2019

Mathematical Analysis of Some Partial Differential Equations with Applications

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Mathematical Analysis of Some Partial Differential Equations with Applications

A Dissertation Presented

by

Kewang Chen

to

The Faculty of the Graduate College

of

The University of Vermont

In Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
Specializing in Mathematical Sciences

May, 2019

Defense Date: March 7th, 2019
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Abstract

In the first part of this dissertation, we produce and study a generalized mathematical model of solid combustion. Our generalized model encompasses two special cases from the literature: a case of negligible heat diffusion in the product, for example, when the burnt product is a foam-like substance; and another case in which diffusivities in the reactant and product are assumed equal. In addition to that, our model pinpoints the dynamics in a range of settings, in which the diffusivity ratio between the burned and unburned materials varies between 0 and 1. The dynamics of temperature distribution and interfacial front propagation in this generalized solid combustion model are studied through both asymptotic and numerical analyses. For asymptotic analysis, we first analyze the linear instability of a basic solution to the generalized model. We then focus on the weakly nonlinear case where a small perturbation of a neutrally stable parameter is taken so that the linearized problem is marginally unstable. Multiple scale expansion method is used to obtain an asymptotic solution for large time by modulating the most linearly unstable mode. On the other hand, we integrate numerically the exact problem by the Crank-Nicolson method. Since the numerical solutions are very sensitive to the derivative interfacial jump condition, we integrate the partial differential equation to obtain an integral-differential equation as an alternative condition. The result system of nonlinear algebraic equations is then solved by the Newton's method, taking advantage of the sparse structure of the Jacobian matrix. By a comparison of our asymptotic and numerical solutions, we show that our asymptotic solution captures the marginally unstable behaviors of the solution for a range of model parameters. Using the numerical solutions, we also delineate the role of the diffusivity ratio between the burned and unburned materials.

In the second part, we study the existence and decay rate of a transmission problem for the plate vibration equation with a memory condition on one part of the boundary. From the physical point of view, the memory effect described by our integral boundary condition can be caused by the interaction of our domain with another viscoelastic element on one part of the boundary. For our mathematical analysis, we first prove the global existence of weak solution by using Faedo-Galerkin's method and compactness arguments. Then, without imposing zero initial conditions on one part of the boundary, two explicit decay rate results are established under two different assumptions of the resolvent kernels. Both of these decay results allow a wider class of relaxation functions and initial data, and thus generalize some previous results existing in the literature.

I dedicate this dissertation to my Mom and Dad.

—K.C.

Acknowledgements

I would first like to thank my advisor Dr. Jun Yu for his guidance, support and everything that he have taught me while completing my graduate studies and conducting research at UVM. I also want to express my appreciation to Dr. Laura K. Gross and Dr. Wenjun Liu for helping me with my research and dissertation.

I am deeply grateful to the Gross family for their love, kindness and friendship. I am blessed to have had Ken, Mary Lou, Kathy, Laura, Karen, David, Matt, Jay, Gabriel and Adam come to my life while I am studying in the United State. Thank you all so much for all the great time.

My profound thanks go out to my Mom and Dad, my sister Huanhuan and my brother Kexing for their consistent support and encouragement. Thank you to Ryan, Sophie, Marcus, Zachary, Hunter, Anton, Tyson, Jeff, Phil, Jack, Francis, James, Tandin, Garvin, Ada, John, Chenyi, Xinkou, Jingyuan, Zhuang, Xiangxiao, Dawei, Jieyu, Shihui and Yichuan for all the good time and friendship. Thank you to all my old friends back in China for continued friendship.

I also owe a large debt of gratitude to the professors at Mathematics Department at the University of Vermont who funded me with a Graduate Teaching Assistantship. Lastly, I send heartfelt thanks to Dr. Danforth, Dr. Sands and Dr. Xia for providing me with their time, understanding and helpful suggestion while serving on my dissertation committee. Thank you all very much.

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PART I

GENERALIZED MODEL OF SOLID COMBUSTION

Chapter 1

INTRODUCTION

1.1 A Brief Historical Overview

Flame propagation is a very complex area of study which combines various physical and chemical disciplines. Recently, with the development of our technologies, more and more experiments were designed and carried out in different types of combustions. However, it is still extremely hard to mathematically describe these observed phenomena. The situation is relatively simpler for condensed phase or solid fuel combustion. For this kind of combustion, a chemical reaction converts a solid fuel directly into solid products without intermediate gas phase formation. In this case, the development and propagation of flame is mainly controlled by the competition between the heat generation by the chemical reactions and heat diffusion in burned and unburned regions, while the complicated gas and liquid dynamical issues can be essentially ignored. A balance exists between the two in some parametric regimes, producing a constant burning rate. In most cases, competition between the reaction and the diffusion results in rich nonuniform behaviors, some leading to chaos.

This condensed phase combustion is a basis for many technological applications. For example, in self-propagating high-temperature synthesis (SHS), a flame wave advances through powdered ingredients, leaving high-quality ceramic materials or metallic alloys in its wake (See, for instance, [16], [38], [40], [61]). The SHS process has many advantages over traditional manufacturing in which mixture is baked in a furnace. The synthesis times are much shorter, the equipment is cheaper and costs less energy. Most importantly, the desired products are more uniform and pure. The SHS technique was first proposed and studied in Soviet Union in late 1960's and early 1970's. Later, these extensive Russian-language literature were translated and collated by Frankhauser and many other researchers. Since then, SHS started to attract researchers' attention in Japan, the USA and elsewhere in the world. For more detailed history of SHS, we refer the readers to [54] and the reference therein. Today, SHS process still remains its popularity in producing advanced materials. Other frontal phonemena include, for example, frontal polymerization [12], [13], [14] and the combustion of degraded energetic materials involves multi-phase flow because of the porosity in the propellant. (See, for instance, [8], [60].)

Experiments have promoted an understanding of the kinds of reaction propagation that result from the interplay between heat generation and heat diffusion in the medium. In some physical contexts, these effects balance to produce a constant burning rate, leading to a more uniform composition of the product. In other cases, the interactions between the reaction and diffusion result in a wide variety of nonuniform behaviors, including chaos (see, for example, [17]). In media containing interstices between explosive particles, convection can play an important and in some cases a dominant role [8].

Numerical simulations have also captured these dynamics. Models have included reaction-diffusion with full Arrhenius kinetics [58], Arrhenius kinetics with a cutoff [5], and concentrated kinetics with large activation energy [9], [33]. See [60], [64] for a review of models and physical experiments.

In this dissertation, we address a broad class of physical problems via a generalized model of solid combustion, encompassing both a case of negligible heat diffusion in the product, for example when the burnt product is a foam-like substance, and a case in which diffusivities in the reactant and product are assumed equal. A free-boundary ("one-sided") model had been applied to the former case and a free-interface ("two-sided") model to the latter. (See, for examples, [24] and [64].) We note that effects of variable material parameters are also analyzed in oscillatory burning in [8], [60] in the context of a solid reactant with pores.

We give a linear stability analysis, carry out a multiple scale expansion and perform simulations on the generalized model, which pinpoint the dynamics in a range of settings. Our numerical study quantitatively predicts the behavior of exothermic reaction fronts in this spectrum of material contexts. The dynamics involve an interplay of competing effects as the diffusivity ratio is tuned to capture different physical systems.

In particular, we take $a_p = \kappa_b/\kappa_u$ as the ratio of the diffusivities κ_b and κ_u in the burned and unburned regions, respectively. The dimensionless parameter, a_p tunes the thermal diffusivity in the product zone from zero—no heat conduction (the one-sided model)—to a maximum of $a_p = 1$, where the diffusivities are the same in the product and fresh mixtures (the two-sided model).

The quality of the product, which is associated with the stability of the uniformly propagating combustion front, has a significant dependence on the diffusivity ratio a_p. In the linear stability analysis we identify the critical value ν_c of a parameter ν related to the activation energy at which a travelling-wave solution is neutrally stable. Our result gives the precise critical value ν_c for the range of physical settings captured by the diffusivity ratio; it depends on a_p. Prediction of loss of stability at this level of detail is valuable; the relative error between the stability thresholds associated with the one- and two-sided models is approximately 30 percent (see chapter two).

We also identify nonlinear instability using Fourier spectra of numerical solutions to show how period-doubling bifurcations arise, as a parameter σ related to the Arrhenius kinetics decreases. We illustrate that to see a period-doubling cascade in σ in the two-sided model in [64] , the parameter ν must deviate by a somewhat larger amount, denoted ϵ^2, from its critical value than is required in the case of the free-boundary model. The nonlinear dynamical behaviors in these two settings are qualitatively similar. The generalized model gives a clear understanding of the quantitative details for how nonlinearities set in for the "intermediate" physical problems.

1.2 Derivation of Mathematical Model

Matkowsky and Sivashinsky [33] consider a system of reaction-diffusion equations

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} + QW(C,T), \quad \frac{\partial C}{\partial t} = -W(C,T) \tag{1.2.1}$$

as a mathematical model of gasless combustion. Here T is the temperature. The fuel concentration C evolves from C_0 in the fresh mixture to zero as the reaction proceeds to completion. The parameter κ is the heat diffusivity; $W > 0$ is the chemical reaction rate; and Q is the heat release.

We assume that the reaction is one-step. It is also first-order: proportional to the concentration. In particular, $W = ZCw(T)$, where Z is a constant, and $w(T)$ is a temperature-dependent factor. We also assume that the mixture does not burn without sufficient heating, namely that at "the cold boundary" $w(T_0) = 0$, where T_0 is the temperature of the fresh mixture. Later in this dissertation, we will show that, under these (as well as under much weaker) assumptions, the system in (1.2.1) supports a traveling wave solution.

We will give the solution explicitly in the case of a free boundary $x = g(t)$, where we effectively replace the reaction rate by a point source, treating W as a δ-function. In particular, we take the reaction rate

$$W = K(T)\delta(x - g(t)).$$

Then the system in (1.2.1) becomes

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} + QK(T)\delta(x - g(t)), \quad \frac{\partial C}{\partial t} = -K(T)\delta(x - g(t)). \tag{1.2.2}$$

As usual, the system in (1.2.2) should be satisfied in the sense of distributions; i.e., the equations are satisfied when sampled with any smooth test function. The boundary conditions far ahead and far behind the front are

$$\lim_{x\to-\infty} |T(x,t)| < \infty, \quad \lim_{x\to\infty} T(x,t) = T_0, \tag{1.2.3}$$

$$\lim_{x\to-\infty} C(x,t) = 0, \quad \lim_{x\to\infty} C(x,t) = C_0.$$

The homogeneous version of the partial differential equation in T in (1.2.2) holds on each side of the interface as

$$\frac{\partial T}{\partial t} = \kappa_b \frac{\partial^2 T}{\partial x^2} \text{ on } x < g(t); \quad \frac{\partial T}{\partial t} = \kappa_u \frac{\partial^2 T}{\partial x^2} \text{ on } x > g(t), \tag{1.2.4}$$

subject to the following jump conditions

$$[T] = 0, \quad \left[\kappa \frac{\partial T}{\partial x}\right] = -QK, \quad \frac{\mathrm{d}g}{\mathrm{d}t}\,[C] = K(T_f) \tag{1.2.5}$$

obtained by integrating equations (1.2.2) across the interface with respect to x. Here κ_u is the diffusivity in the unburned region; κ_b is the diffusivity in the burned region; the square brackets indicate a jump across the interface; and $T_f = T(g(t);t)$ is the front temperature.

As such, the first condition in (1.2.5) is the continuity of T across the front:

$$T|_{\Gamma^+} = T|_{\Gamma^-} = T_f \tag{1.2.6}$$

and the second condition is

$$\left(\kappa \frac{\partial T}{\partial x}\right)\Bigg|_{\Gamma^+} - \left(\kappa \frac{\partial T}{\partial x}\right)\Bigg|_{\Gamma^-} = -QK(T_f),$$

where Γ is the interface given by $x = g(t)$. In particular,

$$\left(\kappa_u \frac{\partial T}{\partial x}\right)\Bigg|_{\Gamma^+} - \left(\kappa_b \frac{\partial T}{\partial x}\right)\Bigg|_{\Gamma^-} = -QK(T_f). \tag{1.2.7}$$

Initial data may be chosen so that the concentration is given by

$$C(x,t) = \begin{cases} 0 & \text{for } x < g(t), \\ C_0 & \text{for } x > g(t). \end{cases}$$

Then the concentration equation $C_t = 0$ is satisfied automatically for both $x < g(t)$ and $x > g(t)$. Thus, the concentration variable can be eliminated from the problem. In addition, $[C] = C_0$ and the last jump condition in (1.2.5) becomes

$$C_0 \frac{\mathrm{d}g}{\mathrm{d}t} = K(T_f). \tag{1.2.8}$$

The form of $K(T_f)$ depends on the details of the kinetic mechanism of reaction. We use Arrhenius kinetics modified for gasless combustion as in [9]:

$$K(T_f) = UC_0 \exp\left[\frac{T_a}{2T_b}\left(1 - \frac{T_b}{T_f}\right)\right], \tag{1.2.9}$$

where U is the traveling-wave velocity; T_a is the activation temperature;

$$T_b = T_0 + QC_0 \tag{1.2.10}$$

is the adiabatic temperature of the combustion products.

Next, we make a change of variables to introduce the dimensionless quantities

$$\tilde{x} = \frac{U}{\kappa_u}x, \quad \tilde{t} = \frac{U^2}{\kappa_u}t, \quad u = \frac{T - T_0}{T_b - T_0}, \quad V = \frac{1}{U}\frac{\mathrm{d}g}{\mathrm{d}t}. \tag{1.2.11}$$

Note that $\mathrm{d}g/\mathrm{d}t$ can be replaced by UV in (1.2.8) and the resulting expression for

$K(T_f)$ can be substituted into (1.2.7) and (1.2.9). Then nondimensionalizing equations (1.2.3), (1.2.4), (1.2.6), (1.2.7) and (1.2.9), gives a system for the dimensionless temperature u and dimensionless front position $f(\tilde{t}) = \frac{U}{\kappa_u} g(t(\tilde{t}))$. Dropping the tildes we get, for $u(x,t)$, $f(t)$, $t > 0$:

$$\frac{\partial u}{\partial t} = a_p \frac{\partial^2 u}{\partial x^2} \text{ on } x < f(t), \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ on } x > f(t), \tag{1.2.12}$$

$$\lim_{x \to -\infty} |u(x,t)| < \infty, \quad \lim_{x \to \infty} u(x,t) = 0, \tag{1.2.13}$$

$$u|_{\Gamma} = u|_{\Gamma^-} = u|_{\Gamma^+} \tag{1.2.14}$$

$$V = \exp\left[\left(\frac{1}{\nu}\right) \frac{u|_{\Gamma} - 1}{\sigma + (1-\sigma)\, u|_{\Gamma}}\right], \tag{1.2.15}$$

$$\left.\frac{\partial u}{\partial x}\right|_{\Gamma^+} - a_p \left.\frac{\partial u}{\partial x}\right|_{\Gamma^-} = -V, \tag{1.2.16}$$

where $V = \frac{\mathrm{d}f}{\mathrm{d}\tilde{t}}$; σ is the temperature ratio

$$\sigma = \frac{T_0}{T_b};$$

ν is the inverse Zel'dovich number, where

$$\frac{1}{\nu} = \frac{T_a}{2T_b}(1-\sigma);$$

and a_p is the ratio of diffusitivies in the product and fresh mixtures

$$a_p = \frac{\kappa_b}{\kappa_u}.$$

Lastly, the problem (1.2.12)–(1.2.16) in a front-attached coordinate frame

$$\eta = x - f\,(t)\,, \quad \tau = t$$

becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2} + \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}\frac{\partial u}{\partial \eta}, \quad \eta > 0, \qquad \frac{\partial u}{\partial \tau} = a_p \frac{\partial^2 u}{\partial \eta^2} + \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}\frac{\partial u}{\partial \eta}, \quad \eta < 0, \tag{1.2.17}$$

$$u|_{\eta=0^-} = u|_{\eta=0^+} = u|_{\eta=0}\,, \tag{1.2.18}$$

$$V = \exp\left[\left(\frac{1}{\nu}\right)\frac{u|_{\eta=0} - 1}{\sigma + (1-\sigma)\,u|_{\eta=0}}\right], \tag{1.2.19}$$

$$\left.\frac{\partial u}{\partial \eta}\right|_{\eta=0^+} - a_p \left.\frac{\partial u}{\partial \eta}\right|_{\eta=0^-} = -V, \tag{1.2.20}$$

$$\lim_{\eta\to-\infty} |u(\eta,\tau)| < \infty, \quad \lim_{\eta\to\infty} u(\eta,\tau) = 0. \tag{1.2.21}$$

Equations (1.2.17)-(1.2.21), together with initial conditions for the temperature and the free interface position, form an initial-boundary value problem. Examples of these initial conditions are given when we solve the problem numerically.

Chapter 2

LINEAR STABILITY ANALYSIS

2.1 Basic Solution and Linearized Equations

We start our analysis by noticing that a basic solution to equations (1.2.17)-(1.2.21) is

$$u_{basic}(\eta,\tau)=\begin{cases} e^{-\eta} & \text{if } \eta\geq 0 \\ 1 & \text{if } \eta<0 \end{cases}, \quad f_{basic}(\tau)=\tau. \tag{2.1.1}$$

And the corresponding basic velocity is

$$V_{basic}=1.$$

We now wish to study the form of small perturbation about the above basic solution. First, we introduce a small, positive, non-dimensional parameter $\epsilon \ll 1$. Then to linearize the problem about the basic solution, we make the following substitution

$$u = u_{basic} + \epsilon w,$$

$$f = f_{basic} + \epsilon \phi$$

for u and f in (1.2.17)-(1.2.21). We also expand $K(V)$ in a Taylor series about $V = 1$, and recall that $K(1) = 0$ and $K'(1) = 1$. Moving all terms to one side of the equation and retaining only terms at order of $O(\epsilon)$, we obtain the following partial differential equations

$$\frac{\partial w}{\partial \tau} + \mathcal{L}(w, \phi) = 0$$

subject to the linear boundary conditions

$$\mathcal{M}(w, \phi) = 0,$$

$$\mathcal{N}(w, \phi) = 0,$$

$$w|_{\eta=0^+} = w|_{\eta=0^-} = w|_{\eta=0},$$

$$\lim_{\eta \to -\infty} |w(\eta, \tau)| < \infty, \ \lim_{\eta \to \infty} |w(\eta, \tau)| = 0,$$

where

$$\mathcal{L}(w, \phi) = \begin{cases} \mathcal{L}_1(w, \phi) = -\dfrac{\partial^2 w}{\partial \eta^2} - \dfrac{\partial w}{\partial \eta} + e^{-\eta} \dfrac{\partial \phi}{\partial \tau}, & \text{if} \ \ \eta > 0, \\ \mathcal{L}_2(w, \phi) = -a_p \dfrac{\partial^2 w}{\partial \eta^2} - \dfrac{\partial w}{\partial \eta}, & \text{if} \ \ \eta < 0, \end{cases} \tag{2.1.2}$$

$$\mathcal{M}(w, \phi) = w|_{\eta=0} - \nu \frac{\partial \phi}{\partial \tau}, \tag{2.1.3}$$

$$\mathcal{N}(w, \phi) = \frac{\partial w}{\partial \eta}\bigg|_{\eta=0^+} - a_p \frac{\partial w}{\partial \eta}\bigg|_{\eta=0^-} + \frac{\partial \phi}{\partial \tau}. \tag{2.1.4}$$

We note here that the above perturbation expansion can be generalized to higher orders later in our weakly nonlinear analysis.

2.2 Eigenvalue Problem

In this section, we derive the normal-mode solutions using a separation of variables for the equations (1.2.17)–(1.2.21). Let us assume

$$w = e^{\lambda\tau} g(\eta) = \begin{cases} e^{\lambda\tau} g_1(\eta), & \text{if } \ \eta > 0 \\ e^{\lambda\tau} g_2(\eta), & \text{if } \ \eta < 0 \end{cases}, \quad \phi = e^{\lambda\tau}. \tag{2.2.1}$$

Substituting (2.2.1) into the linearized problem in previous section gives the eigenvalue problem

$$g_1'' + g_1' - \lambda g_1 = \lambda e^{-\eta}, \ \eta > 0, \tag{2.2.2}$$

$$a_p g_2'' + g_2' - \lambda g_2 = 0, \ \eta < 0, \tag{2.2.3}$$

$$g_1(0) = g_2(0) = \nu\lambda, \tag{2.2.4}$$

$$g_1'|_{\eta=0^+} - a_p g_2'|_{\eta=0^-} = -\lambda, \tag{2.2.5}$$

$$\lim_{\eta \to -\infty} g_2(\eta) < \infty, \ \lim_{\eta \to \infty} g_1(\eta) = 0. \tag{2.2.6}$$

The general solution to the differential equations (2.2.2–2.2.3) is

$$g(\eta) = \begin{cases} g_1(\eta) = c_1 \exp\left(\frac{-1+\sqrt{1+4\lambda}}{2}\eta\right) + c_2 \exp\left(\frac{-1-\sqrt{1+4\lambda}}{2}\eta\right) - \exp(-\eta), \ \eta > 0, \\ g_2(\eta) k_1 \exp\left(\frac{-1+\sqrt{1+4a_p\lambda}}{2a_p}\eta\right) + k_2 \exp\left(\frac{-1-\sqrt{1+4a_p\lambda}}{2a_p}\eta\right) \ , \ \eta < 0. \end{cases} \tag{2.2.7}$$

2.3 Discrete Spectrum

Now the general solution (2.2.7) needs to satisfy the boundary conditions (2.2.4)–(2.2.6) of the eigenvalue problem. We first select $c_1 = k_2 = 0$, which implies that the condition (2.2.6) at infinity holds. Next, condition (2.2.4) dictates that $c_2 = 1 + \nu\lambda$ and $k_1 = \nu\lambda$. Lastly, applying (2.2.5) yields the dispersion relation

$$f(a_p, \nu, \lambda) = 0, \tag{2.3.1}$$

where

$$f(a_p, \nu, \lambda) = \nu\,(2\lambda + 1)\sqrt{1 + 4a_p\lambda} + 2\nu^2\,(1 - a_p)\,\lambda^2 + (4\nu - 2)\,\lambda + \nu.$$

Thus the eigenfunction is

$$g(\eta) = \begin{cases} g_1(\eta) = (1 + \nu\lambda) \exp\left(\frac{-1-\sqrt{1+4\lambda}}{2}\eta\right) - \exp(-\eta), \ \eta > 0, \\ g_2(\eta) = \nu\lambda \exp\left(\frac{-1+\sqrt{1+4a_p\lambda}}{2a_p}\eta\right), \ \eta < 0. \end{cases} \tag{2.3.2}$$

where the eigenvalue λ satisfies the dispersion relation (2.3.1).

If $\Re(\lambda) = 0$, then the basic solution u_{basic} and f_{basic} in (2.1.1) (also known as the traveling-wave solution) is neutrally stable with respect to the small temperature perturbation ϵw and small front perturbation $\epsilon\phi$. Here, w and ϕ are normal modes of the forms (2.2.1). To examine neutral stability, we set λ to be the purely imaginary value $i\omega$ in (2.3.1). The complex equation (2.3.1) is equivalent to two real equations. These can be solved simultaneously for ω and the critical value

$$\nu_c = \nu_c(a_p).$$

Notice that the critical value, ν_c, is a function of the diffusivity ratio, a_p. Therefore, this is the value below which the travelling-wave solution loses stability for the range of physical setting captured by a_p. See Figure 2.1.

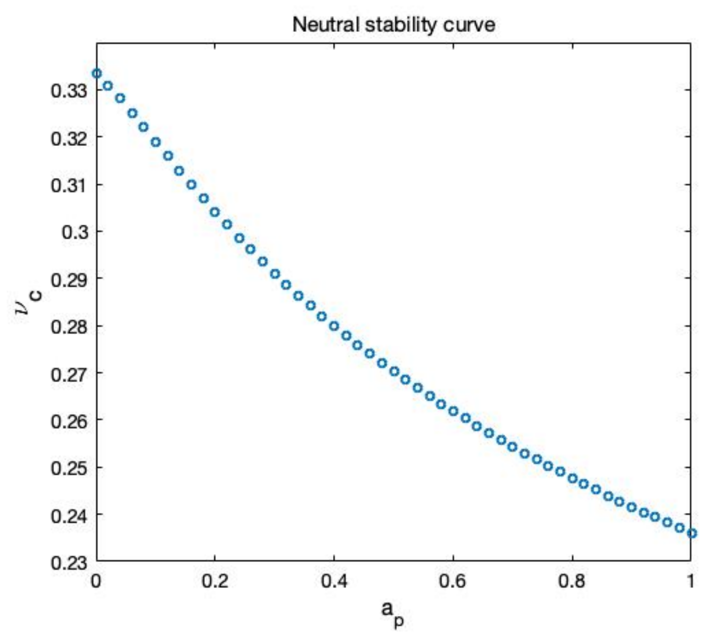

Figure 2.1: Neutral stability curve in the $a_p - \nu_c$ plane.

In the case of negligible heat diffusion behind the front, we set $a_p = 0$ and find

$$\nu_c = \nu_c(0) = 1/3.$$

This is the case modeled on a semi-infinite domain with the one-sided model.

In the case of essentially uniform heat diffusion behind and ahead of the front, we set $a_p = 1$ and find

$$\nu_c = \nu_c(1) = \sqrt{5} - 2.$$

This is the case modeled on the whole real line with the two-sided model. These two values for limiting cases agree with stability thresholds reported in the literature, for examples, [15], [19] and [66].

For these two physical settings, the two critical values differ considerably. For example, if we used the two-sided model where the one-sided model would be more appropriate, the relative error would be

$$\frac{\nu_c(0) - \nu_c(1)}{\nu_c(0)} \approx 0.29.$$

If we used the one-sided model where the two-sided model would be more appropriate, the relative error would be

$$\frac{\nu_c(0) - \nu_c(1)}{\nu_c(1)} \approx 0.41.$$

Rather than choosing $a_p = 0$ or $a_p = 1$ in all cases, we get a substantially more accurate stability threshold by using the value of the diffusivity ratio a_p on the interval $(0, 1)$, as appropriate to the physical setting.

If $\nu < \nu_c(a_p)$, then $\Re\lambda > 0$, and the basic solution in (2.1.1) is linearly unstable. As such, the solution to the linearized equations grows exponentially in time. However, the nonlinear solution is bounded. (See [19].)

In this work, we perform both asymptotic and numerical analysis of the complex nonlinear dynamics for the generalized model. We take

$$\nu = \nu_c(a_p) - \epsilon^2,$$

which is used in our weakly nonlinear analysis. We show in Section (4.5) that in the appropriate regimes, the asymptotic results agree well with the numerical solutions. Moreover, to obtain our numerical solutions, we use the Crank-Nicholson method in Section (4.2) to solve the generalized model in a front-attached coordinate frame.

Chapter 3

WEAKLY NONLINEAR ANALYSIS

3.1 Introduction

After obtaining normal mode solution and the condition for neutral stability in the previous chapter, we are now ready to proceed with the weakly nonlinear analysis. Our goal is to study the evolution of perturbations to the basic solution in the case where $\nu(a_p)$ lies in the weakly unstable region (i.e. set $\nu(a_p) = \nu_c(a_p) - \epsilon^2$). The evolution of a weakly unstable mode in this parameter regime might be modeled well by the neutrally stable linear solution with modulated amplitudes. To seek such a solution which can model the behavior of a perturbation to the basic solution for large times, we employ the method of multiple scale expansion in this chapter. After introducing the multiple time scales: $t_0 = \tau,\ t_1 = \epsilon\tau,\ t_2 = \epsilon^2\tau$, we assume the amplitudes of the modulated eigenmodes depend on t_1 and t_2 (see $A(t_1, t_2), B(t_1, t_2)$ in the following section). Then we show that these amplitude functions $A(t_1, t_2)$ and $B(t_1, t_2)$ can be obtained by satisfying the solvability conditions to our original equations (1.2.17)-(1.2.21). In the following sections, we provide the details, including adjoint eigenvalue problem, introduction of small parameter and the method of multiple scales.

3.2 Adjoint Eigenvalue Problem

Here we define inner product and then use the integration by parts to derive an adjoint eigenvalue problem. The solutions of the adjoint problem will be used later to ensure the solvability of the inhomogeneous equations when applying the perturbation theory. First, the inner product of two functions is defined as

$$(u, v) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{-\infty}^{+\infty} u(\eta, \tau)\overline{v(\eta, \tau)}\, d\eta d\tau,$$

where u and v are bounded measurable functions of η and τ with values in

$$L^2\left((-\infty, +\infty) \times (0, +\infty)\right).$$

Definition 3.2.1 *A function u is in the null space of the adjoint operator if*

$$\left(\frac{\partial v}{\partial t} + \mathcal{L}(v, \phi), u\right) = 0$$

for all v such that

$$\mathcal{M}(v, \phi) = v|_{\eta=0} - \nu \frac{\partial \phi}{\partial \tau} = 0, \tag{3.2.1}$$

$$\mathcal{N}(v, \phi) = \left.\frac{\partial v}{\partial \eta}\right|_{\eta=0^+} - a_p \left.\frac{\partial v}{\partial \eta}\right|_{\eta=0^-} + \frac{\partial \phi}{\partial \tau} = 0. \tag{3.2.2}$$

for $\mathcal{L}$ defined as in §2.

