(a)}^{-\nu} f(t) - \left[\sum_{i=1}^{N-1} H_{\nu-i}(t,a)\nabla^{N-i} f(a) + H_{\nu-N}(t,\rho(a))\right] f(a), \tag{3.1}$$

*for $t \in \mathbb{N}_{a-N+1}$ (note by our convention on sums the second term on the right-hand side is zero when N=1).*

*Proof.* Using the power rule ([8])

$$\nabla_s H_{\nu-1}(t,s) = -H_{\nu-2}(t,\rho(s)),$$

and integrating by parts, we have

$$\begin{aligned}
\nabla_a^{-\nu}\nabla^N f(t) &= \int_a^t H_{\nu-1}(t,\rho(s))\nabla^N f(s)\nabla s \\
&= H_{\nu-1}(t,s)\nabla^{N-1} f(s)\big|_a^t + \int_a^t H_{\nu-2}(t,\rho(s))\nabla^{N-1} f(s)\nabla s \\
&= -H_{\nu-1}(t,a)\nabla^{N-1} f(a) + \int_a^t H_{\nu-2}(t,\rho(s))\nabla^{N-1} f(s)\nabla s.
\end{aligned}$$

By applying integration by parts $N-1$ more times, we get

$$\nabla_a^{-\nu}\nabla^N f(t) = -\sum_{i=1}^{N} H_{\nu-i}(t,a)\nabla^{N-i}f(a) + \int_a^t H_{\nu-N-1}(t,\rho(s))f(s)\nabla s. \tag{3.2}$$

Using Leibniz's rule $N-1$ more times, we get

$$\begin{aligned}
&\nabla^N \nabla_{\rho(a)}^{-\nu} f(t) \\
&= \nabla^N \int_{\rho(a)}^t H_{\nu-1}(t,\rho(s))f(s)\nabla s \\
&= \nabla^{N-1} \int_{\rho(a)}^t H_{\nu-2}(t,\rho(s))f(s)\nabla s \\
&= \int_{\rho(a)}^t H_{\nu-N-1}(t,\rho(s))f(s)\nabla s \\
&= H_{\nu-N-1}(t,\rho(a))f(a) + \int_a^t H_{\nu-N-1}(t,\rho(s))f(s)\nabla s,
\end{aligned} \tag{3.3}$$

and [8, Chapter 3]

$$H_{\nu-N}(t,a) + H_{\nu-N-1}(t,\rho(a)) = H_{\nu-N}(t,\rho(a)). \tag{3.4}$$

From (3.2), (3.3), (3.4), we get that (3.1) holds. This completes the proof. □

Taking $N=1$ in Lemma 3.1, we get that the following corollary holds.

**Corollary 3.2.** *For any $0<\nu<1$, the following equality holds:*

$$\nabla_a^{-\nu}\nabla f(t) = \nabla\nabla_{\rho(a)}^{-\nu} f(t) - H_{\nu-1}(t,\rho(a))f(a),$$

*for $t \in \mathbb{N}_a$.*

**Theorem B.** *Assume $c(t) \le 0$, $0<\nu<1$. Then for all solutions $x(t)$ of the fractional equation*

$$\nabla_{\rho(a)}^{\nu} y(t) = c(t)y(t), \qquad t \in \mathbb{N}_{a+1} \tag{3.5}$$

*satisfying $y(a) > 0$ we have*

$$\lim_{t\to\infty} y(t) = 0.$$

*Proof.* Applying the operator $\nabla_a^{-\nu}$ to each side of equation (3.5) we obtain

$$\nabla_a^{-\nu}\nabla_{\rho(a)}^{\nu} y(t) = \nabla_a^{-\nu} c(t)y(t),$$

which can be written in the form

$$\nabla_a^{-\nu}\nabla\nabla_{\rho(a)}^{-(1-\nu)} y(t) = \nabla_a^{-\nu} c(t)y(t).$$

Using Corollary 3.2, we get that

$$\nabla\nabla_{\rho(a)}^{-\nu}\nabla_{\rho(a)}^{-(1-\nu)} y(t) - \frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}\nabla_{\rho(a)}^{-(1-\nu)} y(t)|_{t=a} = \nabla_a^{-\nu} c(t)y(t).$$

