ERGODIC PROPERTIES OF THE IDEAL GAS MODEL FOR INFINITE BILLIARDS

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Abstract. In this paper we study ergodic properties of the Poisson suspension (the ideal gas model) of the billiard flow $(b_t)_{t\in\mathbb{R}}$ on the plane with a Λ-periodic pattern ($\Lambda \subset \mathbb{R}^2$ is a lattice) of polygonal scatterers. We prove that if the billiard table is additionally rational then for a.e. direction $\theta \in S^1$ the Poisson suspension of the directional billiard flow $(b_t^\theta)_{t\in\mathbb{R}}$ is weakly mixing. This gives the weak mixing of the Poisson suspension of $(b_t)_{t\in\mathbb{R}}$. We also show that for a certain class of such rational billiards (including the periodic version of the classical wind-tree model) the Poisson suspension of $(b_t^\theta)_{t\in\mathbb{R}}$ is not mixing for a.e. $\theta \in S^1$.

1. Introduction

In this paper we deal with billiard dynamical systems on the plane with a Λ-periodic pattern ($\Lambda \subset \mathbb{R}^2$ is a lattice) of polygonal scatterers. We focus only on a rational billiards, i.e. the angles between any pair of sides of the polygons (also different polygons) are rational multiplicities of π. The most celebrated example of such billiard table is a periodic version of the wind-tree model introduced by P. Ehrenfest and T. Ehrenfest in 1912 [10], in which the scatterers are $\mathbb{Z}^2$-translates of the rectangle $[0, a] \times [0, b]$, where $0 < a, b < 1$.

The *billiard flow* $(b_t)_{t\in\mathbb{R}}$ on a polygonal table $\mathcal{T} \subset \mathbb{R}^2$ (the boundary of the table consists of intervals) is the unit speed free motion on the interior of $\mathcal{T}$ with elastic collision (angle of incidence equals to the angle of reflection) from the boundary of $\mathcal{T}$. The phase space $\mathcal{T}^1$ of $(b_t)_{t\in\mathbb{R}}$ consists of points $(x, \theta) \in \mathcal{T} \times S^1$ such that if x belongs to the boundary of $\mathcal{T}$ then $\theta \in S^1$ is an inward direction. The billiard flow preserves the volume measure $\mu \times \lambda$, where μ is the area measure on $\mathcal{T}$ and λ the Lebesgue measure on S^1. For more details on billiards see [24].

Suppose that $\mathcal{T}$ is the table of a Λ-periodic rational polygonal billiard. Then the volume measure is infinite. Since the table is Λ-periodic the set $D \subset S^1$ of directions of all sides in $\mathcal{T}$ is finite. Denote by Γ the group of isometries of S^1 generated by reflections through the axes with directions from D. Since the table is rational, Γ is a finite dihedral group. Therefore the phase space $\mathcal{T}^1$ splits into the family $\mathcal{T}^1_\theta = \mathcal{T} \times \Gamma\theta$, $\theta \in S^1/\Gamma$ of invariant subsets for $(b_t)_{t\in\mathbb{R}}$. The restriction of $(b_t)_{t\in\mathbb{R}}$ to $\mathcal{T}^1_\theta$ is called the *direction billiard flow* in direction θ and is denoted by $(b_t^\theta)_{t\in\mathbb{R}}$. The flow $(b_t^\theta)_{t\in\mathbb{R}}$ preserves μ_θ the product of μ and the counting measure of $\Gamma\theta$; this measure is also infinite. Using the standard unfolding process described in [18] (see also [24]), we obtain a connected translation surface $(M_\mathcal{T}, \omega_\mathcal{T})$ such that the directional linear flow $(\varphi_t^{\mathcal{T},\theta})_{t\in\mathbb{R}}$ on $(M_\mathcal{T}, \omega_\mathcal{T})$ is isomorphic to the flow $(b_t^\theta)_{t\in\mathbb{R}}$ for every $\theta \in S^1$. Moreover, $(M_\mathcal{T}, \omega_\mathcal{T})$ is a $\mathbb{Z}^2$-cover of a compact connected translation surface.

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We are interested in ergodic properties of the directional flows $(b_t^\theta)_{t\in\mathbb{R}}$ (or equivalently $(\varphi_t^{\mathcal{T},\theta})_{t\in\mathbb{R}}$) in typical (a.e.) direction. Recently, some progress has been made in understanding this problem, especially for periodic wind-tree model. In this model, Avila and Hubert in [2] proved the recurrence of $(b_t^\theta)_{t\in\mathbb{R}}$ for a.e. direction. The non-ergodicity for a.e. direction was proved by the author and Ulcigrai in [16]. Delecroix, Hubert and Leliévre proved in [7] that for a.e. direction the diffusion rate of a.e. orbit is 2/3. For more complicated scatterers some related results were obtained in [8, 14, 26]. Ergodic properties for non-periodic wind-three models were also recently studied by Málaga Sabogal and Troubetzkoy in [21, 22].

Unlike the approach presented in the mentioned articles, we does not study the dynamics of a single billiard ball, i.e. the flow $(b_t^\theta)_{t\in\mathbb{R}}$. We are interested in dynamical properties of infinite (countable) configurations of billiard balls without mutual interactions. Formally, we deal with the Poisson suspension of the flow $(b_t^\theta)_{t\in\mathbb{R}}$ which models the ideal gas behavior in $\mathcal{T}$, see [6, Ch. 9]. The main result of the paper is the following:

Theorem 1.1. *Let $(b_t)_{t\in\mathbb{R}}$ be the billiard flow on a Λ-periodic rational polygonal billiard table $\mathcal{T}$. Then for a.e. $\theta \in S^1$ the Poisson suspension of the directional billiard flow $(b_t^\theta)_{t\in\mathbb{R}}$ is weakly mixing. Moreover, the Poisson suspension of $(b_t)_{t\in\mathbb{R}}$ is also weakly mixing.*

In fact, we prove much more general result (Theorem 5.4) concerning $\mathbb{Z}^d$-covers of compact translation surfaces and their directional flows. Since $(b_t^\theta)_{t\in\mathbb{R}}$ can be treated as a directional flow on the translation surface $(M_{\mathcal{T}}, \omega_{\mathcal{T}})$, Theorem 1.1 is a direct consequence of Theorem 5.4. Moreover, in Section 6 we give a criterion (Theorem 6.3) for the absence of mixing for the Poisson suspension of typical directional flows on some $\mathbb{Z}^d$-covers of compact translation surfaces. Its necessary condition (the existence of "good" cylinders) for the absence of mixing coincides with the condition for recurrence provided by [2]. This allows proving the absence of mixing for the Poisson suspension of $(b_t^\theta)_{t\in\mathbb{R}}$ (for a.e. direction) for the standard periodic wind-three model, as well as for other recurrent billiards studied in [14, Sec. 9] and [26, Sec. 8.3].