Integration by parts shows that the inner product in Definition 3.2.1 can be written as

$$\begin{aligned}\left(\frac{\partial v}{\partial t}+\mathcal{L}(v,\phi),u\right) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\Big\{-\int_{0^+}^{+\infty}\Big[v(\bar{u}_\tau+\bar{u}_{\eta\eta}-\bar{u}_\eta)-e^{-\eta}\phi_\tau\bar{u}\Big]\,d\eta \\ &\quad +v_\eta\bar{u}\mid_{\eta=0^+}-v\bar{u}_\eta\mid_{\eta=0^+}+v\bar{u}\mid_{\eta=0^+}\Big\}\,d\tau. \\ &\quad +\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big\{-\int_{-\infty}^{0^-}[v(\bar{u}_\tau+a_p\bar{u}_{\eta\eta}-\bar{u}_\eta)]\,d\eta \\ &\quad -a_pv_\eta\bar{u}\mid_{\eta=0^-}+a_pv\bar{u}_\eta\mid_{\eta=0^-}-v\bar{u}\mid_{\eta=0^-}\Big\}\,d\tau. \\ &\quad +\text{bounday term.}\end{aligned}$$

The boundary terms vanish as T approaches infinity. Using the definition of the inner product and the conditions (3.2.1) and (3.2.2), the above equation becomes

$$\left(\frac{\partial v}{\partial t}+\mathcal{L}(v,\phi),u\right) = \left(v,-\frac{\partial u}{\partial t}+\mathcal{L}^\star u\right)+\lim_{T\to\infty}\frac{1}{T}\mathcal{B}(\phi_\tau,a_p,u,u_\eta), \quad (3.2.3)$$

where

$$\mathcal{L}^\star u=\begin{cases}\mathcal{L}_1^\star u=-\dfrac{\partial^2 u}{\partial\eta^2}+\dfrac{\partial u}{\partial\eta}, & \text{if } \eta>0\\ \mathcal{L}_2^\star u=-a_p\dfrac{\partial^2 u}{\partial\eta^2}+\dfrac{\partial u}{\partial\eta}, & \text{if } \eta<0\end{cases}$$

and

$$\mathcal{B}(\phi_\tau,a_p,u,u_\eta) = \int_0^T\left[\int_{0^+}^\infty e^{-\eta}\phi_\tau\bar{u}\,d\eta-\phi_\tau\bar{u}|_{\eta=0^+}-\nu\phi_\tau\bar{u}_\eta|_{\eta=0^+}+a_p\nu\phi_\tau\,\bar{u}|_{\eta=0^-}\right]d\tau.$$

The operator $-\partial_t + \mathcal{L}^\star$ is adjoint to $\partial_t + \mathcal{L}$ if its domain is defined by the relation

$$\lim_{T\to\infty} \frac{1}{T}\mathcal{B}(\phi_\tau, a_p, u, u_\eta) = 0. \tag{3.2.4}$$

Only the null space of the adjoint operator is of interest for obtaining solvability conditions. For functions with separated variables

$$u = e^{\mu\tau} g(\eta) = \begin{cases} e^{\mu\tau} h_1(\eta), & \text{if } \eta > 0 \\ e^{\mu\tau} h_2(\eta), & \text{if } \eta < 0 \end{cases} \tag{3.2.5}$$

that belong to the null space of the adjoint operator, it is possible to translate (3.2.4) into a boundary condition. To do so, we substitute u into the expression for $\mathcal{B}$. Using the fact that every term has a ϕ_τ, we integrate by parts with respect to τ to obtain

$$\mathcal{B}(\phi_\tau, u, u_\eta) \;=\; B(h_1, h_2, \mu, a_p) \int_0^T \phi(\tau) e^{\bar{\mu}\tau}\, d\tau + \text{boundary terms},$$

where

$$\begin{aligned} & B(h_1, h_2, \mu, a_p) \\ = \; & \left[\bar{\mu}\bar{h}_1(0^+) + \nu\bar{\mu}\bar{h}_1'(0^+) - a_p\nu\bar{\mu}\bar{h}_2'(0^-) - \int_{0^+}^{+\infty} \bar{\mu}\bar{h}_1(\eta) e^{-\eta}\, d\eta\right]. \end{aligned} \tag{3.2.6}$$

Since we require $B(h_1, h_2, \mu, a_p)$ to equal zero, (3.2.3) shows that Definition (3.2.1) can be restated as: A function u is in the null space of the adjoint operator if and only if

$$\left(v, -\frac{\partial u}{\partial t} + \mathcal{L}^\star u\right) = 0$$

for all v satisfying the boundary conditions in Definition 3.2.1. Therefore u is in the null space of the adjoint operator if and only if

$$-\frac{\partial u}{\partial t} + \mathcal{L}^{\star} u = 0$$

for all v satisfying the boundary conditions. Substituting

$$u(\eta, \tau) = e^{\mu\tau} h(\eta) = \begin{cases} e^{\mu\tau} h_1(\eta), & \text{if } \eta > 0 \\ e^{\mu\tau} h_2(\eta), & \text{if } \eta < 0 \end{cases} \tag{3.2.7}$$

into the above equation gives

$$h_1'' - h_1' + \mu h_1 = 0, \tag{3.2.8}$$

$$a_p h_2'' - h_2' + \mu h_2 = 0. \tag{3.2.9}$$

To evaluate the integral in (3.2.6), we perform inner product of $e^{-\eta}$ with equation (3.2.8). Using the integration by parts and requiring that $h_1(\eta)$ and $h_1'(\eta)$ tend to zero as η tends to infinity, we obtain

$$\int_{0^+}^{\infty} \bar{\mu}\bar{h}(\eta) e^{-\eta}\, d\eta = -\bar{h}_1'(0^+)$$

and equation (3.2.6) becomes

$$\left[\bar{\mu}\bar{h}_1(0^+) + \nu\bar{\mu}\bar{h}_1'(0^+) - a_p\nu\bar{\mu}\bar{h}_2'(0^-) - \bar{h}_1'(0^+)\right] = 0. \tag{3.2.10}$$

This condition, together with the differential equation (3.2.8), (3.2.9) and the requirement that

$$\lim_{\eta \to -\infty} h_2(\eta) < \infty, \quad \lim_{\eta \to \infty} h_1(\eta) = 0, \tag{3.2.11}$$

constitute the adjoint eigenvalue problem. We look for $h(\eta)$ in the form

$$h(\eta) = \begin{cases} h_1(\eta) = Ae^{\lambda^+ \eta}, & \text{if} \quad \eta > 0, \\ h_2(\eta) = Ae^{\lambda^- \eta}, & \text{if} \quad \eta < 0, \end{cases} \tag{3.2.12}$$

where A is an arbitrary constant. The condition

$$h(\eta)|_{\eta=0^-}^{\eta=0^+} = h_1(0^+) - h_2(0^-) = 0,$$

is automatically satisfied. Upon substitution of (3.2.12) into the ordinary differential equations (3.2.8) and (3.2.9), we find that λ^+ and λ^- satisfy the equations

$$(\lambda^+)^2 - \lambda^+ + \mu\lambda^+ = 0, \tag{3.2.13}$$

$$a_p(\lambda^-)^2 - \lambda^- + \mu\lambda^- = 0. \tag{3.2.14}$$

We choose the following roots from the above two equations

$$\lambda^+ = \frac{1 - \sqrt{1 - 4\mu}}{2}, \tag{3.2.15}$$

$$\lambda^- = \frac{1 + \sqrt{1 - 4a_p\mu}}{2a_p}, \tag{3.2.16}$$

by enforcing boundedness of solution (3.2.12) as $\eta \to \pm\infty$. If we choose $\mu = -\bar{\lambda}$, i.e.

$\bar{\mu} = -\lambda$, then we obtain from (3.2.10) that

$$-\lambda - \nu\lambda \frac{1-\sqrt{1+4\lambda}}{2} + a_p \nu\lambda \frac{1+\sqrt{1+4a_p\lambda}}{2a_p} - \frac{1-\sqrt{1+4\lambda}}{2} = 0.$$

After simplifying the above equation we end up with

$$\nu\left(2\lambda+1\right)\sqrt{1+4a_p\lambda} + 2\nu^2\left(1-a_p\right)\lambda^2 + \left(4\nu-2\right)\lambda + \nu = 0.$$

which is the same dispersion relation (2.3.1) given in the linear stability analysis of the original linearized problem! Therefore, from (3.2.7) we find that the solutions to the adjoint problem is

$$u(\eta,\tau) = e^{\mu\tau}h(\eta) = \begin{cases} e^{-\bar{\lambda}\tau}h_1(\eta), & \text{if } \eta > 0, \\ e^{-\bar{\lambda}\tau}h_2(\eta), & \text{if } \eta < 0, \end{cases} \tag{3.2.17}$$

where $h_1(x)$ and $h_2(x)$ are given in (3.2.12), (3.2.15) and (3.2.16). Finally, we note that

$$\mu = 0, \;\; h(\eta) = \begin{cases} 1, & \text{if } \eta > 0 \\ e^{\frac{1}{a_p}\eta}, & \text{if } \eta < 0 \end{cases} \tag{3.2.18}$$

also satisfies the adjoint eigenvalue problem (3.2.8)-(3.2.10). Thus

$$u(\eta,\tau) = \begin{cases} 1, & \text{if } \eta > 0 \\ e^{\frac{1}{a_p}\eta}, & \text{if } \eta < 0 \end{cases} \tag{3.2.19}$$

is also an adjoint solution.

Lastly, we give an important lemma which will be used later.

Lemma 3.2.1 *(Fredholm Alternative Theorem) Let X and Y be normed linear spaces and $\mathscr{B}(X,Y)$ be the space of bounded linear operators from X to Y with the usual operator norm. Then for $\mathcal{T} \in \mathscr{B}(X,Y)$,*

$$\bar{\mathscr{R}}_{\mathcal{T}} = \mathscr{N}_{\mathcal{T}^*}^{\perp}$$

where

$\bar{\mathscr{R}}_{\mathcal{T}} =$ the closure of the range of $\mathcal{T}$,

$\mathcal{T}^* =$ the adjoint of $\mathcal{T}$,

$\mathscr{N}_{\mathcal{T}^*}^{\perp} =$ the space orthogonal to the null space of $\mathcal{T}^*$.

In our present work, we define $\mathcal{T}$ as the following linear differential operator:

$$\mathcal{T}(u,\phi) = \frac{\partial u}{\partial t} + \mathcal{L}(u,\phi) = \begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial \eta^2} - \frac{\partial u}{\partial \eta} + e^{-\eta}\frac{\partial \phi}{\partial \tau}, & \text{if } \ \eta > 0, \\ \frac{\partial u}{\partial t} - a_p \frac{\partial^2 u}{\partial \eta^2} - \frac{\partial u}{\partial \eta}, & \text{if } \ \eta < 0. \end{cases} \tag{3.2.20}$$

Here we assume the differential operator $\mathcal{T}$ is bounded and its range is closed (see [24], [31] and [39] for other applications on PDEs of this theorem making these assumptions). One way to try to prove these assumption is to convert the related partial differential equation into an integral equation. Then it can be shown that the associated integral operator is bounded and the kernel of the corresponding integral operator is a Hilbert-Schmidt kernel. Examples of the application of integral methods to the solution of a solid combustion problem can be found in [16] and [17].

3.3 Introduction of Small Parameter

In this study we consider only small deviations from the neutral stability curve in the unstable direction. In particular, define a small parameter ϵ such that

$$\epsilon^2 = \nu_c(a_p) - \nu(a_p) \quad \text{i.e.} \quad \nu(a_p) = \nu_c(a_p) - \epsilon^2. \tag{3.3.1}$$

This choice of the parameter allows for the possibility of a Hopf bifurcation where the magnitude of the solution is on the order of the square root of the bifurcation parameter.

From now on, without ambiguity, we use ν_c and ν to represent $\nu_c(a_p)$ and $\nu(a_p)$, respectively. To find the form of the eigenvalues in the weakly unstable regime, substitute $\nu = \nu_c - \epsilon^2$ into the dispersion relation (2.3.1) to get

$$f(a_p, \nu_c - \epsilon^2, \lambda) = 0.$$

Then we seek eigenvalues of the form

$$\lambda = \lambda_0 + \epsilon^2 \chi + O(\epsilon^3),$$

so

$$f(a_p, \nu_c - \epsilon^2, \lambda_0 + \epsilon^2 \chi) = 0. \tag{3.3.2}$$

Equating $O(1)$ terms gives the equation

$$f(a_p, \nu_c, \lambda_0) = 0,$$

which is the dispersion relation with the critical ν. Therefore, by the definition of critical value, λ_0 is the pure imaginary $i\omega$. Setting $\lambda_0 = i\omega$ and equating the $O(\epsilon^2)$ terms in (3.3.2) gives the equation

$$\chi = \frac{(1+4a_p i\omega)(1+2i\omega) + [(1+4i\omega) - 4\nu_c(1-a_p)\omega^2]\sqrt{1+4a_p i\omega}}{2\nu_c(1+4a_p i\omega) + 4a_p\nu_c(1+2i\omega) + [4\nu_c^2(1-a_p)i\omega + 4\nu_c - 2]\sqrt{1+4a_p i\omega}} \tag{3.3.3}$$

So we know the eigenvalue up to $O(\epsilon^2)$ and we can write the eignenvalues as

$$\lambda = i\omega + \alpha, \tag{3.3.4}$$

where $\alpha = \epsilon^2\chi + O(\epsilon^3)$.

3.4 Method of Multiple Scales

We first introduce the multiple time scales

$$t_0 = \tau, \quad t_1 = \epsilon\tau, \quad t_2 = \epsilon^2\tau, \tag{3.4.1}$$

where ϵ is given by (3.3.1).

Let us seek a solution to the problem (1.2.17)–(1.2.21) as a perturbation about the basic solution, similar to the linearization done in §2, but with $O(\epsilon^2)$ and $O(\epsilon^3)$ terms included as follows:

$$u \sim u_{basic} + \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \cdots, \tag{3.4.2}$$

$$f \sim f_{basic} + \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots. \tag{3.4.3}$$

We substitute the expansions (3.4.2) and (3.4.3) into the problem in the front-attached coordinates (1.2.17)–(1.2.21) and due to (3.4.1), we rewrite $\partial/\partial\tau$ as

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2}.$$

We also expand $K(V)$ in a Taylor series about $V = 1$, replace ν by $\nu_c - \epsilon^2$ and use (3.3.4). Note that the $O(1)$ terms drop out because (u_{basic}, f_{basic}) is a solution to the original problem. Then equating like terms in the problem under consideration, we obtain $O(\epsilon)$, $O(\epsilon^2)$, and $O(\epsilon^3)$ problems.

3.4.1 The $O(\epsilon)$ Problem

The $O(\epsilon)$ problem is the linear problem in §2 with $w = w_1$, $\phi = \phi_1$, and $\tau = t_0$. A solution is of the form

$$w_1 = \begin{cases} A(t_1,t_2)e^{iwt_0}g_1(\eta) + \text{CC}, & \eta > 0, \\ A(t_1,t_2)e^{iwt_0}g_2(\eta) + \text{CC}, & \eta < 0, \end{cases} \tag{3.4.4}$$

$$\phi_1 = \Big\{A(t_1,t_2)e^{i\omega t_0} + \text{CC}\Big\} + B(t_1,t_2), \tag{3.4.5}$$

where we assume that the amplitudes of the modulated eigenmodes depend on t_1 and t_2 and CC are the complex-conjugate terms. In general the time dependent adjoint solutions will be utilized in weakly nonlinear analysis to obtain conditions on the slow time amplitude functions $A(t_1, t_2)$ while the time-independent solution will be used to obtain conditions on the evolution of $B(t_1, t_2)$. From now on we will use the notation A and B to mean $A(t_1, t_2)$ and $B(t_1, t_2)$.

We note here that the functions w_1 and ϕ_1 above "almost" satisfy the eigenvalue problem in §2. There are now small remainder terms that will contribute to the $O(\epsilon^3)$ problem (See Appendix A.1 for detailed derivations).

3.4.2 The $O(\epsilon^2)$ Problem

The $O(\epsilon^2)$ problem will show that the complex amplitude A depends on t_2 only. We also obtain expressions for B, w_2 and ϕ_2 in terms of A. Now let us derive these results!

The $O(\epsilon^2)$ problem consists of the partial differential equation

$$\frac{\partial w_2}{\partial t_0} \quad + \quad \mathcal{L}(w_2, \phi_2) = \begin{cases} -\dfrac{\partial w_1}{\partial t_1} + \dfrac{\partial w_1}{\partial \eta}\dfrac{\partial \phi_1}{\partial t_0} - e^{-\eta}\dfrac{\partial \phi_1}{\partial t_1}, & \eta > 0 \\ -\dfrac{\partial w_1}{\partial t_1} + \dfrac{\partial w_1}{\partial \eta}\dfrac{\partial \phi_1}{\partial t_0}. & \eta < 0 \end{cases} \tag{3.4.6}$$

subject to the boundary conditions

$$\mathcal{M}(w_2, \phi_2) = \nu_c \left(\frac{\partial \phi_1}{\partial t_1} + \frac{K''(1)}{2} \left(\frac{\partial \phi_1}{\partial t_0} \right)^2 \right), \tag{3.4.7}$$

$$\mathcal{N}(w_2, \phi_2) = -\frac{\partial \phi_1}{\partial t_1}, \tag{3.4.8}$$

$$w_2|_{\eta=0^+} = w_2|_{\eta=0^-} = w_2|_{\eta=0},$$

$$\lim_{\eta \to -\infty} w_2 < \infty, \; \lim_{\eta \to \infty} w_2 = 0. \tag{3.4.9}$$

Recall that §2 defines $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$. Given that w_1 and ϕ_1 have the forms (3.4.4) and (3.4.5), we can rewrite the partial differential equation as

$$\frac{\partial w_2}{\partial t_0} + \mathcal{L}(w_2, \phi_2) = R_2(\eta, \mathbf{t}), \tag{3.4.10}$$

where $\mathbf{t} = (t_0, t_1, t_2)$ and

$$R_2(\eta, \mathbf{t}) = \begin{cases} \left(-\frac{\partial A}{\partial t_1} e^{i\omega t_0} \left(g_1(\eta) + e^{-\eta}\right) + i\omega A^2 e^{2i\omega t_0} g_1'(\eta) - A\bar{A} g_1'(\eta) i\omega + \mathrm{CC}\right) - \frac{\partial B}{\partial t_1} e^{-\eta}, & \eta > 0, \\ \left(-\frac{\partial A}{\partial t_1} e^{i\omega t_0} g_2(\eta) + i\omega A^2 e^{2i\omega t_0} g_2'(\eta) - A\bar{A} g_2'(\eta) i\omega + \mathrm{CC}\right), & \eta < 0. \end{cases}$$

Similarly, the conditions at $\eta = 0$ become

$$\mathcal{M}(w_2, \phi_2) = a_2(\mathbf{t}), \tag{3.4.11}$$

$$\mathcal{N}(w_2, \phi_2) = b_2(\mathbf{t}), \tag{3.4.12}$$

where

$$a_2(\mathbf{t}) = \left(\frac{\partial A}{\partial t_1} e^{i\omega t_0} \nu_c - \frac{1}{2} K''(1) \nu_c A^2 e^{2i\omega t_0} + \frac{1}{2} K''(1) \nu_c A\bar{A}\omega^2 + \mathrm{CC}\right) + \frac{\partial B}{\partial t_1} \nu_c,$$

$$b_2(\mathbf{t}) = \left(-\frac{\partial A}{\partial t_1} e^{i\omega t_0} + \mathrm{CC}\right) - \frac{\partial B}{\partial t_1}.$$

Substituting the asymptotic expansion (3.3.4) of λ into the eigenfunction $g(\eta)$ shows

$$g(\eta) \sim g(\eta) \mid_{\lambda = i\omega} + O(\epsilon^2).$$

Starting in this section, the notation $g(\eta)$ will always indicate the eigenfunction evaluated at $\lambda = i\omega$.

We will show that the solvability conditions for the $O(\epsilon^2)$ problem (3.4.10), (3.4.11), and (3.4.11) lead to

$$\frac{\partial A}{\partial t_1} = 0 \tag{3.4.13}$$

and

$$\frac{\partial B}{\partial t_1} = A\bar{A}\, r_0, \tag{3.4.14}$$

where

$$r_0 = -\omega^2 \left[\frac{4}{\sqrt{1 + 4a_p i\omega} + \sqrt{1 - 4a_p i\omega}} + K''(1) \right]. \tag{3.4.15}$$

To derive these conditions, we first change w_2 into a new variable v_2 such that

$$\mathcal{M}(v_2, \phi_2) = \mathcal{N}(v_2, \phi_2) = 0, \tag{3.4.16}$$

i.e. v_2 satisfies the homogenous boundary conditions that define the linearized operator of §2.2. In particular, we set

$$v_2 = w_2 - a_2\mathcal{S}(\eta) - b_2\mathcal{T}(\eta),$$

where a_2 and b_2 are the inhomogeneities in the boundary conditions (3.4.11) and (3.4.12), and $\mathcal{S}$ and $\mathcal{T}$ are functions with the properties that

$$\begin{aligned} \mathcal{S}|_{\eta=0^-}^{\eta=0^+} &= 0, \qquad \mathcal{T}|_{\eta=0^-}^{\eta=0^+} = 0, \\ \mathcal{S}(0) &= 1, \qquad \mathcal{T}(0) = 0, \\ \mathcal{S}'(0^+) - a_p\mathcal{S}'(0^-) &= 0, \qquad \mathcal{T}'(0^+) - a_p\mathcal{T}'(0^-) = 1. \end{aligned} \tag{3.4.17}$$

Also

$$\lim_{\eta \to \pm\infty} \mathcal{S}(\eta) = \lim_{\eta \to \pm\infty} \mathcal{T}(\eta) = 0.$$

We could, for example, take

$$\mathcal{S}(\eta) = \begin{cases} e^{-a_p \eta}, & \eta > 0, \\ (1 - 2\eta)\, e^{\eta}, & \eta < 0. \end{cases}$$

$$\mathcal{T}(\eta) = \begin{cases} \frac{1}{2}\eta e^{-\eta}, & \eta > 0, \\ -\frac{\eta}{2a_p} e^{\eta}, & \eta < 0. \end{cases}$$

Substituting for w_2 with v_2 in equation (3.4.10), we obtain

$$\begin{aligned} \frac{\partial v_2}{\partial t_0} + \mathcal{L}(v_2, \phi_2) &= R_2(\eta, \mathbf{t}) - \Bigg\{ \frac{\partial a_2}{\partial t_0} \mathcal{S}(\eta) \\ &\quad + \frac{\partial b_2}{\partial t_0} \mathcal{T}(\eta) + \mathcal{L}(a_2 \mathcal{S}(\eta), 0) + \mathcal{L}(b_2 \mathcal{T}(\eta), 0) \Bigg\}. \end{aligned} \tag{3.4.18}$$

Then according to the Fredholm Alternative Theorem (see Lemma 3.2.1), the partial differential equation (3.4.18) together with the boundary conditions (3.4.16) has a nonsecular solution (a solution that does not grow in time without bound) if and only if the inhomogeneity is orthogonal to the null space of the adjoint operator. Therefore, the right-hand side RHS of the partial differential equation (3.4.18) must be orthogonal to the adjoint solution

$$u(\eta, \tau) = \begin{cases} e^{-\bar{\lambda} t_0} h_1(\eta), & \eta > 0, \\ e^{-\bar{\lambda} t_0} h_2(\eta), & \eta < 0. \end{cases}$$

By the definition of the inner product given in §3.2, we require

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{-\infty}^{+\infty} \text{RHS}\, \overline{u(\eta,\tau)}\, d\eta d\tau = 0. \tag{3.4.19}$$

To simplify the integral in condition (3.4.19), we note that $-\lambda + i\omega = -\epsilon^2\chi + O(\epsilon^3)$ by (3.3.4), so that $(-\lambda + i\omega)\tau = -\chi t_2 + O(\epsilon^3)$. Thus we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{(-\lambda+ij\omega)\tau}\, d\tau = \begin{cases} e^{-\chi t_2} \neq 0 & \text{if } j = 1, \\ \lim\limits_{T\to\infty} \dfrac{e^{-\chi t_2}}{Ti\omega(j-1)}(e^{i\omega(j-1)T} - 1) = 0 & \text{otherwise.} \end{cases} \tag{3.4.20}$$

Integrating by parts and exploiting our knowledge of the initial and limiting values of $\mathcal{S}(\eta)$ and $\mathcal{T}(\eta)$ shows that

$$\int_{0^+}^{+\infty} (\mathcal{S}''(\eta) + \mathcal{S}'(\eta) - i\omega\mathcal{S}(\eta))\, \bar{h}_1(\eta) d\eta = -\bar{h}_1(0^+)\mathcal{S}'(0^+) + \bar{h}_1'(0^+)\mathcal{S}(0^+) - \bar{h}_1(0^+)\mathcal{S}(0^+)$$

and

$$\int_{-\infty}^{0^-} (\mathcal{S}''(\eta) + \mathcal{S}'(\eta) - i\omega\mathcal{S}(\eta))\, \bar{h}_2(\eta) d\eta = a_p\bar{h}_2(0^-)\mathcal{S}'(0^-) - a_p\bar{h}_2'(0^-)\mathcal{S}(0^-) + \bar{h}_2(0^-)\mathcal{S}(0^-).$$

Therefore, we have

$$\int_{-\infty}^{+\infty} (\mathcal{S}''(\eta) + \mathcal{S}'(\eta) - i\omega\mathcal{S}(\eta))\, \bar{h}(\eta)\, d\eta = \bar{h}_1'(0^+) - a_p\bar{h}_2'(0^-)$$

Similarly, we obtain

$$\int_{-\infty}^{+\infty} (\mathcal{T}''(\eta) + \mathcal{T}'(\eta) - i\omega\mathcal{T}(\eta))\, \bar{h}(\eta)\, d\eta = -\bar{h}_1(0^+)$$

Then condition (3.4.19) reduces to

$$\frac{dA}{dt_1}\Bigg\{ \int_{0^+}^{+\infty} \Big(g_1(\eta) + e^{-\eta}\Big)\, \bar{h}_1(\eta)\, d\eta \\ + \int_{-\infty}^{0^-} g_2(\eta)\bar{h}_2(\eta)\, d\eta + \nu_c \Big(\bar{h}_1'(0^+) - a_p \bar{h}_2'(0^-)\Big) + \bar{h}_1(0^+)\Bigg\} = 0. \quad (3.4.21)$$

Because the second factor is nonzero, we obtain

$$\frac{\partial A}{\partial t_1} = 0.$$

This is equation (3.4.13), which we set out to derive. Note the equation states that the slowly-varying amplitude A depends only on the slowest time scale t_2.

To derive the second condition (3.4.14), the Fredholm Alternative Theorem implies that the right-hand side RHS of the partial differential equation (3.4.18) must be orthogonal also to the adjoint solution (3.2.19) $u(\eta, \tau) = 1$, i.e.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^\infty \mathrm{RHS}\; d\eta d\tau = 0. \quad (3.4.22)$$

To simplify the integral, we note that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{ij\omega\tau} d\tau = \begin{cases} 1, & j = 0, \\ 0, & j = \pm 1, \pm 2, \end{cases}$$

and

$$\int_{-\infty}^{+\infty} \left(\mathcal{S}''(\eta) + \mathcal{S}'(\eta)\right)\, d\eta = \int_{-\infty}^{+\infty} \left(\mathcal{T}''(\eta) + \mathcal{T}'(\eta)\right)\, d\eta = -1.$$

Then condition (3.4.22) becomes

$$\begin{aligned}\frac{\partial B}{\partial t_1} = A\bar{A}r_0 &= -\omega^2\left[\frac{1-\sqrt{1+4a_p i\omega}}{2a_p\omega}i + \frac{1}{2}K''(1) + \text{CC}\right]A\bar{A} \\ &= -\omega^2\left[\frac{4}{\sqrt{1+4a_p i\omega}+\sqrt{1-4a_p i\omega}} + K''(1)\right]A\bar{A} \qquad (3.4.23)\end{aligned}$$

as desired.

Apply conditions (3.4.13) and (3.4.14), we rewrite the $O(\epsilon^2)$ partial differential equation with "legitimate" right-hand side as

$$\frac{\partial w_2}{\partial t_0} + \mathcal{L}(w_2,\phi_2) = \begin{cases}\left(A^2e^{2i\omega t_0}i\omega g_1'(\eta) - A\bar{A}g_1'(\eta)iw + \text{CC}\right) - A\bar{A}r_0e^{-\eta}, & \eta > 0, \\ A^2e^{2i\omega t_0}i\omega g_2'(\eta) - A\bar{A}g_2'(\eta)iw + \text{CC}, & \eta < 0.\end{cases}$$

Similarly, the boundary conditions subject to conditions (3.4.13) and (3.4.14) become

$$\mathcal{M}(w_2,\phi_2) = \left(-\frac{1}{2}K''(1)\nu_c A^2\omega^2 e^{2i\omega t_0} + \frac{1}{2}K''(1)\nu\nu_c A\bar{A}\omega^2 + \text{CC}\right) + A\bar{A}r_0\nu_c, \quad (3.4.24)$$

$$\mathcal{N}(w_2,\phi_2) = -A\bar{A}r_0. \qquad (3.4.25)$$

Let us seek a solution of the same form as the inhomogeneities; namely, let

$$w_2 = \begin{cases} A^2e^{2i\omega t_0}k_2^+(\eta) + A\bar{A}k_0^+(\eta) + \text{CC}, & \eta > 0, \\ A^2e^{2i\omega t_0}k_2^-(\eta) + A\bar{A}k_0^-(\eta) + \text{CC}, & \eta < 0.\end{cases} \qquad (3.4.26)$$

$$\phi_2 = A^2e^{2i\omega t_0}C_2 + A\bar{A}C_0 + \text{CC}. \qquad (3.4.27)$$

Substituting these expressions into the partial differential equation and the boundary conditions (3.4.24) and (3.4.25) and collecting like terms yields the following initial-value problems:

$$\begin{cases} k_0^{''+}(\eta) + k_0^{'+}(\eta) = r_0 e^{-\eta} + i\omega g_1^{'}(\eta), \\ a_p k_0^{''-} + k_0^{'-} = i\omega g_2^{'}(\eta), \\ k_0^{+}(0) = k_0^{-}(0) = \frac{1}{2}K''(1)\nu_c\omega^2 + r_0\nu_c, \\ k_0^{'+}(0^+) - a_p k_0^{'-}(0^-) = -r_0, \\ \lim_{\eta\to\infty} k_0^{+}(\eta) = 0, \quad \lim_{\eta\to-\infty} k_0^{-}(\eta) < \infty \end{cases} \tag{3.4.28}$$

and

$$\begin{cases} k_2^{''+}(\eta) + k_2^{'+}(\eta) - 2i\omega(k_2^2(\eta) + C_2 e^{-\eta}) = -i\omega g_1^{'}(\eta), \\ a_p k_2^{''-} + k_2^{'-} - 2i\omega k_2^{-}(\eta) = -i\omega g_2^{'}(\eta), \\ k_2^{+}(0) = k_2^{-}(0) = -\frac{1}{2}K''(1)\nu_c\omega^2 + 2i\omega r_0\nu_c, \\ k_2^{'+}(0^+) - a_p k_2^{'-}(0^-) = -2i\omega C_2, \\ \lim_{\eta\to\infty} k_2^{+}(\eta) = 0, \quad \lim_{\eta\to-\infty} k_2^{-}(\eta) < \infty. \end{cases} \tag{3.4.29}$$

The solution to problem (3.4.28) and (3.4.29) are given in Appendix A (see A.2).

3.4.3 The $O(\epsilon^3)$ Problem

This section is devoted to deriving the Landau-Stuart equation. The $O(\epsilon^3)$ problem consists of the partial differential equation

$$\begin{aligned} \frac{\partial w_3}{\partial t_0} + \mathcal{L}^+(w_3, \phi_3) &= -\frac{\partial w_1}{\partial t_2} - \frac{\partial w_2}{\partial t_1} + \frac{\partial w_1}{\partial \eta}\left(\frac{\partial \phi_2}{\partial t_0} + \frac{\partial \phi_1}{\partial t_1}\right) \\ &\quad + \frac{\partial w_2}{\partial \eta}\frac{\partial \phi_1}{\partial t_0} - e^{-\eta}\frac{\partial \phi_1}{\partial t_2} - e^{-\eta}\frac{\partial \phi_2}{\partial t_1} \end{aligned}$$

$$
\begin{aligned}
&\quad + \text{contribution from the } O(\epsilon) \text{ problem,} \\
\frac{\partial w_3}{\partial t_0} + \mathcal{L}^-(w_3, \phi_3) \quad &= \quad -\frac{\partial w_1}{\partial t_2} - \frac{\partial w_2}{\partial t_1} + \frac{\partial w_1}{\partial \eta}\left(\frac{\partial \phi_2}{\partial t_0} + \frac{\partial \phi_1}{\partial t_1}\right) + \frac{\partial w_2}{\partial \eta}\frac{\partial \phi_1}{\partial t_0} \\
&\quad + \text{contribution from the } O(\epsilon) \text{ problem,}
\end{aligned}
$$

subject to the boundary conditions

$$
\begin{aligned}
\mathcal{M}(w_3, \phi_3) \quad &= \quad \nu_c \left(\frac{\partial \phi_1}{\partial t_2} + \frac{\partial \phi_2}{\partial t_1} + K''(1)\frac{\partial \phi_1}{\partial t_0}\left(\frac{\partial \phi_2}{\partial t_0} + \frac{\partial \phi_1}{\partial t_1}\right) + \frac{K'''(1)}{6}\left(\frac{\partial \phi_1}{\partial t_0}\right)^3\right) \\
&\quad -\frac{\partial \phi_1}{\partial t_0} + \text{contributions from the } O(\epsilon) \text{ problem,}
\end{aligned}
$$

$$
\mathcal{N}(w_3, \phi_3) \quad = \quad -\frac{\partial \phi_1}{\partial t_2} - \frac{\partial \phi_2}{\partial t_1} + \text{contributions from the } O(\epsilon) \text{ problem,}
$$

$$
w_3|_{\eta=0^+} = w_3|_{\eta=0^-} = w_3|_{\eta=0},
$$

$$
\lim_{\eta \to -\infty} w_3 < \infty, \quad \lim_{\eta \to \infty} w_3 = 0.
$$

The contribution from the $O(\epsilon)$ problem to the right-hand side of the partial differential equation is $-\partial w_1/\partial t_0 - \mathcal{L}(w_1, \phi_1)$, which equals (A.1.3). Substituting the forms of w_1, ϕ_1, w_2 and ϕ_2 from equations (3.4.4), (3.4.5), (3.4.26), and (3.4.27) into the partial differential equation allows us to rewrite the equation as

$$
\frac{\partial w_3}{dt_0} + \mathcal{L}(w_3, \phi_3) = R_3(\eta, \mathbf{t}), \tag{3.4.30}
$$

where

$R_3(\eta, \mathbf{t})$

$$= \left\{ \left(-\frac{\partial A}{\partial t_2} + \chi A\right) \left(g_1(\eta) + e^{-\eta}\right) e^{i\omega t_0} + A^3 e^{3i\omega t_0} P_3^+(\eta) + A^2 \bar{A} e^{i\omega t_0} P_1^+(\eta) + \mathrm{CC} \right\}$$
$$-\frac{\partial B}{\partial t_2} e^{-\eta}, \quad \eta > 0,$$

$$R_3(\eta, \mathbf{t})$$
$$= \left\{ \left(-\frac{\partial A}{\partial t_2} + \chi A\right) g_2(\eta) e^{i\omega t_0} + A^3 e^{3i\omega t_0} P_3^-(\eta) + A^2 \bar{A} e^{i\omega t_0} P_1^-(\eta) + \mathrm{CC} \right\}, \ \eta < 0,$$

where P_3^+ and P_3^- are not important and

$$P_1^+(\eta) = r_0 g_1'(\eta) + 2i\omega C_2 \bar{g}_1'(\eta) + i\omega \left(k_0'^+(\eta) - k_2'^+(\eta)\right),$$

$$P_1^-(\eta) = r_0 g_2'(\eta) + 2i\omega C_2 \bar{g}_2'(\eta) + i\omega \left(k_0'^-(\eta) - k_2'^-(\eta)\right),$$

Similarly, the boundary conditions become

$$\mathcal{M}(w_3, \phi_3) = a_3(\mathbf{t}), \tag{3.4.31}$$

$$\mathcal{N}(w_3, \phi_3) = b_3(\mathbf{t}), \tag{3.4.32}$$

where

$$a_3(\xi, \mathbf{t}) =$$
$$\left\{ \left(\frac{\partial A}{\partial t_2} - \chi A - i\omega A\right) e^{i\omega t_0} \nu_c + A^3 e^{3i\omega t_0} F_3 + A^2 \bar{A} e^{i\omega t_0} F_1 + \mathrm{CC} \right\} + \nu_c \frac{\partial B}{\partial t_2},$$

where F_3 is not important and

$$F_1 = K''(1)\nu_c\left(2\omega^2 C_2 + ir_0\omega\right) + \frac{K'''(1)}{2} i\nu_c\omega^3$$

and

$$b_3(\xi, \mathbf{t}) \;=\; \left\{\left(-\frac{\partial A}{\partial t_2} + \chi A\right)e^{i\omega t_0} + A^3 e^{3i\omega t_0} G_3 + \mathrm{CC}\right\} - \frac{\partial B}{\partial t_2}.$$

where G_3 is not important.

