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SIMPLIFIED KRIPKE STYLE SEMANTICS FOR SOME VERY WEAK MODAL LOGICS

Abstract. In the present paper[1] we examine very weak modal logics **C1**, **D1**, **E1**, **S0.5**°, **S0.5**°+(D), **S0.5** and some of their versions which are closed under replacement of tautological equivalents (rte-versions). We give semantics for these logics, formulated by means of Kripke style models of the form $\langle w, A, V\rangle$, where w is a «distinguished» world, A is a set of *worlds* which are alternatives to w, and V is a valuation which for formulae and worlds assigns the truth-vales such that: (i) for all formulae and all worlds, V preserves classical conditions for truth-value operators; (ii) for the world w and any formula φ, $V(\Box\varphi, w) = 1$ iff $\forall_{x\in A}\, V(\varphi, x) = 1$; (iii) for other worlds formula $\ulcorner\Box\varphi\urcorner$ has an arbitrary value. Moreover, for rte-versions of considered logics we must add the following condition: (iv) $V(\Box\chi, w) = V(\Box\chi[^{\varphi}/_{\psi}], w)$, if φ and ψ are tautological equivalent. Finally, for **C1**, **D1** and **E1** we must add *queer* models of the form $\langle w, V\rangle$ in which: (i) holds and (ii′) $V(\Box\varphi, w) = 0$, for any formula φ. We prove that considered logics are determined by some classes of above models.

Keywords: Simplified Kripke style semantics, very weak modal logics.

1. Preliminaries. Some historical notes

Modal formulae are formed in the standard way from the set At of propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', …; truth-value operators: '¬',

[1] This article is the final version of a draft paper [14], mentioned in the references of the papers [13] and [15].

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'$\vee$', '$\wedge$', '$\supset$', and '$\equiv$' (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); the modal operator '$\Box$' (necessity; the possibility sign '$\Diamond$' is the abbreviation of '$\neg\Box\neg$'); and brackets. Let For be the set of all modal formulae. For any set Γ of formulae we put $\Box\Gamma := \{\ulcorner\Box\varphi\urcorner : \varphi \in \Gamma\}$.

Let Taut be the set of all classical tautologies (without the modal operator) and—as in [3, 4]—let PL be the set of modal formulae which are instances of classical tautologies.

Let Σ be a set of modal formulae. Also as in [3], Σ is a *modal system* iff PL $\subseteq \Sigma$ and Σ is closed under the following rule of detachment for '$\supset$' (*modus ponens*), i.e., for any formulae φ and ψ:

$$\text{if } \varphi \text{ and } \ulcorner\varphi \supset \psi\urcorner \text{ are members of } \Sigma, \text{ so is } \psi. \tag{MP}$$

We say that a modal system is *congruential* iff it is closed under the following rule of congruence:

$$\text{if } \ulcorner\varphi \equiv \psi\urcorner \in \Sigma, \text{ then } \ulcorner\Box\varphi \equiv \Box\psi\urcorner \in \Sigma. \tag{RE}$$

Notice that a modal system Σ is congruential iff it is closed under replacement

$$\text{if } \ulcorner\varphi \equiv \psi\urcorner \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi[{}^{\varphi}/_{\psi}] \in \Sigma, \tag{RRE}$$

or equivalently

$$\text{if } \ulcorner\varphi \equiv \psi\urcorner \in \Sigma, \text{ then } \ulcorner\chi[{}^{\varphi}/_{\psi}] \equiv \chi\urcorner \in \Sigma, \tag{RRE$'$}$$

where $\chi[{}^{\varphi}/_{\psi}]$ is any formula that results from χ by replacing one or more occurrences of φ, in χ, by ψ.

A modal system Σ is called *regular* iff it is closed under the following regularity rule:

$$\text{if } \ulcorner(\varphi \wedge \psi) \supset \chi\urcorner \in \Sigma, \text{ then } \ulcorner(\Box\varphi \wedge \Box\psi) \supset \Box\chi\urcorner \in \Sigma. \tag{RR}$$

A modal system Σ is regular iff it contains all instances of

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \tag{K}$$

and is closed under the following *monotonic* rule

$$\text{if } \ulcorner\varphi \supset \psi\urcorner \in \Sigma \text{ then } \ulcorner\Box\varphi \supset \Box\psi\urcorner \in \Sigma, \tag{RM}$$

iff it is closed under (RM) and contains all instances of

$$(\Box p \wedge \Box q) \supset \Box(p \wedge q) \tag{C}$$

iff it is closed under (RE) and contains all instances of

$$\Box(p \wedge q) \equiv (\Box p \wedge \Box q) \tag{R}$$

We say that a modal system Σ is *normal* iff it contains all instances of (K) and is closed under the following rule:

$$\text{if } \varphi \in \Sigma, \text{ then } \ulcorner \Box \varphi \urcorner \in \Sigma. \tag{RN}$$

A modal system Σ is *normal* iff it is regular and contains the following formula

$$\Box(p \supset p) \tag{N}$$

iff it contains (N) and all instances of (K), and is closed under (RE).

A set Σ of modal formulae is a *logic* iff Σ is a modal system and is closed under the following rule of uniform substitution:

$$\text{if } \varphi \in \Sigma \text{ then } \mathrm{s}\,\varphi \in \Sigma, \tag{US}$$

where $\mathrm{s}\,\varphi$ is the result of uniform substitution of formulae for propositional letters in φ. Of course, the set PL is the smallest modal system and it is a logic.

In [9] Lemmon set out the logic **S0.5** and two groups of non-normal modal logics called the "D" and "E" systems.