2. Poisson point process and Poisson suspension

Let $(X, \mathcal{B}, \mu)$ be a standard σ-finite measure space such that μ has no atom and $\mu(X) = \infty$. Denote by $(X^*, \mathcal{B}^*, \mu^*)$ the associated Poisson point process. For relevant background material concerning Poisson point processes, see [19] and [20]. Then X^* is the space of countable subsets (configurations) of X and the σ-algebra $\mathcal{B}^*$ is generated by the subsets of the form

$$C_{A,n} := \{\overline{x} \in X^* : \operatorname{card}(\overline{x} \cap A) = n\} \text{ for } A \in \mathcal{B} \text{ and } n \geq 0.$$

For every $A \in \mathcal{B}$ denote by $C_A : X^* \to \mathbb{Z}_{\geq 0}$ the measurable map given by $C_A(\overline{x}) = \operatorname{card}(\overline{x} \cap A)$. Then μ^* is a unique probability measure on $\mathcal{B}^*$ such that:

(i) for any pairwise disjoint collection $A_1, \ldots, A_k$ in $\mathcal{B}$ the random variables $C_{A_1}, \ldots, C_{A_k}$ on $(X^*, \mathcal{B}^*, \mu^*)$ are jointly independent;

(ii) for any $A \in \mathcal{B}$ the random variable C_A on $(X^*, \mathcal{B}^*, \mu^*)$ has Poisson distribution with

$$\mu^*(C_{B,n}) = e^{-\mu(A)} \frac{\mu(A)^n}{n!} \quad \text{for} \quad n \geq 0.$$

The existence and uniqueness of the intensity measure μ^* can be found, for instance, in [19].

Poisson suspension is a classical notion introduced in statistical mechanics to model so called ideal gas. For an infinite measure-preserving dynamical system its Poisson suspension is a probability measure-preserving system describing the dynamics of infinite (countable) configurations of particles without mutual interactions. For relevant background material we refer the reader to [6]. More formally, for any $(T_t)_{t\in\mathbb{R}}$ measure preserving flow on $(X,\mathcal{B},\mu)$ by its *Poisson suspension* we mean the flow $(T_t^*)_{t\in\mathbb{R}}$ acting on $(X^*,\mathcal{B}^*,\mu^*)$ by $T_t^*(\overline{x}) = \{T_t y : y \in \overline{x}\}$. Since $(T_t^*)_{t\in\mathbb{R}}$ preserves the measure of any set $C_{A,n}$ and these sets generate the whole σ-algebra, the flow preserves the probability measure μ^*.

Proposition 2.1 (see [27] and [9] for maps)**.** *The flow $(T_t^*)_{t\in\mathbb{R}}$ is ergodic if and only if it is weak mixing and if and only if the flow $(T_t)_{t\in\mathbb{R}}$ has no invariant subset of positive and finite measure.*

The flow $(T_t^*)_{t\in\mathbb{R}}$ is mixing if and only if for all $A\in\mathcal{B}$ with $0<\mu(A)<\infty$ we have $\mu(A\cap T_{-t}A)\to 0$ as $t\to+\infty$.

Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ be two standard σ-finite measure space such that μ and ν have no atoms. Assume that $(T_t)_{t\in\mathbb{R}}$ is a measure preserving flow on $(X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\times\nu)$ such that $T_t(x,y) = (T_t^y x, y)$. Then $(T_t^y)_{t\in\mathbb{R}}$ is a measure-preserving flow on $(X,\mathcal{B},\mu)$ for every $y\in Y$. Applying a standard Fubini argument we have the following result.

Lemma 2.2. *Suppose that for a.e. $y\in Y$ the flow $(T_t^y)_{t\in\mathbb{R}}$ has no invariant subset of positive and finite measure. Then the flow $(T_t)_{t\in\mathbb{R}}$ enjoys the same property.*

3. $\mathbb{Z}^d$-covers of compact translation surfaces

For relevant background material concerning translation surfaces and interval exchange transformations (IETs) we refer the reader to [24], [28], [29] and [30]. Let M be a be a surface (not necessary compact) and let ω be an Abelian differential (holomorphic 1-form) on M. The pair (M,ω) is called a *translation surface*. Denote by $\Sigma\subset M$ the set of zeros of ω. For every $\theta\in S^1=\mathbb{R}/2\pi\mathbb{Z}$ denote by $X_\theta = X_\theta^\omega$ the directional vector field in direction θ on $M\setminus\Sigma$, i.e. $\omega(X_\theta)=e^{i\theta}$ on $M\setminus\Sigma$. Then the corresponding *directional flow* $(\varphi_t^\theta)_{t\in\mathbb{R}} = (\varphi_t^{\omega,\theta})_{t\in\mathbb{R}}$ (also known as *translation flow*) on $M\setminus\Sigma$ preserves the area measure μ_ω ($\mu_\omega(A) = |\int_A \frac{i}{2}\omega\wedge\overline{\omega}|$).

We use the notation $(\varphi_t^v)_{t\in\mathbb{R}}$ for the *vertical flow* (corresponding to $\theta=\frac{\pi}{2}$) and $(\varphi_t^h)_{t\in\mathbb{R}}$ for the *horizontal flow* respectively ($\theta=0$).

Assume that the surface M is compact. Suppose that $\widetilde{M}$ is a $\mathbb{Z}^d$-covering of M and $p:\widetilde{M}\to M$ is its covering map. For any holomorphic 1-form ω on M denote by $\widetilde{\omega}$ the pullback of the form ω by the map p. Then $(\widetilde{M},\widetilde{\omega})$ is a translation surface, called a $\mathbb{Z}^d$-cover of the translation surface $(M.\omega)$.