Proceeding as in previous section, we impose Fredholm Alternative Theorem and obtain the following solvability condition for $O(\epsilon^3)$ problem, which satisfies the Landau-Stuart equation:

$$\frac{dA}{dt_2} = \kappa A + \beta A^2 \bar{A}, \tag{3.4.33}$$

where

$$\kappa = \chi + \frac{S_1}{S_3}, \tag{3.4.34}$$

and

$$\beta = \frac{S_2}{S_3} \tag{3.4.35}$$

for

$$\begin{aligned} S_1 &= -i\omega\mathcal{U}, \\ S_2 &= \int_{0^+}^{\infty} P_1^+(\eta)\bar{h}_1(\eta)\,d\eta + \int_{-\infty}^{0^-} P_1^-(\eta)\bar{h}_2(\eta)\,d\eta + F_1\mathcal{U} \end{aligned}$$

and

$$S_3 \;= \int_{0^+}^{\infty} \left(g_1(\eta) + e^{\eta}\right)\bar{h}_1(\eta)\,d\eta + \int_{-\infty}^{0^-} g_2(\eta)\,d\eta + \nu_c\mathcal{U} - \mathcal{V}.$$

where

$$\mathcal{U} = \bar{h}_1'(0^+) - a_p\bar{h}_2'(0^-), \quad \mathcal{V} = -\bar{h}_1(0^+),$$

$$P_1^+ = r_0 g_1'(\eta) + 2i\omega C_2 \bar{g}_1'(\eta) + i\omega k_0^{'+}(\eta) - i\omega k_2^{'+}(\eta),$$

$$P_1^- = r_0 g_2'(\eta) + 2i\omega C_2 \bar{g}_2'(\eta) + i\omega k_0^{'-}(\eta) - i\omega k_2^{'-}(\eta),$$

$$F_1 = K''(1)\nu_c(2\omega^2 C_2 + r_0 i\omega) + \frac{K'''(1)}{2} i\nu_c\omega^3.$$

Finally, to find $A(t_2)$, we integrate the ordinary differential equation (3.4.33) using a fourth-order Runge-Kutta method in next Section.

Similarly, we obtain that $\frac{\partial B}{\partial t_2} = 0$, which means B is independent of t_2.

Chapter 4

NUMERICAL METHOD

4.1 Introduction to Numerical Method

There are many options for numerically solving PDEs, in this chapter we outline the methods used to obtain the numerical results which are compared to the analytical predictions on the evolution of a perturbed combustion wave.

In this work, we adopt Crank-Nicolson method to numerically solve the exact problem (1.2.17)–(1.2.21). Since the numerical solutions are very sensitive to the derivative interfacial jump condition, we integrate the partial differential equation to obtain an integral-differential equation as an alternative condition. The result system of nonlinear algebraic equations is then solved by the Newton's method, taking advantage of the sparse structure of the Jacobian matrix. Finally, we show that our asymptotic solution captures the marginally unstable behaviors of the solution for a range of model parameters.

Besides, we also apply the fourth-order Runge-Kutta method to obtain our asymptotic solution.

4.2 The Crank-Nicolson Method

The Crank Nicolson finite difference scheme was invented by John Crank and Phyllis Nicolson. They originally applied it to the heat equation and they approximated the solution of the heat equation on some finite grid by approximating the derivatives in space x and time t by finite differences. Much earlier, Richardson devised a finite difference scheme that was easy to compute but was numerically unstable and thus useless. The instability was not recognized until Crank, Nicolson and others carried out lengthy numerical calculations.

The C-N method is known to be unconditionally stable in solving diffusion equations. The basic idea underlying the C-N scheme involves the approximation of second spatial derivatives via the central difference while first spatial and temporal derivatives are approximated via the center difference. The truncation error is then second order in space and time (about an imaginary $n + 1/2$ node). The application of the C-N scheme to our particular problem is described in the following sections. In Appendix C, we provide the numerical code.

4.3 The Discretized Model Equations

In this part, we derive the discretized model equations for the exact problem as given by (1.2.17)–(1.2.21). We first introduce perturbation variables u^* and ϕ^* defined by

$$u = \begin{cases} e^{-\eta} + \epsilon u^* & \text{if } \eta \geq 0 \\ 1 + \epsilon u^* & \text{if } \eta < 0 \end{cases}, \qquad V = \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} = 1 + \epsilon\phi^*. \tag{4.3.1}$$

Substitute them into equation (1.2.17)–(1.2.21), we get

$$u^*_\tau = \begin{cases} u^*_{\eta\eta} + (1+\epsilon\phi^*)u^*_\eta - \phi^* e^{-\eta} & \text{if } \ \eta > 0, \\ a_p u^*_{\eta\eta} + (1+\epsilon\phi^*)u^*_\eta & \text{if } \ \eta < 0, \end{cases} \tag{4.3.2}$$

$$u^*|_{\eta=0} = \frac{\nu K(1+\epsilon\phi^*)}{\epsilon}. \tag{4.3.3}$$

As was pointed out in [17], numerical solutions of (1.2.17)–(1.2.21) are very sensitive to the boundary condition (1.2.20). In order to obtain an alternative condition, we integrate (1.2.17) with respect to η from $-\infty$ to ∞ and apply conditions (1.2.18)–(1.2.21), subsequently. The result equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u^* \mathrm{d}\eta = u^*_\eta\Big|_{\infty} - a_p\, u^*_\eta\Big|_{-\infty} - u^*|_{-\infty} - \epsilon\phi^*\, u^*|_{-\infty}\,. \tag{4.3.4}$$

We use condition (4.3.4) to replace (1.2.20).

To apply the C-N method, for the first partial η derivative terms, we use the center difference formula to get

$$(1+\epsilon\phi^*)u^*_\eta = \frac{(1+\epsilon\phi^{n+1})(U^{n+1}_{i+1} - U^{n+1}_{i-1}) + (1+\epsilon\phi^n)(U^n_{i+1} - U^n_{i-1})}{4\Delta\eta}. \tag{4.3.5}$$

And the second η derivative terms are discretized by using the following central difference formula

$$u^*_{\eta\eta} = \frac{(U^{n+1}_{i+1} - 2U^{n+1}_i + U^{n+1}_{i-1}) + (U^n_{i+1} - 2U^n_i + U^n_{i-1})}{2(\Delta\eta)^2}. \tag{4.3.6}$$

In this way, each will contribute $O((\Delta\eta)^2)$ to the local truncation error. For the first partial time derivative, we apply the center difference formula about an imaginary $n + 1/2$ node to get

$$u_\tau^* = \frac{U_i^{n+1} - U_i^n}{\Delta\tau}. \tag{4.3.7}$$

Here we note that the truncation error in time is also $O((\Delta\tau)^2)$ about an imaginary $n+1/2$ node. Finally, we approximate the integral on the left-hand side of condition (4.3.4) by a composite trapezoidal rule. The computation domain for η is $[-10, 10]$ with $\Delta\tau = \Delta\eta = 0.025$. This produces a nonlinear system of m (= 800) equations.

We solve the nonlinear system of equations using the Newton's method. At each iterating step of the Newton's method, we solve a linear system of equations with

Jacobian matrix that has the following sparse structure:

$$\begin{bmatrix}
\# & \# & \# & \# & \# & \dots & \# & \# & \# & \# & \dots & \# & \# \\
\# & \# & \# & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\# & \# & \# & \# & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\# & 0 & \# & \# & \# & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
\# & 0 & 0 & \dots & \# & \# & \# & 0 & 0 & 0 & \dots & 0 & 0 \\
\# & 0 & 0 & \dots & \dots & \# & \# & \# & 0 & 0 & \dots & 0 & 0 \\
\# & 0 & 0 & \dots & \dots & \dots & 0 & \# & 0 & 0 & \dots & 0 & 0 \\
\# & 0 & 0 & \dots & \dots & \dots & 0 & \# & \# & \# & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & \dots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\# & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \# & \# & \# & 0 \\
\# & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & \# & \# & \# \\
\# & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 & \# & \#
\end{bmatrix} \tag{4.3.8}$$

where # denotes a nonzero element. We take advantage of this sparse structure and apply Gaussian elimination from bottom-up, eliminating the two nonzero elements at $(m-1, m)$ and $(1, m)$ positions first; then the two at $(m-2, m-1)$ and $(1, m-1)$ second; and so on. This reduces the matrix to a lower triangular form. Finally, the solution of the linear system of equations can be obtained by a forward substitution from top-down.

4.4 The Runge-Kutta Method

To calculate the asymptotic solutions, let's first recall the following results from $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$ problems. For $O(\epsilon)$ problem, we have

$$\begin{aligned} w_1 &= \begin{cases} A(t_1, t_2)e^{i\omega t_0} g_1(\eta) + \mathrm{CC}, & \eta > 0, \\ A(t_1, t_2)e^{i\omega t_0} g_2(\eta) + \mathrm{CC}, & \eta < 0, \end{cases} \\ \phi_1 &= \left\{ A(t_1, t_2)e^{i\omega t_0} + \mathrm{CC} \right\} + B(t_1, t_2), \end{aligned}$$

where $g_1(\eta)$, $g_2(\eta)$ are solutions to the related linear problem and CC represents the complex-conjugate terms. A, B can be determined from $O(\epsilon^2)$ and $O(\epsilon^3)$ problems. For $O(\epsilon^2)$ problem, we find that the solvability conditions for this problem are

$$\frac{\partial A}{\partial t_1} = 0$$

and

$$\frac{\partial B}{\partial t_1} = A\bar{A}\, r_0,$$

where

$$r_0 = -\omega^2 \left[\frac{4}{\sqrt{1 - 4a_p i\omega} + \sqrt{1 + 4a_p i\omega}} + K''(1) \right].$$

For $O(\epsilon^3)$ problem, we obtain the following Landau–Stuart equation for the complex amplitude $A(t_2)$

$$\frac{dA}{dt_2} = \kappa A + \beta A^2 \bar{A}, \tag{4.4.1}$$

where

$$\kappa = \chi + \frac{S_1}{S_3},$$

$$\beta = \frac{S_2}{S_3},$$

$$S_1 = -i\omega\mathcal{U},$$

$$S_2 = \int_{0^+}^{\infty} P_1^+(\eta)\bar{h}_1(\eta)\,d\eta + \int_{-\infty}^{0^-} P_1^-\bar{h}_2(\eta)\,d\eta + F_1\mathcal{U},$$

$$S_3 = \int_{0^+}^{\infty} (g_1(\eta) + e^{\eta})\,\bar{h}_1(\eta)\,d\eta + \int_{-\infty}^{0^-} g_2(\eta)\,d\eta - \nu_c\mathcal{U} + \mathcal{V},$$

and

$$\mathcal{U} = \bar{h}_1'(0^+) - a_p\bar{h}_2'(0^-), \quad \mathcal{V} = -\bar{h}_1(0^+).$$

To find $A(t_2)$, we still need to solve the related Landau-Stuart equation (4.4.1). This equation can be numerically solved by using the Runge-Kutta methods.

In numerical analysis, the Runge-Kutta methods are a family of implicit and explicit iterative methods, which include the well-known routine called the Euler Method, used in temporal discretization for the approximate solutions of ordinary differential equations (see [56]). These methods were developed around 1900 by the German mathematicians Carl Runge and Martin Kutta. The most widely known member of the Runge-Kutta family is generally referred to as “RK4”, “classical Runge-Kutta method” or simply as “the Runge-Kutta method”.

The formula for the fourth order Runge-Kutta method (RK4) is given below. We take the following problem as an example:

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0. \end{cases}$$

Define h to be the time step size and $t_i = t_0 + ih$. Then the following formula

$$\begin{cases} w_0 = y_0, \\ k_1 = hf(t_i, w_i), \\ k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right), \\ k_3 = hf(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}), \\ k_4 = hf(t_i + h, w_i + k_3), \\ w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{cases}$$

computes an approximate solution, that is $y(t_i) \approx w_i$. To solve our problem, we just need set $y(t) = A(t_2)$, $f(t, y) = \kappa A + \beta A^2 \bar{A}$ and $y(t_0) = A_0$ in the above scheme. (For related MATLAB code, see Appendix C)

4.5 Comparison between Asymptotics and Numerics

For initial conditions, we use perturbations of the basic solution by linearized solutions, modulated by an amplitude factor A:

$$u(\eta, 0) = e^{-\eta} + \epsilon A(1 + \nu_c i\omega) \exp\left(\frac{-1 - \sqrt{1 + 4i\omega}}{2}\eta\right) + CC, \qquad \eta \geq 0,$$

$$u(\eta, 0) = 1 + \epsilon A \nu_c i\omega \exp\left(\frac{-1 + \sqrt{1 + 4a_p i\omega}}{2a_p}\eta\right) + CC, \qquad \eta < 0,$$

$$V(0) = 1 + \epsilon A i\omega + CC,$$

where CC are complex-conjugate terms.

To start, take $\sigma = 0.46$ in the kinetics function (1.2.15). Throughout this paper, the amplitude A is taken to be 0.1.

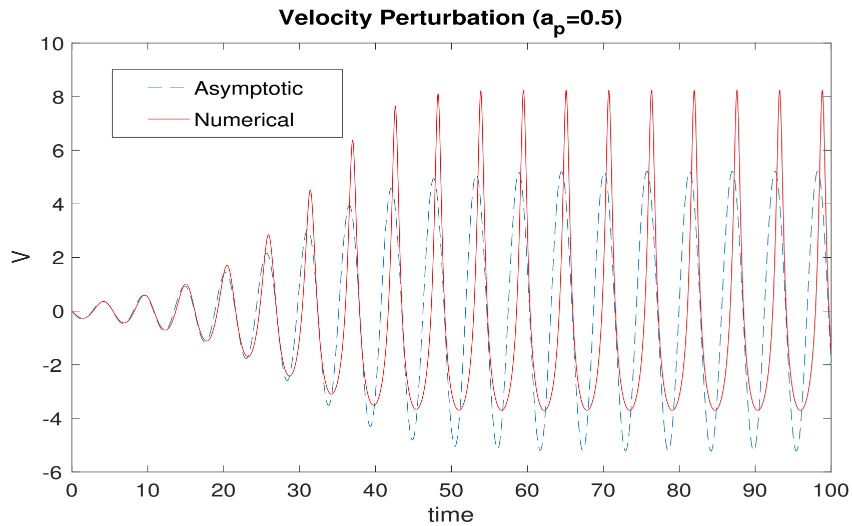

Figure 4.1: Comparison between numerical (solid line) and asymptotic (dashed line) for Arrhenius kinetics: $a_p = 0.5$, $\sigma = 0.46$, $\epsilon = 0.1$, $A(0) = 0.1$, $\nu_c \approx 0.2703$.

For $a_p = 0.5$, Figure 4.1 reveals that from $t = 0$ to about $t = 30$, the small front speed perturbation is linearly unstable, and its amplitude grows exponentially in time. As this amplitude becomes large, nonlinearity takes effect. At around $t = 30$, the front speed perturbation has reached steady oscillation. The two solutions are slightly out of phase, and the asymptotic solution oscillates symmetrically about the time axis, while the numerical solution has spiky peaks. The asymptotic solution accurately captures the period in both transient behavior for $t = 0$ to 30 and the long-time behavior after $t = 30$. This is an example in which the weakly nonlinear approach describes rather well the marginally unstable large-time behavior: A single modulated temporal mode captures the period of fluctuations (at frequency $\omega = 1.1406$ for $a_p = 0.5$) in velocity perturbation. Similar results for $a_p = 1$ and $a_p = 0.2$ are obtained in Appendix B (See Figure B.1 and Figure B.2).

We then numerically calculated the velocity perturbation data on the time interval $50 < t < 100$, using the parameter values as in Figure 4.1. As shown in Figure 4.2, the discrete Fourier transform of the data reveals the dominance of one mode. However, the subsequent modes do contribute to the solution as well. The second spike in Figure 4.2 is about 4/9 the height of the first, and the third is less than 1/2 the height of the second. Contributions of higher-order modes may explain some quantitative discrepancies between the numerical and asymptotic solutions in Figure 4.1.

For $\epsilon = 0.1$ and $a_p = 0.5$, Figure 4.3 summarizes the Fourier transformed velocity data for all physical values of $\sigma \in (0, 1)$. For each σ value and each frequency, the color indicates the corresponding amplitude, with the red end of the spectrum standing for larger numbers than the violet end, as the legend to the right of the figure illustrates. As predicted by weakly nonlinear analysis, Figure 4.3 shows that one mode dominates strongly for all physical values of $\sigma \in (0, 1)$ when $\nu = \nu_c - \epsilon^2$ is sufficiently close to the neutrally stable value.

We have done both asymptotic and numerical analyses for a generalized solid combustion model. For the chosen parameters (e.g., $\sigma = 4.6, \epsilon = 0.1$ and etc.), our asymptotic solution with a single modulated temporal mode describes well the marginally unstable solution behavior. The Crank-Nicolson finite difference method with the Newton's method, using the sparse structure of the Jacobian matrix, provides an efficient numerical solver for the nonlinear free boundary problem.

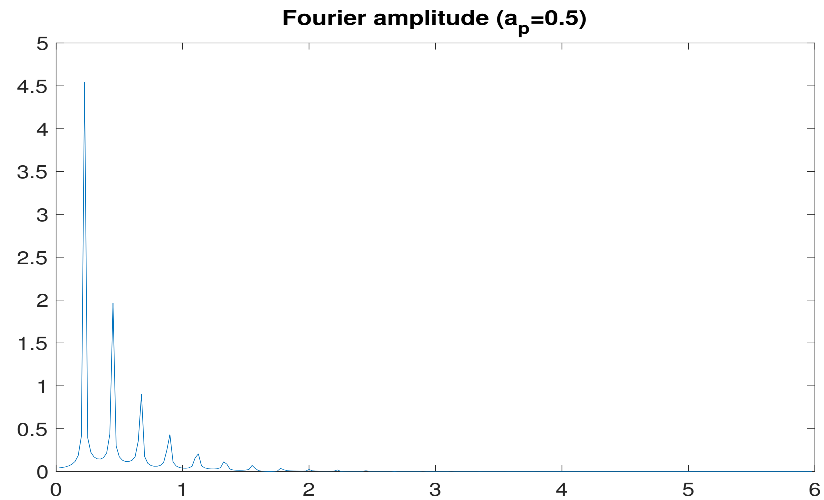

Figure 4.2: Fourier amplitude of the numerical steady-state velocity perturbation: $a_p = 0.5$, $\sigma = 0.46$, $\epsilon = 0.1$, $A(0) = 0.1$, $50 < t < 100$, $\nu_c \approx 0.2703$.

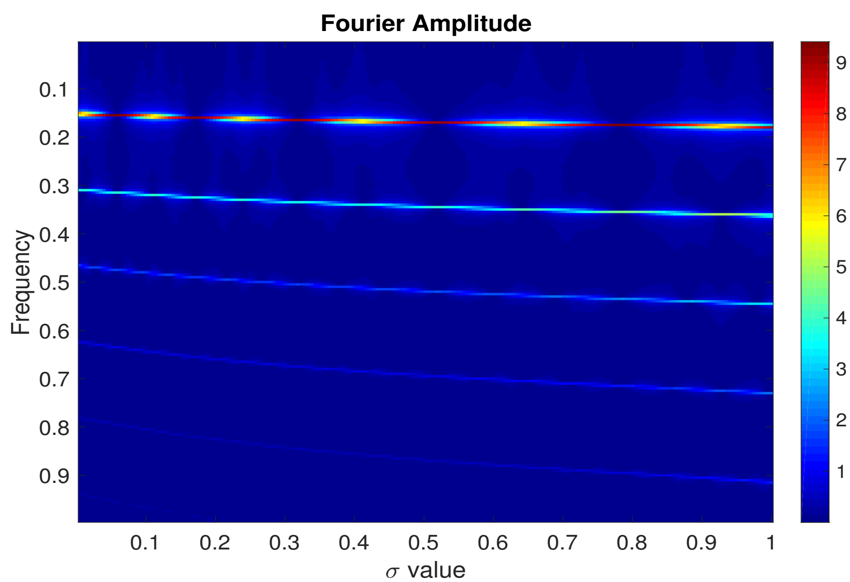

Figure 4.3: Amplitudes corresponding to each frequency of the Fourier transformed velocity perturbation data for the Arrhenius kinetics parameter: $a_p = 0.5$, $\sigma \in (0, 1)$, $\epsilon = 0.1$, $A(0) = 0.1$, $\nu_c \approx 0.2703$ *and* $1000 < t < 1500$.

Chapter 5

RESULTS AND DISCUSSION

5.1 Discussion

In this section, we summarize more interesting results obtained from our numerical solutions and delineate the role of the diffusivity ratio between the burned and unburned materials. First, let's look at the dynamics associated with the model include a cascade of period-doubling bifurcations. For example, for a given value of ϵ and a_p, we observe period doubling, quadrupling, and octupling as we decrease σ. Figures 5.1 and 5.2 illustrate these dynamics as time plots and phase portraits, respectively, for the case $\epsilon = 0.2$, $a_p = 0.5$ on the time scale $1350 < t < 1500$.

Figure 5.3 depicts the amplitudes corresponding to each frequency of the Fourier transformed velocity perturbation data in this case ($\epsilon = 0.2$, $a_p = 0.5$). It confirms that a period-doubling bifurcation occurs at approximately $\sigma = 0.72$.

We see similar dynamics for other choices of the thermal diffusivity ratio a_p (see Appendix B). Figures 5.4 and 5.5 depict the amplitudes corresponding to each frequency of the Fourier transformed velocity perturbation data for the cases $a_p = 0.2$

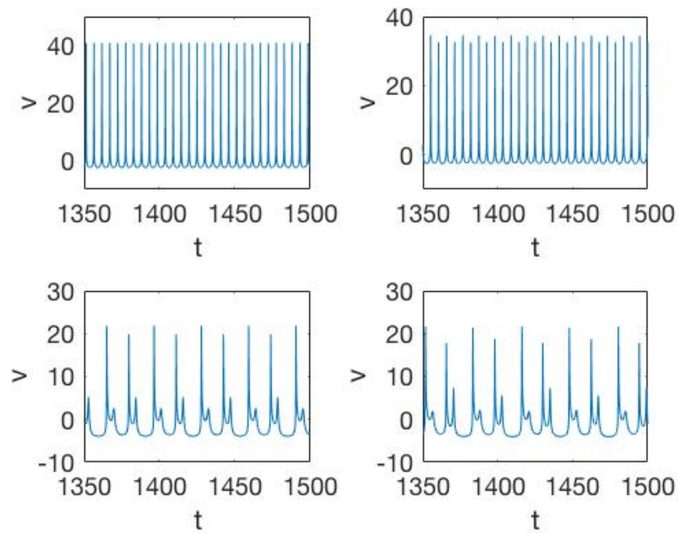

Figure 5.1: Velocity perturbations versus time ($\epsilon = 0.2$, $a_p = 0.5$, $A(0) = 0.1$, $\nu = \nu_c - \epsilon^2$) upper left: quasi-periodic solution for $\sigma = 0.78$, upper right: period doubling ($\sigma = 0.72$); lower left: period quadrupling ($\sigma = 0.030$), lower right: period octupling ($\sigma = 0.027$)

and $a_p = 0.8$, respectively, when $\epsilon = 0.2$ (for more cases, see Appendix B). Note the period-doubling bifurcations occur at approximately $\sigma = 0.80$ in Figure 5.4 for the case $a_p = 0.2$ and at approximately $\sigma = 0.62$ in Figure 5.5 for the case $a_p = 0.8$. The early stages of the period doubling can be used to identify the onset of nonlinear dynamics. We also look at the analogous stage of bifurcation for two additional values of a_p, namely 0.35 and 0.65. Table 5.1 provides a summary. Pairs in Table 5.1 are captured in Figure 5.6.

Note that the curve corresponding to $a_p = 0.2$ contains the point $(\sigma, \epsilon) = (0.8, 0.2)$; the curve corresponding to $a_p = 0.35$ contains the point $(\sigma, \epsilon) = (0.77, 0.2)$; etc.

In Figure 5.6 , the diffusivity ratio a_p varies from 0 to 1. Analysis of the limiting cases of 0 and 1 are described in the one-sided and two-sided models in the literature, respectively, see [19], [64] and [67]. While the evolution of the critical values of ν

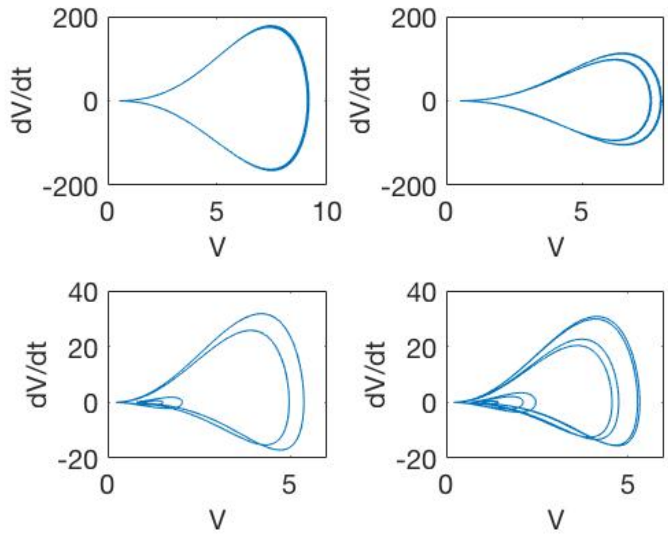

Figure 5.2: Phase plots of the four solutions of Figure 5.1 for $a_p = 0.5$, $1350 < t < 1500$: velocity perturbations $v(t)$ versus dv/dt.

changes in an unsurprising way (Figure 2.1), there is significant nonlinear behavior in the evolution of period-doubling bifurcations. The profiles in Figure 5.6 show significant change as the thermal diffusivity varies from 0 to 1.

There is some consistency. For example, all the curves in Figure 5.6 are increasing on $0 < \sigma < 1$. As

$$\sigma = \frac{T_0}{T_b} = \frac{T_0}{T_0 + QC_0}$$

gets larger, we must increase ϵ to get period doubling: We have to go deeper into the instability region to get a period-doubling bifurcation. This phenomenon is the stabilizing effect associated with less discrepancy between the burned temperature T_b and the fresh temperature T_0. Such scenarios arise when QC_0, the product of the heat release Q and concentration of reactant C_0, is small. All the curves in Figure 5.6 show this stabilization as σ increases.

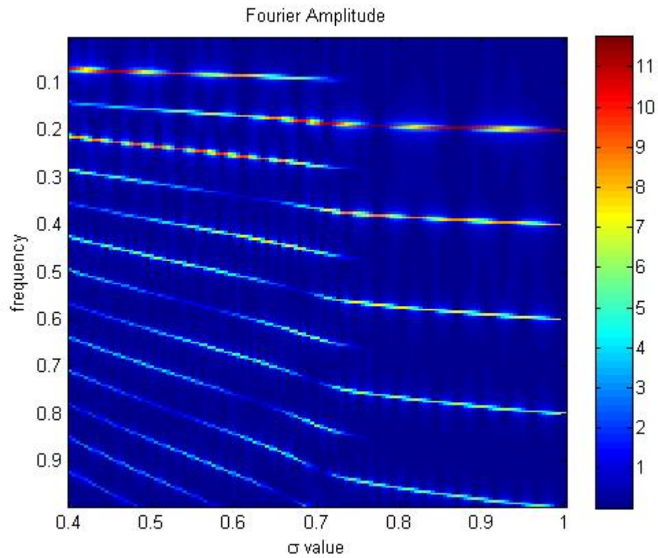

Figure 5.3: Amplitudes corresponding to each frequency of the Fourier transformed velocity perturbation data for $a_p = 0.5$, the Arrhenius kinetics parameter σ in the interval $(0.4, 1)$, $\epsilon = 0.2$, $A(0) = 0.1$, $1000 < t < 1500$ ($\nu = \nu_c - \epsilon^2$)

One might expect that increasing a_p would have a stabilizing effect, as well. With less discrepancy between the diffusivity κ_b in the product and the diffusivity κ_u in the unburned mixture, the heat equilibrates more readily across the interface.

As κ_b approaches κ_u, we do see the stabilizing effect of the decreasing discrepancy in diffusivities across the interface for σ values smaller than about 0.2. In this regime, the graphs do shift higher, perhaps with reduced slope, as a_p increases. (See Figure 5.6.)

That is, for σ values smaller than about 0.2, Figure 5.6 shows that as $a_p = \dfrac{\kappa_b}{\kappa_u}$ increases, period-doubling bifurcations occur at increasing values of ϵ: We have to go deeper into the instability region to get a period-doubling bifurcation. For example, for $\sigma = 0.15$, the period-doubling bifurcations occur at the approximate parameter values shown in Table 5.2.