Using

$$\nabla_{\rho(a)}^{-(1-\nu)} y(t)|_{t=a} = \int_{\rho(a)}^a H_{-\nu}(a,\rho(s))y(s)\nabla s = H_{-\nu}(a,\rho(a))y(a) = y(a),$$

we get that

$$\nabla\nabla_{\rho(a)}^{-\nu}\nabla_{\rho(a)}^{-(1-\nu)}y(t)=\frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}y(a)+\nabla_a^{-\nu}c(t)y(t).$$

Using the composition rule, ([8, Chapter 3]) $\nabla_{\rho(a)}^{-\nu}\nabla_{\rho(a)}^{-(1-\nu)}y(t)=\nabla_{\rho(a)}^{-1}y(t)$ and $\nabla\nabla_{\rho(a)}^{-1}y(t)=y(t)$, we get that

$$y(t)=\frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}y(a)+\nabla_a^{-\nu}c(t)y(t).$$

That is

$$\begin{aligned}y(t)&=\frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}y(a)+\int_a^t H_{\nu-1}(t,\rho(s))c(s)y(s)\nabla s\\&=\frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}y(a)+\sum_{s=a+1}^{t}H_{\nu-1}(t,\rho(s))c(s)y(s)\\&=\frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}y(a)+\sum_{s=a+1}^{t}\frac{(t-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)}c(s)y(s).\end{aligned}\tag{3.6}$$

From $y(a)>0$, $0<\nu<1$, $c(t)\le 0$ and (2.7), using the strong induction principle, it is easy to prove $y(t)>0$ for $t\in\mathbb{N}_a$. Since

$$\frac{(t-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)}=\frac{\Gamma(\nu+t-s)}{\Gamma(t-s+1)\Gamma(\nu)}>0$$

for $t\ge s$ and $c(s)\le 0$, from (3.6) we get that (taking $t=a+k$)

$$\begin{aligned}&0<y(a+k)\\&\le\frac{(k+1)^{\overline{\nu-1}}}{\Gamma(\nu)}y(a)\\&\le\frac{\Gamma(\nu+k)}{\Gamma(k+1)\Gamma(\nu)}\\&=\frac{(\nu+k-1)(\nu+k-2)\cdots(\nu+1)\nu}{\Gamma(k+1)}\\&=\frac{(\nu+k-1)(\nu+k-2)\cdots(\nu+1)\nu}{k!}.\end{aligned}\tag{3.7}$$

From Lemma 2.7, we have

$$\frac{1}{\Gamma(\nu)}=\lim_{k\to\infty}\frac{(\nu+k-1)(\nu+k-2)\cdots(\nu+1)\nu}{(k-1)^{\nu}(k-1)!},$$

so also using $0<\nu<1$, we have that

$$\begin{aligned}&\lim_{k\to\infty}\frac{(\nu+k-1)(\nu+k-2)\cdots(\nu+1)\nu}{k!}\\&\quad=\lim_{k\to\infty}\frac{(\nu+k-1)(\nu+k-2)\cdots(\nu+1)\nu}{(k-1)!(k-1)^{\nu}}\cdot\frac{(k-1)^{\nu}}{k}=0.\end{aligned}$$

Therefore from (3.7) we have

$$\lim_{k\to\infty}y(k+a)=0.$$

□

From Lemma 2.4 and Theorem B, we can obtain the following corollary.

**Corollary 3.3.** *Assume that* $0 < b < 1$, $0 < \nu < 1$. *Then*

$$\lim_{t\to\infty} E_{-b,\nu,\nu-1}(t,\rho(a)) = 0$$

*where* $E_{-b,\nu,\nu-1}(t,\rho(a)) = \sum_{k=0}^{\infty}(-b)^k \frac{(t-a+1)^{\overline{\nu k+\nu-1}}}{\Gamma(\nu k+\nu)}$ *is the discrete Mittag-Leffler function.*

**Remark 3.4.** The above corollary is not obvious, since $E_{-b,\nu,\nu-1}(t,\rho(a))$ is an infinite series whose terms change sign.