Firstly, the logic **S0.5** is the smallest modal logic which includes $\Box$Taut, and contains (K) and the following formula:

$$\Box p \supset p \tag{T}$$

The logic **S0.5°** is associated with Lemmon's **S0.5** (for these logics see e.g. [4, 9, 16]). **S0.5°** is the smallest logic which includes $\Box$Taut and contains (K). Thus, **S0.5** is **S0.5°** plus (T). Of course, by (US), **S0.5** and **S0.5°** include the set $\Box$PL, and **S0.5°** $\subsetneq$ **S0.5** (see Fact 4.1).

Secondly, Lemmon "consider a series of Lewis modal systems E1, E2, E3, E4, and E5, which are intended as possible epistemic counterparts to the five systems S0.5, S2, S3, S4, and S5. A distinguishing mark of E-systems is that in none of them is there *any* thesis of the form $L\alpha$" [9, p. 181–182] (in our text $L\alpha := \Box\varphi$). All E-systems—just like all S-systems—are logics that contain (K) and (T), include the set Taut, and are closed under the rules: (MP) and (US) (so they include the set PL). Moreover, the logics **E2**–**E5** are regular. For example, **E2** is the smallest regular modal logic which contains (T). **E3** is the smallest modal logic which is closed under the rule RM and contains (T) and the following formula:

$$\Box(p \supset q) \supset \Box(\Box p \supset \Box q) \tag{sK}$$

Thus, by PL, (sK) and (T), the logic **E3** contains (K). So it is regular.

The logic **E1** is closed neither under (RM) nor under (RR). It is the smallest logic which contains (K) and (T), and includes the following set of formulae:

$$\mathrm{M}_{\mathrm{Taut}} := \{\ulcorner \Box \varphi \supset \Box \psi \urcorner : \ulcorner \varphi \supset \psi \urcorner \in \mathrm{Taut}\}.$$

Thus, **E1** also includes the following sets of formulae.

$$\begin{aligned} \mathrm{M}_{\mathrm{PL}} &:= \{\ulcorner \Box \varphi \supset \Box \psi \urcorner : \ulcorner \varphi \supset \psi \urcorner \in \mathrm{PL}\}, \\ \mathrm{R}_{\mathrm{PL}} &:= \{\ulcorner (\Box \varphi \wedge \Box \psi) \supset \Box \chi \urcorner : \ulcorner (\varphi \wedge \psi) \supset \chi \urcorner \in \mathrm{PL}\}, \\ \mathrm{E}_{\mathrm{PL}} &:= \{\ulcorner \Box \varphi \equiv \Box \psi \urcorner : \ulcorner \varphi \equiv \psi \urcorner \in \mathrm{PL}\}. \end{aligned}$$

We have **E1** $\subsetneq$ **S0.5** (see Fact 4.1).

Thirdly, the five D-logics, **D1**, **D2**, **D3**, **D4** and **D5**, were associated with the five E-logics. "The distinguishing feature of D-systems is that axiom (T) of the corresponding E-systems is weakened to (D)" [9, p. 184]

$$\Box p \supset \neg \Box \neg p \tag{D}$$

Precisely, **D1** is the smallest logic which contains (K) and (D), and includes the set $\mathrm{M}_{\mathrm{Taut}}$. Thus, the logic **D1** also includes the sets M_{PL}, R_{PL} and E_{PL}. We have **D1** $\subsetneq$ **E1** (see Fact 4.1). The logics **D2**–**D5** are regular, e.g. **D2** is the smallest regular modal logic which contains (D). We have **D2** $\subsetneq$ **E2**.

In [10] the logic **C2** is examined. It is **E2** without (T) and (D). Precisely, **C2** is the smallest regular logic. We have **C2** $\subsetneq$ **D2**.

By analogy to **C2**, in [16] by '**C1**' Routley denoted the system **E1** without (T) and (D), i.e., **C1** is the smallest modal logic which contains (K) and includes the set $\mathrm{M}_{\mathrm{Taut}}$. So **C1** includes M_{PL}, R_{PL} and E_{PL}. We have **C1** $\subsetneq$ **D1** and **C1** $\subsetneq$ **S0.5°** (see Fact 4.1).

As in [2, 4], we say that a modal system Σ is *closed under replacement of tautological equivalents* iff for all $\varphi, \psi, \chi \in$ For:

$$\text{if } \ulcorner \varphi \equiv \psi \urcorner \in \mathrm{PL} \text{ and } \chi \in \Sigma, \text{ then } \chi[^{\varphi}/_{\psi}] \in \Sigma. \tag{rte}$$

or equivalently

$$\text{if } \ulcorner \varphi \equiv \psi \urcorner \in \mathrm{PL}, \text{ then } \chi \in \Sigma \text{ iff } \chi[^{\varphi}/_{\psi}] \in \Sigma. \tag{rte$'$}$$

Thus, by PL, a modal system is closed under (rte) iff it includes the following set of formulae:

$$\mathrm{RE}_{\mathrm{PL}} := \{\ulcorner\chi[{}^{\varphi}/_{\psi}] \equiv \chi\urcorner : \ulcorner\varphi \equiv \psi\urcorner \in \mathrm{PL}\}\,.$$

In [2] a modal logic is called *classical modal* iff it contains (K) and (N), and is closed under (rte).[2] Notice that

LEMMA 1.1. *If Σ is closed under* (rte) *and* (N) $\in \Sigma$, *then* $\Box\mathrm{PL} \subseteq \Sigma$.

PROOF. For any $\tau \in \mathrm{PL}$ we have that $\ulcorner(p \supset p) \equiv \tau\urcorner \in \mathrm{PL}$. Hence $\ulcorner\Box\,\tau\urcorner \in \Sigma$, by (rte) for $\chi :=$ (N), $\varphi :=$ '$p \supset p$' and $\psi :=$ 'τ'. ⊣

The non-congruential logics **S0.9**°, **S0.9**, **S1**°, **S1**, **S2**°, **S2**, **S3** and **S3.5** are examples of "classical modal logics" in the sense of [2]. For details concerning these logics see [4, 9] and Appendix A.