All $\mathbb{Z}^d$-covers of M up to isomorphism are in one-to-one correspondence with $H_1(M,\mathbb{Z})^d$. For any pair ξ_1,ξ_2 in $H_1(M,\mathbb{Z})$ denote by $\langle\xi_1,\xi_2\rangle$ the algebraic intersection number of ξ_1 with ξ_2. Then the $\mathbb{Z}^d$-cover $\widetilde{M}_\gamma$ determined by $\gamma\in H_1(M,\mathbb{Z})^d$ has the following properties: if $\sigma:[t_0,t_1]\to M$ is a close curve in M and

$$n := \langle\gamma,[\sigma]\rangle = (\langle\gamma_1,[\sigma]\rangle,\ldots,\langle\gamma_d,[\sigma]\rangle)\in\mathbb{Z}^d$$

($[\sigma]\in H_1(M,\mathbb{Z})$), then σ lifts to a path $\widetilde{\sigma}:[t_0,t_1]\to\widetilde{M}_\gamma$ such that $\sigma(t_1)=n\cdot\sigma(t_0)$, where $\cdot$ denotes the action of $\mathbb{Z}^d$ by deck transformations on $\widetilde{M}_\gamma$.

Let (M,ω) be a compact translation surface and let $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ be its $\mathbb{Z}^d$-cover. Let us consider the vertical flow $(\widetilde{\varphi}_t^v)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ such that the flow $(\varphi_t^v)_{t\in\mathbb{R}}$ on (M,ω) is uniquely ergodic. Let $I\subset M\setminus\Sigma$ be a horizontal interval in (M,ω) with no self-intersections. Then the Poincaré (first return) map $T:I\to I$ for the flow $(\widetilde{\varphi}_t^v)_{t\in\mathbb{R}}$ is a uniquely ergodic interval exchange transformation (IET). Denote by $(I_\alpha)_{\alpha\in\mathcal{A}}$ the family of exchanged intervals. Let $\tau:I\to\mathbb{R}_{>0}$ be the corresponding first return time map. Then τ is constant on each interval I_α, $\alpha\in\mathcal{A}$.

For every $\alpha \in \mathcal{A}$ we denote by $\xi_\alpha = \xi_\alpha(\omega, I) \in H_1(M, \mathbb{Z})$ the homology class of any loop formed by the segment of orbit for $(\varphi_t^v)_{t\in\mathbb{R}}$ starting at any $x \in \operatorname{Int} I_\alpha$ and ending at Tx together with the segment of I that joins Tx and x.

Proposition 3.1 (see [16] for $d = 1$). *Let $I \subset M \setminus \Sigma$ be a horizontal interval in (M, ω) with no self-intersections. Then for every $\gamma \in H_1(M, \mathbb{Z})^d$ the vertical flow $(\widetilde{\varphi}_t^v)_{t\in\mathbb{R}}$ on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ has a special representation over the skew product $T_{\psi_{\gamma,I}} : I \times \mathbb{Z}^d \to I \times \mathbb{Z}^d$ of the form $T_{\psi_{\gamma,I}}(x, m) = (Tx, m + \psi_{\gamma,I}(x))$, where $\psi_{\gamma,I} : I \to \mathbb{Z}^d$ is a piecewise constant function given by*

$$\psi_{\gamma,I}(x) = \langle \gamma, \xi_\alpha \rangle = \big(\langle \gamma_1, \xi_\alpha \rangle, \ldots, \langle \gamma_d, \xi_\alpha \rangle\big)$$

if $x \in I_\alpha$ for $\alpha \in \mathcal{A}$. Moreover, the roof function $\widetilde{\tau} : I \times \mathbb{Z}^d \to \mathbb{R}_{>0}$ is given by $\widetilde{\tau}(x, m) = \tau(x)$ for $(x, m) \in I \times \mathbb{Z}^d$.

Remark 3.2. Since the roof function $\widetilde{\tau}$ is bounded and uniformly separated from zero, the absence of invariant set of finite and positive measure for the flow $(\widetilde{\varphi}_t^v)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ is equivalent the absence of invariant set of finite and positive measure for the skew product $T_{\psi_{\gamma,I}}$.

Cocycles for transformations and essential values. Given an ergodic automorphism T of a standard probability space $(X, \mathcal{B}, \mu)$, a locally compact abelian second countable group G and a measurable map $\psi : X \to G$, called a *cocycle* for T, consider the skew-product extension T_ψ acting on $(X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times m_G)$ ($\mathcal{B}_G$ is the Borel σ-algebra on G) by

$$T_\psi(x, y) = (Tx, y + \psi(x)).$$

Clearly T_ψ preserves the product of μ and the Haar measure m_G on G. Moreover, for any $n \in \mathbb{Z}$ we have

$$T_\psi^n(x, y) = (T^n x, y + \psi^{(n)}(x)),$$

where

$$\psi^{(n)}(x) = \begin{cases} \sum_{0\leq j<n} \psi(T^j x) & \text{if } \ n \geq 0 \\ -\sum_{n\leq j<0} \psi(T^j x) & \text{if } \ n < 0. \end{cases}$$

The cocycle $\psi : X \to G$ is called a *coboundary* for T if there exists a measurable map $h : X \to G$ such that $\psi = h - h \circ T$. Then $\psi^{(n)} = h - h \circ T^n$ for every $n \in \mathbb{Z}$.

An element $g \in G$ is said to be an *essential value* of $\psi : X \to G$, if for each open neighborhood V_g of g in G and each $B \in \mathcal{B}$ with $\mu(B) > 0$, there exists $n \in \mathbb{Z}$ such that

$$\mu\big(B \cap T^{-n}B \cap \{x \in X : \psi^{(n)}(x) \in V_g\}\big) > 0.$$

Proposition 3.3 (see [25]). *The set of essential values $E_G(\psi)$ is a closed subgroup of G. If ψ is a coboundary then $E_G(\psi) = \{0\}$.*

Proposition 3.4 (see [3]). *If T is an ergodic automorphism of $(X, \mathcal{B}, \mu)$ then the cocycle $\psi : X \to G$ for T is a coboundary if and only if the skew product $T_\psi : X \times G \to X \times G$ has an invariant set of positive and finite measure.*

Proposition 3.5 (see [5]). *Let $\mathcal{B}$ be the σ–algebra of Borel sets of a compact metric space (X, d) and let μ be a probability Borel measure on $\mathcal{B}$. Suppose that T is an ergodic measure–preserving automorphism of $(X, \mathcal{B}, \mu)$ for which there exist a sequence of Borel sets $(C_n)_{n\geq 1}$ and an increasing sequence of natural numbers $(h_n)_{n\geq 1}$ such that*

$$\mu(C_n) \to \alpha > 0, \ \mu(C_n \triangle T^{-1}C_n) \to 0 \ \ \textit{and} \ \ \sup_{x\in C_n} d(x, T^{h_n}x) \to 0.$$

If $\psi : X \to G$ is a measurable cocycle such that $\psi^{(h_n)}(x) = g_n$ for all $x \in C_n$ and $g_n \to g$, then $g \in E(\psi)$.