Note that if we let $x(t)$ be a solution of the $\nu$-th order fractional nabla equation

$$\nabla^{\nu}_{\rho(a)}x(t) = c(t)x(t), \tag{3.8}$$

satisfying $x(a) < 0$ and if we set $y(t) = -x(t)$, then using Theorem A and Theorem B, we can get the following theorems.

**Theorem Â.** *Assume* $0 < \nu < 1$ *and there exists a constant* $b$ *such that* $0 < b \le c(t) < 1$. *Then the solutions of the equation* (3.8) *satisfy*

$$\lim_{t\to\infty} x(t) = -\infty.$$

**Theorem B̂.** *Assume* $0 < \nu < 1$ *and* $c(t) \le 0$. *Then the solutions of the equation* (5.5) *satisfy*

$$\lim_{t\to\infty} x(t) = 0.$$

# 4 Asymptotic behavior, delta case, $c(t) \ge b > 0$

In this section we will be concerned with the asymptotic behavior of solutions of the $\nu$-th order delta fractional difference equation

$$\Delta^{\nu}_{a+\nu-1}x(t) = c(t)x(t+\nu-1), \qquad t \in \mathbb{N}_a. \tag{4.1}$$

Let $\Gamma(x)$ denote the gamma function. Then we define the falling function (see [10]) by

$$t^{\underline{r}} := \frac{\Gamma(t+1)}{\Gamma(t+1-r)}$$

respectively, for those values of $t$ and $r$ such that the right-hand sides of these equations are well defined. We also use the standard extensions of their domains to define these functions to be zero when the numerators are well defined, but the denominator is not defined. The delta fractional Taylor monomial of degree $\nu$ based at $a$ (see [8]) is defined by

$$h_{\nu}(t,a) := \frac{(t-a)^{\underline{\nu}}}{\Gamma(\nu+1)}. \tag{4.2}$$

First we will establish a useful comparison theorem. The following lemma is from [8].

**Lemma 4.1.** *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$, *and* $\nu > 0$ *be given, with* $N-1 < \nu < N$. *Then*

$$\Delta^{\nu}_a f(t) = \int_a^{t+\nu+1} h_{-\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau,$$

*for* $t \in \mathbb{N}_{a+N-\nu}$.

The following comparison theorem plays an important role in proving our main result.

**Theorem 4.2.** *Assume $c_1(t) \geq c_2(t) \geq -\nu$, $0 < \nu < 1$, and $x(t)$, $y(t)$ are solutions of the equations*

$$\Delta^{\nu}_{a+\nu-1} x(t) = c_1(t) x(t+\nu-1), \tag{4.3}$$

*and*

$$\Delta^{\nu}_{a+\nu-1} y(t) = c_2(t) y(t+\nu-1), \tag{4.4}$$

*respectively, for $t \in \mathbb{N}_a$ satisfying $x(a+\nu-1) \geq y(a+\nu-1) > 0$. Then*

$$x(t) \geq y(t),$$

*for $t \in \mathbb{N}_{a+\nu-1}$.*

*Proof.* For simplicity, we let $a = 0$. From Lemma 4.1, we have for $t = (\nu-1)+1-\nu+k = k$, $k \geq 0$

$$\begin{aligned} \Delta^{\nu}_{\nu-1} x(t) &= \int_{\nu-1}^{t+\nu+1} h_{-\nu-1}(t, \sigma(s)) x(s) \Delta s \\ &= \sum_{s=\nu-1}^{\nu+k} h_{-\nu-1}(k, s+1) x(s) \\ &= x(\nu+k) - \nu x(\nu+k-1) - \frac{\nu(-\nu+1)}{2} x(\nu+k-2) \\ &\quad - \cdots - \frac{\nu(-\nu+1)\cdots(-\nu+k)}{(k+1)!} x(\nu-1). \end{aligned}$$