2. Some very weak systems

2.1. Very weak t-regular systems

Any modal system which includes the set R_{PL} we will call *t-regular*. Thus, the set R_{PL} replaces the rule (RR) in the formulation of regular systems. Of course, if Σ is a t-regular system and Σ' is a modal system such that $\Sigma \subseteq \Sigma'$, then Σ' is also a t-regular.

LEMMA 2.1. *All t-regular systems include the sets* M_{PL} *and* E_{PL}.

PROOF. If $\ulcorner\varphi \supset \psi\urcorner \in \mathrm{PL}$, then also $\ulcorner(\varphi \wedge \varphi) \supset \psi\urcorner \in \mathrm{PL}$. So we use R_{PL} and PL. Moreover, If $\ulcorner\varphi \equiv \psi\urcorner \in \mathrm{PL}$, then also $\ulcorner\varphi \supset \psi\urcorner \in \mathrm{PL}$ and $\ulcorner\psi \supset \varphi\urcorner \in \mathrm{PL}$. So we use M_{PL} and PL. ⊣

LEMMA 2.2. *All instances of* (K), (C), (R) *and*

$$(\Box(p \supset q) \wedge \Box(q \supset r)) \supset \Box(p \supset r) \qquad (\mathtt{X})$$

are members of all t-regular systems.

PROOF. Since $\ulcorner((\varphi \supset \psi) \wedge \varphi) \supset \psi\urcorner$, $\ulcorner(\varphi \wedge \psi) \supset \varphi\urcorner$, $\ulcorner(\varphi \wedge \psi) \supset \psi\urcorner$, $\ulcorner(\varphi \wedge \psi) \supset (\varphi \wedge \psi)\urcorner$ and $\ulcorner((\varphi \supset \psi) \wedge (\psi \supset \chi)) \supset (\varphi \supset \chi)\urcorner$ belong to PL and all t-regular systems include R_{PL} and M_{PL}. ⊣

[2] In [3, 4] the expression 'classical modal' was referred to 'congruential'.

LEMMA 2.3. *For any system Σ the following conditions are equivalent*:

(a) *Σ is t-regular,*

(b) *Σ contains all instances of* (K) *and includes the set* M_{PL},

(c) *Σ contains all instances of* (C) *and includes the set* M_{PL},

(d) *Σ contains all instances of* (X) *and includes the set* M_{PL}.

PROOF. "(a) $\Rightarrow$ (b)", "(a) $\Rightarrow$ (c)", "(a) $\Rightarrow$ (d)" By lemmas 2.1 and 2.2.

"(c) $\Rightarrow$ (a)" If $\ulcorner(\varphi \wedge \psi) \supset \chi\urcorner \in$ PL, then $\ulcorner\Box(\varphi \wedge \psi) \supset \Box\chi\urcorner \in \Sigma$, since $M_{PL} \subseteq \Sigma$. Hence $\ulcorner(\Box\varphi \wedge \Box\psi) \supset \Box\chi\urcorner \in \Sigma$, by (C) and PL.

"(b) $\Rightarrow$ (a)" If $\ulcorner(\varphi \wedge \psi) \supset \chi\urcorner \in$ PL, then $\ulcorner\varphi \supset (\psi \supset \chi)\urcorner \in$ PL, by PL. Hence $\ulcorner\Box\varphi \supset \Box(\psi \supset \chi)\urcorner \in \Sigma$, by $M_{PL} \subseteq \Sigma$. So $\ulcorner\Box\varphi \supset (\Box\psi \supset \Box\chi)\urcorner \in \Sigma$, by (K) and PL.

"(d) $\Rightarrow$ (b)" By (X), $\ulcorner(\Box(\tau \supset \varphi) \wedge \Box(\varphi \supset \psi)) \supset \Box(\tau \supset \psi)\urcorner \in \Sigma$, for any $\tau \in$ Taut. Since $\ulcorner\varphi \equiv (\tau \supset \varphi)\urcorner \in$ PL and $E_{PL} \subseteq \Sigma$, so $\ulcorner\Box\varphi \equiv \Box(\tau \supset \varphi)\urcorner \in \Sigma$. Similarly for ψ. Hence $\ulcorner\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)\urcorner \in \Sigma$, by PL. ⊣

All t-regular systems contain all instances of the following formulae:

$$\Diamond p \equiv \neg\Box\neg p \tag{df $\Diamond$}$$

$$\Box p \equiv \neg\Diamond\neg p \tag{df $\Box$}$$

$$\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q) \tag{$\mathtt{R}^{\Diamond}$}$$

$$\Diamond(p \supset q) \equiv (\Box p \supset \Diamond q) \tag{$\mathtt{R}^{\Diamond\Box}$}$$

The logics **C1**, **D1** and **E1** are t-regular (for these logics see p. 274). The logic **C1** is the smallest t-regular system.

Notice that **E1** contains the following formula:

$$p \supset \Diamond p \tag{$\mathtt{T}^{\Diamond}$}$$

and (D). Moreover, by ($\mathtt{R}^{\Diamond\Box}$), **D1** contains the following formula:

$$\Diamond(p \supset p) \tag{P}$$

In this paper by **C1**+($\mathtt{T_q}$) we denote the smallest t-regular logic which contains the following formula

$$\Box p \supset (p \vee \Box q) \tag{$\mathtt{T_q}$}$$

For t-regular logics the formula ($\mathtt{T_q}$) may be replace by

$$\neg\Box(q \wedge \neg q) \supset (\Box p \supset p) \qquad (\mathtt{T'_q})$$

$$\Diamond(q \supset q) \supset (\Box p \supset p) \qquad (\mathtt{T''_q})$$

The name '$\mathtt{T_q}$' is an abbreviation for 'quasi-$\mathtt{T}$', because ($\mathtt{T}$) and ($\mathtt{T_q}$) are valid in all reflexive and quasi-reflexive standard models, respectively.[3] We have that **C1** $\subsetneq$ **D1** $\subsetneq$ **E1** and **C1** $\subsetneq$ **C1**+($\mathtt{T_q}$) $\subsetneq$ **E1** (see Fact 4.1).