4. Teichmüller flow and Kontsevich-Zorich cocycle

Given a compact connected oriented surface M, denote by $\mathrm{Diff}^+(M)$ the group of orientation-preserving homeomorphisms of M. Denote by $\mathrm{Diff}_0^+(M)$ the subgroup of elements $\mathrm{Diff}^+(M)$ which are isotopic to the identity. Let $\Gamma(M) := \mathrm{Diff}^+(M)/\mathrm{Diff}_0^+(M)$ be the *mapping-class* group. We will denote by $\mathcal{T}(M)$ the *Teichmüller space of Abelian differentials*, that is the space of orbits of the natural action of $\mathrm{Diff}_0^+(M)$ on the space of all Abelian differentials on M. We will denote by $\mathcal{M}(M)$ the *moduli space of Abelian differentials*, that is the space of orbits of the natural action of $\mathrm{Diff}^+(M)$ on the space of Abelian differentials on M. Thus $\mathcal{M}(M) = \mathcal{T}(M)/\Gamma(M)$.

The group $SL(2,\mathbb{R})$ acts naturally on $\mathcal{T}(M)$ and $\mathcal{M}(M)$ as follows. Given a translation structure ω, consider the charts given by local primitives of the holomorphic 1-form. The new charts defined by postcomposition of this charts with an element of $SL(2,\mathbb{R})$ yield a new complex structure and a new differential which is Abelian with respect to this new complex structure, thus a new translation structure. We denote by $g\cdot\omega$ the translation structure on M obtained acting by $g\in SL(2,\mathbb{R})$ on a translation structure ω on M. The *Teichmüller flow* $(g_t)_{t\in\mathbb{R}}$ is the restriction of this action to the diagonal subgroup $(\mathrm{diag}(e^t,e^{-t}))_{t\in\mathbb{R}}$ of $SL(2,\mathbb{R})$ on $\mathcal{T}(M)$ and $\mathcal{M}(M)$. We will deal also with the rotations $(r_\theta)_{\theta\in S^1}$ that acts on $\mathcal{T}(M)$ and $\mathcal{M}(M)$ by $r_\theta\omega = e^{i\theta}\omega$. Then the flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$ on (M,ω) coincides with the vertical flow on $(M, r_{\pi/2-\theta}\omega)$. Moreover, for any $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ the directional flow $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ on $\widetilde{(M_\gamma,\widetilde{\omega}_\gamma)}$ in the direction $\theta\in S^1$ coincides with the vertical flow $(\widetilde{\varphi}_t^v)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,(r_{\pi/2-\theta}\omega)_\gamma)$.

Kontsevich-Zorich cocycle. The *Kontsevich-Zorich (KZ) cocycle* $(A_g)_{g\in SL(2,\mathbb{R})}$ is the quotient of the product action $(g\times\mathrm{Id})_{g\in SL(2,\mathbb{R})}$ on $\mathcal{T}(M)\times H_1(M,\mathbb{R})$ by the action of the mapping-class group $\Gamma(M)$. The mapping class group acts on the fiber $H_1(M,\mathbb{R})$ by induced maps. The cocycle $(A_g)_{g\in SL(2,\mathbb{R})}$ acts on the homology vector bundle

$$\mathcal{H}_1(M,\mathbb{R}) = (\mathcal{T}(M)\times H_1(M,\mathbb{R}))/\Gamma(M)$$

over the $SL(2,\mathbb{R})$-action on the moduli space $\mathcal{M}(M)$.

Clearly the fibers of the bundle $\mathcal{H}_1(M,\mathbb{R})$ can be identified with $H_1(M,\mathbb{R})$. The space $H_1(M,\mathbb{R})$ is endowed with the symplectic form given by the algebraic intersection number. This symplectic structure is preserved by the action of the mapping-class group and hence is invariant under the action of $(A_g)_{g\in SL(2,\mathbb{R})}$.

The standard definition of KZ-cocycle bases on cohomological bundle. The identification of the homological and cohomological bundle and the corresponding KZ-cocycles is established by the Poincaré duality $\mathcal{P}: H_1(M,\mathbb{R})\to H^1(M,\mathbb{R})$. This correspondence allow us to define so called Hodge norm (see [13] for cohomological bundle) on each fiber of the bundle $\mathcal{H}_1(M,\mathbb{R})$. The norm on the fiber $H_1(M,\mathbb{R})$ over $\omega\in\mathcal{M}(M)$ will be denoted by $\|\cdot\|_\omega$.

Generic directions. Let $\omega\in\mathcal{M}(M)$ and denote by $\mathcal{M} = \overline{SL(2,\mathbb{R})\omega}$ the closure of the $SL(2,\mathbb{R})$-orbit of ω in $\mathcal{M}(M)$. The celebrated result of Eskin, Mirzakhani and Mohammadi, proved in [12] and [11], says that $\mathcal{M}\subset\mathcal{M}(M)$ is an affine $SL(2,\mathbb{R})$-invariant submanifold. Denote by $\nu_{\mathcal{M}}$ the corresponding affine $SL(2,\mathbb{R})$-invariant probability measure supported on $\mathcal{M}$. The measure $\nu_{\mathcal{M}}$ is ergodic under the action of the Teichmüller flow.

Theorem 4.1 (see [4]). *For every $\phi\in C_c(\mathcal{M})$ and almost all $\theta\in S^1$ we have*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\phi(g_t r_\theta\omega)\,dt = \int_{\mathcal{M}}\phi\,d\nu_{\mathcal{M}}. \tag{4.1}$$

Theorem 4.2 (see [23]). *For a.e. direction $\theta \in S^1$ the directional flows $(\varphi_t^v)_{t\in\mathbb{R}}$ and $(\varphi_t^h)_{t\in\mathbb{R}}$ on $(M, r_\theta\omega)$ are uniquely ergodic.*

All directions $\theta \in S^1$ for which the assertion of Theorems 4.1 and 4.2 holds are called *Birkhoff-Masur generic* for the translation surface (M,ω).