Table 5.1: Approximate values at which period-doubling bifurcations occur for simulations with $\epsilon = 0.2$ and $A(0) = 0.1$

a_p	σ	ϵ
0.1	0.9	0.2
0.15	0.84	0.2
0.2	0.8	0.2
0.25	0.79	0.2
0.3	0.78	0.2
0.35	0.77	0.2
0.5	0.72	0.2
0.65	0.67	0.2
0.8	0.62	0.2
1.0	0.57	0.2

Table 5.2: Approximate values at which period-doubling bifurcations occur for simulations with $\sigma = 0.15$ and $A(0) = 0.1$

a_p	σ	ϵ
0	0.15	0.088
0.05	0.15	0.107
0.1	0.15	0.117
0.15	0.15	0.125
0.2	0.15	0.132
0.25	0.15	0.137
0.3	0.15	0.142
0.35	0.15	0.146
0.5	0.15	0.152
0.65	0.15	0.154
0.8	0.15	0.155
1.0	0.15	0.155

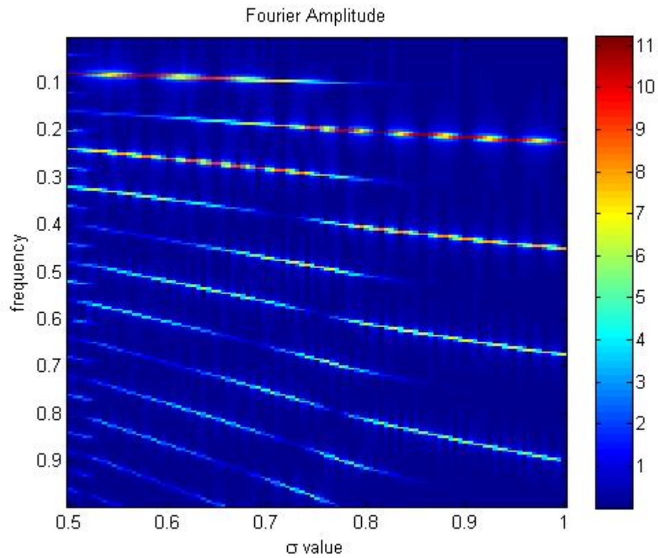

Figure 5.4: Amplitudes corresponding to each frequency of the Fourier transformed velocity perturbation data for $a_p = 0.2$*, the Arrhenius kinetics parameter* σ *in the interval* $(0.5, 1)$*,* $\epsilon = 0.2$*,* $A(0) = 0.1$*,* $1000 < t < 1500$ *(*$\nu = \nu_c - \epsilon^2$*)*

Figure 5.6 also shows that stabilization slows down as we continue to increase a_p from 0. That is the gap between the curves decrease, as a_p increase from 0, and eventually the curves pile up closely for lager a_p.

The stabilizing effect of increasing a_p does not persist as we increase σ. Figure 5.6 shows "cross-over" of the curves as σ increases beyond about 0.2. There seems to be a subtle interplay among the nonlinear effects in the problem.

In particular, one does not see a simple translation of curves in the $\sigma\epsilon$-plane upward on the whole interval $0 < \sigma < 1$ as a_p increases, given the nonlinearity of the curves and the nonlinearity in the problem. The rate of change of ϵ with respect to σ is non-constant for each choice of a_p. Depending on the curve in question, the slopes in Figure 5.6 may decrease and increase as σ increases.

We also examine this nonlinear behavior by focusing closely on the interval 0.2 <

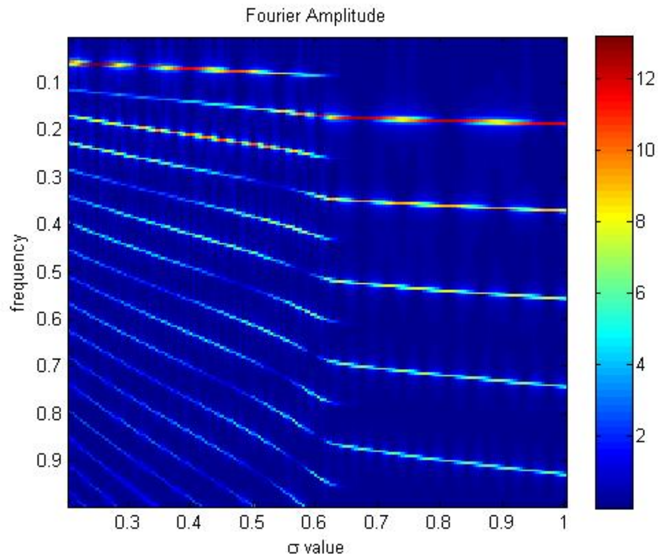

Figure 5.5: Amplitudes corresponding to each frequency of the Fourier transformed velocity perturbation data for $a_p = 0.8$, the Arrhenius kinetics parameter σ in the interval $(0.2, 1)$, $\epsilon = 0.2$, $A(0) = 0.1$, $1000 < t < 1500$ ($\nu = \nu_c - \epsilon^2$)

$\sigma < 0.8$. Figure 5.7 shows the region of overlapping profiles parameter space with three new points between every pair of points used in the previous Figure. This Figure reveals the evolution of a possible point of inflection as a_p changes. The curves may change from concave down to concave up, and this phenomenon becomes more pronounced as a_p increases.

5.2 Summary and Future Work

In summary, we have developed a generalized model of solid combustion in the first part of this dissertation. Our generalized model pinpoints the dynamics in a range of settings, in which the diffusivity ratio between the burned and unburned materials varies between 0 and 1. The dynamics involve an interplay of competing effects as

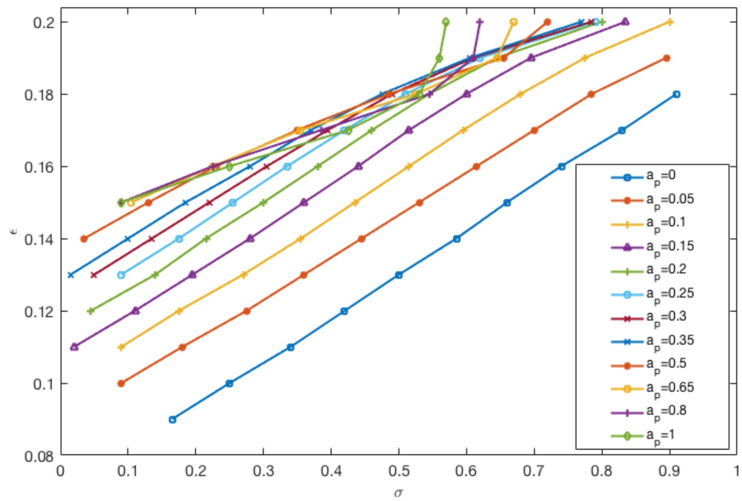

Figure 5.6: Approximate locations of pairs (σ, ϵ) *at which period-doubling bifurcations occur for* $a_p = 0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.5, 0.65,\ 0.8$ *and* 1.0 *(for simulations with* $A(0) = 0.1,\ \nu = \nu_c - \epsilon^2$*).*

the diffusivity ratio is tuned to capture different physical systems. Here we first analyze the linear instability of a basic solution to the generalized model. Then multiple scale expansions are utilized to describe changes in the amplitude and phase of perturbations to the basic solution in slow-time when the physical parameters are set in the weakly nonlinear regime. We have considered an inverse Zel'dovich number ν that deviates by a relatively small number ϵ^2 from the neutrally stable value into the unstable regime and simulated complex nonlinear dynamics for the generalized model. We have delineated the roles of various parameters.

In particular, Figure 5.6 shows that as the temperature ratio σ increases, ϵ increases for solutions to reach the same stage of period doubling. This stabilizing effect in σ is quite linear when ϵ is less than about 0.16. For larger values of ϵ, the behavior is more nonlinear, as expected. (See Figure 5.6.)

Increasing the diffusivity ratio a_p has a varying effect in different regimes. For σ

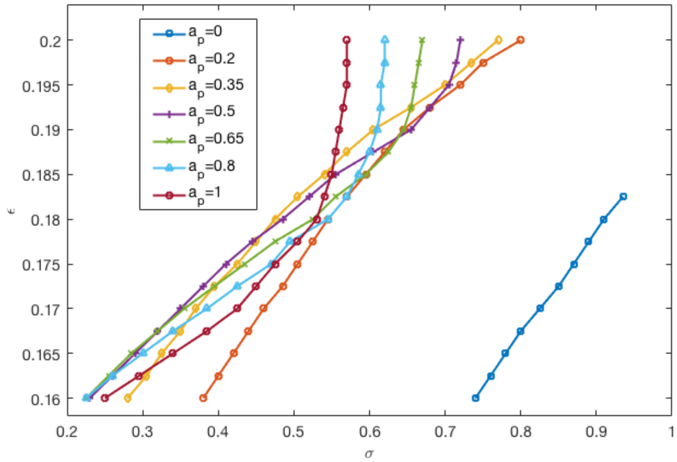

Figure 5.7: Approximate locations of pairs (σ, ϵ) *at which period-doubling bifurcations occur for* $a_p = 0, 0.2, 0.35, 0.5, 0.65,$ *0.8 and* 1.0 *(for simulations with* $A(0) = 0.1$*,* $\nu = \nu_c - \epsilon^2$*).*

values smaller than about 0.2, increasing a_p has a stabilizing effect. For larger values of σ, the graphs in Figure 5.6 evolve in a complex way as a_p increases, developing points of inflection and manifesting increasingly sharp stabilization in σ. The various competing effects produce the "cross-over" phenomenon that we see in Figure 5.6 as σ increases beyond about 0.2.

Future work will include:

1) Use more modes of linear solution in the asymptotic procedure in order to cover more variabilities in the nonlinear solution.

2) Extend the present work to 2D and 3D combustion models.

3) Develop new and more accurate combustion models, for example, the solid propellant combustion model as in [32] and [51].

PART II

TRANSMISSION PROBLEM OF THE PLATE

Chapter 6

INTRODUCTION

6.1 A Brief Historical Overview

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_2$ for Γ_0 and Γ_2 are closed, nonnull and disjoint, Γ_0 is the boundary of a small circle $C(x_0)$ containing $x_0 \in \mathbb{R}^2$, $\Omega_2 \subset \Omega$ is a subdomain with smooth boundary $\Gamma_0 \cup \Gamma_1$ in the outside of $C(x_0)$, $\Omega_1 = \Omega \setminus (\bar{\Omega}_2 \cup C(x_0))$ is a subdomain with smooth boundary $\Gamma_1 \cup \Gamma_2$, $\nu = (\nu_1, \nu_2)$ represents the outward unit normal vector to $\partial\Omega$ and $\tau = (-\nu_2, \nu_1)$ is the corresponding unit tangent vector, in cases of common boundary Γ_1 the vector ν is outward for Ω_1 (see Figure 6.1). We investigate the following transmission problem for the plate equation :

$$\begin{cases} K(x)u_{tt} + \Delta^2 u + f_1(u) = 0, & (x,t) \in \Omega_1 \times (0,\infty), \\ K(x)v_{tt} + \Delta^2 v + f_2(v) = 0, & (x,t) \in \Omega_2 \times (0,\infty), \\ v = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \Gamma_0 \times [0,\infty), \\ u = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}, \quad \mathscr{B}_1 u = \mathscr{B}_1 v, \quad \mathscr{B}_2 u = \mathscr{B}_2 v, & (x,t) \in \Gamma_1 \times [0,\infty), \\ -u + \int_0^t g_1(t-s)\mathscr{B}_2 u(s)\mathrm{d}s = \frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\mathscr{B}_1 u(s)\mathrm{d}s = 0, & (x,t) \in \Gamma_2 \times [0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega_1, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega_2, \end{cases} \tag{6.1.1}$$

where $K(x) \in C^1(\bar{\Omega})$ and $K(x) \geq 0$ for all $x \in \Omega$ which satisfies some appropriate conditions. The relaxation functions g_1, g_2 are positive and nonincreasing, the function $f_1, f_2 \in C^1(\mathbb{R})$ and

$$\mathscr{B}_1 u = \Delta u + (1-\mu)B_1 u, \quad \mathscr{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \tau} \tag{6.1.2}$$

with

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx}, \quad B_2 u = (\nu_1^2 - \nu_2^2)u_{xy} + \nu_1\nu_2(u_{yy} - u_{xx}),$$

and the constant $\mu \in \left(0, \frac{1}{2}\right)$, represents Poisson's ratio.

From the physical point of view, we know that the memory effect described in integral equation $(6.1.1)_5$ can be caused by the interaction with another viscoelastic element. In fact, the boundary conditions $(6.1.1)_3 - (6.1.1)_5$ mean that Ω is com-

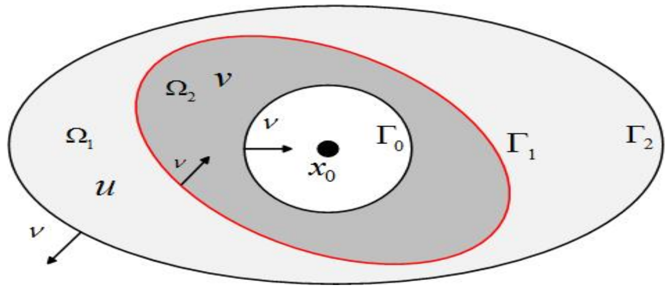

Figure 6.1: The configuration.

posed of two different materials with different boundary conditions on Γ_2 and Γ_0, respectively. Transmission problems related to (6.1.1) are interesting not only from the point of view of PDE general theory, but also due to its application in mechanics.

Recently, various decay results for classical wave equations were studied by many authors (see [6,10,11,21,26–29,34,46–49,59,62,63,65]). Messaoudi and Soufyane [36] considered the following wave equation with a boundary condition of memory type:

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u = 0, & (x,t) \in \Gamma_0 \times [0,\infty), \\ u + \displaystyle\int_0^t g(t-s)\frac{\partial u}{\partial \nu}\mathrm{d}s = 0, & (x,t) \in \Gamma_1 \times [0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{6.1.3}$$

By establishing some relations between the relaxation function g and the corresponding resolvent kernel, they proved a general decay result, which is more general than those usually found in the literature. In [57], Shin and Kang considered the following plate equation with a memory condition on the boundary:

$$\begin{cases} K(x)u_{tt} + \Delta^2 u + f(u) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \Gamma_0 \times [0,\infty), \\ -u + \int_0^t g_1(t-s)\mathscr{B}_2 u(s)\mathrm{d}s = \frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\mathscr{B}_1 u(s)\mathrm{d}s = 0, & (x,t) \in \Gamma_1 \times [0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{6.1.4}$$

If k_i denote the resolvent kernels of $-\frac{g_i'}{g_i(0)}$, $i = 1, 2$ and satisfy

$$k_i(0) > 0, \quad \lim_{t\to\infty} k_i(t) = 0, \quad k_i'(t) \le 0, \quad k_i''(t) \ge -\zeta_i(t)k_i'(t),\ \forall\ t \ge 0,\ i = 1, 2, \tag{6.1.5}$$

where $\zeta_i : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing continuous functions, a general decay result was established in their work, from which the usual exponential and polynomial decay rates are only special cases. More recently, Mustafa and Abusharkh [43] studied system (6.1.4) with $K(x) = 1$, $f(u) = 0$ and k_i satisfying

$$k_i(0) > 0, \quad \lim_{t\to\infty} k_i(t) = 0, \quad k_i'(t) \le 0, \quad k_i''(t) \ge H(-k_i'(t)), \quad \forall\ t > 0,\ i = 1, 2, \tag{6.1.6}$$

where H is a positive function, which is linear or strictly increasing and strictly convex of class C^2 on $(0, r]$, $r < 1$, and $H(0) = 0$. Under the assumption that $u_0 = \frac{\partial u_0}{\partial \nu} = 0$ on Γ_1, they obtained an explicit energy decay formula, which is not necessarily of exponential-type or polynomial-type. For more general decay results related to condition (6.1.6), we refer the readers to [7, 42] and the references therein.

On the other hand, the existence, regularity and decay results for transmission problems were studied by many authors recently [1,2,4,22,35]. In [1], Andrade et al. considered the following transmission problem:

$$
\begin{cases}
\rho_1 u_{tt} - \gamma_1 \Delta u + f_1(u) = 0, & (x,t) \in \Omega_1 \times (0,\infty), \\
\rho_2 v_{tt} - \gamma_2 \Delta v + f_2(v) = 0, & (x,t) \in \Omega_2 \times (0,\infty), \\
v = 0, & (x,t) \in \Gamma_0 \times [0,\infty), \\
u = v, \quad \gamma_1 \dfrac{\partial u}{\partial \nu} = \gamma_2 \dfrac{\partial v}{\partial \nu}, & (x,t) \in \Gamma_1 \times [0,\infty), \\
u + \displaystyle\int_0^t g(t-s) \frac{\partial u}{\partial \nu} \mathrm{d}s = 0, & (x,t) \in \Gamma_2 \times [0,\infty), \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega_1, \\
v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega_2.
\end{cases} \tag{6.1.7}
$$

They proved the existence of the global solution and showed that its solution decays exponentially when the relaxation function decays exponentially. Bae [4] studied the transmission problem for wave equation with boundary condition

$$
\begin{cases}
u_{tt} - \|\nabla u\|_{\Omega_1}^2 \Delta u + f_1(u) = 0, & (x,t) \in \Omega_1 \times (0,\infty), \\
v_{tt} - \|\nabla v\|_{\Omega_2}^2 \Delta v + f_2(v) = 0, & (x,t) \in \Omega_2 \times (0,\infty), \\
v = 0, & (x,t) \in \Gamma_0 \times [0,\infty), \\
u = v, \quad \|\nabla u\|_{\Omega_1}^2 \dfrac{\partial u}{\partial \nu} = \|\nabla v\|_{\Omega_2}^2 \dfrac{\partial v}{\partial \nu}, & (x,t) \in \Gamma_1 \times [0,\infty), \\
u + \displaystyle\int_0^t g(t-s) \|\nabla u\|_{\Omega_1}^2 \frac{\partial u}{\partial \nu} \mathrm{d}s = 0, & (x,t) \in \Gamma_2 \times [0,\infty), \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega_1, \\
v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega_2
\end{cases} \tag{6.1.8}
$$

and proved the exponential and polynomial decay rates when the kernel function decays exponentially and polynomially, respectively. More recently, Park [44] investigated the decay rate of the solution for the following transmission problem:

$$
\begin{cases}
u_{tt} - \left(1 + \|\nabla u\|^2_{\Omega_1}\right) \Delta u = 0, & (x,t) \in \Omega_1 \times (0,\infty), \\
v_{tt} - \left(1 + \|\nabla v\|^2_{\Omega_2}\right) \Delta v = 0, & (x,t) \in \Omega_2 \times (0,\infty), \\
v = 0, & (x,t) \in \Gamma_0 \times [0,\infty), \\
u = v, \quad \left(1 + \|\nabla u\|^2_{\Omega_1}\right) \dfrac{\partial u}{\partial \nu} = \left(1 + \|\nabla v\|^2_{\Omega_2}\right) \dfrac{\partial v}{\partial \nu}, & (x,t) \in \Gamma_1 \times [0,\infty), \\
u + \displaystyle\int_0^t g(t-s) \left(1 + \|\nabla u\|^2_{\Omega_1}\right) \dfrac{\partial u}{\partial \nu} \mathrm{d}s = 0, & (x,t) \in \Gamma_2 \times [0,\infty), \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega_1, \\
v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega_2.
\end{cases}
\tag{6.1.9}
$$

They established a general decay rate of the solution for the above transmission problem, which generalized the earlier decay results of the solution to problem (6.1.9).

Motivated by these results, in the present work we investigate the existence and decay of the solution for the transmission problem (6.1.1) by combining the frameworks of Guesmia [20], Shin and Kang [57], Park [44,45], Mustafa and Abusharkh [43] and Boulanouar and Drabla [7] with some necessary modifications due to the nature of the problem treated here. More precisely, firstly, to overcome the technical difficulties in dealing with $(6.1.1)_3$, we use the idea in [50,57] to transform it into a more general condition (see (6.2.4) and (6.2.5) below) by using the inverse Volterra's operator. Then, by using Faedo-Galerkin's method and compactness arguments as used in [45], we prove the global existence of weak solution. After that, for the resolvent kernels satisfy (6.1.5), by introducing suitable energy and Lyapunov functionals, we show that the energy decays at the rate similar to the relaxation functions, which are not necessarily decaying like polynomial or exponential functions; while for the resolvent kernels satisfy (6.1.6), due to there is no restriction of $u_0 = \frac{\partial u_0}{\partial \nu} = 0$ on Γ_2, the energy E defined by (6.2.13) not necessarily dissipative, see (7.2.2) below. In order to overcome this difficult to obtain general energy decay estimate, we shall use the multiplier method and construct suitable perturbed Lyapunov functionals, combing the frameworks of [7] with necessary modifications. This general decay result also allows a larger class of relaxation functions and initial data, and hence generalizes some previous results existing in the literature.

6.2 Preliminaries and Main Results

In this section, we present some materials needed in the proof of our results. To begin with, let's recall some useful definitions.

Definition 6.2.1 *Let X be a measurable set of $\mathbb{R}^n$, we say a function f is in $L^p(X)$ (Lebesgue spces) if f is measurable and $\int_X |f|^p \mathrm{d}x < \infty$.*

Remark 1 *The integral here is in the Lebsegue sense and $0 < p < +\infty$. Also, we equip $L^p(X)$ with the following norm*

$$||f||_p = \left(\int_X |f|^p \mathrm{d}x \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty$$

Definition 6.2.2 *Let X be a measurable set of $\mathbb{R}^n$, we call $L^\infty(X)$ the set of all functions which are bounded on X, except maybe on a subset of measure zero.*

Remark 2 *The norm for* $L^\infty(X)$ *is defined by*

$$||f||_\infty = \operatorname{ess\,sup}|f|.$$

Remark 3 *The* $L^p(X)$, $1 \le p \le \infty$, *equipped with the above norms is complete (Banach space).*

Remark 4 *The* $L^2(X)$ *equipped with the following inner product*

$$< f, g >= \int_X fg\mathrm{d}x.$$

is a Hilbert space.

Definition 6.2.3 *(Weak derivative) Let* f *and* g *are in space* $L^1_{loc}(X)$ *of locally integrable functions for some open set* $X \in \mathbb{R}^n$ *and if* α *is a multi-index, we say that* g *is the* α^{th}*-weak derivative of* f *if*

$$\int_X f \cdot D^\alpha \phi \mathrm{d}x = (-1)^\alpha \int_X g \cdot \phi \mathrm{d}x,$$

for all $\phi \in C_c^\infty(X)$, *that is, for all infinitely differentiable functions* ϕ *with compact support in* X. *Here* $D^\alpha\phi$ *is defined as*

$$D^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Definition 6.2.4 *(Sobolev space) The Sobolev space* $W^{k,p}(X)$, $1 \le p \le \infty$ *is the space of all locally summable functions* $f : X \longrightarrow \mathbb{R}$ *such that, for every multi-index* α *with* $|\alpha| \le k$, *the weak derivative* $D^\alpha f$ *exists and belongs to* $L^p(X)$.

Remark 5 *If for any* $f \in W^{k,p}(X)$, $1 \leq p < \infty$ *we equip the above Sobolev space with the following norm:*

$$\|f\|_{k,p} = \left(\sum_{i=0}^{k} \left\|f^{(i)}\right\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{i=0}^{k} \int \left|f^{(i)}(t)\right|^p dt\right)^{\frac{1}{p}}.$$

then $W^{k,p}$ *becomes a Banach space. It turns out that the norm defined by*

$$\left\|f^{(k)}\right\|_p + \|f\|_p$$

is equivalent to the norm above.

Remark 6 *If* $k = 0$, *then* $W^{0,p}(X) = L^p(X)$.

Remark 7 *Sobolev spaces with* $p = 2$ *are especially important because they form a Hilbert space (equipped with normal inner product). A special notation has arisen to cover this case, since the space is a Hilbert space:*

$$H^k(X) = W^{k,2}(X).$$

Next, we list several important inequalities that are frequently used in our proofs.

Lemma 6.2.1 (*Jensen's inequality*) *If* F *is a convex function on* $[a, b]$, $f : \Omega \to [a, b]$ *and* h *are integrable functions on* Ω, $h(x) \geq 0$, *and* $\int_\Omega h(x)\mathrm{d}x = p > 0$, *then Jensen's inequality sates that*

$$F\left[\frac{1}{p}\int_\Omega f(x)h(x)\mathrm{d}x\right] \leq \frac{1}{p}\int_\Omega F[f(x)]h(x)\mathrm{d}x.$$

Lemma 6.2.2 *($H\ddot{o}lder's$ inequality) Let p and q be nonnegative real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(X)$ and $g \in L^q(X)$, then $fg \in L^1(X)$ and*

$$\int_X |fg| \mathrm{d}x \leq ||f||_p ||g||_q.$$

Lemma 6.2.3 *($Young's$ inequality) Let a, b be positive real numbers and $p > 1$ and $q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq a^p + b^q.$$

More generally, for some small value $\epsilon > 0$, we have,

$$ab \leq \epsilon a^p + C(\epsilon) b^q.$$

Remark 8 *The special case $p = q = 2$ is known as Cauchy's inequality.*

For our purpose, throughout this dissertation, we define

$$W = \left\{ \phi \in H^2(\Omega_2) : \phi = \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\},$$

$$V = \left\{ (\varphi, \phi) \in H^2(\Omega_1) \times W(\Omega_2) : \varphi = \phi, \frac{\partial \varphi}{\partial \nu} = \frac{\partial \phi}{\partial \nu} \text{ on } \Gamma_1 \right\},$$

and

$$(\varphi, \phi)_{\Omega_i} = \int_{\Omega_i} \varphi(x)\phi(x)\mathrm{d}x \text{ for } i = 1, 2, \ (\varphi, \phi)_{\Gamma_j} = \int_{\Gamma_j} \varphi(x)\phi(x)\mathrm{d}\Gamma \text{ for } j = 0, 1, 2.$$

As in [50, 57], let us define the bilinear form $a_i(\cdot,\cdot)$ $(i=1,2)$ as follows:

$$a_i(\varphi,\phi)=\int_{\Omega_i}\{\varphi_{xx}\phi_{xx}+\varphi_{yy}\phi_{yy}+\mu\left(\varphi_{xx}\phi_{yy}+\varphi_{yy}\phi_{xx}\right)+2(1-\mu)\varphi_{xy}\phi_{xy}\}\,\mathrm{d}x\mathrm{d}y,$$

and we know that $a_i(\varphi,\varphi)$ are equivalent to the $H^2(\Omega)$ norm; that is

$$c\|\varphi\|^2_{H^2(\Omega_i)}\le a_i(\varphi,\varphi)\le C\|\varphi\|^2_{H^2(\Omega_i)},\ \ i=1,2, \tag{6.2.1}$$

where c and C are positive constants. As in [57], a simple calculation, based on the integration by parts formula, yields

$$(\Delta^2\varphi,\phi)_{\Omega_i}=a_i(\varphi,\phi)+(\mathscr{B}_2\varphi,\phi)_{\partial\Omega_i}-\left(\mathscr{B}_1\varphi,\frac{\partial\phi}{\partial\nu}\right)_{\partial\Omega_i},\ \ i=1,2. \tag{6.2.2}$$

Besides, set x^0 be a fixed point in $\mathbb{R}^2$, $m=x-x^0$ and $R=\max\{\mid x-x^0\mid: x\in\bar{\Omega}\}$. Assume that there exists a small positive constant δ such that

$$\Gamma_2=\{x\in\Gamma: m\cdot\nu\ge\delta>0\}\text{ and }\Gamma_0=\{x\in\Gamma: m\cdot\nu\le 0\}. \tag{6.2.3}$$

First, following the idea in [50, 57], we shall use $(6.1.1)_3$ to estimate the values $\mathscr{B}_1$ and $\mathscr{B}_2$ on Γ_2. Denoting by

$$(g*\varphi)(t):=\int_0^t g(t-s)\varphi(s)\mathrm{d}s,$$

the convolution product operator, and differentiating $(6.1.1)_5$ with respect to t, we obtain the following Volterra equations:

$$\mathscr{B}_2 u + \frac{1}{g_1(0)} g_1' * \mathscr{B}_2 u = \frac{1}{g_1(0)} u_t,$$

$$\mathscr{B}_1 u + \frac{1}{g_2(0)} g_2' * \mathscr{B}_1 u = -\frac{1}{g_2(0)} \frac{\partial u_t}{\partial \nu}.$$

Then using the Volterra's inverse operator, we get

$$\mathscr{B}_2 u = \frac{1}{g_1(0)} \{u_t + k_1 * u_t\},$$

$$\mathscr{B}_1 u = -\frac{1}{g_2(0)} \left\{ \frac{\partial u_t}{\partial \nu} + k_2 * \frac{\partial u_t}{\partial \nu} \right\},$$

where the resolvent kernels satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i', \ i = 1, 2.$$

Denoting by $\tau_1 = \frac{1}{g_1(0)}$ and $\tau_2 = \frac{1}{g_2(0)}$, we obtain

$$\mathscr{B}_2 u = \tau_1 \{u_t + k_1(0) u - k_1(t) u_0 + k_1' * u\}, \tag{6.2.4}$$

and

$$\mathscr{B}_1 u = -\tau_2 \left\{ \frac{\partial u_t}{\partial \nu} + k_2(0) \frac{\partial u}{\partial \nu} - k_2(t) \frac{\partial u_0}{\partial \nu} + k_2' * \frac{\partial u}{\partial \nu} \right\}. \tag{6.2.5}$$

Reciprocally, considering that the initial data satisfies $u_0 = \frac{\partial u_0}{\partial \nu} = 0$ on Γ_2, (6.2.4) and (6.2.5) imply $(6.1.1)_5$. Therefore, we use equation (6.2.4) and (6.2.5) instead of the boundary conditions $(6.1.1)_5$. Then, we get the following more broader problem:

$$\begin{cases} K(x)u_{tt} + \Delta^2 u + f_1(u) = 0, & (x,t) \in \Omega_1 \times (0,\infty), \\ K(x)v_{tt} + \Delta^2 v + f_2(v) = 0, & (x,t) \in \Omega_2 \times (0,\infty), \\ v = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \Gamma_0 \times [0,\infty), \\ u = v, \quad \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu}, \quad \mathscr{B}_1 u = \mathscr{B}_1 v, \quad \mathscr{B}_2 u = \mathscr{B}_2 v, & (x,t) \in \Gamma_1 \times [0,\infty), \\ \mathscr{B}_2 u = \tau_1 \{u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\}, & (x,t) \in \Gamma_2 \times [0,\infty), \\ \mathscr{B}_1 u = -\tau_2 \left\{ \dfrac{\partial u_t}{\partial \nu} + k_2(0)\dfrac{\partial u}{\partial \nu} - k_2(t)\dfrac{\partial u_0}{\partial \nu} + k_2' * \dfrac{\partial u}{\partial \nu} \right\}, & (x,t) \in \Gamma_2 \times [0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega_1, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega_2. \end{cases} \tag{6.2.6}$$

We note that when $u_0 = \dfrac{\partial u_0}{\partial \nu} = 0$ on Γ_2, problem (6.2.6) is equivalent to problem (6.1.1).

Let us denote

$$(g\Box\varphi)(t) := \int_0^t g(t-s)|\varphi(t) - \varphi(s)|^2 \mathrm{d}s.$$

The following lemma gives an important property of the convolution operator.

Lemma 6.2.4 *[41] For $g, \varphi \in C^1([0,\infty) : \mathbb{R})$, we have*

$$(g * \varphi)\varphi_t = -\frac{1}{2}g(t)|\varphi(t)|^2 + \frac{1}{2}g'\Box\varphi - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[g\Box\varphi - \left(\int_0^t g(s)\mathrm{d}s\right)|\varphi|^2\right].$$

The proof of this lemma follows by differentiating the term $g\Box\varphi$.

Lemma 6.2.5 *[23] Suppose that $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ and $h \in H^{3/2}(\Gamma)$, then* *any solution of*

$$a(\varphi, w) = \int_\Omega f w \mathrm{d}x + \int_\Gamma g w \mathrm{d}\Gamma + \int_\Gamma h \frac{\partial w}{\partial \nu} \mathrm{d}\Gamma, \ \forall w \in W$$

satisfies $\varphi \in H^4(\Omega)$.

Now, we state our main assumptions:

(A_1) Let $f_i \in C^1(\mathbb{R})$ satisfy

$$f_i(s)s \geq 0, \ \forall \ s \in \mathbb{R}, \ i = 1, 2.$$

Additionally, we suppose that $f_1 = f_2$ on Γ_1 and f_i are superlinear, that is,

$$f_i(s)s \geq (2 + \eta_i)F_i(s), \quad F_i(z) = \int_0^z f_i(s)ds, \ \forall \ s \in \mathbb{R}, i = 1, 2, \tag{6.2.7}$$

for some $\eta_i > 0$ with the following growth condition:

$$|f_i(x) - f_i(y)| \leq c \left(1 + |x|^{\rho-1} + |y|^{\rho-1}\right) |x - y|, \ \forall \ x, y \in \mathbb{R},$$

for some $c > 0$ and $\rho \geq 1$ such that $(n-2)\rho \leq n$.