Notice that the logic **C1** plus two axioms ($\mathtt{D}$) and ($\mathtt{T_q}$) equals **E1** (i.e. **E1** = **C1**+($\mathtt{D}$)+($\mathtt{T_q}$) = **D1**+($\mathtt{T_q}$)). Indeed, by **C1** and ($\mathtt{D}$) we obtain ($\mathtt{P}$). Hence we have ($\mathtt{T}$), by ($\mathtt{T''_q}$), (MP) and (US).

In this paper we prove that the logics **C1**, **D1**, **C1**+($\mathtt{T_q}$) and **E1** are not closed under (rte). For example, the formula '$\Box\Box p \equiv \Box\Box\neg\neg p$' is not a member of these logics (see Remark 3.2 and Fact 4.1).

2.2. Very weak t-normal systems

Any modal system which contains all instances of ($\mathtt{K}$) and includes the set $\Box$PL will be called *t-normal*. Thus, the set $\Box$PL replaces the rule (RN) in the formulation of normal systems. Of course, if Σ is a t-normal system and Σ' is a modal system such that $\Sigma \subseteq \Sigma'$, then Σ' is also a t-normal.

Lemma 2.4. *For any system Σ the following conditions are equivalent*:

(a) *Σ is t-normal,*

(b) *Σ is t-regular and contains* ($\mathtt{N}$).

Proof. "(a) $\Rightarrow$ (b)" ($\mathtt{N}$) $\in \Box$PL. Moreover, if $\ulcorner(\varphi \wedge \psi) \supset \chi\urcorner \in$ PL, then $\ulcorner\varphi \supset (\psi \supset \chi)\urcorner \in$ PL, by PL and (MP). Hence $\ulcorner\Box(\varphi \supset (\psi \supset \chi))\urcorner \in \Sigma$, since $\Box$PL $\subseteq \Sigma$. So $\ulcorner\Box\varphi \supset (\Box\psi \supset \Box\chi)\urcorner \in \Sigma$ and $\ulcorner(\Box\varphi \wedge \Box\psi) \supset \Box\chi\urcorner \in \Sigma$, by ($\mathtt{K}$), PL and (MP).

"(b) $\Rightarrow$ (a)" By Lemma 2.3, Σ contains all instances of ($\mathtt{K}$) and includes the set $\mathrm{M_{PL}}$, Let $\tau \in$ PL. Then $\ulcorner(p \supset p) \supset \tau\urcorner \in$ PL. So $\ulcorner(\mathtt{N}) \supset \Box\tau\urcorner \in \Sigma$, since $\mathrm{M_{PL}} \subseteq \Sigma$. Thus, $\Box$PL $\subseteq \Sigma$. ⊣

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[3] In any quasi-reflexive standard frame an *accessibility* relation R on a set W of worlds is such that $\forall_{x,y\in W}(x\,R\,y \Rightarrow x\,R\,x)$. See [3, p. 92, Exercise 3.51], where instead of 'quasi-reflexive' the term 'reverse secondary reflexive' is used.

The logic $\mathbf{S0.5}^\circ$ is the smallest t-normal system; **S0.5** is the smallest t-normal logic which contains (T) (for these logics see p. 273). Of course, **S0.5** contains ($\mathtt{T}^\diamond$) and (D).

In the present paper by $\mathbf{S0.5}^\circ$+(D) we denote the smallest t-normal logic which contains (D), i.e. $\mathbf{S0.5}^\circ$ plus (D). Of course, $\mathbf{S0.5}^\circ$+(D) contains (P). Moreover, by $\mathbf{S0.5}^\circ$+($\mathtt{T_q}$) we denote the smallest t-normal logic which contains ($\mathtt{T_q}$), i.e. $\mathbf{S0.5}^\circ$ plus the axiom ($\mathtt{T_q}$).

We have that $\mathbf{S0.5}^\circ \subsetneq \mathbf{S0.5}^\circ$+(D) $\subsetneq$ **S0.5**, besides $\mathbf{S0.5}^\circ \subsetneq \mathbf{S0.5}^\circ$+($\mathtt{T_q}$) $\subsetneq$ **S0.5** and **C1**+($\mathtt{T_q}$) $\subsetneq \mathbf{S0.5}^\circ$+($\mathtt{T_q}$) (see Fact 4.1).

Notice that the logic $\mathbf{S0.5}^\circ$ plus two axioms (D) and ($\mathtt{T_q}$) is equals **S0.5** (i.e. **S0.5** = $\mathbf{S0.5}^\circ$+(D)+($\mathtt{T_q}$)). Indeed, from $\mathbf{S0.5}^\circ$ and (D) we obtain (P), and hence (T), by ($\mathtt{T''_q}$), (MP) and (US).

In this paper we prove that $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ$+($\mathtt{T_q}$), $\mathbf{S0.5}^\circ$+(D) and **S0.5** are not closed under (rte). For example, the formula '$\Box\Box p \equiv \Box\Box\neg\neg p$' is not a member of these logics (see Remark 3.2 and Fact 4.1).