5. Directional flows on $\mathbb{Z}^d$-covers and weak mixing of their Poisson suspensions

Suppose that the direction $0 \in S^1$ is Birkhoff-Masur generic for (M,ω). Then the vertical and horizontal flows $(\varphi_t^v)_{t\in\mathbb{R}}$, $(\varphi_t^h)_{t\in\mathbb{R}}$ on (M,ω) is uniquely ergodic. Let $I \subset M \setminus \Sigma$ (Σ is the set of zeros of ω) be a horizontal interval. Then the interval I has no self-intersections and the Poincaré return map $T : I \to I$ for the flow $(\varphi_t^v)_{t\in\mathbb{R}}$ is a uniquely ergodic IET. Denote by I_α, $\alpha \in \mathcal{A}$ the intervals exchanged by T. Let $\lambda_\alpha(\omega, I)$ stands for the length of the interval I_α.

Denote by $\tau : I \to \mathbb{R}_{>0}$ the map of the first return time to I for the flow $(\varphi_t^v)_{t\in\mathbb{R}}$. Then τ is constant on each I_α and denote by $\tau_\alpha = \tau_\alpha(\omega, I) > 0$ its value on I_α, $\alpha \in \mathcal{A}$. Let us denote by $\delta(\omega, I) > 0$ the maximal number $\Delta > 0$ for which the set $\mathcal{R}^\omega(I,\Delta) := \{\varphi_t^v x : t \in [0,\Delta), x \in I\}$ is a rectangle in (M,ω) without any singular point (from Σ).

Suppose that $J \subset I$ is a subinterval. Denote by $S : J \to J$ the Poincaré return map to J for the flow $(\varphi_t^v)_{t\in\mathbb{R}}$. Then S is also an IET and suppose it exchanges intervals $(J_\alpha)_{\alpha\in\mathcal{A}}$. The IET S is the induced transformation of T on J. Moreover, all elements of J_α have the same time of the first return to J for the transformation T and let us denote this return time by $h_\alpha \geq 0$ for $\alpha \in \mathcal{A}$. Then I is the union of disjoint towers $\{T^j J_\alpha : 0 \leq j < h_\alpha\}$, $\alpha \in \mathcal{A}$.

The following result follows directly from Lemmas 4.12 and 4.13 in [15].

Lemma 5.1. *Assume that for some $\Delta > 0$ the set $\mathcal{R}^\omega(J,\Delta)$ is a rectangle in (M,ω) without any singular point. Let $h = \left[\Delta / \max_{\alpha\in\mathcal{A}} \tau_\alpha\right]$. Then for every $\gamma \in H_1(M,\mathbb{Z})$ we have*

$$\psi_{\gamma,I}^{(h_\alpha)}(x) = \langle \gamma, \xi_\alpha(\omega, J)\rangle \ \text{and}\ |T^{h_\alpha}x - x| \leq |J| \quad \text{for} \quad x \in C_\alpha := \bigcup_{0\leq j\leq h} T^j J_\alpha. \tag{5.1}$$

The following result follows directly from Lemmas A.3 and A.4 in [14].

Lemma 5.2. *If $0 \in S^1$ is Birkhoff-Masur generic for (M,ω) then there exist positive constants $A, C, c > 0$, a sequence of nested horizontal intervals $(I_k)_{k\geq 0}$ in (M,ω) and an increasing divergent sequence of real numbers $(t_k)_{k\geq 0}$ with $t_0 = 0$ such that for every $k \geq 0$ we have*

$$\frac{1}{c}\|\xi\|_{g_{t_k}\omega} \leq \max_\alpha |\langle \xi_\alpha(g_{t_k}\omega, I_k), \xi\rangle| \leq c\|\xi\|_{g_{t_k}\omega} \quad \text{for every} \quad \xi \in H_1(M,\mathbb{R}), \tag{5.2}$$

$$\lambda_\alpha(g_{t_k}\omega, I_k)\,\delta(g_{t_k}\omega, I_k) \geq A \ \text{and}\ \frac{1}{C} \leq \tau_\alpha(g_{t_k}\omega, I_k) \leq C \ \text{for any}\ \alpha \in \mathcal{A}. \tag{5.3}$$

Lemma 5.3. *If $0 \in S^1$ is Birkhoff-Masur generic for (M,ω) then for every non-zero $\gamma \in H_1(M,\mathbb{Z})$ the cocycle $\psi_{\gamma,I} : I \to \mathbb{Z}$ ($I := I_0$ come from Lemma 5.2) is not a coboundary.*

Proof. By Lemma 5.2, there exist a sequence of nested horizontal intervals $(I_k)_{k\geq 0}$ in (M,ω) and an increasing divergent sequence of real numbers $(t_k)_{k\geq 0}$ such that (5.2) and (5.3) hold for $k \geq 0$ and $t_0 = 0$. Let $I := I_0$ and denote by $T : I \to I$ the Poincaré return map to I for the vertical flow $(\varphi_t^v)_{t\in\mathbb{R}}$. Suppose, contrary to our claim, that $\psi_{\gamma,I} : I \to \mathbb{Z}$ is a coboundary with a measurable transfer function $u : I \to \mathbb{R}$, i.e. $\psi_{\gamma,I} = u - u \circ T$.

For every $k\geq 1$ the Poincaré return map $T_k:I_k\to I_k$ to I_k for the vertical flow $(\varphi^v_t)_{t\in\mathbb{R}}$ on (M,ω) is an IET exchanging intervals $(I_k)_\alpha$, $\alpha\in\mathcal{A}$. The length of $(I_k)_\alpha$ in (M,ω) is equal to $\lambda_\alpha(\omega,I_k)=e^{-t_k}\lambda_\alpha(g_{t_k}\omega,I_k)$ for $\alpha\in\mathcal{A}$. In view of (5.3), the length of I_k in (M,ω) is

$$|I_k|=\sum_{\alpha\in\mathcal{A}}e^{-t_k}\lambda_\alpha(g_{t_k}\omega,I_k)\leq Ce^{-t_k}\sum_{\alpha\in\mathcal{A}}\lambda_\alpha(g_{t_k}\omega,I_k)\tau_\alpha(g_{t_k}\omega,I_k)=Ce^{-t_k}\mu_\omega(M).$$

By the definition of δ, the set $\mathcal{R}^\omega(I_k,e^{t_k}\delta(g_{t_k}\omega,I_k))=\mathcal{R}^{g_{t_k}\omega}(I_k,\delta(g_{t_k}\omega,I_k))$ is a vertical rectangle in $(M,g_{t_k}\omega)$ without any singular point. It follows that the set $\mathcal{R}^\omega(I_k,e^{t_k}\delta(g_{t_k}\omega,I_k))$ is a rectangle in (M,ω) without any singular point.