(A_2) $K \in C^1(\bar{\Omega})$; $H_0^2(\Omega) \cap L^\infty(\Omega)$ with $K(x) \geq 0$, for all $x \in \Omega$, and satisfies the following condition:

$$\nabla K \cdot m \geq 0, \ x \in \Omega. \tag{6.2.8}$$

(A_3) $k_i : \mathbb{R}^+ \to \mathbb{R}^+$, for $i = 1, 2$, are C^2 functions such that

$$k_i(0) > 0, \ \lim_{t \to \infty} k_i(t) = 0, \ k_i'(t) \leq 0,$$

and there exist nonincreasing continuous functions $\zeta_i : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying

$$k_i''(t) \geq -\zeta_i(t) k_i'(t), \ \forall\, t \geq 0, \ i = 1, 2. \tag{6.2.9}$$

(A_4) $k_i : \mathbb{R}^+ \to \mathbb{R}^+$, for $i = 1, 2$, are C^2 functions such that

$$k_i(0) > 0, \ \lim_{t\to\infty} k_i(t) = 0, \ k_i'(t) \leq 0,$$

and there exists a positive function $H \in C^1(\mathbb{R}^+)$, where H is linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, with $H(0) = H'(0) = 0$, such that

$$k_i''(t) \geq H(-k_i'(t)), \ \forall\, t > 0, \ i = 1, 2. \tag{6.2.10}$$

According to previous results in the literature (see e.g. [45, 57]), we can state the following well-posedness.

Theorem 6.2.6 *Consider assumptions* (A_1)-(A_2) *and let* $k_i \in C^2(\mathbb{R}^+)$ *be such that*

$$k_i, \ -k_i', \ k_i'' \geq 0, \ i = 1, 2. \tag{6.2.11}$$

If $(u_0, v_0) \in H^4(\Omega_1) \times (W \cap H^4(\Omega_2))$, $(u_1, v_1) \in V$, *satisfying the compatibility condition*

$$\mathscr{B}_1 u_0 = -\tau_2 \frac{\partial u_1}{\partial \nu}, \quad \mathscr{B}_2 u_0 = \tau_1 u_1 \text{ on } \Gamma_2, \tag{6.2.12}$$

then problem (6.2.6) *has a unique solution in the class:*

$$(u, v) \in L^\infty\left(0, T; H^4(\Omega_1) \times \left(W \cap H^4(\Omega_2)\right)\right)$$

To state our decay result, we introduce the following energy functional:

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\Omega_1} K(x)|u_t|^2\mathrm{d}x + \frac{1}{2}\int_{\Omega_2} K(x)|v_t|^2\mathrm{d}x + \frac{1}{2}a_1(u,u) + \frac{1}{2}a_2(v,v) \\ &+ \int_{\Omega_1} F_1(u)\mathrm{d}x + \int_{\Omega_2} F_2(v)\mathrm{d}x + \frac{\tau_1}{2}\int_{\Gamma_2}\left(k_1(t)|u|^2 - k_1'\Box u\right)\mathrm{d}\Gamma \\ &+ \frac{\tau_2}{2}\int_{\Gamma_2}\left(k_2(t)\left|\frac{\partial u}{\partial \nu}\right|^2 - k_2'\Box\frac{\partial u}{\partial \nu}\right)\mathrm{d}\Gamma. \end{aligned} \tag{6.2.13}$$

If the resolvent kernels satisfy (A_3) and (A_4), respectively, then we have the following two general decay results for problem (6.2.6):

Theorem 6.2.7 *Suppose that (A_1), (A_2) and (A_3) hold. Then, for some t_0 large enough, there exist constants $\omega, C > 0$ such that:*

(i) If $u_0 = \dfrac{\partial u_0}{\partial \nu} = 0$ on Γ_2, then

$$E(t) \le CE(0)e^{-\omega\int_0^t \zeta(s)\mathrm{d}s}, \forall\ t \ge t_0. \tag{6.2.14}$$

(ii) Otherwise,

$$E(t) \le C\left(E(0) + \int_0^t k_0(s)e^{\omega\int_0^s \zeta(\tau)\mathrm{d}\tau}\mathrm{d}s\right)e^{-\omega\int_0^t \zeta(s)\mathrm{d}s}, \forall\ t \ge t_0, \tag{6.2.15}$$

where

$$\zeta(t) = \min\{\zeta_1(t), \zeta_2(t)\},\ \ k_0(t) = \int_{\Gamma_2} k_1^2(t)|u_0|^2\mathrm{d}\Gamma + \int_{\Gamma_2} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma.$$

Remark 9 *We note that the exponential and the polynomial decay estimates are only particular cases of (6.2.14). In fact, we obtain exponential decay for $\zeta(t) \equiv a$ and polynomial decay for $\zeta(t) = a(1+t)^{-1}$, where a is a constant.*

Theorem 6.2.8 *Suppose that* (A_1), (A_2) *and* (A_4) *hold. Then, for some* t_1 *large enough, there exist some positive constants* c_0, c_1, c_2 *and* C *such that:*

(*i*) *In the special case of* $H(t) = ct^p$, *where c is a positive constant and* $1 \leq p < \frac{3}{2}$, *the solution of* (6.2.6) *satisfies*

$$
\begin{aligned}
& E(t) \\
\leq \; & \left(\frac{c_0 + c_1 \int_{t_1}^{t} \left[k_1(s) \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma \right]^{2p-1} \mathrm{d}s + c_2 \int_{t_1}^{t} \left[k_2(s) \int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma \right]^{2p-1} \mathrm{d}s}{t} \right)^{\frac{1}{2p-1}} \\
& - \frac{\tau_1}{2} \left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma \right) \int_t^{\infty} k_1^2(s) \mathrm{d}s - \frac{\tau_2}{2} \left(\int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma \right) \int_t^{\infty} k_2^2(s) \mathrm{d}s, \quad \forall \; t \geq t_1.
\end{aligned}
\tag{6.2.16}
$$

(*ii*) *In the general case, the solution of* (6.2.6) *satisfies*

$$
\begin{aligned}
& E(t) \\
\leq \; & C H_1^{-1} \left(\frac{c_0 + c_1 \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma \int_{t_1}^{t} H_0(k_1(s)) \mathrm{d}s + c_2 \int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma \int_{t_1}^{t} H_0(k_2(s)) \mathrm{d}s}{t} \right) \\
& - \frac{\tau_1}{2} \left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma \right) \int_t^{\infty} k_1^2(s) \mathrm{d}s - \frac{\tau_2}{2} \left(\int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma \right) \int_t^{\infty} k_2^2(s) \mathrm{d}s, \quad \forall \; t \geq t_1.
\end{aligned}
\tag{6.2.17}
$$

where

$$
H_1(t) = t H_0'(\varepsilon_0 t), \quad H_0(t) = H(D(t)),
$$

D is a positive C^1 function with $D(0) = 0$, and H_0 is a strictly increasing and strictly convex C^2 function on $(0, r]$ satisfying

$$\int_0^\infty \frac{-k_i'(s)}{H_0^{-1}(k_i''(s))} \mathrm{d}s < +\infty, \ i = 1, 2. \tag{6.2.18}$$

Remark 10 (1) *If $u_0 = \frac{\partial u_0}{\partial \nu} = 0$ on Γ_2, then we have*

(*i*) *In the special case of $H(t) = ct^p$, where c is a positive constant and $1 \le p < \frac{3}{2}$, then the solution of* (6.2.6) *satisfies*

$$E(t) \le \left(\frac{c_0}{t}\right)^{\frac{1}{2p-1}}, \quad \forall\, t \ge 0.$$

(*ii*) *In the general case, the solution of* (6.2.6) *satisfies*

$$E(t) \le C H_1^{-1}\left(\frac{c_0}{t}\right), \qquad \forall\, t \ge 0.$$

These results are similar to [43].

(2) *If $\int_0^\infty H_0(k_i(s))\mathrm{d}s < +\infty$, $i = 1, 2$, then from* (6.2.17), *we have*

$$E(t) \le C H_1^{-1}\left(\frac{c}{t}\right) - \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s - \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s,$$

which clearly shows that $\lim_{t\to\infty} E(t) = 0$.

(3) *The usual exponential and polynomial decay rate estimates, which have already been proved for k_i satisfying $k_i'' \ge d(-k_i')^p$, $i = 1, 2$ and $1 \le p < \frac{3}{2}$, are special cases of our result. The above condition assumes that $-k_i'(t) \le \omega e^{-dt}$ for $p = 1$, and $-k_i'(t) \le \frac{\omega}{t^{\frac{1}{p-1}}}$ for $1 < p < \frac{3}{2}$. Our result allows resolvent kernels whose derivatives are*

not necessarily of exponential or polynomial decay. We give an example in Appendix A (see A.2).

(4) *As in [7, 43], for* $i = 1, 2$, *since* $\lim_{t \to +\infty} k_i(t) = 0$ *and* $-k_i'(t)$ *are nonnegative and nonincreasing, then we can easily deduce that* $\lim_{t \to +\infty} (-k_i'(t)) = 0$. *Similarly, assuming the existence of the limit, we find that* $\lim_{t \to +\infty} k_i''(t) = 0$. *Hence, there exists* $t_1 > 0$ *large enough such that* $k_i'(t_1) < 0$ *and*

$$\max\{k_i(t), -k_i'(t), k_i''(t)\} < \min\{r, H(r), H_0(r)\}, \ \forall \ t \geq t_1, \ i = 1, 2. \tag{6.2.19}$$

Noting that k_i' *are nondecreasing,* $k_i'(0) < 0$ *and* $k_i'(t_1) < 0$, *then* $k_i'(t) < 0$ *for any* $t \in [0, t_1]$ *and*

$$0 < -k_i'(t_1) \leq -k_i'(t) \leq k_i'(0), \ \forall \ t \in [0, t_1], \ i = 1, 2.$$

Also, since H *is a positive continuous function, then*

$$a \leq H(-k_i'(t)) \leq b, \ \forall \ t \in [0, t_1], \ i = 1, 2$$

for some positive constants a *and* b. *Therefore, we have*

$$k_i''(t) \geq H(-k_i'(t)) \geq a = \frac{a}{k_i'(0)} k_i'(0) \geq \frac{a}{k_i'(0)} k_i'(t), \ \forall \ t \in [0, t_1], \ i = 1, 2,$$

which implies, for some positive constant d,

$$k_i''(t) \geq d(-k_i'(t)), \ \forall \ t \in [0, t_1], \ i = 1, 2 \tag{6.2.20}$$

Chapter 7

PROOF OF MAIN THEOREMS

7.1 Existence: Proof of Theorem 6.2.6

In this section we, briefly prove the existence of global weak solution to problem (6.2.6). For convenience, from now on, we shall omit x and t in some functions if there is no ambiguity, C denotes an arbitrary positive constant, which may be different from line to line.

Since $K \geq 0$, before applying directly the Faedo-Galerkin's method, we first need to perturb problem (6.2.6) with terms $\epsilon u''$ and $\epsilon v''$ $(0 < \epsilon < 1)$.

Let $\{w_j\}_{j\in\mathbb{N}}$ be the basis of W which is orthogonal and normalized in $L^2(\Omega)$. For each $m \in \mathbb{N}$, let $U_m = \text{span}\{w_1, w_2, ..., w_m\}$. For each $\epsilon \in (0,1)$, $m \in \mathbb{N}$ and any $T > 0$, standard results on ordinary differential equations guarantee that, for $0 < T_m \leq T$, there exists only one local solution

$$u_{m\epsilon}(x,t) = \sum_{j=1}^{m} \alpha_{jm}(t) w_j(x), \quad x \in \Omega_1 \text{ and } t \in [0, T_m],$$

$$v_{m\epsilon}(x,t)=\sum_{j=1}^{m}\beta_{jm}(t)w_j(x), \quad x\in\Omega_2 \ \text{ and } \ t\in[0,T_m]$$

satisfy the approximate perturbed problem

$$\begin{cases} \quad ((K+\epsilon)u''_{m\epsilon}(t),w)_{\Omega_1}+a_1(u_{m\epsilon}(t)),w)+(f_1(u_{m\epsilon}),w)_{\Omega_1} \\ =(\mathscr{B}_2u_{m\epsilon},w)_{\Gamma_1}-(\mathscr{B}_1u_{m\epsilon},w)_{\Gamma_1}-(\mathscr{B}_2u_{m\epsilon},w)_{\Gamma_2}+(\mathscr{B}_1u_{m\epsilon},w)_{\Gamma_2} \\ \quad ((K+\epsilon)v''_{m\epsilon}(t),w)_{\Omega_2}+a_2(v_{m\epsilon}(t)),w)+(f_2(v_{m\epsilon}),w)_{\Omega_2} \\ =-(\mathscr{B}_2v_{m\epsilon},w)_{\Gamma_1}+(\mathscr{B}_1v_{m\epsilon},w)_{\Gamma_1} \end{cases} \tag{7.1.1}$$

for all $w\in U_m$, where

$$\begin{cases} (\mathscr{B}_2u_{m\epsilon},w)_{\Gamma_1}=\tau_1\left(u'_{m\varepsilon}+k_1(0)u_{m\epsilon}-k_1(t)u_{m\epsilon}(0)+k'_1*u_{m\varepsilon},w\right)_{\Gamma_1} \\ (\mathscr{B}_1u_{m\epsilon},w)_{\Gamma_1}=-\tau_2\left(\dfrac{\partial u'_{m\epsilon}}{\partial\nu}+k_2(0)\dfrac{\partial u_{m\epsilon}}{\partial\nu}-k_2(t)\dfrac{\partial u'_{m\epsilon}(0)}{\partial\nu}+k'_2*\dfrac{\partial u_{m\epsilon}}{\partial\nu},\dfrac{\partial w}{\partial\nu}\right)_{\Gamma_1} \\ (\mathscr{B}_2u_{m\epsilon},w)_{\Gamma_2}=\tau_1\left(u'_{m\varepsilon}+k_1(0)u_{m\epsilon}-k_1(t)u_{m\epsilon}(0)+k'_1*u_{m\varepsilon},w\right)_{\Gamma_2} \\ (\mathscr{B}_1u_{m\epsilon},w)_{\Gamma_2}=-\tau_2\left(\dfrac{\partial u'_{m\epsilon}}{\partial\nu}+k_2(0)\dfrac{\partial u_{m\epsilon}}{\partial\nu}-k_2(t)\dfrac{\partial u'_{m\epsilon}(0)}{\partial\nu}+k'_2*\dfrac{\partial u_{m\epsilon}}{\partial\nu},\dfrac{\partial w}{\partial\nu}\right)_{\Gamma_2} \\ (\mathscr{B}_2v_{m\epsilon},w)_{\Gamma_1}=\tau_1\left(v'_{m\varepsilon}+k_1(0)v_{m\epsilon}-k_1(t)v_{m\epsilon}(0)+k'_1*v_{m\varepsilon},w\right)_{\Gamma_1} \\ (\mathscr{B}_1v_{m\epsilon},w)_{\Gamma_1}=-\tau_2\left(\dfrac{\partial v'_{m\epsilon}}{\partial\nu}+k_2(0)\dfrac{\partial v_{m\epsilon}}{\partial\nu}-k_2(t)\dfrac{\partial v'_{m\epsilon}(0)}{\partial\nu}+k'_2*\dfrac{\partial v_{m\epsilon}}{\partial\nu},\dfrac{\partial w}{\partial\nu}\right)_{\Gamma_1} \end{cases} \tag{7.1.2}$$

with the transmission conditions

$$\begin{aligned} &u_{m\varepsilon}=v_{m\epsilon}, \quad \frac{\partial u_{m\epsilon}}{\partial\nu}=\frac{\partial v_{m\varepsilon}}{\partial\nu}, \\ &\mathscr{B}_1u_{m\epsilon}=\mathscr{B}_1v_{m\varepsilon}, \quad \mathscr{B}_2u_{m\epsilon}=\mathscr{B}_2v_{m\varepsilon}, \quad (x,t)\in\Gamma_1\times[0,\infty) \end{aligned} \tag{7.1.3}$$

with initial data

$$\begin{aligned}(u_{m\epsilon}(0), v_{m\epsilon}(0)) &= (u_{m0}, v_{m0}) \quad \text{strongly in} \quad H^4(\Omega_1) \times \left(W \cap H^4(\Omega_2)\right), \\ (u'_{m\epsilon}(0), v'_{m\epsilon}(0)) &= (u_{m1}, v_{m1}) \quad \text{strongly in} \quad V.\end{aligned} \tag{7.1.4}$$

Now we need estimates which allow us to extend the solutions to the whole interval $[0, T]$ and pass to limit as $m \to \infty$ and $\epsilon \to 0$. Therefore, uniform estimates with respect to m and ϵ are needed.

The First Estimate: Letting $w = u'_{m\epsilon}(t)$ in equation $(7.1.1)_1$ and $w = v'_{m\epsilon}(t)$ in equation $(7.1.1)_2$, respectively and integrating over $(0, t)$, we get by using the transmission condition (7.1.3) that

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega_1}(K(x)+\epsilon)|u'_{m\epsilon}|^2\mathrm{d}x + \int_{\Omega_2}(K(x)+\epsilon)|v'_{m\epsilon}|^2\mathrm{d}x + a_1(u_{m\varepsilon}, u_{m\epsilon}) \right. \\ & \left. + a_2(v_{m\epsilon}, v_{m\epsilon}) + 2\int_{\Omega_1} F_1(u_{m\epsilon})\mathrm{d}x + 2\int_{\Omega_2} F_2(v_{m\epsilon})\mathrm{d}x\right\} \\ = & -\int_{\Gamma_1}(\mathscr{B}_2 u_{m\epsilon})u'_{m\epsilon}\mathrm{d}\Gamma + \int_{\Gamma_1}(\mathscr{B}_1 u_{m\epsilon})\frac{\partial u'_{m\epsilon}}{\partial \nu}\mathrm{d}\Gamma + \int_{\Gamma_1}(\mathscr{B}_2 v_{m\epsilon})v'_{m\epsilon}\mathrm{d}\Gamma \\ & -\int_{\Gamma_1}(\mathscr{B}_1 v_{m\epsilon})\frac{\partial v'_{m\epsilon}}{\partial \nu}\mathrm{d}\Gamma - \int_{\Gamma_2}(\mathscr{B}_2 u_{m\epsilon})u'_{m\epsilon}\mathrm{d}\Gamma + \int_{\Gamma_2}(\mathscr{B}_1 u_{m\epsilon})\frac{\partial u'_{m\epsilon}}{\partial \nu}\mathrm{d}\Gamma \\ = & -\int_{\Gamma_2}(\mathscr{B}_2 u_{m\epsilon})u'_{m\epsilon}\mathrm{d}\Gamma + \int_{\Gamma_2}(\mathscr{B}_1 u_{m\epsilon})\frac{\partial u'_{m\epsilon}}{\partial \nu}\mathrm{d}\Gamma. \end{aligned} \tag{7.1.5}$$

By using (7.1.2), we get from Lemma 6.2.4 that

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega_1}(K(x)+\epsilon)|u'_{m\epsilon}|^2\mathrm{d}x + \int_{\Omega_2}(K(x)+\epsilon)|v'_{m\epsilon}|^2\mathrm{d}x + a_1(u_{m\varepsilon}, u_{m\epsilon}) \right. \\ & + a_2(v_{m\epsilon}, v_{m\epsilon}) + 2\int_{\Omega_1} F_1(u_{m\epsilon})\mathrm{d}x + 2\int_{\Omega_2} F_2(v_{m\epsilon})\mathrm{d}x + \tau_1 k_1(t)\int_{\Gamma_2}|u_{m\epsilon}|^2\mathrm{d}\Gamma \\ & \left. -\tau_1\int_{\Gamma_2} k'_1\Box u_{m\epsilon}\mathrm{d}\Gamma + \tau_2 k_2(t)\int_{\Gamma_2}\left|\frac{\partial u_{m\epsilon}}{\partial \nu}\right|^2\mathrm{d}\Gamma - \tau_2\int_{\Gamma_2} k'_2\Box\frac{\partial u_{m\epsilon}}{\partial \nu}\mathrm{d}\Gamma\right\} \end{aligned}$$

$$
\begin{aligned}
= \quad & -\tau_1 \int_{\Gamma_2} |u'_{m\epsilon}|^2 \mathrm{d}\Gamma + \tau_1 k_1(t) \int_{\Gamma_2} u_{m0} u'_{m\epsilon} \mathrm{d}\Gamma + \frac{\tau_1}{2} k'_1(t) \int_{\Gamma_1} |u_{m\epsilon}|^2 \mathrm{d}\Gamma \\
& - \frac{\tau_1}{2} \int_{\Gamma_2} k''_1 \Box u_{m\epsilon} \mathrm{d}\Gamma - \tau_2 \int_{\Gamma_2} \left| \frac{\partial u_{m\epsilon}}{\partial \nu} \right|^2 \mathrm{d}\Gamma + \tau_2 k_2(t) \int_{\Gamma_2} \frac{\partial u_{m0}}{\partial \nu} \frac{\partial u'_{m\epsilon}}{\partial \nu} \mathrm{d}\Gamma \\
& + \frac{\tau_2}{2} k'_2(t) \int_{\Gamma_2} \left| \frac{\partial u_{m\epsilon}}{\partial \nu} \right|^2 \mathrm{d}\Gamma - \frac{\tau_2}{2} \int_{\Gamma_2} k''_2 \Box \frac{\partial u_{m\epsilon}}{\partial \nu} \mathrm{d}\Gamma. \qquad (7.1.6)
\end{aligned}
$$

Integrating (7.1.6) over $(0, t)$, using the assumption (A_1), (A_2) and (6.2.11), and taking the convergence in (7.1.4) into consideration, we have

$$
\begin{aligned}
& \int_{\Omega_1} K(x) |u'_{m\epsilon}(x)|^2 \mathrm{d}x + \int_{\Omega_2} K(x) |v'_{m\epsilon}(x)|^2 \mathrm{d}x + \int_{\Omega_1} \epsilon |u'_{m\epsilon}|^2 \mathrm{d}x + \int_{\Omega_2} \epsilon |v'_{m\epsilon}|^2 \mathrm{d}x \\
& + a_1(u_{m\epsilon}, u_{m\epsilon}) + a_2(v_{m\epsilon}, v_{m\epsilon}) \le C_1, \qquad (7.1.7)
\end{aligned}
$$

where C_1 is constant independent on m, ϵ, and $t \in [0, T]$.

The Second Estimate: First, we estimate the initial data $\|u''_{m\epsilon}(0)\|^2$ and $\|v''_{m\epsilon}(0)\|^2$ in the L^2 norm. Taking $t = 0$ in (7.1.1), and using the compatibility conditions (6.2.12), we get

$$
\begin{aligned}
& (K(0) + \epsilon)(\|u''_{m\epsilon}(0)\|^2 + \|v''_{m\epsilon}(0)\|^2) \\
\le \quad & \|\Delta^2 u_{m\epsilon}(0)\|_2 \|u''_{m\epsilon}(0)\|_2 + \|\Delta^2 v_{m\epsilon}(0)\|_2 \|v''_{m\epsilon}(0)\|_2 \\
\le \quad & \|f_1(u_{m\epsilon})(0)\|_2 \|u''_{m\epsilon}(0)\|_2 + \|f_2(u_{m\epsilon})(0)\|_2 \|v''_{m\epsilon}(0)\|_2. \qquad (7.1.8)
\end{aligned}
$$

Using the definition of the initial condition (7.1.4), we conclude that

$$
(K(0) + \epsilon)(\|u''_{m\epsilon}(0)\|^2 + \|v''_{m\epsilon}(0)\|^2) \le C_2, \qquad (7.1.9)
$$

where C_2 is independent of m and ϵ.

Differentiating (7.1.1) with respect to t and Letting $w = u''_{m\epsilon}(t)$ in equation $(7.1.1)_1$ and $w = v''_{m\epsilon}(t)$ in equation $(7.1.1)_2$, respectively, we obtain

$$\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_{\Omega_1}(K(x)+\epsilon)|u''_{m\epsilon}|^2\mathrm{d}x + \int_{\Omega_2}(K(x)+\epsilon)|v''_{m\epsilon}|^2\mathrm{d}x \\
& +a_1(u'_{m\varepsilon}, u'_{m\epsilon}) + a_2(v'_{m\epsilon}, v'_{m\epsilon})\Big\} \\
= & -\int_{\Omega_1} f'_1(u_{m\epsilon})u'_{m\epsilon}u''_{m\epsilon}\mathrm{d}\Gamma - \int_{\Omega_1} f'_2(v_{m\epsilon})v'_{m\epsilon}v''_{m\epsilon}\mathrm{d}\Gamma - \tau_1\int_{\Omega_1}|u'_{m\epsilon}|^2\mathrm{d}\Gamma \\
& -\tau_1\int_{\Gamma_1} k_1(0)u'_{m\epsilon}u''_{m\epsilon}\mathrm{d}\Gamma + \tau_1 k'_1(t)\int_{\Gamma_1} u_{m\epsilon}(0)u''_{m\epsilon}\mathrm{d}\Gamma - \tau_1\int_{\Gamma_1}(k'_1 * u_{m\epsilon})'u''_{m\epsilon}\mathrm{d}\Gamma \\
& -\tau_2\int_{\Gamma_2}\left|\frac{\partial u''_{m\epsilon}}{\partial\nu}\right|^2\mathrm{d}\Gamma - \tau_2\int_{\Gamma_2} k_2(0)\frac{\partial u'_{m\epsilon}}{\partial\nu}\frac{\partial u''_{m\epsilon}}{\partial\nu}\mathrm{d}\Gamma + \tau_2 k'_2(t)\int_{\Gamma_2}\frac{\partial u_{m\epsilon}(0)}{\partial\nu}\frac{\partial u''_{m\epsilon}}{\partial\nu}\mathrm{d}\Gamma \\
& -\tau_2\int_{\Gamma_2}\left(k'_2 * \frac{\partial u_{m\epsilon}}{\partial\nu}\right)'\frac{\partial u''_{m\epsilon}(0)}{\partial\nu}\mathrm{d}\Gamma. \qquad (7.1.10)
\end{aligned}$$

Using Lemma 6.2.4 and noting that

$$(k'_i * u_{m\epsilon})' = k'_i(t)u_{m0} + \int_0^t k'_i(t-s)u'_{m\epsilon}(s)\mathrm{d}s,$$

we have

$$\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_{\Omega_1}(K(x)+\epsilon)|u''_{m\epsilon}|^2\mathrm{d}x + \int_{\Omega_2}(K(x)+\epsilon)|v''_{m\epsilon}|^2\mathrm{d}x \\
& +a_1(u'_{m\varepsilon}, u'_{m\epsilon}) + a_2(v'_{m\epsilon}, v'_{m\epsilon}) + \tau_1 k_1(t)\int_{\Gamma_2}|u'_{m\epsilon}|^2\mathrm{d}\Gamma \\
& -\tau_1\int_{\Gamma_2} k'_1\Box u'_{m\epsilon}\mathrm{d}\Gamma + \tau_2 k_2(t)\int_{\Gamma_2}\left|\frac{\partial u'_{m\epsilon}}{\partial\nu}\right|^2\mathrm{d}\Gamma - \tau_2\int_{\Gamma_2} k'_2\Box\frac{\partial u'_{m\epsilon}}{\partial\nu}\mathrm{d}\Gamma\Big\} \\
= & -\int_{\Omega_1} f'_1(u_{m\epsilon})u'_{m\epsilon}u''_{m\epsilon}\mathrm{d}\Gamma - \int_{\Omega_1} f'_2(v_{m\epsilon})v'_{m\epsilon}v''_{m\epsilon}\mathrm{d}\Gamma - \tau_1\int_{\Gamma_2}|u''_{m\epsilon}|^2\mathrm{d}\Gamma \\
& +\tau_1 k_1(t)\int_{\Gamma_2} u'_{m0}u'_{m\epsilon}\mathrm{d}\Gamma + \frac{\tau_1}{2}k'_1(t)\int_{\Gamma_1}|u'_{m\epsilon}|^2\mathrm{d}\Gamma - \frac{\tau_1}{2}\int_{\Gamma_2} k''_1\Box u'_{m\epsilon}\mathrm{d}\Gamma \\
& -\tau_2\int_{\Gamma_2}\left|\frac{\partial u''_{m\epsilon}}{\partial\nu}\right|^2\mathrm{d}\Gamma + \tau_2 k_2(t)\int_{\Gamma_2}\frac{\partial u'_{m0}}{\partial\nu}\frac{\partial u'_{m\epsilon}}{\partial\nu}\mathrm{d}\Gamma
\end{aligned}$$

$$+\frac{\tau_2}{2}k_2'(t)\int_{\Gamma_2}\left|\frac{\partial u'_{m\epsilon}}{\partial\nu}\right|^2\mathrm{d}\Gamma-\frac{\tau_2}{2}\int_{\Gamma_2}k_2''\Box\frac{\partial u'_{m\epsilon}}{\partial\nu}\mathrm{d}\Gamma. \tag{7.1.11}$$

Letting $p_n=\frac{2n}{n-2}$, from the assumption (A_1) and using Sobolev imbedding, we get

$$\begin{aligned}
&\int_{\Omega_1} f_1'(u_{m\epsilon})u'_{m\epsilon}u''_{m\epsilon}\mathrm{d}\Gamma\\
\leq\ & c\int_{\Omega_1}(1+2|u_{m\epsilon}|^{\rho-1})|u'_{m\epsilon}||u''_{m\epsilon}|\mathrm{d}x\\
\leq\ & c\left[\int_{\Omega_1}(1+2|u_{m\epsilon}|^{\rho-1})^n\mathrm{d}x\right]^{\frac{1}{n}}\left[\int_{\Omega_1}|u'_{m\epsilon}|^{p_n}\mathrm{d}x\right]^{\frac{1}{p_n}}\left[\int_{\Omega_1}|u''_{m\epsilon}|^2\mathrm{d}x\right]^{\frac{1}{2}}\\
\leq\ & c\left[\int_{\Omega_1}(1+|\Delta u_{m\epsilon}|^2)^n\mathrm{d}x\right]^{\frac{\rho-1}{2}}\left[\int_{\Omega_1}|\Delta u'_{m\epsilon}|^2\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{\Omega_1}|u''_{m\epsilon}|^2\mathrm{d}x\right]^{\frac{1}{2}}.
\end{aligned} \tag{7.1.12}$$

Taking into the first estimate (7.1.7) we get

$$\begin{aligned}
\int_{\Omega_1} f_1'(u_{m\epsilon})u'_{m\epsilon}u''_{m\epsilon}\mathrm{d}\Gamma &\leq c\left[\int_{\Omega_1}\|\Delta u'_{m\epsilon}\|^2\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{\Omega_2}|u''_{m\epsilon}|^2\mathrm{d}x\right]^{\frac{1}{2}}\\
&\leq c\left\{\int_{\Omega_1}\|\Delta u'_{m\epsilon}\|^2\mathrm{d}x+\int_{\Omega_2}|u''_{m\epsilon}|^2\mathrm{d}x\right\}.
\end{aligned} \tag{7.1.13}$$

Similarly, we have

$$\begin{aligned}
\int_{\Omega_2} f_2'(v_{m\epsilon})v'_{m\epsilon}v''_{m\epsilon}\mathrm{d}\Gamma &\leq c\left[\int_{\Omega_1}\|\Delta v'_{m\epsilon}\|^2\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{\Omega_2}|v''_{m\epsilon}|^2\mathrm{d}x\right]^{\frac{1}{2}}\\
&\leq c\left\{\int_{\Omega_1}\|\Delta v'_{m\epsilon}\|^2\mathrm{d}x+\int_{\Omega_2}|v''_{m\epsilon}|^2\mathrm{d}x\right\}.
\end{aligned} \tag{7.1.14}$$

Integrating (7.1.11) over $(0,t)$, using the above inequalities and employing Gronwall's inequality we get

$$\int_{\Omega_1}K(x)|u''_{m\epsilon}(x)|^2\mathrm{d}x+\int_{\Omega_2}K(x)|v''_{m\epsilon}(x)|^2\mathrm{d}x+\int_{\Omega_1}\epsilon|u''_{m\epsilon}|^2\mathrm{d}x+\int_{\Omega_2}\epsilon|v''_{m\epsilon}|^2\mathrm{d}x$$

$$+a_1(u'_{m\epsilon}, u'_{m\epsilon}) + a_2(v'_{m\epsilon}, v'_{m\epsilon}) \le C_3, \tag{7.1.15}$$

where C_3 is constant independent on m, ϵ, and $t \in [0, T]$.