2.3. Very weak t-normal rte-systems

By *rte-systems* we mean modal systems which are closed under (rte). By Lemma 1.1 we have

Lemma 2.5. *If a rte-system contains* (N) *and all instances of* (K), *then it is t-normal.*

Let $\mathbf{S0.5}^\circ_{\mathbf{rte}}$, $\mathbf{S0.5}_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$+(D) and $\mathbf{S0.5}^\circ_{\mathbf{rte}}$+($\mathtt{T_q}$) be, respectively, such versions of the logics $\mathbf{S0.5}^\circ$, **S0.5**, $\mathbf{S0.5}^\circ$+(D) and $\mathbf{S0.5}^\circ$+($\mathtt{T_q}$) that are closed under (rte). Thus, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$ is the smallest t-normal rte-system, and $\mathbf{S0.5}_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$+(D) and $\mathbf{S0.5}^\circ_{\mathbf{rte}}$+($\mathtt{T_q}$) are the smallest t-normal rte-logics which contain (T), (D) and ($\mathtt{T_q}$), respectively.[4] We have that $\mathbf{S0.5}^\circ_{\mathbf{rte}} \subsetneq \mathbf{S0.5}^\circ_{\mathbf{rte}}$+(D) $\subsetneq \mathbf{S0.5}_{\mathbf{rte}}$ and $\mathbf{S0.5}^\circ_{\mathbf{rte}} \subsetneq \mathbf{S0.5}^\circ_{\mathbf{rte}}$+($\mathtt{T_q}$) $\subsetneq \mathbf{S0.5}_{\mathbf{rte}}$ (see Fact 4.1).

2.4. Very weak t-regular rte-systems

Let $\mathbf{C1}_{\mathbf{rte}}$, $\mathbf{D1}_{\mathbf{rte}}$, $\mathbf{E1}_{\mathbf{rte}}$ and $\mathbf{E1}_{\mathbf{rte}}$+($\mathtt{T_q}$) be, respectively, such versions of the logics **C1**, **D1**, **E1** and **C1**+($\mathtt{T_q}$) that are closed under (rte). The

[4]Thus, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$ is the smallest classical modal logic in the sense of [2], and $\mathbf{S0.5}_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$+(D) and $\mathbf{S0.5}^\circ_{\mathbf{rte}}$+($\mathtt{T_q}$) are the smallest classical modal logics (in the sense of [2]) which contain (T), (D) and ($\mathtt{T_q}$), respectively.

logic $\mathbf{C1_{rte}}$ is the smallest t-regular rte-system. The logics $\mathbf{D1_{rte}}$, $\mathbf{E1_{rte}}$ and $\mathbf{E1_{rte}}+(\mathtt{T_q})$ are smallest t-regular rte-logics which contain (T), (D) and ($\mathtt{T_q}$), respectively. We have that $\mathbf{C1_{rte}} \subsetneq \mathbf{D1_{rte}} \subsetneq \mathbf{E1_{rte}}$ and $\mathbf{C1_{rte}} \subsetneq \mathbf{E1_{rte}}+(\mathtt{T_q}) \subsetneq \mathbf{E1_{rte}}$ (see Fact 4.1).

3. Semantics for very weak systems

3.1. Models for very weak t-normal and t-regular systems

For very weak t-normal modal systems we are using the following semantics, which consists of "t-normal models".

A *model for very weak t-normal systems* (or *t-normal model*) is any triple $\langle w, A, V\rangle$ in which

1. w is a «distinguished» (normal) world,
2. A is a set of *worlds* which are alternatives to the world w,
3. V is a valuation from For $\times$ $(\{w\} \cup A)$ to $\{0,1\}$:
   (i) for all formulae and all worlds, V preserves classical conditions for truth-value operators,
   (ii) for the world w and any $\varphi \in$ For
   $$(V_\Box) \qquad V(\Box\varphi, w) = 1 \text{ iff } \forall_{x\in A}\, V(\varphi, x) = 1,$$
   (iii) for every world from $A \setminus \{w\}$, formulae $\ulcorner\Box\varphi\urcorner$ have arbitrary values.

A formula φ is *true* in a t-normal model $\langle w, A, V\rangle$ iff $V(\varphi, w) = 1$. We say that a formula is *t-normal valid* iff it is true in all t-normal models.

We say that a t-normal model $\langle w, A, V\rangle$ is *self-associate* (resp. *empty*, *non-empty*) iff $w \in A$ (resp. $A = \emptyset$, $A \neq \emptyset$). Let nM be the class of all t-normal models. Moreover, let $\mathsf{nM^{sa}}$ (resp. $\mathsf{nM^{\varnothing}}$, $\mathsf{nM^{+}}$) be the class of t-normal models which are self-associate (resp. empty, non-empty). Of course, $\mathsf{nM^{sa}} \subsetneq \mathsf{nM^{+}}$ and $\mathsf{nM^{\varnothing}} \cap \mathsf{nM^{+}} = \emptyset$.

Remark 3.1. We may also use the class of models of the form $\langle W, w, A, V\rangle$, where W is a non-empty set of *worlds*, $w \in W$, $A \subseteq W$, and w, A and V are as mentioned above. Of course, the triple $\langle w, A, V\rangle$ may be identified with the quadruple $\langle W, w, A, V\rangle$ such that $W = \{w\} \cup A$. ⊣

In the case of very weak t-regular systems we broaden the class of t-normal models by the class of *queer* models of the form $\langle w, V\rangle$ with only one (queer) world w and a valuation $V\colon \mathrm{For}\times\{w\}\to\{0,1\}$ which satisfies classical conditions for truth-value operators and such that

(ii′) for the world w and any $\varphi\in\mathrm{For}$

$$V(\Box\varphi, w) = 0.$$

Of course, a queer model $\langle w, V\rangle$ may be identified with the valuation $V\colon \mathrm{For}\to\{0,1\}$ such that $V(\varphi) = V(\varphi, w)$, for any φ from For.

Let **qM** be the class of all queer models and we put $\mathbf{rM} := \mathbf{nM}\cup\mathbf{qM}$, i.e. **rM** is the class of models for very weak t-regular systems.

A formula φ is *true* in a queer model $\langle w, V\rangle$ iff $V(\varphi, w) = 1$. We say that a formula is *t-regular valid* iff it is true in all models from **rM**. We have the following lemmas.