Using (4.3) and (4.4), we get that

$$\begin{aligned} x(\nu+k) = [\nu + c_1(k)] x(\nu+k-1) + \frac{\nu(-\nu+1)}{2} x(\nu+k-2) \\ + \cdots + \frac{\nu(-\nu+1)\cdots(-\nu+k)}{(k+1)!} x(\nu-1), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} y(\nu+k) = [\nu + c_2(k)] y(\nu+k-1) + \frac{\nu(-\nu+1)}{2} y(\nu+k-2) \\ + \cdots + \frac{\nu(-\nu+1)\cdots(-\nu+k)}{(k+1)!} y(\nu-1). \end{aligned} \tag{4.6}$$

We will prove $x(\nu+k-1) \geq y(\nu+k-1) \geq 0$ for $k \in \mathbb{N}_0$ by using the principle of strong induction. When $i = 0$, from the assumption, the result holds. Suppose that $x(\nu+i-1) \geq y(\nu+i-1) \geq 0$, for $i = 1, 2, \ldots, k-1$. Since

$$\frac{\nu(-\nu+1)\cdots(-\nu+i-1)}{i!} > 0$$

for $i = 2, 3, \ldots, k-1$, from (4.5), (4.6) and $c_2(t) \geq c_1(t) \geq -\nu$ we have

$$x(\nu+k) \geq y(\nu+k) \geq 0.$$

This completes the proof. □

The following theorem appears in [6] and [1, equation (3.7)].

**Theorem 4.3.** *Assume* $0 < \nu < 1$, $b$ *is a constant and* $a_0 \in \mathbb{R}$. *Then the IVP*

$$\Delta^{\nu}_{a+\nu-1} y(t) = b y(t+\nu-1), \qquad t \in \mathbb{N}_a, \tag{4.7}$$

$$y(a+\nu-1) = \Delta^{\nu-1}_{a+\nu-1} y(t)|_{t=a} = a_0, \tag{4.8}$$

*has a unique solution given by*

$$y(t) = a_0 \sum_{i=0}^{\infty} \frac{b^i}{\Gamma((i+1)\nu)} (t-a+i(\nu-1))^{\underline{i\nu+\nu-1}}, \tag{4.9}$$

*for* $t \in \mathbb{N}_{a+\nu-1}$.

Note that if we let $\nu = 1$ in Theorem 4.3 we get the known result that $y(t) = a_0 e_b(t,a)$ is the unique solution of the IVP

$$\Delta y(t) = b y(t), \qquad t \in \mathbb{N}_a$$
$$y(a) = a_0.$$

**Remark 4.4.** In [1], page 987, the "$i-1$" in equation (3.7) should be replaced by "$i$".

In the following corollary (see [6]) we give a simplification of the formula for the solution given in Theorem 4.3.

**Corollary 4.5.** *Assume* $0 < \nu < 1$, $b$ *is a constant and* $a_0 \in \mathbb{R}$. *Then the solution of the IVP* (4.7), (4.8) *is given by*

$$y(t) = a_0 \sum_{i=0}^{t-a-\nu+1} b^i h_{i\nu+\nu-1}(t, a-i(\nu-1)), \qquad t \in \mathbb{N}_{a+\nu-1}. \tag{4.10}$$

*Proof.* From Theorem 4.3 we have that the solution of the IVP (4.7), (4.8) is given by

$$\begin{aligned} y(t) &= a_0 \sum_{i=0}^{\infty} \frac{b^i}{\Gamma((i+1)\nu)} (t-a+i(\nu-1))^{\underline{i\nu+\nu-1}}, \\ &= a_0 \sum_{i=0}^{\infty} b^i h_{i\nu+\nu-1}(t, a-i(\nu-1)) \\ &= a_0 \sum_{i=0}^{t-a-\nu+1} b^i h_{i\nu+\nu-1}(t, a-i(\nu-1)) \\ &\quad + y(a+\nu-1) \sum_{i=t-a-\nu+2}^{\infty} b^i h_{i\nu+\nu-1}(t, a-i(\nu-1)) \\ &= a_0 \sum_{i=0}^{t-a-\nu+1} b^i h_{i\nu+\nu-1}(t, a-i(\nu-1)), \end{aligned}$$

since

$$\begin{aligned} h_{i\nu+\nu-1}(t, a-i(\nu-1)) &= \frac{(t-a+i(\nu-1))^{\underline{i\nu+\nu-1}}}{\Gamma((i+1)\nu)} \\ &= \frac{\Gamma(t-a+i(\nu-1)+1)}{\Gamma(t-a-i-\nu+2)\Gamma((i+1)\nu)} = 0 \end{aligned}$$

and since $i \geq t-a-\nu+2$ implies that the integer $t-a-i-\nu+2 \leq 0$ and the numerator in this last expression is well defined. □

