Logic and Logical Philosophy
Volume 27 (2018), 143–147
DOI: 10.12775/LLP.2018.003

BOOK REVIEWS

Sequent calculi for beginners and professionals

Andrzej Indrzejczak, **Rachunki sekwentowe w logice klasycznej** (Sequent calculi for classical logic), Wydawnictwo Uniwersytetu Łódzkiego, Łódź, 2013, 299 pages, ISBN 978-83-7525-812-7.

The book under review provides a detailed description of sequent calculi for classical logic. It may be read as an introduction but equally as a technical monograph that summarizes and discusses the most important results in this area. The author of the book has written many papers on analyses of sequent calculi. In most of them, a preliminary section discusses classical logic but the focus is on non-classical logics, especially modal and hybrid logics. This time, however, classical logic is his sole concern.

The first chapter concerns Classical Propositional Logic (in short: CPL). It introduces some notations used in the book and well known metalogical facts which the author makes use of in later chapters. In this chapter readers find a presentation of a Hilbert-style system and a semantics for CPL. A method of natural deduction is also introduced. One of the most interesting parts concerns various strategies for a proof of completeness theorem for CPL. The author discusses a non-constructive proof by Henkin and also some constructive proofs introduced respectively by Asser, Hintikka, Smullyan and Post.

The second chapter introduces an original typology of sequent calculi with respect to structural parameters and an analysis of rules of sequent calculi. With respect to structural parameters the following aspects are distinguished:

Published online February 2, 2018

1. the type of argument of sequents (sets, multisets, sequences)
2. the quantity of formulas in arguments of sequents
3. the structure of sequents (e.g., a quantity of arguments of sequents)
4. the character of objects consisting of arguments of sequents (e.g., sequents as a set of sequents).

And in the case of the analysis of rules of sequent calculi the following aspects are distinguished:

1. the admissible form of logical rules (rules might introduce something into the antecedent or consequent of a sequent, or eliminate something from the antecedent or consequent of a sequent)
2. the type of logical rules (context-sharing rules, context-free rules)
3. the eliminability of structural rules
4. the proportion of sequents to rules of sequents in a sequent calculus.

In this chapter some standard interpretations of sequents are discussed, especially one which associates sequents with a notion of logical consequence. Some differences between a theory of logical consequence via sequent calculi and Tarski's analysis are also discussed.

The next chapter is basically devoted to Gentzen's sequent calculus for CPL. However, some of the concepts introduced in this chapter are defined in such a manner to be used in further chapters, such as a rule and its elements, a proof treated as a tree-like structure and for example measures like length, height and size of a proof. Next, the author proves that the calculus is a sound and complete with respect to the deductive system for CPL. The last issue concerns a semantic interpretation of the calculus.

The fourth chapter concerns various sequent calculi for CPL different to Gentzen's calculus. Subsequent calculi are introduced by means of some modifications of the rules. In this context the author analyzes such problems as the eliminability and reversibility of the introduced rules. Some examples of deductive systems other than Gentzen's are calculi based on so-called int-sequents, which were originally introduced in order to provide a calculus for Intuitionistic Logic. What distinguishes such sequents is that their consequences contain only one formula. Other issues discussed concern properties of rules and a method of proof of the equivalence of certain rules. The issue of properties of rules is especially important in the context of the cut-elimination theorem.

The cut-elimination theorem is one of the most important results of the proof theory based on sequent calculi. Various proofs of this theorem are discussed in the fifth chapter. The author emphasizes a difference between the eliminability and the accessibility of the cut-rule. It is important issue, since in some cases the cut-elimination theorem is considered as a theorem about accessibility rather than eliminability. It obviously depends on whether we consider a calculus with or without cut-rule. The author presents and discusses Gentzen's proof of the elimination of cut-rule and proofs of the accessibility of cut-rule which were originally constructed by Dragalin, Smullyan, Schütte, Curry and Buss respectively. In order to discuss the first four proofs, a method for the local transformation of a proof is discussed. And in order to explain the last two proofs a method of global transformation is discussed. The comparison of proofs discussed in the chapter is the topic of the last section. A detailed explanation given by the author not only provides an elegant presentation of these proofs but also enables to learn how to construct such proofs.