LEMMA 3.1. 1. *If* $\varphi\in\mathrm{PL}$, *then* $V(\varphi, x) = 1$, *for any world* x *in any model from* **rM**. *So all formulae from* PL *are t-regular valid.*

2. *All formulae from* $\Box\mathrm{PL}$ *are t-normal valid.*

3. *All formulae from the sets* M_{PL} R_{PL} *and* E_{PL} *are t-regular valid.*

LEMMA 3.2. 1. *All instances of formulae* (K) *and* (R) *are t-regular valid.*

2. *All instances of the formulae* (T) *and* ($\mathrm{T_q}$) *are true in any model from* $\mathbf{nM}^{\mathbf{sa}}\cup\mathbf{qM}$.

3. *All instances of the formula* (D) *are true in all models from* $\mathbf{nM}^{+}\cup\mathbf{qM}$.

4. *All instances of the formula* ($\mathrm{T_q}$) *are true in all models from* $\mathbf{nM}^{\varnothing}$.

FACT 3.3. *Let* $\ulcorner\varphi\equiv\psi\urcorner\in\mathrm{PL}$. *Then for any classical formula* χ *(without the modal operator) following holds:* $V(\chi, x) = V(\chi[{}^{\varphi}/_{\psi}], x)$, *for any world* x *in any model from* **rM**.

Remark 3.2. Let $w\neq a$, $A := \{w, a\}$ and V be an arbitrary valuation such that $V(\Box p, a) = 1$ and $V(\Box\neg\neg p, a) = 0$. Then $\langle w, A, V\rangle$ belongs to $\mathbf{nM}^{\mathbf{sa}}$ and the formula '$\Box\Box p\equiv\Box\Box\neg\neg p$' is not true in this model. ⊣

3.2. Models for very weak t-normal and t-regular rte-systems

For very weak t-normal rte-systems we are using t&rte-normal models, where by a *t&rte-normal model* we mean a t-normal model $\langle w, A, V\rangle$ which satisfies the following condition:

(iv) for all formulae φ, ψ and χ: if $\ulcorner\varphi \equiv \psi\urcorner \in \mathrm{PL}$ and $\forall_{x\in A}\, V(\chi, x) = 1$, then $\forall_{x\in A}\, V(\chi[{}^{\varphi}/_{\psi}], x) = 1$.

Of course, the condition (iv) is equivalent to the following:

(iv′) for all formulae φ, ψ and χ: if $\ulcorner\varphi \equiv \psi\urcorner \in \mathrm{PL}$, then $\forall_{x\in A}\, V(\chi, x) = 1$ iff $\forall_{x\in A}\, V(\chi[{}^{\varphi}/_{\psi}], x) = 1$.

Moreover, by $(V_\Box)$, the condition (iv) is equivalent to the following one:

(iv″) for all formulae φ, ψ and χ: if $\ulcorner\varphi \equiv \psi\urcorner \in \mathrm{PL}$, then $V(\Box\,\chi, w) = V(\Box\,\chi[{}^{\varphi}/_{\psi}], w)$.

Let $\mathbf{nM_{rte}}$ be the class of all t&rte-normal models. Moreover, let $\mathbf{nM^{sa}_{rte}}$ (resp. $\mathbf{nM^{\emptyset}_{rte}}$, $\mathbf{nM^{+}_{rte}}$) be the class of t&rte-normal models which are self-associate (resp. empty, non-empty).

In the case of very weak t-regular rte-systems we broaden the class of t&rte-normal models by queer models. We put $\mathbf{rM_{rte}} := \mathbf{nM_{rte}} \cup \mathbf{qM}$, i.e. $\mathbf{rM_{rte}}$ is the class of models for very weak t&rte-regular systems.

We say that a formula is *t&rte-normal valid* (resp. *t&rte-regular valid*) iff it is true in all models from $\mathbf{nM_{rte}}$ (resp. $\mathbf{rM_{rte}}$).

We have the following lemma.

Lemma 3.4. *If* $\ulcorner\varphi \equiv \psi\urcorner \in \mathrm{PL}$, *then* $V(\chi, w) = V(\chi[{}^{\varphi}/_{\psi}], w)$ *in all t&rte-normal models and all queer models. So all formulae from* $\mathrm{RE_{PL}}$ *are t&rte-regular valid.*

4. Determination theorems

Let $\boldsymbol{C}$ be any class of considered models. We say that a formula φ is $\boldsymbol{C}$*-valid* (written $\models_{\boldsymbol{C}} \varphi$) iff φ is true in all models from $\boldsymbol{C}$.

Let Σ be an arbitrary modal system. We say that Σ is *sound* with respect to $\boldsymbol{C}$ iff $\Sigma \subseteq \{\varphi \in \mathrm{For} : \models_{\boldsymbol{C}} \varphi\}$. We say that Σ is *complete* with respect to $\boldsymbol{C}$ iff $\Sigma \supseteq \{\varphi \in \mathrm{For} : \models_{\boldsymbol{C}} \varphi\}$. We say that Σ is *determined* by $\boldsymbol{C}$ iff $\Sigma = \{\varphi \in \mathrm{For} : \models_{\boldsymbol{C}} \varphi\}$, i.e., Σ is sound and complete with respect to $\boldsymbol{C}$.

4.1. Soundness

By lemmas 3.1, 3.2 and 3.4 we obtain the following facts.

FACT 4.1. 1. **C1** *is sound with respect to the class* rM.