**Theorem C.** *Assume $0 < b \le c(t)$, $0 < \nu < 1$ and $x(t)$ is the solution of the initial value problem*

$$\begin{aligned} &\Delta^{\nu}_{a+\nu-1}x(t) = c(t)x(t+\nu-1), \qquad t \in \mathbb{N}_a \\ &x(a+\nu-1) > 0. \end{aligned} \tag{4.11}$$

*Then*

$$\lim_{t\to\infty} x(t) = \infty.$$

*Proof.* For simplicity, we let $a = 0$. In (4.4), take $c_2(t) = b$, $y(\nu-1) = \frac{1}{2}x(\nu-1)$. Using (4.6). we have

$$\begin{aligned} y(\nu+k) = (\nu+b)y(\nu+k-1) + \frac{\nu(-\nu+1)}{2}y(\nu+k-2) \\ + \cdots + \frac{\nu(-\nu+1)\cdots(-\nu+k)}{(k+1)!}y(\nu-1). \end{aligned} \tag{4.12}$$

From the strong induction principle, it is easy to prove $y(\nu+k) > 0$. Similarly, from (4.5) we have also $x(\nu+k) > 0$, for $k \in \mathbb{N}_0$. Then $x(t)$ and

$$y(t) = \frac{x(\nu-1)}{2}\sum_{i=0}^{\infty} b^i h_{i\nu+\nu-1}(t, -i(\nu-1))$$

satisfy

$$\Delta^{\nu}_{\nu-1}x(t) = c(t)x(t+\nu-1), \tag{4.13}$$

and

$$\Delta^{\nu}_{\nu-1}y(t) = by(t+\nu-1), \tag{4.14}$$

respectively, for $t \in \mathbb{N}_{\nu-1}$ and

$$x(\nu-1) > \frac{x(\nu-1)}{2} = y(\nu-1) > 0.$$

From the comparison theorem (Theorem 4.2), we get that

$$x(t) \ge \frac{x(\nu-1)}{2}y(t),$$

for $t \in \mathbb{N}_{\nu-1}$. We now show that

$$\lim_{t\to\infty} y(t) = \infty.$$

Letting $t = k+\nu-1$, for fixed $i$, then when $k > i$, we have

$$\begin{aligned} h_{i\nu+\nu-1}(t, -i(\nu-1)) &= \frac{\Gamma(\nu(i+1)+k-i)}{\Gamma(k-i+1)\Gamma((i+1)\nu)} \\ &= \frac{(\nu(i+1)+k-i-1)(\nu(i+1)+k-i-2)\cdots(\nu(i+1))}{(k-i)!}. \end{aligned} \tag{4.15}$$

From Lemma 2.7, we have

$$\frac{1}{\Gamma(\nu(i+1))} = \lim_{k\to\infty} \frac{((\nu(i+1)+k-i-1)((\nu(i+1)+k-i-2)\cdots(\nu(i+1))}{(k-i-1)^{\nu(i+1)}(k-i-1)!},$$

for the real part of $\nu(i+1) > 0$. Using this formula for $\nu(i+1) > 0$ we have

$$\begin{aligned}&\lim_{k\to\infty} \frac{(\nu(i+1)+k-i-1)(\nu(i+1)+k-i-2)\cdots(\nu(i+1))}{(k-i)!} \\ &\quad = \lim_{k\to\infty} \frac{(\nu(i+1)+k-i-1)\cdots(\nu(i+1))}{(k-i-1)!(k-i-1)^{\nu(i+1)}} \cdot \frac{(k-i-1)^{\nu(i+1)}}{(k-i)}.\end{aligned} \tag{4.16}$$