Denote by $h^k_\alpha\geq 0$ the first return time of the interval $(I_k)_\alpha$ to I_k for the IET T. Let

$$h_k:=\big[e^{t_k}\delta(g_{t_k}\omega,I_k)/\max_{\alpha\in\mathcal{A}}\tau_\alpha(\omega,I)\big]\text{ and }C^k_\alpha:=\bigcup_{0\leq j\leq h_k}T^j(I_k)_\alpha.$$

Now Lemma 5.1 applied to $J=I_k$ and $\Delta=e^{t_k}\delta(g_{t_k}\omega,I_k)$ gives

$$\psi^{(h^k_\alpha)}_{\gamma,I}(x)=\langle\gamma,\xi_\alpha(\omega,I_k)\rangle\text{ and }|T^{h^k_\alpha}x-x|\leq|I_k|\leq Ce^{-t_k}\mu_\omega(M)\quad\text{for}\quad x\in C^k_\alpha\tag{5.4}$$

for every $k\geq 1$ and $\alpha\in\mathcal{A}$. Moreover, by (5.3),

$$Leb(C^k_\alpha)=(h_k+1)|(I_k)_\alpha|\geq\frac{e^{t_k}\delta(g_{t_k}\omega,I_k)}{\max_{\alpha\in\mathcal{A}}\tau_\alpha}e^{-t_k}\lambda_\alpha(g_{t_k}\omega,I_k)\geq\frac{A}{\max_{\alpha\in\mathcal{A}}\tau_\alpha}=:a>0.$$

By assumption, in view of (5.2), we have

$$\|\gamma\|_{g_{t_k}\omega}\leq c\max_{\alpha\in\mathcal{A}}|\langle\gamma,\xi_\alpha(g_{t_k}\omega,I_k)\rangle|.$$

Choose $B>0$ such that $Leb(U_B)<a/2$ for $U_B=\{x\in I:|u(x)|>B\}$. For every $m\geq 1$ let $J_m:=I\setminus(U_B\cup T^{-m}U_B)$. Then $Leb(I\setminus J_m)<a$ and for every $x\in J_m$ we have both $|u(x)|\leq B$, $|u(T^mx)|\leq B$. As $Leb(I\setminus J_{h^k_\alpha})<a$ and $Leb(C^k_\alpha)\geq a$, there exists $x^k_\alpha\in C^k_\alpha\cap J_{h^k_\alpha}$. Therefore, by (5.4), for all $k\geq 1$ and $\alpha\in\mathcal{A}$ we have

$$|\langle\gamma,\xi_\alpha(\omega,I_k)\rangle|=|\psi^{(h^k_\alpha)}_{\gamma,I}(x^k_\alpha)|=|u(x^k_\alpha)-u(T^{h^k_\alpha}x^k_\alpha)|\leq|u(x^k_\alpha)|+|u(T^{h^k_\alpha}x^k_\alpha)|\leq 2B.$$

Since $\langle\gamma,\xi_\alpha(\omega,I_k)\rangle\rangle\in\mathbb{Z}$, passing to a subsequence, if necessary, we can assume that for every $\alpha\in\mathcal{A}$ the sequence $(\langle\gamma,\xi_\alpha(\omega,I_k)\rangle)_{k\geq1}$ is constant. Since (5.4) holds and $Leb(C^k_\alpha)\geq a>0$ for $k\geq 1$ and $\alpha\in\mathcal{A}$, we can apply Proposition 3.5 to $\psi=\psi_{\gamma,I}$, $C_k=C^k_\alpha$ and $h_k=h^k_\alpha$. This gives $\langle\gamma,\xi_\alpha(\omega,I_k)\rangle\in E(\psi_{\gamma,I})$ for all $k\geq 1$ and $\alpha\in\mathcal{A}$. In view of Proposition 3.3, as $\psi_{\gamma,I}$ is a coboundary, we have $E(\psi_{\gamma,I})=\{0\}$, so $\langle\gamma,\xi_\alpha(\omega,I_k)\rangle=0$ for all $k\geq 1$ and $\alpha\in\mathcal{A}$. Since $\langle\gamma,\xi_\alpha(g_{t_k}\omega,I_k)\rangle=\langle\gamma,\xi_\alpha(\omega,I_k)\rangle$, this gives

$$\|\gamma\|_{g_{t_k}\omega}\leq c\max_{\alpha\in\mathcal{A}}|\langle\gamma,\xi_\alpha(g_{t_k}\omega,I_k)\rangle|=0.$$

It follows that $\gamma=0$, contrary to $\gamma\neq 0$. Consequently, the cocycle $\psi_{\gamma,I}$ is not a coboundary for the IET $T:I\to I$. □

Theorem 5.4. *Let (M,ω) be a compact connected translation surface and let $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ be its non-trivial $\mathbb{Z}^d$-cover (i.e. $\gamma\in H_1(M,\mathbb{Z})^d$ is non-zero). Then for a.e. $\theta\in S^1$ the Poisson suspension of the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ flow on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ is weakly mixing.*

Proof. By Theorems 4.1 and 4.2, the set $\Theta\subset S^1$ of all $\theta\in S^1$ for which $\pi/2-\theta$ is Birkhoff-Masur generic for (M,ω) has full Lebesgue measure in S^1. We show that for every $\theta\in\Theta$ the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ flow on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ has no invariant set of positive and finite measure. In view of Proposition 2.1, this proves the theorem.

Suppose that $\theta \in \Theta$. Then $0 \in S^1$ is a Birkhoff-Masur generic direction for $(M, r_{\pi/2-\theta}\omega)$ and the flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ coincides with the vertical flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{(r_{\pi/2-\theta}\omega)}_\gamma)$.