2. **D1** *is sound with respect to the class* $\mathsf{nM}^{+} \cup \mathsf{qM}$.
3. **E1** *is sound with respect to the class* $\mathsf{nM}^{\mathsf{sa}} \cup \mathsf{qM}$.
4. **C1**$+(\mathtt{T_q})$ *is sound with respect to the class* $\mathsf{nM}^{\mathsf{sa}} \cup \mathsf{nM}^{\emptyset} \cup \mathsf{qM}$.
5. **S0.5**$^{\circ}$ *is sound with respect to the class* nM.
6. **S0.5**$^{\circ}+(\mathtt{D})$ *is sound with respect to the class* nM^{+}.
7. **S0.5** *is sound with respect to the class* $\mathsf{nM}^{\mathsf{sa}}$.
8. **S0.5**$^{\circ}+(\mathtt{T_q})$ *is sound with respect to the class* $\mathsf{nM}^{\mathsf{sa}} \cup \mathsf{nM}^{\emptyset}$.
9. **C1**$_{\mathsf{rte}}$ *is sound with respect to the class* $\mathsf{rM}_{\mathsf{rte}}$.
10. **D1**$_{\mathsf{rte}}$ *is sound with respect to the class* $\mathsf{nM}^{+}_{\mathsf{rte}} \cup \mathsf{qM}$.
11. **E1**$_{\mathsf{rte}}$ *is sound with respect to the class* $\mathsf{nM}^{+}_{\mathsf{rte}} \cup \mathsf{qM}$.
12. **E1**$_{\mathsf{rte}}+(\mathtt{T_q})$ *is sound with respect to the class* $\mathsf{nM}^{+}_{\mathsf{rte}} \cup \mathsf{qM}$.
13. **S0.5**$^{\circ}_{\mathsf{rte}}$ *is sound with respect to the class* $\mathsf{nM}_{\mathsf{rte}}$.
14. **S0.5**$^{\circ}_{\mathsf{rte}}+(\mathtt{D})$ *is sound with respect to the class* $\mathsf{nM}^{+}_{\mathsf{rte}}$.
15. **S0.5**$_{\mathsf{rte}}$ *is sound with respect to the class* $\mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}}$.
16. **S0.5**$^{\circ}_{\mathsf{rte}}+(\mathtt{T_q})$ *is sound with respect to the class* $\mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}} \cup \mathsf{nM}^{\emptyset}_{\mathsf{rte}}$.

For completeness of considered very weak logics we use canonical models metod.

4.2. Notions and facts concerning maximal consistent sets

For the following definitions see, for example, [3, 2.4 and 2.6]. Let Σ and Σ' be any modal systems, and $\Gamma \subseteq \mathrm{For}$.

Σ is *consistent* iff $\Sigma \neq \mathrm{For}$; equivalently in the light of PL, iff '$p \wedge \neg p$' does not belong to Σ. For example, all modal logics from Section 2 are consistent.

A formula φ is *deducible* from Γ in Σ (written $\Gamma \vdash_{\Sigma} \varphi$) iff for some $\{\psi_1, \ldots, \psi_n\} \subseteq \Gamma$ ($n \geqslant 0$) we have $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \Sigma$. We have $\vdash_{\mathrm{PL}} \subseteq \vdash_{\Sigma}$. Moreover, $\Sigma \vdash_{\Sigma} \varphi$ iff $\varphi \in \Sigma$ iff $\emptyset \vdash_{\Sigma} \varphi$.

A set Γ is *Σ-consistent* iff for some $\varphi \in \mathrm{For}$, $\Gamma \nvdash_{\Sigma} \varphi$; equivalently in the light of PL, iff $\Gamma \nvdash_{\Sigma} p \wedge \neg p$. We have (see e.g. [3]):

- If Γ is Σ-consistent, then Σ is consistent.
- Σ is consistent iff Σ is Σ-consistent.
- If Γ is Σ-consistent and $\Sigma' \subseteq \Sigma$, then Γ is Σ'-consistent; so, Γ is PL-consistent.

We say that Γ is *Σ-maximal* iff Γ is Σ-consistent and Γ has only Σ-inconsistent proper extensions. Let Max_Σ be the set of all Σ-maximal sets.

LEMMA 4.2 ([3]). *Let $\Gamma \in \mathrm{Max}_\Sigma$. Then*

1. *$\Sigma \subseteq \Gamma$ and Γ is a modal system.*
2. *$\Gamma \vdash_\Sigma \varphi$ iff $\varphi \in \Gamma$.*
3. *$\ulcorner\neg\varphi\urcorner \in \Gamma$ iff $\varphi \notin \Gamma$.*
4. *$\ulcorner\varphi \wedge \psi\urcorner \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.*
5. *$\ulcorner\varphi \vee \psi\urcorner \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.*
6. *$\ulcorner\varphi \supset \psi\urcorner \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.*
7. *$\ulcorner\varphi \equiv \psi\urcorner \in \Gamma$ iff either $\varphi, \psi \in \Gamma$ or $\varphi, \psi \notin \Gamma$.*

LEMMA 4.3. *If $\Gamma \in \mathrm{Max}_\Sigma$ and $\Sigma' \subseteq \Sigma$, then $\Gamma \in \mathrm{Max}_{\Sigma'}$. So $\Gamma \in \mathrm{Max}_{\mathrm{PL}}$.*

PROOF. Let $\Gamma \in \mathrm{Max}_\Sigma$ and $\Sigma' \subseteq \Sigma$. Then Γ is Σ'-consistent and PL-consistent. Moreover, suppose that $\Gamma \cup \{\varphi\}$ is Σ'-consistent. Then $\Gamma \cup \{\varphi\}$ is also PL-consistent. So $\ulcorner\neg\varphi\urcorner \notin \Gamma$. Therefore $\varphi \in \Gamma$, by Lemma 4.2.3. Hence $\Gamma \cup \{\varphi\} = \Gamma$. Thus Γ be Σ'-maximal. ⊣

LEMMA 4.4 ([3]). 1. *$\Gamma \vdash_\Sigma \varphi$ iff $\varphi \in \Delta$, for any Δ such that $\Delta \in \mathrm{Max}_\Sigma$ and $\Gamma \subseteq \Delta$.*

2. *$\varphi \in \Sigma$ iff $\varphi \in \Delta$, for any $\Delta \in \mathrm{Max}_\Sigma$.*

4.3. Canonical models

For completeness of very weak logics we need two following auxiliary lemmas.