Making use of Aubin-Loins theorem [25], we deduce that there exists a subsequence of $\{(u_{m\epsilon}, v_{m\epsilon})\}$ such that

$$(u_{m\epsilon}, v_{m\epsilon}) \to (u_\epsilon, v_\epsilon) \text{ strongly in } L^2(0, T; W(\Omega_1) \times W(\Omega_2)).$$

Then from the estimates (7.1.7) and (7.1.15), there exists a subsequence $\{(u_{m\epsilon}, v_{m\epsilon})\}$ which we also denote as $\{(u_{m\epsilon}, v_{m\epsilon})\}$ such that

$$\begin{aligned}
&(u_{m\epsilon}, v_{m\epsilon}) \to (u, v) \text{ weak star in } L^\infty(0, T; W(\Omega_1) \times W(\Omega_2)),\\
&(u'_{m\epsilon}, v'_{m\epsilon}) \to (u', v') \text{ weak star in } L^\infty(0, T; H^2(\Omega_1) \times H^2(\Omega_2)),\\
&(u''_{m\epsilon}, v''_{m\epsilon}) \to (u'', v'') \text{ weak star in } L^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2)),\\
&(Ku''_{m\epsilon}, Kv''_{m\epsilon}) \to (Ku'', Kv'') \text{ weak star in } L^\infty(0, T; L^2(\Omega_1) \times L^2(\Omega_2)).
\end{aligned} \tag{7.1.16}$$

The above convergences are sufficient to pass to the limit in (7.1.1). Then from Lemma 6.2.5, we get

$$(u, v) \in L^\infty(0, T; H^4(\Omega_1) \times \Big(W \cap H^4(\Omega_2)\Big)).$$

The proof of the uniqueness is a routine, here we omit it. $\square$

7.2 General Decay I: Proof of Theorem 6.2.7

In this section, we shall prove the general decay rate of problem (6.2.6) as stated in Theorem 6.2.7. For convenience, from now on, we shall omit x and t in some functions if there is no ambiguity and let c be an arbitrary positive constant, which may be different from line to line. First, we prove some useful lemmas.

Lemma 7.2.1 *For every $v \in H^4(\Omega)$ and for every $\mu \in \mathbb{R}$, one has*

$$\begin{aligned}\int_\Omega (m\cdot\nabla v)\Delta^2 v \mathrm{d}x &= a(v,v) + \int_\Gamma \left[(\mathscr{B}_2 v) m\cdot\nabla v - (\mathscr{B}_1 v)\frac{\partial}{\partial\nu}(m\cdot\nabla v)\right]\mathrm{d}\Gamma \\ &\quad + \frac{1}{2}\int_\Gamma m\cdot\nu\left[v_{xx}^2 + v_{yy}^2 + 2\mu v_{xx}v_{yy} + 2(1-\mu)v_{xy}^2\right]\mathrm{d}\Gamma \quad (7.2.1)\end{aligned}$$

For the proof of this lemma, we refer the readers to [23].

Lemma 7.2.2 *Let (u,v) be the solution of* (6.2.6). *Then the energy functional $E(t)$ satisfies*

$$\begin{aligned}E'(t) &\leq -\frac{\tau_1}{2}\int_{\Gamma_2}\left(|u_t|^2 - k_1^2(t)|u_0|^2 - k_1'(t)|u|^2 + k_1''\circ u\right)\mathrm{d}\Gamma \\ &\quad -\frac{\tau_2}{2}\int_{\Gamma_2}\left(\left|\frac{\partial u_t}{\partial\nu}\right|^2 - k_2^2(t)\left|\frac{\partial u_0}{\partial\nu}\right|^2 - k_2'(t)\left|\frac{\partial u}{\partial\nu}\right|^2 + k_2''\circ\frac{\partial u}{\partial\nu}\right)\mathrm{d}\Gamma. \quad (7.2.2)\end{aligned}$$

Proof. Multiplying $(6.2.6)_1$ by u_t, $(6.2.6)_2$ by v_t and integrating over Ω_1 and Ω_2, respectively, using $(6.2.6)_3$-$(6.2.6)_6$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega_1} K(x)|u_t|^2\mathrm{d}x + \int_{\Omega_2} K(x)|v_t|^2\mathrm{d}x + a_1(u,u) + a_2(v,v)\right.$$

$$
\begin{aligned}
& +2\int_{\Omega_1} F_1(u)\mathrm{d}x + 2\int_{\Omega_2} F_2(v)\mathrm{d}x\Big\} \\
= \;& -\int_{\Gamma_1}(\mathscr{B}_2 u)u_t\mathrm{d}\Gamma + \int_{\Gamma_1}(\mathscr{B}_1 u)\frac{\partial u_t}{\partial \nu}\mathrm{d}\Gamma + \int_{\Gamma_1}(\mathscr{B}_2 v)v_t\mathrm{d}\Gamma - \int_{\Gamma_1}(\mathscr{B}_1 v)\frac{\partial v_t}{\partial \nu}\mathrm{d}\Gamma \\
& -\int_{\Gamma_2}(\mathscr{B}_2 u)u_t\mathrm{d}\Gamma + \int_{\Gamma_2}(\mathscr{B}_1 u)\frac{\partial u_t}{\partial \nu}\mathrm{d}\Gamma \\
= \;& -\int_{\Gamma_2}(\mathscr{B}_2 u)u_t\mathrm{d}\Gamma + \int_{\Gamma_2}(\mathscr{B}_1 u)\frac{\partial u_t}{\partial \nu}\mathrm{d}\Gamma. \qquad (7.2.3)
\end{aligned}
$$

Substituting the boundary terms by (6.2.4) and (6.2.5) and using Lemma 6.2.4 and Young's inequality, our conclusion follows. □

Let us consider the following binary operator:

$$
(k \diamond \varphi)(t) := \int_0^t k(t-s)(\varphi(t) - \varphi(s))\mathrm{d}s.
$$

Applying the Hölder's inequality for $0 \le \sigma \le 1$, we have

$$
|(k \diamond \varphi)(t)|^2 \le \left[\int_0^t |k(s)|^{2(1-\sigma)}\mathrm{d}s\right]\left(|k|^{2\sigma} \circ \varphi\right)(t). \qquad (7.2.4)
$$

The following lemma plays an important role in constructing the Lyapunov functional.

Lemma 7.2.3 *The functional*

$$
\Phi(t) := \int_{\Omega_1}\left\{m\cdot\nabla u + \left(\frac{n}{2} - \theta\right)u\right\}Ku_t\mathrm{d}x + \int_{\Omega_2}\left\{m\cdot\nabla v + \left(\frac{n}{2} - \theta\right)v\right\}Kv_t\mathrm{d}x \qquad (7.2.5)
$$

satisfies, along the solution of problem (6.2.6), *for any* $\epsilon > 0$,

$$
\begin{aligned}
\Phi'(t) \;\le\; & \frac{1}{2}\int_{\Gamma_2} m\cdot\nu K|u_t|^2\mathrm{d}\Gamma - \left(1 + \frac{n}{2} - \theta - \epsilon\lambda_0\right)a_1(u,u) - \left(1 + \frac{n}{2} - \theta\right)a_2(v,v) \\
& -\left(\frac{n\eta_1}{2} - 2\theta - \eta_1\theta\right)\int_{\Omega_1} F_1(u)\mathrm{d}x - \left(\frac{n\eta_2}{2} - 2\theta - \eta_2\theta\right)\int_{\Omega_2} F_2(v)\mathrm{d}x
\end{aligned}
$$

$$
-\left(\frac{1}{2}-\frac{\epsilon\lambda_0}{\delta}\right)\int_{\Gamma_2} m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]\mathrm{d}\Gamma
$$

$$
+\frac{2\tau_2^2}{\epsilon}\int_{\Gamma_2}\left\{\left|\frac{\partial u_t}{\partial\nu}\right|^2+k_2^2(t)\left|\frac{\partial u}{\partial\nu}\right|^2+k_2^2(t)\left|\frac{\partial u_0}{\partial\nu}\right|^2+\left|k_2'\diamond\frac{\partial u}{\partial\nu}\right|^2\right\}\mathrm{d}\Gamma
$$

$$
+\frac{2\tau_1^2}{\epsilon}\int_{\Gamma_2}\left\{|u_t|^2+k_1^2(t)|u|^2+k_1^2(t)|u_0|^2+|k_1'\diamond u|^2\right\}\mathrm{d}\Gamma, \tag{7.2.6}
$$

where $\theta,\tau_1,\tau_2,\eta_1$ and η_2 are positive constants.

Proof. Using (6.2.6) and Lemma 7.2.1, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega_1}\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}Ku_t\mathrm{d}x\right\}\\
=\;&\int_{\Omega_1}\left\{m\cdot\nabla u_t+\left(\frac{n}{2}-\theta\right)u_t\right\}Ku_t\mathrm{d}x+\int_{\Omega_1}\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}Ku_{tt}\mathrm{d}x\\
=\;&\frac{1}{2}\int_{\Gamma_1}m\cdot\nu K|u_t|^2\mathrm{d}\Gamma+\frac{1}{2}\int_{\Gamma_2}m\cdot\nu K|u_t|^2\mathrm{d}\Gamma-\theta\int_{\Omega_1}K|u_t|^2\mathrm{d}x\\
&-\frac{1}{2}\int_{\Omega_1}\nabla K\cdot m|u_t|^2\mathrm{d}x-\left(1+\frac{n}{2}-\theta\right)a_1(u,u)\\
&+n\int_{\Omega_1}F_1(u)\mathrm{d}x-\left(\frac{n}{2}-\theta\right)\int_{\Omega_1}f_1(u)u\mathrm{d}x\\
&+\int_{\Gamma_1}(\mathscr{B}_1u)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)\frac{\partial u}{\partial\nu}\right]\mathrm{d}\Gamma\\
&-\int_{\Gamma_1}(\mathscr{B}_2u)\left[(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)u\right]\mathrm{d}\Gamma\\
&+\int_{\Gamma_2}(\mathscr{B}_1u)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)\frac{\partial u}{\partial\nu}\right]\mathrm{d}\Gamma\\
&-\int_{\Gamma_2}(\mathscr{B}_2u)\left[(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)u\right]\mathrm{d}\Gamma\\
&-\frac{1}{2}\int_{\Gamma_1}m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]\mathrm{d}\Gamma\\
&-\frac{1}{2}\int_{\Gamma_2}m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]\mathrm{d}\Gamma.
\end{aligned}\tag{7.2.7}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega_2}\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}Kv_t\mathrm{d}x\right\}\\
=\ & -\frac{1}{2}\int_{\Gamma_1} m\cdot\nu K|v_t|^2\mathrm{d}\Gamma+\frac{1}{2}\int_{\Gamma_0} m\cdot\nu K|v_t|^2\mathrm{d}\Gamma-\theta\int_{\Omega_2}K|v_t|^2\mathrm{d}x\\
&-\frac{1}{2}\int_{\Omega_2}\nabla K\cdot m|v_t|^2\mathrm{d}x-\left(1+\frac{n}{2}-\theta\right)a_2(v,v)\\
&+n\int_{\Omega_2}F_2(v)\mathrm{d}x-\left(\frac{n}{2}-\theta\right)\int_{\Omega_2}f_2(v)v\mathrm{d}x\\
&-\int_{\Gamma_1}(\mathscr{B}_1 v)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla v)+\left(\frac{n}{2}-\theta\right)\frac{\partial v}{\partial\nu}\right]\mathrm{d}\Gamma\\
&+\int_{\Gamma_1}(\mathscr{B}_2 v)\left[(m\cdot\nabla v)+\left(\frac{n}{2}-\theta\right)v\right]\mathrm{d}\Gamma\\
&+\int_{\Gamma_0}(\mathscr{B}_1 v)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla v)+\left(\frac{n}{2}-\theta\right)\frac{\partial v}{\partial\nu}\right]\mathrm{d}\Gamma\\
&-\int_{\Gamma_0}(\mathscr{B}_2 v)\left[(m\cdot\nabla v)+\left(\frac{n}{2}-\theta\right)v\right]\mathrm{d}\Gamma\\
&+\frac{1}{2}\int_{\Gamma_1} m\cdot\nu\left[v_{xx}^2+v_{yy}^2+2\mu v_{xx}v_{yy}+2(1-\mu)v_{xy}^2\right]\mathrm{d}\Gamma\\
&-\frac{1}{2}\int_{\Gamma_0} m\cdot\nu\left[v_{xx}^2+v_{yy}^2+2\mu v_{xx}v_{yy}+2(1-\mu)v_{xy}^2\right]\mathrm{d}\Gamma. \qquad (7.2.8)
\end{aligned}
$$

Noting that $v=\dfrac{\partial v}{\partial\nu}=0$ on Γ_0, we have $B_1v=B_2v=0$ on Γ_0. Therefore, we rewrite (7.2.8) as

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega_2}\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}Kv_t\mathrm{d}x\right\}\\
=\ &\int_{\Omega_2}\left\{m\cdot\nabla v_t+\left(\frac{n}{2}-\theta\right)v_t\right\}Kv_t\mathrm{d}x+\int_{\Omega_2}\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}Kv_{tt}\mathrm{d}x\\
=\ & -\frac{1}{2}\int_{\Gamma_1} m\cdot\nu K|v_t|^2\mathrm{d}\Gamma-\theta\int_{\Omega_2}K|v_t|^2\mathrm{d}x-\frac{1}{2}\int_{\Omega_2}\nabla K\cdot m|v_t|^2\mathrm{d}x\\
&-\left(1+\frac{n}{2}-\theta\right)a_2(v,v)+n\int_{\Omega_2}F_2(v)\mathrm{d}x-\left(\frac{n}{2}-\theta\right)\int_{\Omega_2}f_2(v)v\mathrm{d}x\\
&-\int_{\Gamma_1}(\mathscr{B}_1 v)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla v)+\left(\frac{n}{2}-\theta\right)\frac{\partial v}{\partial\nu}\right]\mathrm{d}\Gamma
\end{aligned}
$$

$$+\int_{\Gamma_1}(\mathscr{B}_2 v)\left[(m\cdot\nabla v)+\left(\frac{n}{2}-\theta\right)v\right]\mathrm{d}\Gamma$$
$$+\frac{1}{2}\int_{\Gamma_1} m\cdot\nu\left[v_{xx}^2+v_{yy}^2+2\mu v_{xx}v_{yy}+2(1-\mu)v_{xy}^2\right]\mathrm{d}\Gamma. \tag{7.2.9}$$

From (7.2.7) and (7.2.9), and noticing that $f_1 = f_2$ on Γ_1, we have

$$\begin{aligned}
\Phi'(t) \;=\;& \int_{\Omega_1}\left\{m\cdot\nabla u_t+\left(\frac{n}{2}-\theta\right)u_t\right\}Ku_t\mathrm{d}x+\int_{\Omega_1}\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}Ku_{tt}\mathrm{d}x\\
&+\int_{\Omega_2}\left\{m\cdot\nabla v_t+\left(\frac{n}{2}-\theta\right)v_t\right\}Kv_t\mathrm{d}x+\int_{\Omega_2}\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}Kv_{tt}\mathrm{d}x\\
=\;& \frac{1}{2}\int_{\Gamma_2} m\cdot\nu K|u_t|^2\mathrm{d}\Gamma-\theta\int_{\Omega_1}K|u_t|^2\mathrm{d}x-\theta\int_{\Omega_2}K|v_t|^2\mathrm{d}x\\
&-\frac{1}{2}\int_{\Omega_1}\nabla K\cdot m|u_t|^2\mathrm{d}x-\frac{1}{2}\int_{\Omega_2}\nabla K\cdot m|v_t|^2\mathrm{d}x-\left(1+\frac{n}{2}-\theta\right)a_1(u,u)\\
&-\left(1+\frac{n}{2}-\theta\right)a_2(v,v)+n\int_{\Omega_1}F_1(u)\mathrm{d}x+n\int_{\Omega_2}F_2(v)\mathrm{d}x\\
&-\left(\frac{n}{2}-\theta\right)\int_{\Omega_1}f_1(u)u\mathrm{d}x-\left(\frac{n}{2}-\theta\right)\int_{\Omega_2}f_2(v)v\mathrm{d}x\\
&+\int_{\Gamma_2}(\mathscr{B}_1 u)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)\frac{\partial u}{\partial\nu}\right]\mathrm{d}\Gamma\\
&-\int_{\Gamma_2}(\mathscr{B}_2 u)\left[(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)u\right]\mathrm{d}\Gamma\\
&-\frac{1}{2}\int_{\Gamma_2} m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]\mathrm{d}\Gamma.
\end{aligned}\tag{7.2.10}$$

Using Young's inequality, we get

$$\begin{aligned}
&\left|\int_{\Gamma_2}(\mathscr{B}_2 u)\left[(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)u\right]\mathrm{d}\Gamma\right|\\
\le\;& \frac{1}{2\epsilon}\int_{\Gamma_2}|\mathscr{B}_2 u|^2\mathrm{d}\Gamma+\epsilon\int_{\Gamma_2}\left(|m\cdot\nabla u|^2+\left(\frac{n}{2}-\theta\right)^2|u|^2\right)\mathrm{d}\Gamma,
\end{aligned}\tag{7.2.11}$$

and

$$\left|\int_{\Gamma_2}(\mathscr{B}_1 u)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)\frac{\partial u}{\partial\nu}\right]\mathrm{d}\Gamma\right|$$

$$\leq \frac{1}{2\epsilon}\int_{\Gamma_2}|\mathscr{B}_1 u|^2\mathrm{d}\Gamma+\epsilon\int_{\Gamma_2}\left(\left|\frac{\partial}{\partial\nu}(m\cdot\nabla u)\right|^2+\left(\frac{n}{2}-\theta\right)^2\left|\frac{\partial u}{\partial\nu}\right|^2\right)\mathrm{d}\Gamma, \quad (7.2.12)$$

where ϵ is a positive constant. Since the bilinear form $a_1(u,u)$ is strictly coercive on $H^2(\Omega)$, using the trace theory, we obtain

$$\begin{aligned}
&\int_{\Gamma_2}\left(|m\cdot\nabla u|^2+\left(\frac{n}{2}-\theta\right)^2|u|^2\right)\mathrm{d}\Gamma+\int_{\Gamma_2}\left(\left|\frac{\partial}{\partial\nu}(m\cdot\nabla u)\right|^2+\left(\frac{n}{2}-\theta\right)^2\left|\frac{\partial u}{\partial\nu}\right|^2\right)\mathrm{d}\Gamma\\
\leq\ & \lambda_0 a_1(u,u)+\frac{\lambda_0}{\delta}\int_{\Gamma_2}m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]\mathrm{d}\Gamma, \qquad (7.2.13)
\end{aligned}$$

where λ_0 is a constant depending on Ω_1, μ, θ and n. Substituting inequalities (7.2.11)-(7.2.13) into (7.2.10) and taking into account that $m\cdot\nu\leq 0$ on Γ_0, as well as (A_1) and (A_2), we have

$$\begin{aligned}
&\Phi'(t)\\
\leq\ & \frac{1}{2}\int_{\Gamma_2}m\cdot\nu K|u_t|^2\mathrm{d}\Gamma-\theta\int_{\Omega_1}K|u_t|^2\mathrm{d}x-\theta\int_{\Omega_2}K|v_t|^2\mathrm{d}x-\frac{1}{2}\int_{\Omega_1}\nabla K\cdot m|u_t|^2\mathrm{d}x\\
&-\frac{1}{2}\int_{\Omega_2}\nabla K\cdot m|v_t|^2\mathrm{d}x-\left(1+\frac{n}{2}-\theta-\epsilon\lambda_0\right)a_1(u,u)-\left(1+\frac{n}{2}-\theta\right)a_2(v,v)\\
&-\left(\frac{n\eta_1}{2}-2\theta-\eta_1\theta\right)\int_{\Omega_1}F_1(u)\mathrm{d}x-\left(\frac{n\eta_2}{2}-2\theta-\eta_2\theta\right)\int_{\Omega_2}F_2(v)\mathrm{d}x\\
&-\left(\frac{1}{2}-\frac{\epsilon\lambda_0}{\delta}\right)\int_{\Gamma_2}m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]\mathrm{d}\Gamma\\
&+\frac{1}{2\epsilon}\int_{\Gamma_2}\left(|\mathscr{B}_1 u|^2+|\mathscr{B}_2 u|^2\right)\mathrm{d}\Gamma. \qquad (7.2.14)
\end{aligned}$$

Noticing that the boundary conditions (6.2.4) and (6.2.5) can be written as

$$\mathscr{B}_2 u=\tau_1\{u_t+k_1(t)u-k_1(t)u_0-k_1'\diamond u\},$$

and

$$\mathscr{B}_1 u = -\tau_2 \left\{ \frac{\partial u_t}{\partial \nu} + k_2(t)\frac{\partial u}{\partial \nu} - k_2(t)\frac{\partial u_0}{\partial \nu} - k_2' \diamond \frac{\partial u}{\partial \nu} \right\},$$

our conclusion follows. □

Now, let us define the perturbed energy functional

$$G(t) = NE(t) + \Phi(t).$$

It is easy to check that

$$G(t) \sim E(t) \ \text{ for appropriately large } N > 0. \tag{7.2.15}$$

Lemma 7.2.4 *If $N > 0$ is appropriately large, for some large t_1, there exist positive constants θ_0, θ_1, θ_2, θ_3, θ_4 verifying*

$$\begin{aligned} G'(t) \ \leq \ & -\theta_0 E(t) + \theta_1 \int_{\Gamma_2} k_1^2(t)|u_0|^2 \mathrm{d}\Gamma + \theta_2 \int_{\Gamma_2} k_2^2(t) \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma \\ & -\theta_3 \int_{\Gamma_2} k_1' \circ u \mathrm{d}\Gamma - \theta_4 \int_{\Gamma_2} k_2' \circ \frac{\partial u}{\partial \nu} \mathrm{d}\Gamma, \ \forall \ t \geq t_1. \end{aligned} \tag{7.2.16}$$

Proof. From (7.2.2) and (7.2.6) and applying (7.2.4) with $\sigma = 1/2$, we get

$$\begin{aligned} G'(t) \ \leq \ & \frac{1}{2}\int_{\Gamma_2} m \cdot \nu K |u_t|^2 \mathrm{d}\Gamma - \left(1 + \frac{n}{2} - \theta - \epsilon\lambda_0\right) a_1(u,u) - \left(1 + \frac{n}{2} - \theta\right) a_2(v,v) \\ & - \left(\frac{n\eta_1}{2} - 2\theta - \eta_1\theta\right) \int_{\Omega_1} F_1(u)\mathrm{d}x - \left(\frac{n\eta_2}{2} - 2\theta - \eta_2\theta\right) \int_{\Omega_2} F_2(v)\mathrm{d}x \\ & - \left(\frac{1}{2} - \frac{\epsilon\lambda_0}{\delta}\right) \int_{\Gamma_2} m \cdot \nu \left[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2\right] \mathrm{d}\Gamma \\ & + \frac{2\tau_2^2}{\epsilon} \int_{\Gamma_2} \left\{ \left|\frac{\partial u_t}{\partial \nu}\right|^2 + k_2^2(t)\left|\frac{\partial u}{\partial \nu}\right|^2 + k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 - k_2(0)k_2' \circ \frac{\partial u}{\partial \nu} \right\} \mathrm{d}\Gamma \end{aligned}$$

$$
\begin{aligned}
&-\frac{\tau_2 N}{2}\int_{\Gamma_2}\left(\left|\frac{\partial u_t}{\partial \nu}\right|^2 - k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 - k_2'(t)\left|\frac{\partial u}{\partial \nu}\right|^2 + k_2'' \circ \frac{\partial u}{\partial \nu}\right)\mathrm{d}\Gamma \\
&+\frac{2\tau_1^2}{\epsilon}\int_{\Gamma_2}\left\{|u_t|^2 + k_1^2(t)|u|^2 + k_1^2(t)|u_0|^2 - k_1(0)k_1'(t)\circ u\right\}\mathrm{d}\Gamma \\
&-\frac{\tau_1 N}{2}\int_{\Gamma_2}\left(|u_t|^2 - k_1^2(t)|u_0|^2 - k_1'(t)|u|^2 + k_1''\circ u\right)\mathrm{d}\Gamma. \qquad (7.2.17)
\end{aligned}
$$

First, we take θ and ϵ so small such that

$$
1+\frac{n}{2}-\theta>0,\ 1+\frac{n}{2}-\theta-\varepsilon\lambda_0>0,
$$

$$
\frac{n\eta_1}{2}-2\theta-\eta_1\theta>0,\ \frac{n\eta_2}{2}-2\theta-\eta_2\theta>0,\ \frac{1}{2}-\frac{\epsilon\lambda_0}{\delta}>0.
$$

Noticing that $K(x)\geq 0$, $K\in L^\infty(\Omega)$ and $\lim_{t\to\infty}k_i(t)=0$, for $i=1,2$, then for large t_1, by choosing N large enough, our conclusion follows. □

Continuity of the proof of Theorem 6.2.7. Let $\zeta(t)=\min\{\zeta_1(t),\zeta_2(t)\}$, $t\geq t_0$, then, using (A_3) and (7.2.2), we get

$$
\begin{aligned}
\zeta(t)\frac{\mathrm{d}}{\mathrm{d}t}G(t) \leq\ & -\theta_0\zeta(t)E(t)+\theta_1\zeta(t)\int_{\Gamma_2}k_1^2(t)|u_0|^2\mathrm{d}\Gamma+\theta_2\zeta(t)\int_{\Gamma_2}k_2^2(t)\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma \\
& -\theta_3\zeta(t)\int_{\Gamma_2}k_1'\circ u\mathrm{d}\Gamma-\theta_4\zeta(t)\int_{\Gamma_2}k_2'\circ\frac{\partial u}{\partial\nu}\mathrm{d}\Gamma \\
\leq\ & -\theta_0\zeta(t)E(t)+\theta_1\zeta(t)\int_{\Gamma_2}k_1^2(t)|u_0|^2\mathrm{d}\Gamma+\theta_2\zeta(t)\int_{\Gamma_2}k_2^2(t)\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma \\
& +\theta_3\int_{\Gamma_2}k_1''\circ u\mathrm{d}\Gamma+\theta_4\int_{\Gamma_2}k_2''\circ\frac{\partial u}{\partial\nu}\mathrm{d}\Gamma \\
\leq\ & -\theta_0\zeta(t)E(t)+\theta_1\zeta(t)\int_{\Gamma_2}k_1^2(t)|u_0|^2\mathrm{d}\Gamma+\theta_2\zeta(t)\int_{\Gamma_2}k_2^2(t)\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma \\
& -c\frac{\mathrm{d}}{\mathrm{d}t}E(t),\ \forall\ t\geq t_0, \qquad (7.2.18)
\end{aligned}
$$

where c is a positive constant. Noting that ζ is a nonincreasing continuous function as ζ_1 and ζ_2 are nonincreasing, and so ζ is differentiable with $\zeta'(t) \le 0$ for a.e. t, then we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\zeta(t)G(t) + cE(t)\right) \\
\le\ & -\theta_0\zeta(t)E(t) + \theta_1\zeta(t)\int_{\Gamma_2} k_1^2(t)|u_0|^2 \mathrm{d}\Gamma + \theta_2\zeta(t)\int_{\Gamma_2} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma,\ \forall\ t \ge t_0.
\end{aligned} \tag{7.2.19}$$

We define

$$\mathcal{G}(t) = \zeta(t)G(t) + cE(t). \tag{7.2.20}$$

Since $\zeta(t)$ is a nonincreasing positive function, we can easily observe that $\mathcal{G}(t)$ is equivalent to $E(t)$ by using (7.2.15). Thus, for some positive constants ω, C_1 and C_2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t) \le -\omega\zeta(t)E(t) + C_1\int_{\Gamma_2} k_1^2(t)|u_0|^2 \mathrm{d}\Gamma + C_2\int_{\Gamma_2} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma,\ \forall\, t \ge t_0. \tag{7.2.21}$$

Case (i): If $u_0 = \dfrac{\partial u_0}{\partial \nu} = 0$ on Γ_2, (7.2.21) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t) \le -\omega\zeta(t)\mathcal{G}(t),\ \forall\ t \ge t_0.$$

Integration this over (t_0, t) gives

$$\mathcal{G}(t) \le \mathcal{G}(t_0)e^{-\omega\int_{t_0}^{t}\zeta(s)\mathrm{d}s},\ \forall\ t \ge t_0. \tag{7.2.22}$$

By using (7.2.2) and (7.2.20), we then obtain that, for some positive constant C,

$$E(t) \leq CE(0)e^{-\omega \int_0^t \zeta(s)\mathrm{d}s}, \forall\ t \geq t_0. \tag{7.2.23}$$

Case (ii): If $\left(u_0, \dfrac{\partial u_0}{\partial \nu}\right) \neq (0,0)$ on Γ_2, we set

$$\mathcal{F}(t) := \mathcal{G}(t) - Ce^{-\omega \int_0^t \zeta(s)\mathrm{d}s} \int_0^t k_0(s) e^{\omega \int_0^s \zeta(\tau)\mathrm{d}\tau} \mathrm{d}s, \forall\ t \geq t_0,$$

where

$$k_0(t) = \int_{\Gamma_2} k_1^2(t)|u_0|^2 \mathrm{d}\Gamma + \int_{\Gamma_2} k_2^2(t) \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma.$$

Then, with suitable choice of C, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \leq -\omega\zeta(t)\mathcal{F}(t), \forall\ t \geq t_0,$$

and hence

$$\mathcal{F}(t) \leq \mathcal{F}(t_0) e^{-\omega \int_{t_0}^t \zeta(s)\mathrm{d}s}.$$

This yields that

$$\mathcal{G}(t) \leq \left[\mathcal{G}(t_0) + C\int_{t_0}^t k_0(s) e^{\omega \int_{t_0}^s \zeta(\tau)\mathrm{d}\tau} \mathrm{d}s\right] e^{-\omega \int_{t_0}^t \zeta(s)\mathrm{d}s}. \tag{7.2.24}$$

Hence, from (7.2.20), we deduce

$$E(t) \leq C\left(E(0) + \int_0^t k_0(s) e^{\omega \int_0^s \zeta(\tau)\mathrm{d}\tau} \mathrm{d}s\right) e^{-\omega \int_0^t \zeta(s)\mathrm{d}s}, \forall\ t \geq t_0. \tag{7.2.25}$$

This completes the proof of Theorem 6.2.7. $\square$

7.3 General Decay II: Proof of Theorem 6.2.8

In this section, we shall prove the general decay rate of problem (6.2.6) as stated in Theorem 6.2.8. Now by using (6.2.20) and (7.2.2), we deduce that, for some positive constant c and all $t \geq t_1$,

$$
\begin{aligned}
& -\int_0^{t_1} k_1'(s) \int_{\Gamma_2} |u(t) - u(t-s)|^2 \mathrm{d}\Gamma \mathrm{d}s \\
\leq & \ \frac{1}{d} \int_0^{t_1} k_1''(s) \int_{\Gamma_2} |u(t) - u(t-s)|^2 \mathrm{d}\Gamma \mathrm{d}s \\
\leq & \ -c \left[E'(t) - \frac{\tau_1}{2} k_1^2(t) \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma - \frac{\tau_2}{2} k_2^2(t) \int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma \right]
\end{aligned}
\tag{7.3.1}
$$

and

$$
\begin{aligned}
& -\int_0^{t_1} k_2'(s) \int_{\Gamma_2} \left| \frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu} \right|^2 \mathrm{d}\Gamma \mathrm{d}s \\
\leq & \ \frac{1}{d} \int_0^{t_1} k_2''(s) \int_{\Gamma_2} \left| \frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu} \right|^2 \mathrm{d}\Gamma \mathrm{d}s \\
\leq & \ -c \left[E'(t) - \frac{\tau_1}{2} k_1^2(t) \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma - \frac{\tau_2}{2} k_2^2(t) \int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma \right].
\end{aligned}
\tag{7.3.2}
$$

Next, we take $F(t) = G(t) + cE(t)$, we easily observe that $F(t)$ is equivalent to $E(t)$ by using (7.2.15). Then using (7.3.1) and (7.3.2), we obtain that, for all $t \geq t_1$,

$$
\begin{aligned}
F'(t) \leq & -\theta_0 E(t) + c_1 k_1^2(t) \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + c_2 k_2^2(t) \int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma \\
& -c_3 \int_{t_1}^{t} k_1'(s) \int_{\Gamma_2} |u(t) - u(t-s)|^2 \mathrm{d}\Gamma \mathrm{d}s
\end{aligned}
$$

$$-c_4\int_{t_1}^{t} k_2'(t)\int_{\Gamma_2}\left|\frac{\partial u(t)}{\partial \nu}-\frac{\partial u(t-s)}{\partial \nu}\right|^2 \mathrm{d}\Gamma\mathrm{d}s. \tag{7.3.3}$$

where θ_0, c_1, c_2, c_3 and c_4 are some positive constants.