Assume that $\gamma = (\gamma_1, \ldots, \gamma_d)$ and $\gamma_j \in H_1(M, \mathbb{Z})$ is non-zero for some $1 \le j \le d$. By Lemma 5.2 and 5.3, there exists a horizontal interval in $(M, r_{\pi/2-\theta}\omega)$ such that $\psi_{\gamma_j,I} : I \to \mathbb{Z}$ is not a coboundary for the Poincaré return map $T : I \to I$ for the vertical flow on $(M, r_{\pi/2-\theta}\omega)$. Since $\psi_{\gamma_j,I}$ is the j-th coordinate function of $\psi_{\gamma,I} : I \to \mathbb{Z}^d$, the latter is also not a coboundary for T. In view of Proposition 3.4, the skew product $T_{\psi_{\gamma,I}}$ on $I \times \mathbb{Z}^d$ has no invariant set of positive and finite measure. By Proposition 3.1 and Remark 3.2, the vertical flow on $(\widetilde{M}_\gamma, \widetilde{(r_{\pi/2-\theta}\omega)}_\gamma)$ has no invariant set of positive and finite measure as well. As the vertical flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{(r_{\pi/2-\theta}\omega)}_\gamma)$ coincides with the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$, this completes the proof. □

Proof of Theorem 1.1. The first part of Theorem 1.1 follows directly from Theorem 5.4 applied to the $\mathbb{Z}^2$-cover $(M_\mathcal{T}, \omega_\mathcal{T})$. Non-triviality of the $\mathbb{Z}^2$-cover follows from the connectivity of $M_\mathcal{T}$.

The second part of Theorem 1.1 is based on the fact that the billiard flow $(b_t)_{t\in\mathbb{R}}$ of $\mathcal{T}^1$ is metrically isomorphic to the flow $(\varphi^\mathcal{T}_t)_{t\in\mathbb{R}}$ on $M_\mathcal{T} \times S^1/\Gamma$ given by $\varphi^\mathcal{T}_t(x,\theta) \mapsto (\varphi^{\mathcal{T},\theta}_t x, \theta)$. By Theorem 5.4, for a.e. $\theta \in S^1/\Gamma$ the flow $(\varphi^{\mathcal{T},\theta}_t)_{t\in\mathbb{R}}$ has no invariant subset of positive and finite measure. In view Lemma 2.2, the flow $(\varphi^\mathcal{T}_t)_{t\in\mathbb{R}}$ enjoys the same property. The proof is completed by applying Proposition 2.1. □

6. Absence of mixing

Let (M, ω) be a compact connected translation surface and let $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ be its $\mathbb{Z}^d$-cover determined by $\gamma \in H_1(M, \mathbb{Z})^d$. Denote by $p_\gamma : \widetilde{M}_\gamma \to M$ the covering map. Let d^ω_γ be the geodesic distance on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$. Of course, $d^\omega_\gamma = d^{r_\theta\omega}_\gamma$ for every $\theta \in S^1$. Denote by $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ the vertical flow on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$.

Definition (cf. [2]). Given real numbers $c, L, \delta > 0$ the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ is called *(c, L, δ)-recurrent* if there exist a horizontal interval $I \subset M \setminus \Sigma$ such that the set $\mathcal{R}^\omega(I, L) = \{\varphi^v_t x : x \in I, t \in [0, L)\}$ is a vertical rectangle (without any singularity) in (M, ω) with $\mu_\omega(\mathcal{R}^\omega(I, L)) \ge c$ and for every $\widetilde{x} \in p_\gamma^{-1}(\mathcal{R}^\omega(I, L))$ the points $\widetilde{x}$ and $\widetilde{\varphi}^v_L \widetilde{x}$ belong to the same horizontal leaf on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ and the distance between them along this leaf is smaller than δ.

Let $\mathcal{M} = \overline{SL(2,\mathbb{R})\omega}$ and let us consider the bundle $\mathcal{H}^\mathcal{M}_1(M, \mathbb{R}) \to \mathcal{M}$ which is the restriction of the homological bundle to $\mathcal{M}$. Assume that

$$\mathcal{H}^\mathcal{M}_1(M, \mathbb{R}) = \mathcal{K} \oplus \mathcal{K}^\perp \tag{6.1}$$

is a continuous symplectic orthogonal splitting of the bundle which is $(A_g)_{g\in SL(2,\mathbb{R})}$-invariant. Denote by $H_1(M, \mathbb{R}) = K_{\omega'} \oplus K^\perp_{\omega'}$ the corresponding splitting of the fiber over any $\omega' \in \mathcal{M}$.

A cylinder C on (M, ω) is a maximal open annulus filled by homotopic simple closed geodesics. The direction of C is the direction of these geodesics and the homology class of them is denoted by $\sigma(C) \in H_1(M, \mathbb{Z})$. A cylinder C on $(M, \omega') \in \mathcal{M}$ is called *$\mathcal{K}$-good* if $\sigma(C) \in K^\perp_{\omega'} \cap H_1(M, \mathbb{Z})$. If a cylinder C on (M, ω) is $\mathcal{K}$-good and $\gamma \in (K_\omega \cap H_1(M, \mathbb{Z}))^d$ then C lifts to a cylinder on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$.

Proposition 6.1 (see the proof of Proposition 2 in [2]). *Suppose that $(M, \omega_*) \in \mathcal{M}$ has a vertical $\mathcal{K}$-good cylinder. If the positive $(g_t)_{t\in\mathbb{R}}$ orbit of (M, ω) accumulates on*

(M,ω_*) then for any $\gamma \in (K_\omega \cap H_1(M,\mathbb{Z}))^d$ there exists $c>0$ and two sequences of positive numbers $(L_n)_{n\geq 1}$, $(\delta_n)_{n\geq 1}$ such that $L_n \to +\infty$, $\delta_n \to 0$ and the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ is (c, L_n, δ_n)-recurrent for $n \geq 1$.

For every $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ let $D^\omega_\gamma \subset \widetilde{M}_\gamma$ be a fundamental domain for the deck group action so that the boundary of D^ω_γ is a finite union of intervals. Then, $\mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma) = \mu_\omega(M) \in (0,+\infty)$. Moreover, choose the fundamental domains such that $D^\omega_\gamma = D^{r_\theta\omega}_\gamma$ for every $\theta \in S^1$.