An obvious question now comes to mind: why is the cut-elimination theorem so important? The author gives an answer in the sixth chapter. He explains how by means of the cut-elimination theorem proofs of the consistency and decidability of CPL or a proof of the interpolation theorem might be constructed. It appears that an quite important role for these proofs is played by the subformulas property. The author discusses this property in the context of rules, sequent calculi and proofs.[1] The analysis leads in a natural way to the problem of decidability. Some observations on an algorithm for conducting proofs within the framework of a sequent calculus are presented. It is emphasized that this algorithm is used in proofs of many of the facts and lemmas presented in the book.

The seventh chapter is entirely devoted to various proofs of a completeness theorem for an analytical version of a sequent calculus, which was introduced in the fourth chapter. The author discusses three types of proofs and makes some remarks about normal forms and Post-style proofs. He starts with a Henkin-style proof. The second one is a Hintikka-style proof and the third one is in some sense a combination of both. In the case of that last proof, the first step is to define a notion of

[1] A rule has such a property just in a case all formulas which appear in sequents-premises are subformulas of formulas form sequents-consequences. In turn, if all rules of a given calculus have the property we say that this calculus has the property. And if that is the case, then in any proof of a given sequent s, all sequents from this proof will be built by means of subformulas which appear in s. In such cases we say that a proof has the subformulas property.

relative maximality in a set. Next, we need to prove that if a pair of sets (Γ, Δ) is consistent and relative maximal in set of subformulas of $\Gamma \cup \Delta$, then there is a pair (Γ_n, Δ_n) such that $\Gamma \subseteq \Gamma_n$ and $\Delta \subseteq \Delta_n$. This fact should be considered as a counterpart of Lindenbaum's lemma. Finally, the author proves that the pair (Γ_n, Δ_n) is saturated.

The eighth chapter plays a similar role to the first one but this time First Order Logic (in short: FOL) is analysed. The author discusses well-known aspects of the syntax of FOL, especially a problem of substitution for free variables. A deductive system for FOL is presented by means of a Hilbert calculus and some examples of first order theories like Peano Arithmetic or Lattice theory are also discussed. After a section concerning the interpretation and the model structure of FOL a proof of soundness and a Henkin-style proof of completeness are presented. In this chapter we also find some remarks concerning the problem of decidability and the rules of natural deduction for FOL.

In the ninth chapter the results presented in chapters from three to seven for CPL are applied to FOL and first-order theories. In particular a sequent calculus for FOL, also in an analytical version, is introduced. The author, however, considers only certain strategies or approaches to some of the problems or proofs explored in previous chapters, while the rest of them are left as exercises for readers. For instance, only Gentzen's proof of cut-elimination theorem is discussed. In this chapter readers also find a proof of the strong cut-elimination theorem which might be recognised as a version of Herbrand's theorem. The last issue discussed concerns sequent calculi for first-order theories.

The tenth chapter concerns non-standard types of sequent calculi for FOL. The deductive systems considered are grouped into three types of calculi:

- Gentzen-like calculi, in which rules determine the meanings of logical constants
- Hertz-like calculi, in which sequents determine the meanings of logical constants
- a mixed type, in which both rules and sequents characterize logical constants.

The author discusses the following systems of the first type: (i) Gentzen's sequential system of natural deduction, (ii) systems introduced by Ershow, Palyutin, Lavrov, Maksimova, (iii) Andrews's system, (iii) a system of natural deduction defined by Suppes and Lemmon, (iv) Hermes's calculus and a system defined by Ebbinghaus, Flum and Thomas,

(v) Leblanc's system and (vi) Došen's system. As regards the second type Suszko's system is analysed and in the case of the third type the following systems are considered: sequential systems of natural deduction determined by Kleene, Hasenjaeger's system, and Rieger's system. The last section is devoted to a short comparison of sequent calculi with other deductive systems.

The last (eleventh) chapter is an appendix which establishes notations and various facts concerning sets, relations and functions. Some aspects of mathematical induction are also discussed. The author illustrates his remarks by using mathematical induction in the proofs of some well-known metalogical facts such as the deduction theorem.

Mateusz Klonowski
Department of Logic
Faculty of Humanities
Nicolaus Copernicus University in Toruń, Poland
matklon@doktorant.umk.pl