Similarly to [43], we consider the following two cases:

Case I: $H(t) = ct^p$, c is a positive constant and $1 \le p < \dfrac{3}{2}$.

If $1 < p < \dfrac{3}{2}$, one can easily verify that $\displaystyle\int_0^{\infty}[-k_i'(s)]^{1-\delta_0}\mathrm{d}s < +\infty$ for all $\delta_0 < 2-p$ and $i = 1, 2$. Using this fact, (7.2.2), the trace theory and noting that

$$E(t) \le E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^t k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_0^t k_2^2(s)\mathrm{d}s \le M, \tag{7.3.4}$$

for some $M > 0$, we obtain that, for all $t \ge t_1$ (choosing t_1 even large if needed),

$$\begin{aligned}\xi(t) \quad &:= \int_{t_1}^{t}[-k_1'(s)]^{1-\delta_0}\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s\\ &\le 2\int_{t_1}^{t}[-k_1'(s)]^{1-\delta_0}\int_{\Gamma_2}\left(|u(t)|^2+|u(t-s)|^2\right)\mathrm{d}\Gamma\mathrm{d}s\\ &\le cM\int_0^{\infty}[-k_1'(s)]^{1-\delta_0}\mathrm{d}s < 1 \end{aligned}\tag{7.3.5}$$

and

$$\begin{aligned}\gamma(t) \quad &:= \int_{t_1}^{t}[-k_2'(s)]^{1-\delta_0}\int_{\Gamma_2}\left|\frac{\partial u(t)}{\partial \nu}-\frac{\partial u(t-s)}{\partial \nu}\right|^2\mathrm{d}\Gamma\mathrm{d}s\\ &\le 2\int_{t_1}^{t}[-k_2'(s)]^{1-\delta_0}\int_{\Gamma_2}\left(\left|\frac{\partial u(t)}{\partial \nu}\right|^2+\left|\frac{\partial u(t-s)}{\partial \nu}\right|^2\right)\mathrm{d}\Gamma\mathrm{d}s\\ &\le cM\int_0^{\infty}[-k_2'(s)]^{1-\delta_0}\mathrm{d}s < 1. \end{aligned}\tag{7.3.6}$$

Then, using Jensen's inequality, (7.2.2), (7.3.5) and hypothesise (A_4), we deduce that

$$\begin{aligned}
& -\int_{t_1}^{t} k_1'(s)\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s \\
= & \int_{t_1}^{t}[-k_1'(s)]^{\delta_0}[-k_1'(s)]^{1-\delta_0}\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s \\
= & \int_{t_1}^{t}[-k_1'(s)]^{(p-1+\delta_0)\left(\frac{\delta_0}{p-1+\delta_0}\right)}[-k_1'(s)]^{1-\delta_0}\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s \\
\leq & \ \xi(t)\left[\frac{1}{\xi(t)}\int_{t_1}^{t}[-k_1'(s)]^{p-1+\delta_0}[-k_1'(s)]^{1-\delta_0}\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s\right]^{\frac{\delta_0}{p-1+\delta_0}} \\
\leq & \left[\int_{t_1}^{t}[-k_1'(s)]^{p}\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s\right]^{\frac{\delta_0}{p-1+\delta_0}} \\
\leq & \ c\left[\int_{t_1}^{t}k_1''(s)\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s\right]^{\frac{\delta_0}{p-1+\delta_0}} \\
\leq & \ c\left[-E'(t)+\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{\frac{\delta_0}{p-1+\delta_0}}. \qquad (7.3.7)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& -\int_{t_1}^{t} k_2'(s)\int_{\Gamma_2}\left|\frac{\partial u(t)}{\partial\nu}-\frac{\partial u(t-s)}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}s \\
\leq & \ c\left[-E'(t)+\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{\frac{\delta_0}{p-1+\delta_0}}. \qquad (7.3.8)
\end{aligned}$$

Using (7.3.7), (7.3.8) and choosing $\delta_0=\dfrac{1}{2}$ in particular, (7.3.3) becomes

$$\begin{aligned}
F'(t) \leq & -\theta_0E(t)+c\left[-E'(t)+\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{\frac{1}{2p-1}} \\
& +c_1k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma, \qquad (7.3.9)
\end{aligned}$$

for all $t \geq t_1$. On the other side, we have

$$\begin{aligned}
&\left(F(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right)'\\
\leq\ & F'(t)\\
\leq\ & -\theta_0E(t)+c\left[-E'(t)+\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{\frac{1}{2p-1}}\\
&+c_1k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma. \qquad (7.3.10)
\end{aligned}$$

Thus, for all $t \geq t_1$, we have

$$\begin{aligned}
&\left(F(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right)'\\
\leq\ & -\theta_0\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]\\
&+\theta_0\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\theta_0\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\\
&+c\left[-E'(t)+\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{\frac{1}{2p-1}}\\
&+c_1k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma. \qquad (7.3.11)
\end{aligned}$$

Using the hypothesis (A_4), for all $t \geq t_1$, we have

$$\int_t^\infty k_i^2(s)\mathrm{d}s\leq k_i(t)\int_t^\infty k_i(s)\mathrm{d}s\leq k_i(t)\int_0^\infty k_i(s)\mathrm{d}s\quad\text{and}\quad k_i^2(t)\leq ck_i(t),\quad i=1,2. \qquad (7.3.12)$$

Then, (7.3.11) becomes, for some new positive constants c_1 and c_2

$$\begin{aligned}
&\left(F(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right)' \\
\leq\ & -\theta_0\left[E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right] \\
&+c\left[-E'(t) + \frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + \frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right]^{\frac{1}{2p-1}} \\
&+c_1 k_1(t)\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + c_2 k_2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma. \qquad (7.3.13)
\end{aligned}$$

Multiplied by

$$\left[E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-2},$$

and noting that

$$E'(t) \leq \frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + \frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma, \qquad (7.3.14)$$

we get

$$\begin{aligned}
&\left(\left[F(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]\right. \\
&\left.\times\left[E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-2}\right)' \\
\leq\ & \left(F(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right)'
\end{aligned}$$

$$
\begin{aligned}
&\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-2}\\
\leq\ & -\theta_0\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-1}\\
&+c\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-2}\\
&\times\left[-E'(t)+\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma+\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right]^{\frac{1}{2p-1}}\\
&+\left(c_1k_1(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\\
&\times\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-2}
\end{aligned}
\tag{7.3.15}
$$

Using Young's inequality with $\lambda = 2p-1$ and $\lambda' = \frac{2p-1}{2p-2}$, we have, for some new positive constants c_1 and c_2

$$
\begin{aligned}
&\Bigg(\left[F(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]\\
&\times\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-2}\Bigg)'\\
\leq\ & -\theta_0\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-1}\\
&+2\varepsilon\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-1}\\
&+c_\varepsilon\left[-E'(t)+\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma+\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right]\\
&+\left(c_1k_1(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)^{2p-1}.
\end{aligned}
\tag{7.3.16}
$$

Consequently, for $2\varepsilon < \theta_0$, we have

$$
\begin{aligned}
& F_0'(t) \\
\leq \ & -\tilde{\theta}\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_2^2(s)\mathrm{d}s\right]^{2p-1} \\
& +\left(c_1 k_1(t)\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma + c_2 k_2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)^{2p-1}. \qquad (7.3.17)
\end{aligned}
$$

where $\tilde{\theta}$ is a positive constant and

$$
\begin{aligned}
& F_0(t) \\
= \ & \left[F(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_2^2(s)\mathrm{d}s\right] \\
& \times\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_2^2(s)\mathrm{d}s\right]^{2p-2} \\
& +c_\varepsilon\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_2^2(s)\mathrm{d}s\right]. \qquad (7.3.18)
\end{aligned}
$$

Also, it is easy to show that inequality (7.3.17) is true for $p=1$. Once again, we use (7.3.14) to deduce that, for all $t \geq t_1$,

$$
\begin{aligned}
& \left(t\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_2^2(s)\mathrm{d}s\right]^{2p-1}\right)' \\
\leq \ & \left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^{\infty} k_2^2(s)\mathrm{d}s\right]^{2p-1} \\
\leq \ & -\frac{1}{\tilde{\theta}}F_0'(t)+\frac{1}{\tilde{\theta}}\left[c_1 k_1(t)\int_{\Gamma_2}|u_0|^2 \mathrm{d}\Gamma + c_2 k_2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right]^{2p-1}. \qquad (7.3.19)
\end{aligned}
$$

A simple integration over (t_1, t) yields

$$\begin{aligned}
& t\left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-1} \\
\leq\ & \frac{1}{\tilde{\tilde{\theta}}}F_0(t_1)+\frac{1}{\tilde{\tilde{\theta}}}\int_{t_1}^t\left[c_1k_1(s)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2(s)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{2p-1}\mathrm{d}s+t_1\times \\
& \left[E_1(t_1)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_{t_1}^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_{t_1}^\infty k_2^2(s)\mathrm{d}s\right]^{2p-1} \\
\leq\ & c_0+\frac{1}{\tilde{\tilde{\theta}}}\int_{t_1}^t\left[c_1k_1(s)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2(s)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{2p-1}\mathrm{d}s, \qquad (7.3.20)
\end{aligned}$$

where c_0 is a positive constant. Hence, for some new positive constants c_1 and c_2, we get

$$\begin{aligned}
& \left[E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right]^{2p-1} \\
\leq\ & \frac{c_0}{t}+\frac{c_1}{t}\int_{t_1}^t\left[k_1(s)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right]^{2p-1}\mathrm{d}s+\frac{c_2}{t}\int_{t_1}^t\left[k_2(s)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{2p-1}\mathrm{d}s. \qquad (7.3.21)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& E(t) \\
\leq\ & \left(\frac{c_0+c_1\int_{t_1}^t\left[k_1(s)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right]^{2p-1}\mathrm{d}s+c_2\int_{t_1}^t\left[k_2(s)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]^{2p-1}\mathrm{d}s}{t}\right)^{\frac{1}{2p-1}} \\
& -\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s-\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s, \quad \forall\ t\geq t_1.
\end{aligned}$$

$$(7.3.22)$$

Case II: The general case. As in [43], we define

$$I(t) := \int_{t_1}^{t} \frac{-k_1'(s)}{H_0^{-1}(k_1''(s))} \int_{\Gamma_2} |u(t) - u(t-s)|^2 \mathrm{d}\Gamma \mathrm{d}s, \tag{7.3.23}$$

where H_0 is such that (6.2.18) is satisfied. From (7.3.5), we have, for all $t \geq t_1$, $I(t)$ satisfies

$$I(t) < 1. \tag{7.3.24}$$

We also assume, without loss of generality that $I(t) \geq \beta_0 > 0$, for all $t \geq t_1$, otherwise (7.3.3) yields an explicit decay. In addition, we define $\psi(t)$ by

$$\psi(t) := \int_{t_1}^{t} k_1''(s) \frac{-k_1'(s)}{H_0^{-1}(k_1''(s))} \int_{\Gamma_2} |u(t) - u(t-s)|^2 \mathrm{d}\Gamma \mathrm{d}s, \tag{7.3.25}$$

and infer from (A_4) and the properties of H_0 and D that

$$\frac{-k_1'(s)}{H_0^{-1}(k_1''(s))} \leq \frac{-k_1'(s)}{H_0^{-1}(H(-k_1'(s)))} = \frac{-k_1'(s)}{D^{-1}(-k_1'(s))} \leq k_0, \tag{7.3.26}$$

for some positive constant k_0. Then, using (6.2.19), (7.2.2) and (7.3.4), one can easily see that $\psi(t)$ satisfies, for all $t \geq t_1$,

$$\begin{aligned} \psi(t) &\leq k_0 \int_{t_1}^{t} k_1''(s) \int_{\Gamma_2} |u(t) - u(t-s)|^2 \mathrm{d}\Gamma \mathrm{d}s \\ &\leq cM \int_{t_1}^{t} k_1''(s) \mathrm{d}s \leq cM(-k_1'(t_1)) < \min\{r, H(r), H_0(r)\}, \end{aligned} \tag{7.3.27}$$

for t_1 even larger if needed. In addition, we can easily prove that

$$\psi(t) \leq -c\left[E'(t) - \frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma - \frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right], \quad \forall\ t \geq t_1. \quad (7.3.28)$$

Since H_0 is strictly convex on $(0, r]$, and $H_0(0) = 0$, we have

$$H_0(\alpha x) \leq \alpha H_0(x), \quad (7.3.29)$$

provided that $0 \leq \alpha \leq 1$, and $x \in (0, r]$. Using (7.3.24), (7.3.27), (7.3.29) and Jensen's inequality, we obtain that

$$\begin{aligned} \psi(t) &= \frac{1}{I(t)}\int_{t_1}^{t} I(t)H_0[H_0^{-1}(k_1''(s))]\frac{-k_1'(s)}{H_0^{-1}(k_1''(s))}\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s \\ &\geq \frac{1}{I(t)}\int_{t_1}^{t} H_0[I(t)H_0^{-1}(k_1''(s))]\frac{-k_1'(s)}{H_0^{-1}(k_1''(s))}\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s \\ &\geq H_0\left(\frac{1}{I(t)}\int_{t_1}^{t} I(t)H_0^{-1}(k_1''(s))\frac{-k_1'(s)}{H_0^{-1}(k_1''(s))}\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s\right) \\ &= H_0\left(-\int_{t_1}^{t} k_1'(s)\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s\right), \quad \forall\ t \geq t_1. \quad (7.3.30) \end{aligned}$$

This implies that

$$-\int_{t_1}^{t} k_1'(s)\int_{\Gamma_2}|u(t)-u(t-s)|^2\mathrm{d}\Gamma\mathrm{d}s \leq H_0^{-1}(\psi(t)), \quad \forall\ t \geq t_1.$$

Also, we define

$$\varpi(t) := \int_{t_1}^{t}\frac{-k_2'(s)}{H_0^{-1}(k_2''(s))}\int_{\Gamma_2}\left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2\mathrm{d}\Gamma\mathrm{d}s,$$

and

$$\chi(t) := \int_{t_1}^{t} k_2''(s) \frac{-k_2'(s)}{H_0^{-1}(k_2''(s))} \int_{\Gamma_2} \left| \frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu} \right|^2 \mathrm{d}\Gamma \mathrm{d}s.$$

Similarly, we deduce that, for all $t \geq t_1$,

$$\varpi(t) \leq 1, \tag{7.3.31}$$

and

$$\chi(t) < \min\{r, H(r), H_0(r)\}. \tag{7.3.32}$$

It is also easy to see that

$$\chi(t) \leq -c \left[E'(t) - \frac{\tau_1}{2} k_1^2(t) \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma - \frac{\tau_2}{2} k_2^2(t) \int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma \right], \quad \forall\, t \geq t_1. \tag{7.3.33}$$

Repeating the above steps, we arrive at

$$-\int_{t_1}^{t} k_2'(s) \int_{\Gamma_2} \left| \frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu} \right|^2 \mathrm{d}\Gamma \mathrm{d}s \leq H_0^{-1}(\chi(t)). \tag{7.3.34}$$

Therefore, (7.3.3) becomes, for some new positive constants c, c_1 and c_2,

$$\begin{aligned} & F'(t) \\ \leq\; & -\theta_0 E(t) + c_1 k_1^2(t) \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + c_2 k_2^2(t) \int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma + c H_0^{-1}(\psi(t)) + c H_0^{-1}(\chi(t)) \\ \leq\; & -\theta_0 E(t) + c_1 k_1^2(t) \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + c_2 k_2^2(t) \int_{\Gamma_2} \left| \frac{\partial u_0}{\partial \nu} \right|^2 \mathrm{d}\Gamma + c H_0^{-1}(\Lambda(t)), \qquad (7.3.35) \end{aligned}$$

where

$$\Lambda(t) = \max_{t \geq t_1}\{\psi(t), \chi(t)\} < \min\{r, H(r), H_0(r)\}. \tag{7.3.36}$$

Also, we have

$$\Lambda(t) \leq -c\left[E'(t) - \frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma - \frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right], \quad \forall\, t \geq t_1. \quad (7.3.37)$$

Now, for $\varepsilon_0 < r$, $C_0 > 0$, using the fact that $H_0' > 0$, $H_0'' > 0$ and (7.3.14), we obtain that the functional

$$\begin{aligned} &F_1(t) = F(t) \times \\ &H_0'\left(\varepsilon_0 \frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_2^2(s)\mathrm{d}s}\right) \\ &+C_0\left(E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_2^2(s)\mathrm{d}s\right) \end{aligned} \quad (7.3.38)$$

satisfies

$$\begin{aligned} &F_1'(t) \\ = &\left(\varepsilon_0 \frac{E'(t) - \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)k_1^2(t) - \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)k_2^2(t)}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_2^2(s)\mathrm{d}s}\right)F(t) \\ \times\ &H_0''\left(\varepsilon_0 \frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_2^2(s)\mathrm{d}s}\right) \\ +\ &H_0'\left(\varepsilon_0 \frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_2^2(s)\mathrm{d}s}\right)F'(t) \end{aligned}$$

$$
\begin{aligned}
&+C_0\left[E'(t)-\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma-\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]\\
\le\ & F'(t)\times\\
&H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)\\
&+C_0\left[E'(t)-\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma-\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right],\quad\forall\ t\ge t_1.
\end{aligned}
\tag{7.3.39}
$$

Hence, using (7.3.35), we obtain, for all $t\ge t_1$,

$$
\begin{aligned}
&F_1'(t)\le +cH_0^{-1}(\Lambda(t))-\theta_0E(t)\\
&\times H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)\\
&\times H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)\\
&+H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)\\
&\times\left(c_1k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\\
&+C_0\left[E'(t)-\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma-\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right],\quad\forall\ t\ge t_1.
\end{aligned}
\tag{7.3.40}
$$

Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [3], pp.61-64), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad \text{if} \quad s \in (0, H_0'(r)], \tag{7.3.41}$$

and H_0^* satisfies the following Young's inequality:

$$AB \le H_0^*(A) + H_0(B), \quad \text{if } A \in (0, H_0'(r)],\ B \in (0, r] \tag{7.3.42}$$

with

$$A = H_0'\left(\varepsilon_0 \frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right),$$

$$B = H_0^{-1}(\Lambda(t)).$$

Using (7.3.36), (7.3.37) and (7.3.40)-(7.3.42), we obtain that, for all $t \ge t_1$,

$$\begin{aligned}
&F_1'(t) \le -\theta_0 E(t) A + cH_0^*(A) \\
&+A\left(c_1 k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma + c_2 k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right) + c\Lambda(t) \\
&+C_0\left[E'(t) - \frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma - \frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2\mathrm{d}\Gamma\right] \\
\le\ & -\theta_0 E(t) \\
&\times H_0'\left(\varepsilon_0 \frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)
\end{aligned}$$

$$
\begin{aligned}
&+H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)\\
&\times c\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)\\
&+H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)\\
&\times\left(c_1k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\\
&-c\left[E'(t)-\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma-\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]\\
&+C_0\left[E'(t)-\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma-\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right]. \qquad (7.3.43)
\end{aligned}
$$

Consequently, with a suitable choice of C_0, we have, for all $t \geq t_1$,

$$
\begin{aligned}
&F_1'(t)\leq\\
&-\theta_0\left(\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)\\
&\times H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)
\end{aligned}
$$

$$+\theta_0\left(\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s\right)$$

$$\times H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$

$$+H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$

$$\times c\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$

$$+H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$

$$\times\left(c_1k_1^2(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2^2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right). \tag{7.3.44}$$

Using (7.3.12), for some new positive constants c_1 and c_2, we have

$$F_1'(t)\leq$$

$$-\theta_0\left(\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$

$$\times H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$
$$+H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$
$$\times c\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$
$$+H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$
$$\times\left(c_1k_1(t)\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma+c_2k_2(t)\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right). \tag{7.3.45}$$

Similarly, using (7.3.41) and (7.3.42), we find that, for all $t \geq t_1$ (t_1 could be even large if needed),

$$k_i(t)\times$$
$$H_0'\left(\varepsilon_0\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s}\right)$$
$$\leq$$

$$
\left(\varepsilon_0 \frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s} \right)
$$

$$
\times H_0' \left(\varepsilon_0 \frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s} \right)
$$

$$
+H_0(k_i(t)), \qquad i = 1, 2.
$$

Therefore, with suitable choice of ε_0, (7.3.45) becomes

$$
\begin{aligned}
& F_1'(t) \leq \\
& -\theta' \left(\frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s} \right) \\
& \times H_0' \left(\varepsilon_0 \frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s} \right) \\
& +c_1 H_0(k_1(t))\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + c_2 H_0(k_2(t))\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma \\
= \;& -\theta' H_1 \left(\frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right)\int_0^\infty k_2^2(s)\mathrm{d}s} \right) \\
& +c_1 H_0(k_1(t))\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + c_2 H_0(k_2(t))\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma, \qquad (7.3.46)
\end{aligned}
$$

where $H_1(t) = tH_0'(\varepsilon_0 t)$ and θ' is a positive constant.

Noting that $H_1'(t) = H_0'(\varepsilon_0 t) + \varepsilon_0 t H_0''(\varepsilon_0 t)$, then using the strict convexity of H_0 on $(0, r]$, we find that $H_1'(t) > 0$, $H_1(t) > 0$, on $(0, 1]$. Thus, from (7.3.14) and (7.3.44), we obtain that, for all $t \geq t_1$,

$$
\begin{aligned}
&\left[tH_1 \left(\frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right) \int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right) \int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right) \int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right) \int_0^\infty k_2^2(s)\mathrm{d}s} \right) \right]' \\
\leq\ & H_1 \left(\frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right) \int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right) \int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right) \int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right) \int_0^\infty k_2^2(s)\mathrm{d}s} \right) \\
\leq\ & -\frac{1}{\theta'} F_1'(t) + \frac{c_1}{\theta'} H_0(k_1(t)) \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma + \frac{c_2}{\theta'} H_0(k_2(t)) \int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma. \qquad (7.3.47)
\end{aligned}
$$

A simple integration over (t_1, t) yields

$$
\begin{aligned}
& tH_1 \left(\frac{E(t) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right) \int_t^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right) \int_t^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right) \int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right) \int_0^\infty k_2^2(s)\mathrm{d}s} \right) \\
\leq\ & t_1 H_1 \left(\frac{E(t_1) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right) \int_{t_1}^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right) \int_{t_1}^\infty k_2^2(s)\mathrm{d}s}{E(0) + \frac{\tau_1}{2}\left(\int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma\right) \int_0^\infty k_1^2(s)\mathrm{d}s + \frac{\tau_2}{2}\left(\int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma\right) \int_0^\infty k_2^2(s)\mathrm{d}s} \right) \\
& + \frac{1}{\theta'} F_1(t_1) + \frac{c_1}{\theta'} \int_{\Gamma_2} |u_0|^2 \mathrm{d}\Gamma \int_{t_1}^t H_0(k_1(s))\mathrm{d}s + \frac{c_2}{\theta'} \int_{\Gamma_2} \left|\frac{\partial u_0}{\partial \nu}\right|^2 \mathrm{d}\Gamma \int_{t_1}^t H_0(k_2(s))\mathrm{d}s.
\end{aligned}
$$

(7.3.48)

This gives, for some new positive constants c_0, c_1 and c_2, and for all $t \geq t_1$

$$H_1\left(\frac{E(t)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_2^2(s)\mathrm{d}s}{E(0)+\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_1^2(s)\mathrm{d}s+\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_0^{\infty}k_2^2(s)\mathrm{d}s}\right)$$
$$\leq \frac{c_0}{t}F_1(t_1)+\frac{c_1}{t}\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\int_{t_1}^{t}H_0(k_1(s))\mathrm{d}s+\frac{c_2}{t}\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\int_{t_1}^{t}H_0(k_2(s))\mathrm{d}s. \tag{7.3.49}$$

Therefore, we obtain, for some positive constant C,

$$E(t)\leq CH_1^{-1}\left(\frac{c_0+c_1\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\int_{t_1}^{t}H_0(k_1(s))\mathrm{d}s+c_2\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\int_{t_1}^{t}H_0(k_2(s))\mathrm{d}s}{t}\right)$$
$$-\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_1^2(s)\mathrm{d}s-\frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^{\infty}k_2^2(s)\mathrm{d}s. \tag{7.3.50}$$

This completes the proof of Theorem 6.2.8.

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Appendix A

SELECTED DERIVATION

A.1 Some Derivations in $O(\epsilon)$ Problem

The $O(\epsilon)$ problem is the linear problem in §2 with $w = w_1$, $\phi = \phi_1$, and $\tau = t_0$. A solution is

$$
\begin{aligned}
w_1 &= \begin{cases} Ae^{\lambda t_0} g_1(\eta), & \eta > 0 \\ Ae^{\lambda t_0} g_2(\eta), & \eta < 0 \end{cases}, \\
\phi_1 &= Ae^{\lambda t_0} + B,
\end{aligned}
$$

where A and B are constants. Because pure imaginary eigenvalues mark a transition from a stable to an unstable regime, we are interested in λ expands at $i\omega$. Also, we will try to produce a solution with the same structure to the nonlinear problem by allowing A and B to vary slowly. In other words, let us modulate the amplitudes of the linearly unstable modes by functions that depend only on t_1 and t_2. Then we are considering functions of the form

$$w_1 = \begin{cases} A(t_1,t_2)e^{iwt_0}g_1(\eta) + \text{CC}, & \eta > 0, \\ A(t_1,t_2)e^{iwt_0}g_2(\eta) + \text{CC}, & \eta < 0, \end{cases} \tag{A.1.1}$$

$$\phi_1 = \Big\{A(t_1,t_2)e^{i\omega t_0} + \text{CC}\Big\} + B(t_1,t_2), \tag{A.1.2}$$

where CC are the complex-conjugate terms. From now on we will use the notation A and B to mean $A(t_1,t_2)$ and $B(t_1,t_2)$.

The functions w_1 and ϕ_1 above "almost" satisfy the eigenvalue problem in §2. In particular,

$$\frac{\partial w_1}{\partial t_0} + \mathcal{L}(w_1,\phi_1) = \begin{cases} -Ae^{i\omega t_0}(g_1'' + g_1' - i\omega g_1 - i\omega e^{-\eta}) + \text{CC}, & \eta > 0 \\ -Ae^{i\omega t_0}(a_p g_2'' + g_2' - i\omega g_2) + \text{CC}. & \eta < 0 \end{cases}$$

$$\frac{\partial w_1}{\partial t_0} + \mathcal{L}(w_1,\phi_1) = \begin{cases} -Ae^{i\omega t_0}(g_1'' + g_1' - i\omega g_1 - i\omega e^{-\eta}) + \text{CC}, & \eta > 0, \\ -Ae^{i\omega t_0}(a_p g_2'' + g_2' - i\omega g_2) + \text{CC}. & \eta < 0. \end{cases}$$

Recalling that the eigenvalue $\lambda = \alpha + i\omega$, add and subtract α terms to rewrite the right-hand side above as

$$\begin{cases} -Ae^{i\omega t_0}\left(g_1'' + g_1' - (\alpha+i\omega)g_1 - (\alpha+i\omega)e^{-\eta}\right) - A\alpha e^{i\omega t_0}(g_1 + e^{-\eta}) + \text{CC}, & \eta > 0, \\ -Ae^{i\omega t_0}\left(a_p g_2'' + g_2' - (\alpha+i\omega)g_2\right) - A\alpha e^{i\omega t_0}g_2 + \text{CC}, & \eta < 0, \end{cases}$$

which equals

$$\begin{cases} -A\alpha e^{i\omega t_0}(g_1 + e^{-\eta}) + \text{CC}, & \eta > 0, \\ -A\alpha e^{i\omega t_0}g_2 + \text{CC}, & \eta < 0, \end{cases}$$

because $\lambda = \alpha + i\omega$ is an exact eigenvalue. Therefore by (3.3.4), we obtain

$$\frac{\partial w_1}{\partial t_0} + \mathcal{L}(w_1, \phi_1) = \begin{cases} -\epsilon^2 \chi A e^{i\omega t_0}(g_1 + e^{-\eta}) + \text{CC} + O(\epsilon^3), & \eta > 0, \\ -\epsilon^2 \chi A e^{i\omega t_0} g_2 + \text{CC} + O(\epsilon^3), & \eta < 0. \end{cases} \tag{A.1.3}$$

Similarly, substituting w_1 and ϕ_1 into the boundary conditions, we find

$$\mathcal{M}(w_1, \phi_1) = \epsilon^2 \chi A e^{i\omega t_0} \nu_c + \text{CC} + O(\epsilon^3) \tag{A.1.4}$$

and

$$\mathcal{N}(w_1, \phi_1) = -\epsilon^2 \chi A e^{i\omega t_0} + \text{CC} + O(\epsilon^3). \tag{A.1.5}$$

The right-hand sides of (A.1.3)–(A.1.5) will contribute to the $O(\epsilon^3)$ problem.