Theorem 6.2. *Suppose that (M,ω) has a $\mathcal{K}$-good cylinder C. If $\pi/2-\theta \in S^1$ is a Birkhoff generic direction then for every $\gamma \in (K_\omega \cap H_1(M,\mathbb{Z}))^d$ we have*

$$\liminf_{t\to+\infty} \mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma \cap \widetilde{\varphi}^\theta_t D^\omega_\gamma) > 0.$$

Proof. Denote by $\theta_0 \in S^1$ the direction of the cylinder C on (M,ω). Since the splitting (6.1) is $(A_g)_{g\in SL(2,\mathbb{R})}$-invariant, C is a vertical $\mathcal{K}$-good cylinder on the translation surface $(M, r_{\pi/2-\theta_0}\omega) \in \mathcal{M}$. Since $\pi/2-\theta \in S^1$ is Birkhoff generic, applying (4.1) to a sequence $(\phi_k)_{k\geq 1}$ in $C_c(\mathcal{M})$ such that $(\mathrm{supp}(\phi_k))_{k\geq 1}$ is a decreasing nested sequence of non-empty compact subsets with the intersection $\{r_{\pi/2-\theta_0}\omega\}$, there exists $t_n \to +\infty$ such that $g_{t_n}(r_{\pi/2-\theta}\omega) \to r_{\pi/2-\theta_0}\omega$. By Proposition 6.1, there exists $c>0$ and two sequences of positive numbers $(L_n)_{n\geq 1}$, $(\delta_n)_{n\geq 1}$ such that $L_n \to +\infty$, $\delta_n \to 0$ and the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{r_{\pi/2-\theta}\omega_\gamma})$ is (c, L_n, δ_n)-recurrent for $n \geq 1$. Let us denote by $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ the vertical flow on $(\widetilde{M}_\gamma, \widetilde{r_{\pi/2-\theta}\omega_\gamma})$ which coincides with the flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ in direction $\theta \in S^1$ on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$. Then there exists a sequence $(I_n)_{n\geq 1}$ of horizontal intervals in $(M, r_{\pi/2-\theta}\omega)$ such that $\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n, L_n)$ is a rectangle in $(M, r_{\pi/2-\theta}\omega)$ such that $\mu_\omega(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n)) = \mu_{r_{\pi/2-\theta}\omega}(\mathcal{R}^{r_{\pi/2-\theta}}(I_n,L_n)) > c$ and

$$\text{for every } \widetilde{x} \in p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n)) \text{ we have } d^\omega_\gamma(\widetilde{x}, \widetilde{\varphi}^v_{L_n}\widetilde{x}) < \delta_n. \tag{6.2}$$

As $D^\omega_\gamma \subset \widetilde{M}_\gamma$ is a fundamental domain for the $\mathbb{Z}^d$-action of the deck group, we have

$$\mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma \cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n))) = \mu_\omega(\mathcal{R}^{r_{\pi/2-\theta}}(I_n,L_n)) > c. \tag{6.3}$$

For every $\delta>0$ denote by $\partial_\delta D^\omega_\gamma$ the δ-neighborhood in $(\widetilde{M}_\gamma, d^\omega_\gamma)$ of the boundary ∂D^ω_γ. Since $\mu_{\widetilde{\omega}_\gamma}(\partial D^\omega_\gamma) = 0$, we have

$$\mu_{\widetilde{\omega}_\gamma}(\partial_\delta D^\omega_\gamma) \to 0 \quad \text{as} \quad \delta \to 0. \tag{6.4}$$

In view of (6.2), we obtain

$$\widetilde{\varphi}^v_{L_n}\big((D^\omega_\gamma \cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n))) \setminus \partial_{\delta_n} D^\omega_\gamma\big) \subset D^\omega_\gamma.$$

It follows that

$$\begin{aligned}
\mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma \cap \widetilde{\varphi}^\theta_{L_n} D^\omega_\gamma) &= \mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma \cap \widetilde{\varphi}^v_{L_n} D^\omega_\gamma) \\
&\geq \mu_{\widetilde{\omega}_\gamma}\big(\widetilde{\varphi}^v_{L_n}\big((D^\omega_\gamma \cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n))) \setminus \partial_{\delta_n} D^\omega_\gamma\big)\big) \\
&= \mu_{\widetilde{\omega}_\gamma}\big((D^\omega_\gamma \cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n))) \setminus \partial_{\delta_n} D^\omega_\gamma\big) \\
&\geq \mu_{\widetilde{\omega}_\gamma}\big(D^\omega_\gamma \cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n))\big) - \mu_{\widetilde{\omega}_\gamma}(\partial_{\delta_n} D^\omega_\gamma).
\end{aligned}$$

By (6.3) and (6.4), this gives $\liminf_{n\to+\infty} \mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma \cap \widetilde{\varphi}^\theta_{L_n} D^\omega_\gamma) \geq c > 0$, which completes the proof. ∎

In view of Proposition 2.1 and Theorem 4.1, this leads to the following result:

Theorem 6.3. *Suppose that (M,ω) is a compact connected translation surface with a $\mathcal{K}$-good cylinder. Then for every $\gamma \in (K_\omega \cap H_1(M,\mathbb{Z}))^d$ and for a.e. $\theta \in S^1$ the Poisson suspension of the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ is not mixing.*

The notion of $\mathcal{K}$-good cylinder was introduced in [2] and applied to prove recurrence for a.e. directional billiard flow in the standard periodic wind tree model. The existence of $\mathcal{K}$-good cylinders was also shown in more complicated billiards on periodic tables in [14] and [26]. The paper [26] deal with $\mathbb{Z}^2$-periodic patterns of scatterers of right-angled polygonal shape with horizontal and vertical sides; the obstacles are horizontally and vertically symmetric. Some Λ-periodic patterns of scatterers with horizontal and vertical sides are considered in [14] for any lattice $\Lambda \subset \mathbb{R}^2$; here obstacles are centrally symmetric. Among others, the existence of $\mathcal{K}$-good cylinders was shown for Λ_λ-periodic wind tree model (obstacles are rectangles), where Λ_λ is any lattice of the form $(1,\lambda)\mathbb{Z} + (0,1)\mathbb{Z}$. In view of Theorem 6.3, we have the absence of mixing for the Poisson suspension of the directional billiard flows $(b_t^\theta)_{t\in\mathbb{R}}$ for a.e. $\theta \in S^1$ on all billiards tables considered in [2, 14, 26].

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