Take $i$ sufficiently large, such that $\nu(i+1) > 1$. From (4.15) and (4.16), we get that when $\nu(i+1) > 1$,

$$\lim_{k\to\infty} h_{i\nu+\nu-1}(t, -i(\nu-1)) = \infty. \tag{4.17}$$

When $\nu(i+1) < 1$,

$$\lim_{k\to\infty} \frac{(k-i-1)^{\nu(i+1)}}{(k-i)} = 0. \tag{4.18}$$

When $\nu(i+1) = 1$,

$$\lim_{k\to\infty} \frac{(k-i-1)^{\nu(i+1)}}{(k-i)} = 1. \tag{4.19}$$

Note that there are only a finite number of $i$ which satisfy $\nu(i+1) \le 1$. So from (4.17), (4.18), (4.19), we get that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y(a+\nu-1) \sum_{i=0}^{\infty} b^i h_{i\nu+\nu-1}(t, -i(\nu-1)) = \infty.$$

Since $x(t) \ge y(t)$ we get the desired result $\lim_{t\to\infty} x(t) = \infty$ and the proof is complete. □

## 5 Asymptotic behavior, delta case, $-\nu < c(t) \le 0$

The following lemma appears in [1,8,9].

**Lemma 5.1.** *Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Then*

$$\Delta_{a+1-\nu}^{-\nu} \Delta_a^{\nu} f(t) = f(t) - h_{\nu-1}(t-1+\nu, a)\Delta_a^{-(1-\nu)} f(a+1-\nu), \tag{5.1}$$

*for $t \in \mathbb{N}_{a+1}$.*

**Theorem D.** *Assume $-\nu < c(t) \le 0$ and $0 < \nu < 1$. Then for all solutions $x(t)$ of the fractional equation*

$$\Delta_{a+\nu-1}^{\nu} y(t) = c(t) y(t+\nu-1), \qquad t \in \mathbb{N}_a \tag{5.2}$$

*satisfying $y(a+\nu-1) > 0$, we have*

$$\lim_{t\to\infty} y(t) = 0.$$

*Proof.* Assume $y(t)$ is as in the statement of this theorem. Then applying the operator $\Delta_{a+\nu-1+1-\nu}^{-\nu} = \Delta_a^{-\nu}$ to each side of (5.2) we obtain

$$\Delta_a^{-\nu} \Delta_{a+\nu-1}^{\nu} y(t) = \Delta_a^{-\nu} c(t) y(t+\nu-1)$$

Using Lemma 5.1, we have

$$y(t) - h_{\nu-1}(t-1+\nu, a+\nu-1)\Delta_{a+\nu-1}^{-(1-\nu)} y(a+\nu-1+1-\nu) = \Delta_a^{-\nu} c(t) y(t+\nu-1).$$

That is

$$y(t) - h_{\nu-1}(t-1+\nu, a+\nu-1)\Delta_{a+\nu-1}^{-(1-\nu)} y(a) = \Delta_a^{-\nu} c(t) y(t+\nu-1).$$

Then since

$$\begin{aligned}\Delta_{a+\nu-1}^{-(1-\nu)} y(t)|_{t=a} &= \int_{a+\nu-1}^{a-(1-\nu)+1} h_{(1-\nu)-1}(a, \sigma(s)) y(s) \Delta s \\ &= h_{-\nu}(a, a+\nu) y(a+\nu-1) \\ &= y(a+\nu-1),\end{aligned}$$

we have

$$y(t) - h_{\nu-1}(t-1+\nu, a+\nu-1) y(a+\nu-1) = \Delta_a^{-\nu} c(t) y(t+\nu-1).$$

So

$$y(t) = h_{\nu-1}(t-1+\nu, a+\nu-1) y(a+\nu-1) + \Delta_a^{-\nu} c(t) y(t+\nu-1), \tag{5.3}$$