A.2 Some Derivations in $O(\epsilon^2)$ Problem

The solution to problem (3.4.28) and (3.4.29) are given as follows:

$$g_1(\eta) = (1 + \nu_c \lambda) \exp\left(\frac{-1 - \sqrt{1 + 4\lambda}}{2} \eta\right) - e^{-\eta},$$

$$g_2(\eta) = \nu_c \lambda \exp\left(\frac{-1 + \sqrt{1 + 4a_p \lambda}}{2a_p} \eta\right),$$

$$k_0^+(\eta) = (c_1^* - r_0 \eta)\, e^{-\eta} + g_1'(\eta),$$

$$k_0^-(\eta) = c_2^* e^{-\frac{\eta}{a_p}} + g_2' + c_3^*,$$

where

$$c_1^* = \frac{1}{2}K''(1)\nu_c\omega^2 + r_0\nu_c - g_1'(0),$$

$$c_2^* = -\left(g_1''(0) - c_1^* - a_p g_2''(0)\right),$$

$$c_3^* = \frac{1}{2}K''(1)\nu_c\omega^2 + r_0\nu_c - c_2^* - g_2'(0).$$

And

$$k_2^+(\eta) = c_4^* \exp\left(\frac{-1-\sqrt{1+8i\omega}}{2}\eta\right) + g_1'(\eta) - \left(\frac{1}{2} + c_2\right)e^{-\eta},$$

$$k_2^-(\eta) = c_5^* \exp\left(\frac{-1+\sqrt{1+8a_p i\omega}}{2a_p}\eta\right) + g_2'(\eta),$$

where

$$c_4^* = \left(-\frac{1}{2}K''(1)\nu_c\omega^2 - g_1'(0) + \frac{1}{2}\right) + (1+2\nu_c i\omega)\,C_2,$$

$$c_5^* = \left(-\frac{1}{2}K''(1)\nu_c\omega^2 - g_2'(0)\right) + 2\nu_c i\omega C_2,$$

$$C_2 = \frac{\text{Top}}{\text{Bottom}},$$

$$\begin{aligned}\text{Top} \quad = \quad & \left(\frac{1}{2}K''(1)\nu_c\omega^2 + g_1'(0) - \frac{1}{2}\right)\frac{-1-\sqrt{1+8i\omega}}{2} - \frac{1}{2} - \left(g_1''(0) - a_p g_2''(0)\right) \\ & -\left(\frac{1}{2}K''(1)\nu_c\omega^2 + g_2'(0)\right)\frac{-1+\sqrt{1+8a_p i\omega}}{2},\end{aligned}$$

$$\text{Bottom} = (1+2\nu_c i\omega)\frac{-1-\sqrt{1+8i\omega}}{2} - 2\nu_c i\omega\frac{-1+\sqrt{1+8a_p i\omega}}{2} + 2i\omega + 1.$$

A.3 Example

For instance as in [43], if

$$k_i'(t) = -\exp(-t^q),\ i = 1, 2, \quad \text{where} \quad 0 < q < 1$$

then

$$k_i''(t) = H(-k_i'(t)), i = 1, 2, \quad \text{where} \quad H(t) = \frac{qt}{[\ln\left(\frac{1}{t}\right)]^{\frac{1}{q}-1}}, \text{ for } t \in (0, r], r < 1.$$

Since

$$H'(t) = \frac{(1-q) + q\ln\left(\frac{1}{t}\right)}{[\ln\left(\frac{1}{t}\right)]^{\frac{1}{q}}} \quad \text{and} \quad H''(t) = \frac{(1-q)\left[\ln\left(\frac{1}{t}\right) + \frac{1}{q}\right]}{[\ln\left(\frac{1}{t}\right)]^{\frac{1}{q}+1}},$$

the function H satisfies hypothesis (A_4) on the interval $(0, r]$ for any $0 < r < 1$. Also, we can easily verify that (6.2.18) is satisfied for any $\alpha > 1$ by talking $D(t) = t^\alpha$. Thus, we can obtain an explicit rate of decay as stated in Theorem 6.2.8. Noting that the function $H_0(t) = H(t^\alpha)$ has the following derivative:

$$H_0'(t) = \frac{q\alpha t^{\alpha-1}\left[\frac{1}{q} - 1 + \ln\left(\frac{1}{t^\alpha}\right)\right]}{[\ln\left(\frac{1}{t^\alpha}\right)]^{\frac{1}{q}}}.$$

Then, we do some direct calculations and use (6.2.17) to obtain that

$$\begin{aligned} E(t) \quad &\leq \quad C\left(\frac{c_0 + c_1\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\int_{t_1}^t H_0(k_1(s))\mathrm{d}s + c_2\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\int_{t_1}^t H_0(k_2(s))\mathrm{d}s}{t}\right)^{1/(2\alpha)} \\ &\quad -\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s - \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s, \quad \forall\ t \geq t_1, \end{aligned}$$

(A.3.1)

for any $\alpha > 1$, where

$$H_0(k_i(s)) = \frac{q(k_i(s))^\alpha}{\ln\left(\frac{1}{(k_i(s))^\alpha}\right)}, \quad i = 1, 2.$$

Therefore, taking $\alpha \to 1$, the energy decays at the following rate:

$$\begin{aligned} E(t) \;\; &\leq \;\; C\left(\frac{c_0 + c_1\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\int_{t_1}^t H_0(k_1(s))\mathrm{d}s + c_2\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\int_{t_1}^t H_0(k_2(s))\mathrm{d}s}{t}\right)^{1/2} \\ &\quad -\frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s - \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s, \;\forall\; t \geq t_1. \end{aligned} \tag{A.3.2}$$

If $\displaystyle\int_0^\infty H(k_i(s))\mathrm{d}s < +\infty,\; i = 1, 2$, then for all $t \geq t_1$, equation (A.3.2) is reduced to

$$E(t) \leq \frac{C}{t^{\frac{1}{2}}} - \frac{\tau_1}{2}\left(\int_{\Gamma_2}|u_0|^2\mathrm{d}\Gamma\right)\int_t^\infty k_1^2(s)\mathrm{d}s - \frac{\tau_2}{2}\left(\int_{\Gamma_2}\left|\frac{\partial u_0}{\partial\nu}\right|^2\mathrm{d}\Gamma\right)\int_t^\infty k_2^2(s)\mathrm{d}s, \;\forall\; t \geq t_1.$$

Appendix B

SELECTED FIGURES

B.1 Figure for Numerical and Asymptotic Solutions ($a_p = 1$)

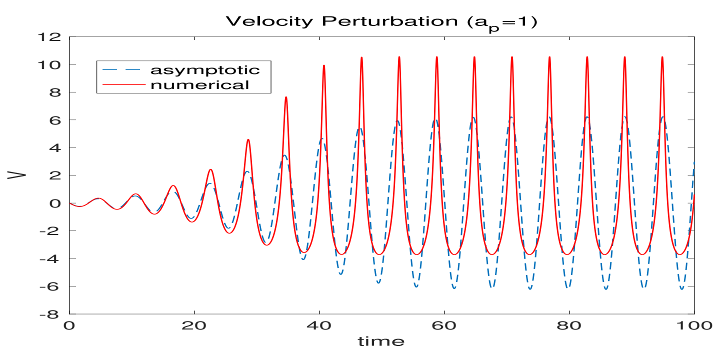

Figure B.1: Comparison between numerical (solid line) and asymptotic (dashed line) for Arrhenius kinetics: $a_p = 1$, $\sigma = 0.46$, $\epsilon = 0.1$, $A(0) = 0.1$, $\nu_c \approx 0.2361$.

B.2 Figure for Numerical and Asymptotic Solutions ($a_p = 0.2$)

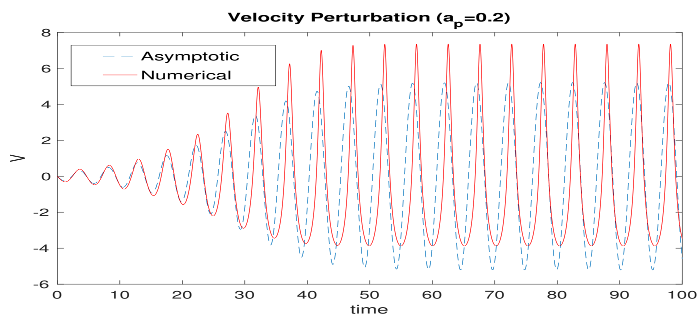

Figure B.2: Comparison between numerical (solid line) and asymptotic (dashed line) for Arrhenius kinetics: $a_p = 0.2$, $\sigma = 0.46$, $\epsilon = 0.1$, $A(0) = 0.1$, $\nu_c \approx 0.3041$.

B.3 Figure for Velocity ($a_p = 0.2$)

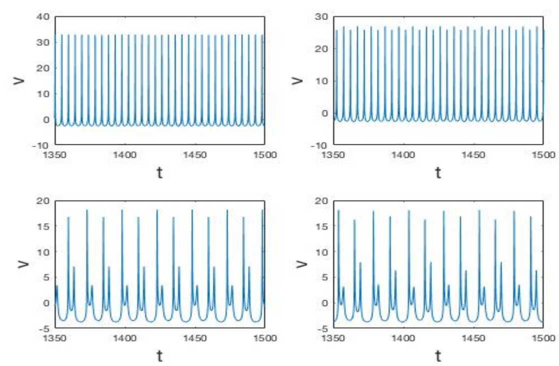

Figure B.3: Velocity perturbations versus time ($\epsilon = 0.2$, $a_p = 0.2$, $A(0) = 0.1$, $\nu = \nu_c - \epsilon^2$), for different σ, we get: upper left: quasi-periodic solution, upper right: period doubling; lower left: period quadrupling, lower right: period octupling

B.4 Figure for Phase Plots ($a_p = 0.2$)

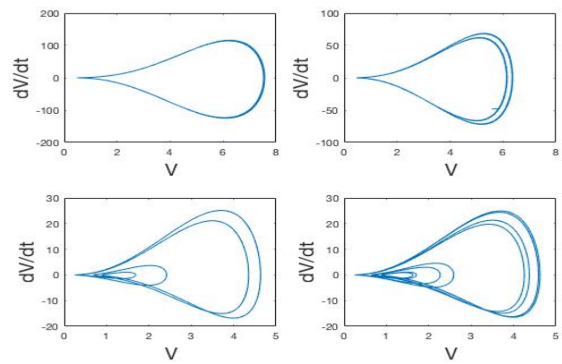

Figure B.4: Phase plots of the four solutions of Figure B.3 for $a_p = 0.2$, $1350 < t < 1500$: velocity perturbations $v(t)$ versus dv/dt.

B.5 Figure for Velocity ($a_p = 0.8$)

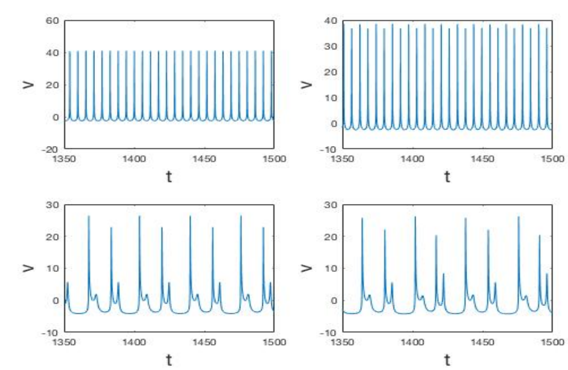

Figure B.5: Velocity perturbations versus time ($\epsilon = 0.2$, $a_p = 0.8$, $A(0) = 0.1$, $\nu = \nu_c - \epsilon^2$) for different σ, we get: upper left: quasi-periodic solution, upper right: period doubling; lower left: period quadrupling, lower right: period octupling

B.6 Figure for Phase Plots ($a_p = 0.8$)

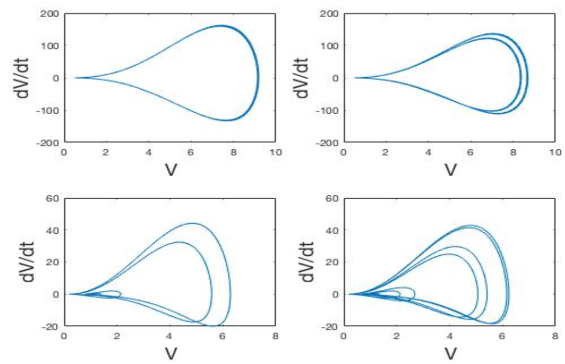

Figure B.6: Phase plots of the four solutions of Figure B.5 for $a_p = 0.8$, $1350 < t < 1500$: velocity perturbations $v(t)$ versus dv/dt.

B.7 Figure for Fourier Plots ($a_p = 1$)

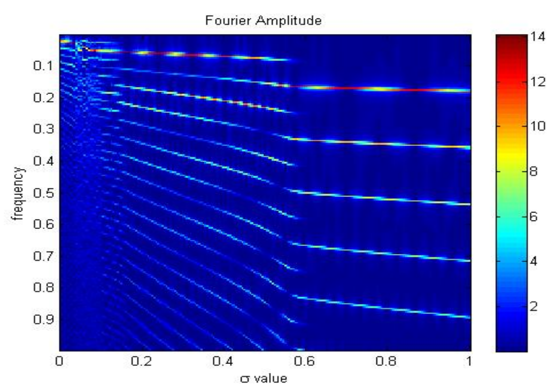

Figure B.7: Fourier amplitude for $a_p = 1$.

B.8 Figure for Fourier Plots ($a_p = 0.8$)

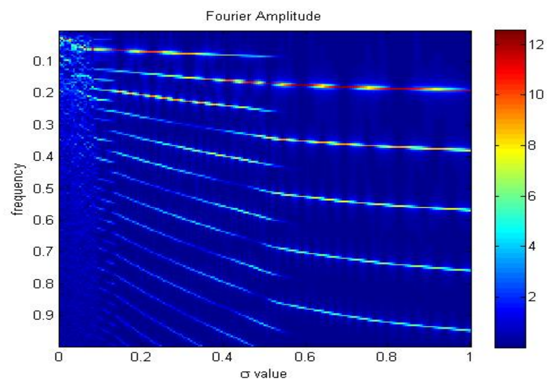

Figure B.8: Fourier amplitude for $a_p = 0.8$.

B.9 Figure for Fourier Plots ($a_p = 0.6$)

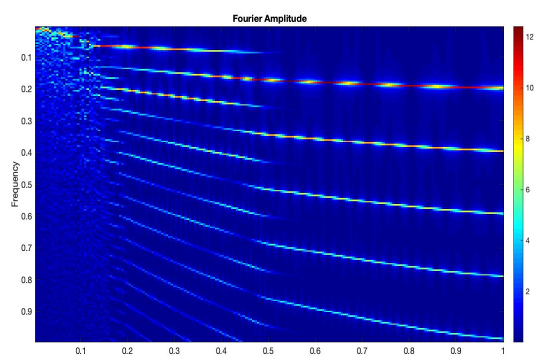

Figure B.9: Fourier amplitude for $a_p = 0.6$. $0 < \sigma < 1$

Appendix C

SELECTED NUMERICAL CODE

C.1 MATLAB Code for Numerical Solution

```
% Kewang -- Numerical solution / update: 03/05/2018
%equatiions:
%partial_u/partial_t-V*partial_u/partial_x
%     =a_b*partial^2_u/partial_x^2, for x<0,                      (1)
%%partial_u/partial_t-V*partial_u/partial_x
%     =partial^2_u/partial_x^2,     for x>0,                      (2)
%boundary conditions:
%partial_u(0+,t)/parti_x-a_b*partial_u(0-,t)/parti_x=-V(t),       (3)
%u(0+,t)=u(0-,t)=u_0(t)=1+neo*K(V(t)),                           (4)
%u(L,t)=0,                                                        (5)
%partial_u(-L,t)/partial_x=0,                                     (6)
% where
%K(V)=log(V)/(1-(1-sigma)*neo*ln(V)),                             (7)
%0<=a_b<=1,  0<=sigma<=1                                          (8)
%neo=neo_c-ep*ep                                                  (9)
%with neo_c determined by
%(1-a_b)^2*(3+a_b)*(neo_c)^4-(1-a_b)*(19a_b-11)*(neo_c)^3
%+2*(4*a_b*a_b+9*a_b-11)*(neo_c)^2+4*(3+a_b)*neo_c-2*(1+a_b)=0,  (10)
%initial condition
%V(0)=1+ep*ap,                                                   (11)
%u(x>=0,0)=exp(-x)+ep*ap*2*real(Gn*exp(miu_n*x)-exp(-x)),        (12)
```

```
%u(x<=0,0)=1+Gp*ep*ap*2*real(exp(miu_p*x)),                              (13)
%where
%Gn=(neo_c*(1+a_b*miu_p)-1)/(neo_c*(a_b*miu_p-miu_n)-1),                 (14)
%Gp=neo_c*(1+miu_n)/(neo_c*(a_b*miu_p-miu_n)-1),                         (15)
%miu_p=2.0*omega_c*1.0i/(1+sqrt(1+4.0*a_b*omega_c*1.0i)),                (16)
%miu_n=(-1-sqrt(1+4.0*omega_c*1.0i))/2.0,                                (17)
%omega_c^2=((1-a_b)*neo_c-1)/(2*neo_c*(2*(1-a_b)*neo_c-1-a_b)).          (18)
clear;clc
a_p=0.2;
%a_p=1;
%a_p=0.8;
sigma=0.46;ep=0.1;
dt=0.005; tmax=100; t=(0:0.005:tmax); n=length(t);
% Perturbate Velocity
% calculate neo_c and omega
coe_neo=zeros(1,5);
coe_neo(1)=(1-a_p)^2;
coe_neo(2)=(1-a_p)*(11-19*a_p)/(3+a_p);
coe_neo(3)=2*(-11+9*a_p+4*a_p*a_p)/(3+a_p);
coe_neo(4)=4;
coe_neo(5)=-2*(1+a_p)/(3+a_p);
x_neo_c=roots(coe_neo);

I_neo_c=0;
for i_x=1:length(x_neo_c)
    if imag(x_neo_c(i_x))<=1.0e-6
    % neo_c value
    neo_c=real(x_neo_c(i_x));
    %omega_c value
    omega_c_sq=((1-a_p)*neo_c-1)/(2*neo_c*(2*(1-a_p)*neo_c-1-a_p));
    if  omega_c_sq>1.0e-6
        omega_c=sqrt(omega_c_sq);
        miu_n=-(1+sqrt(1+4.0*omega_c*1.0i))/2.0;
        miu_p=2.0*omega_c*1.0i/(1+sqrt(1+4.0*a_p*omega_c*1.0i));
        %check non-trivial solution condition
        non_trival=(1.0i*omega_c*neo_c*miu_n+1.0i*omega_c+miu_n+1)...
            *(neo_c+neo_c*a_p*miu_p-1)...
            -(1.0i*omega_c*neo_c+1)*a_p*miu_p*neo_c*(miu_n+1);
        if abs(non_trival)<1.0e-6
        I_neo_c=I_neo_c+1;
        end
    end
    end
end
if abs(I_neo_c-1)>0.1
    disp('problem for solving critical value neo_c')
    pause
end
```

```
omega=omega_c;
disp('a_p,  neo_c,  omega')
disp([a_p neo_c omega])
%%%%%%%%%%%%%%--genealized model  %%%%%%%%%%%
% let \eta be x
dx=0.025; %step for x
x_neg=-10+dx:0.025:0-dx;
x_pos=0+dx:0.025:10-dx;
%\exp(-\eta)
E=exp(-(0:0.025:10-dx));
% initial amplitude
A_0=0.1;
g1=2*real((1+neo_c*1i*omega)*exp((-1-sqrt(1+4*1i*omega))...
    /2*x_pos)-exp(-x_pos));
g2=2*real(neo_c*1i*omega*exp((-1+sqrt(1+4*a_p*1i*omega))/(2*a_p)*x_neg));

% U_0 is the initial values for [\phi,u_2,....,u_m0_...u_M-1]
U_0=[0,A_0*g2,0,A_0*g1];
size_n=size(U_0);
%solution
% Initial guess for Newton's method.
%U=zeros(size_n);
U_initial=U_0;
% d is the first column of Df(x); r is the first row of Df(x)
% a is the main digonal; c is the upper digonal
% b is the lower digonal;  f is nonhomogenious terms
% initialize
d=zeros(size_n);r=zeros(size_n);a=zeros(size_n);b=zeros(size_n-1);
c=zeros(size_n-1);f=ones(size_n);udif=ones(size_n);inf=1;
V=zeros(1,n); V(1)=0;
for k=1:n-1

while inf<=100 && (norm(udif)>0.0000001 || norm(f)>0.0000001)
    inf=inf+1;
   r(1)=ep*U_0(2);r(2)=(dx/dt+(1+ep*U_0(1))-a_p/dx);r(3)=dx/dt...
       +a_p/dx; r(4:798)=dx/dt;r(799)=dx/dt+1/dx;r(800)=dx/dt-1/dx;

   a(1)=0;a(2:400)=-(1+dt/(dx)^2*a_p);a(401)=1;a(402:800)=-(1+dt/(dx)^2);

   c(1)=r(2);c(2:400)=(dt/(2*(dx)^2)*a_p+dt/(4*dx)*(1+ep*U_initial(1)));
   c(401)=0;c(402:799)=(dt/(2*(dx)^2)+dt/(4*dx)*(1+ep*U_initial(1)));

   d(1)=ep*U_0(2);d(2)=ep*dt/(4*dx)*U_initial(3);
   b(1)=d(2);b(2:399)=dt/(2*(dx)^2)*a_p-dt/(4*dx)*(1+ep*U_initial(1));
   b(400)=0;b(401:799)=dt/(2*(dx)^2)-dt/(4*dx)*(1+ep*U_initial(1));
   f(1)=-(dx/dt*(sum(U_initial(2:800)))-dx/dt*(sum(U_0(2:800))));
   Part1_A=-((1+dt/(dx)^2*a_p)*U_initial(2)+(dt/(2*(dx)^2)*a_p...
       +dt/(4*dx)*(1+ep*U_initial(1)))*U_initial(3));
```

```
Part1_B=(1-dt/(dx)^2*a_p)*U_0(2)+(dt/(2*(dx)^2)*a_p...
    +dt/(4*dx)*(1+ep*U_0(1)))*U_0(3);
f(2)=-(Part1_A+Part1_B);

for i=3:400
    d(i)=ep*dt/(4*dx)*(U_initial(i+1)-U_initial(i-1));
    Part2_A=(dt/(2*(dx)^2)*a_p-dt/(4*dx)...
        *(1+ep*U_initial(1)))*U_initial(i-1)...
        -(1+dt/(dx)^2*a_p)*U_initial(i)...
        +(dt/(2*(dx)^2)*a_p+dt/(4*dx)...
        *(1+ep*U_initial(1)))*U_initial(i+1);
    Part2_B=(dt/(2*(dx)^2)*a_p-dt/(4*dx)*(1+ep*U_0(1)))...
        *U_0(i-1)+(1-dt/(dx)^2*a_p)*U_0(i)...
        +(dt/(2*(dx)^2)*a_p+dt/(4*dx)*(1+ep*U_0(1)))*U_0(i+1);
    f(i)=-(Part2_A+Part2_B);
end
d(401)=(-neo_c*1/((1+ep*U_initial(1))*(1-neo_c*(1-sigma)...
    *log(1+ep*U_initial(1)))^2));
f(401)=-(U_initial(401)-(neo_c*log(1+ep*U_initial(1))...
    /(1-neo_c*(1-sigma)*log(1+ep*U_initial(1))))/ep);
for i=402:799
    d(i)=ep*dt/dx/4*(U_initial(i+1)-U_initial(i-1))-dt*E(i-400)/2;
    Part3_A=(dt/(2*(dx)^2)-dt/(4*dx)*(1+ep*U_initial(1)))...
        *U_initial(i-1)-(1+dt/(dx)^2)*U_initial(i)...
        +(dt/(2*(dx)^2)+dt/(4*dx)*(1+ep*U_initial(1)))*U_initial(i+1);
    Part3_B=(dt/(2*(dx)^2)-dt/(4*dx)*(1+ep*U_0(1)))...
        *U_0(i-1)+(1-dt/(dx)^2)*U_0(i)...
        +(dt/(2*(dx)^2)+dt/(4*dx)*(1+ep*U_0(1)))*U_0(i+1);
    Part3_C=(U_initial(1)+U_0(1))*dt/2*E(i-400);
    f(i)=-(Part3_A+Part3_B-Part3_C);
end
d(800)=-ep*dt/dx/4*U_initial(798)-dt*E(400)/2;
Part4_A=(dt/(2*(dx)^2)-dt/(4*dx)*(1+ep*U_initial(1)))...
    *U_initial(799)-(1+dt/(dx)^2)*U_initial(800);
Part4_B=(dt/(2*(dx)^2)-dt/(4*dx)*(1+ep*U_0(1)))...
    *U_0(799)+(1-dt/(dx)^2)*U_0(800);
Part4_C=(U_initial(1)+U_0(1))*dt/2*E(400);
f(800)=-(Part4_A+Part4_B-Part4_C);


for j=800:-1:3
    a(j-1)=a(j-1)-b(j-1)*c(j-1)/a(j);
    d(j-1)=d(j-1)-d(j)*c(j-1)/a(j);
    f(j-1)=f(j-1)-f(j)*c(j-1)/a(j);

    r(j-1)=r(j-1)-b(j-1)*r(j)/a(j);
    f(1)=f(1)-f(j)*r(j)/a(j);
    d(1)=d(1)-d(j)*r(j)/a(j); a(1)=d(1);
```

```
        r(1)=d(1);
        c(j-1)=0;r(j)=0;
    end
    a(1)=a(1)-d(2)*c(1)/a(2);
    f(1)=f(1)-f(2)*c(1)/a(2);
    c(1)=0;r(2)=c(1);
    b(1)=d(2);r(1)=a(1);d(1)=a(1);

    udif(1)=f(1)/a(1);udif(2)=(f(2)-d(2)*udif(1))/a(2);
    for i=3:800
        j=i-1;
        udif(i)=(f(i)-d(i)*udif(1)-b(j)*udif(j))/a(i);
    end
    U_initial=U_initial+udif;
end
udif=ones(size_n);f=ones(size_n);
clear a b c d f r
clear Part1_A Part2_A Part3_A Part4_A Part1_B Part2_B ...
    Part3_B Part4_B Part3_C Part4_C
U_0=U_initial;
%U_initial=U;
V(k+1)=U_0(1);
inf=1;
end
tp=(0:dt:tmax);
plot(tp,V);
```

C.2 MATLAB Code for Asymptotic Solution

```
% Kewang : asymptotic solution/ update: 03/05/2018
%a_p=1;
%a_p=0.2;
a_p=0.5;
%a_p=0.8;
sigma=0.46;ep=0.1;
dt=0.00025; tmax=1; t=(0:0.00025:tmax); n=length(t);

% calculate neo_c and omega
coe_neo=zeros(1,5);
coe_neo(1)=(1-a_p)^2;
```

```
coe_neo(2)=(1-a_p)*(11-19*a_p)/(3+a_p);
coe_neo(3)=2*(-11+9*a_p+4*a_p*a_p)/(3+a_p);
coe_neo(4)=4;
coe_neo(5)=-2*(1+a_p)/(3+a_p);
x_neo_c=roots(coe_neo);

I_neo_c=0;
for i_x=1:length(x_neo_c)
    if imag(x_neo_c(i_x))<=1.0e-6
    % neo_c value
    neo_c=real(x_neo_c(i_x));
    %omega_c value
    omega_c_sq=((1-a_p)*neo_c-1)/(2*neo_c*(2*(1-a_p)*neo_c-1-a_p));
    if omega_c_sq>1.0e-6

        omega_c=sqrt(omega_c_sq);
        miu_n=-(1+sqrt(1+4.0*omega_c*1.0i))/2.0;
        miu_p=2.0*omega_c*1.0i/(1+sqrt(1+4.0*a_p*omega_c*1.0i));
        %check non-trivial solution condition
        non_trival=(1.0i*omega_c*neo_c*miu_n+1.0i*omega_c+miu_n+1)...
            *(neo_c+neo_c*a_p*miu_p-1)...
            -(1.0i*omega_c*neo_c+1)*a_p*miu_p*neo_c*(miu_n+1);
        if abs(non_trival)<1.0e-6
        I_neo_c=I_neo_c+1;
        end
    end
    end
end

if abs(I_neo_c-1)>0.1
    disp('problem for solving critical value neo_c')
    pause
end
omega=omega_c;
disp('a_p,  neo_c,  omega')
disp([a_p neo_c omega])
%%%%%%%%%%%%%%%-Kewang--genealized model  %%%%%%%%%%%%
% let \eta be x
syms x
%K(x)=\ln(V)/(1-neo_c(1-sigma)\ln(V))
%K_2 is the second order derivative,K_3 third order derivative at V=1
K_2=-1+2*neo_c*(1-sigma);
K_3=6*neo_c^2*(1-sigma)^2-6*neo_c*(1-sigma)+2;
%r0
%r0=-omega^2*(4/(sqrt(1-4*a_p*1i*omega)+sqrt(1+4*a_p*1i*omega))+K_2);
r0=-omega^2*((1-sqrt(1+4*a_p*1i*omega))/(2*a_p*omega)*1i-1/2*K_2);

% let x=\eta y=sigma
```

```
N_Big_kai=((1+4*a_p*1i*omega)*(2*1i*omega+1)-4*neo_c*(1-a_p)...
    *omega^2*sqrt(1+4*a_p*1i*omega)+(1+4*1i*omega)*sqrt(1+4*a_p*1i*omega));
D_Big_kai=(4*a_p*neo_c*(1+2*1i*omega)+2*neo_c*(1+4*a_p*1i*omega)...
    +(4*(1-a_p)*neo_c^2*1i*omega+4*neo_c-2)*sqrt(1+4*a_p*1i*omega));
Big_kai=N_Big_kai/D_Big_kai;

%%%%% g1, g2, g1' and g2'
g1=(1+neo_c*1i*omega)*exp((-1-sqrt(1+4*1i*omega))/2*x)-exp(-x);
g2=neo_c*1i*omega*exp((-1+sqrt(1+4*a_p*1i*omega))/(2*a_p)*x);

%g1_1 is the value of the first derivative of g1 at zero; g1_2 --second
%derivative at zero
g1_1=subs(diff(g1),0);
g1_2=subs(diff(diff(g1)),0);
g2_1=subs(diff(g2),0);
g2_2=subs(diff(diff(g2)),0);

% coeficents for k0_1(x) and k0_2(x); x is \eta
c=a_p+2/3;
c1=(g1_2-1i*omega-a_p*g2_2);
c2=1/2*K_2*neo_c*omega^2+1/2*r0*neo_c-g1_1-c1;
c3=1/2*K_2*neo_c*omega^2+1/2*r0*neo_c-g2_1;

% k0_1(x) and k0_2(x)
k0_1=(c1-(1/2*r0+1i*omega)*x)*exp(-x)+diff(g1)+c2;
k0_2=diff(g2)+c3;

% C_2, c4,c5 are coeficents for k2_1(x) and k2_2(x)
% C_2, see the paper.
N_C_2=(1/2*K_2*neo_c*omega^2+g1_1-1/2)*(-1-sqrt(1+8*1i*omega))...
    /2-(1/2*K_2*neo_c*omega^2-g2_1)...
    *(-1+sqrt(1+8*1i*omega))/2-(g1_2-a_p*g2_2)-1/2;
D_C_2=(1+2*neo_c*1i*omega)*(-1-sqrt(1+8*1i*omega))...
    /2-2*neo_c*1i*omega*(-1+sqrt(1+8*a_p*1i*omega))/2+2*1i*omega+1;
C_2=N_C_2/D_C_2;

c4=(-1/2*K_2*neo_c*omega^2-g1_1)+(1+2*neo_c*1i*omega)*C_2;
c5=(-1/2*K_2*neo_c*omega^2-g2_1)+2*neo_c*1i*omega*C_2;

% k2_1(x) and k2_2(x)
k2_1=c4*exp((-1-sqrt(1+8*1i*omega))/2*x)+diff(g1)-(1/2+C_2)*exp(-x);
k2_2=c5*exp((-1+sqrt(1+8*1i*a_p*omega))/(2*a_p)*x)+diff(g2);

% F1, P1 is P1_+ ; P2 is P1_-
F1=K_2*neo_c*(2*omega^2*C_2+1i*r0*omega)+K_3/2*1i*neo_c*omega^3;
P1=r0*diff(g1)+2*1i*omega*C_2*conj(diff(g1))...
    +1i*omega*real(diff(k0_1))-1i*omega*real(diff(k2_1));
P2=r0*diff(g2)+2*1i*omega*C_2*conj(diff(g2))...
```

```
    +1i*omega*real(diff(k0_2))-1i*omega*real(diff(k2_2));
% h1, h2
h1=exp((1-sqrt(1+4*1i*omega))/2*x);
h2=exp((1+sqrt(1+4*a_p*1i*omega))/(2*a_p)*x);

%U, V
U=-subs(conj(diff(h1)),0)-a_p*subs(conj(diff(h2)),0);
V=-subs(conj(h1),0);

% calcualte Small_kai and Beta_0 in Landau?Stuart equation
N_Beta_0=int(P1*conj(h1),0,inf)+int(P2*conj(h2),-inf,0)+F1*U;
D_Beta_0=int((g1+exp(-x))*conj(h1),0,inf)...
    +int(g2*conj(h2),-inf,0)-neo_c*U+V;
Beta_0=N_Beta_0/D_Beta_0;
Small_kai=Big_kai-1i*omega*U/(D_Beta_0);
%%%%%%%%%%%%%%%%%%%%  Runge kutta Method
y0=0.1;ay=t;alp=1;
bea=double(Small_kai);beb=double(Beta_0);
y=y0; ay(1)=y;
for k=1:n-1
    tk1=dt*y*(alp*bea+beb*y*conj(y));
    y1=y+tk1/2;
    tk2=dt*y1*(alp*bea+beb*y1*conj(y1));
    y1=y+tk2/2;
    tk3=dt*y1*(alp*bea+beb*y1*conj(y1));
    y1=y+tk3;
    tk4=dt*y1*(alp*bea+beb*y1*conj(y1));
    y=y+(tk1+2*tk2+2*tk3+tk4)/6;
    ay(k+1)=y;
end
vvlin=(1i*omega*c)*exp((1i*omega)*t/ep/ep);
vvlin=real(ay.*vvlin+conj(ay.*vvlin));
% Plot the solution
plot(t/ep/ep,vvlin,'--')
title('Asymptotic solution for the velocity')
xlabel(['time'],'fontsize',15)
ylabel(['V'],'fontsize',15)
```