$t \in \mathbb{N}_{a+\nu-1}$. Using (4.6), $c(t)+\nu > 0$, and $y(\nu-1) > 0$, we can apply the strong induction principle to get

$$y(t+\nu-1) > 0, \qquad t \in \mathbb{N}_0.$$

Now consider

$$\begin{aligned}\Delta_a^{-\nu} c(t) y(t+\nu-1) &= \int_a^{t-\nu+1} h_{\nu-1}(t, \tau+1) c(\tau) y(\tau+\nu-1) \Delta\tau \\ &= \sum_{\tau=a}^{t-\nu} \frac{\Gamma(t-\tau)}{\Gamma(t-\tau-\nu+1)\Gamma(\nu)} c(\tau) y(\tau+\nu-1).\end{aligned}$$

Since $\Gamma(t-\tau) \geq 0$, $\Gamma(t-\tau-\nu+1) \geq 0$, and $c(t) \leq 0$ we get

$$\Delta_a^{-\nu} c(t) y(t+\nu-1) \leq 0.$$

From this inequality and (5.3), we get

$$y(t) \leq h_{\nu-1}(t-1+\nu, a+\nu-1) y(a+\nu-1).$$

Taking $t = a+\nu-1+k$, $k \geq 0$ we have

$$\begin{aligned}0 < y(a+\nu-1+k) &\leq h_{\nu-1}(a+2\nu-1+k, a+\nu-1) y(a+\nu-1) \\ &= \frac{(\nu+k)^{\underline{\nu-1}}}{\Gamma(\nu)} y(a+\nu-1) \\ &= \frac{\Gamma(\nu+k+1)}{\Gamma(k+2)\Gamma(\nu)} y(a+\nu-1) \\ &= \frac{(\nu+k)(\nu+k-1)\cdots(\nu+1)\nu}{(k+1)!} y(a+\nu-1).\end{aligned} \tag{5.4}$$

From Lemma 2.7, we have that

$$\frac{1}{\Gamma(\nu)} = \lim_{k\to\infty} \frac{(\nu+k)(\nu+k-1)\cdots(\nu+1)\nu}{k!k^\nu},$$

for the real part $\nu > 0$. Using this formula for $0 < \nu < 1$, we have

$$\lim_{k\to\infty} \frac{(\nu+k)(\nu+k-1)\cdots(\nu+1)\nu}{(k+1)!} = \lim_{k\to\infty}\left[\frac{(\nu+k)(\nu+k-1)\cdots(\nu+1)\nu}{k!k^\nu} \cdot \frac{k^\nu}{k+1}\right] = 0.$$

Therefore from (5.4) we have

$$\lim_{k\to\infty} y(a+\nu-1+k) = 0.$$

This completes the proof. □

From Theorem 4.3 and Theorem D, we can obtain the following corollary.

**Corollary 5.2.** *Assume that* $-\nu < -b < 0$, $0 < \nu < 1$. *Then for* $t \in \mathbb{N}_{a+\nu-1}$, *we have*

$$\lim_{t\to\infty} \sum_{i=0}^{\infty} \frac{(-b)^i}{\Gamma((i+1)\nu)} (t-a+i(\nu-1))^{\underline{i\nu+\nu-1}} = 0.$$

Now we consider solutions of the following $\nu$-th order fractional delta equation

$$\Delta^{\nu}_{a+\nu-1} x(t) = c(t)x(t+\nu-1), \qquad t \in \mathbb{N}_a, \tag{5.5}$$

satisfying $y(a+\nu-1) < 0$. By making the transformation $x(t) = -y(t)$ and using Theorem C and Theorem D, we can get the following theorems.

**Theorem Ĉ.** *Assume* $0 < \nu < 1$ *and there exists a constant* $b$ *such that* $c(t) \geq b > 0$. *Then the solutions of the equation* (5.5) *satisfying* $x(a+\nu-1) < 0$, *satisfy*

$$\lim_{t\to\infty} x(t) = -\infty.$$

**Theorem D̂.** *Assume* $0 < \nu < 1$ *and* $-\nu \leq c(t) < 0$. *Then the solutions of the equation* (5.5) *satisfying* $x(a+\nu-1) < 0$, *satisfy*

$$\lim_{t\to\infty} x(t) = 0.$$

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