Lemma 4.5. *Let Σ be a t-regular consistent system and let Γ be a Σ-maximal set such that $\Gamma \cap \Box\mathrm{For} \neq \emptyset$, i.e. $\{\psi \in \mathrm{For} : \ulcorner\Box\psi\urcorner \in \Gamma\} \neq \emptyset$.*[5] *Then for every $\varphi \in \mathrm{For}$ the following conditions are equivalent:*

(a) $\ulcorner\Box\varphi\urcorner \in \Gamma$.

(b) $\Gamma \vdash_\Sigma \Box\varphi$.

(c) $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \vdash_{\mathrm{PL}} \varphi$.

(d) $\varphi \in \Delta$, *for any* PL*-maximal set Δ such that $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \subseteq \Delta$.*

Proof. "(a) $\Leftrightarrow$ (b)" Lemma 4.2.2.

"(a) $\Rightarrow$ (d)" It is trivial, since for any $\Gamma, \Delta \subseteq \mathrm{For}$, if $\ulcorner\Box\varphi\urcorner \in \Gamma$ and $\{\psi \in \mathrm{For} : \ulcorner\Box\psi\urcorner \in \Gamma\} \subseteq \Delta$, then $\varphi \in \Delta$.

"(d) $\Leftrightarrow$ (c)" By Lemma 4.4.1.

"(c) $\Rightarrow$ (b)" Ether $\varphi \in \mathrm{PL}$ or for some $\psi_1, \ldots, \psi_n \in \{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\}$, $n > 0$, we have $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \mathrm{PL}$. But the first case entails the second case. Hence $\ulcorner(\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \supset \Box\varphi\urcorner \in \Sigma$, since $\mathrm{R}_{\mathrm{PL}} \subseteq \Sigma$. But Γ contains each of $\ulcorner\Box\psi_1\urcorner, \ldots, \ulcorner\Box\psi_n\urcorner$, so $\Gamma \vdash_\Sigma \Box\varphi$. ⊣

Let Σ be a t-regular system, $\Gamma \in \mathrm{Max}_\Sigma$ and $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \neq \emptyset$. We say that $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is a *canonical model for Σ and Γ* iff it satisfies these conditions:

- $w_\Gamma := \Gamma$,
- $A_\Gamma := \{\Delta \in \mathrm{Max}_{\mathrm{PL}} : \forall_{\psi\in\mathrm{For}}(\ulcorner\Box\psi\urcorner \in \Gamma \Rightarrow \psi \in \Delta)\}$,
- $V_\Gamma \colon \mathrm{For} \times (\{w_\Gamma\} \cup A_\Gamma) \to \{0, 1\}$ is the valuation such that for all $\varphi \in \mathrm{For}$ and $\Delta \in \{w_\Gamma\} \cup A_\Gamma$

$$V_\Gamma(\varphi, \Delta) := \begin{cases} 1 & \text{if } \varphi \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

Lemma 4.6. *For any t-regular system Σ and any $\Gamma \in \mathrm{Max}_\Sigma$ such that $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \neq \emptyset$ it holds that:*

(a) *$\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is a t-normal model.*

(b) *If Σ contains all instances of* (T)*, then $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is self-associate.*

[5] Notice that all t-normal systems satisfy these assumptions. Firstly, all t-normal systems are t-regular. Secondly, for any t-normal system Σ, if Γ is Σ-maximal, then $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \neq \emptyset$, since $\Box\mathrm{PL} \subseteq \Sigma \subseteq \Gamma$, by Lemma 4.2.1.

(c) *If Σ contains all instances of* (D), *then* $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is non-empty.*

(d) *If Σ contains all instances of* ($\mathtt{T_q}$), *then* $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is either empty or self-associate.*

(e) *If Σ is a rte-system, then* $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is t&rte-normal model.*

Proof. (a) Thanks to properties of maximal sets (see Lemma 4.2), for every $\Delta \in \{w_\Gamma\} \cup A_\Gamma$ the assignment $V_\Gamma(\cdot, \Delta)$ preserves classical conditions for truth-value operators. We prove that for w_Γ the assignment $V_\Gamma(\cdot, w_\Gamma)$ satisfies the condition $(V_\Box)$.

For any $\varphi \in$ For: $V_\Gamma(\Box\varphi, w_\Gamma) = 1$ iff $\ulcorner\Box\varphi\urcorner \in \Gamma$ (by definition of V_Γ) iff for every $\Delta \in \mathrm{Max}_{\mathrm{PL}}$ for which $\{\psi \in \mathrm{For} : \ulcorner\Box\psi\urcorner \in \Gamma\} \subseteq \Delta$ we have $\varphi \in \Delta$ (by Lemma 4.5) iff for every $\Delta \in A_\Gamma$, $\varphi \in \Delta$ (by definition of A_Γ) iff for every $\Delta \in A_\Gamma$, $V_\Gamma(\varphi, \Delta) = 1$ (by definition of V_Γ).

(b) We show that $w_\Gamma \in A_\Gamma$. Firstly, by Lemma 4.3, $\Gamma \in \mathrm{Max}_{\mathrm{PL}}$-maximal. Secondly, for any $\psi \in$ For, $\ulcorner\Box\psi \supset \psi\urcorner \in \Gamma$, by Lemma 4.2.1. So, if $\ulcorner\Box\psi\urcorner \in \Gamma$, then $\psi \in \Gamma$, by Lemma 4.2.6.

(c) For some φ_0 we have $\ulcorner\Box\varphi_0\urcorner \in \Gamma$. By Lemma 4.2.1, $\ulcorner\Box\varphi_0 \supset \neg\Box\neg\varphi_0\urcorner \in \Gamma$. Hence, by lemmas 4.2.6 and 4.2.1, $\ulcorner\neg\Box\neg\varphi_0\urcorner \in \Gamma$ and $\ulcorner\Box\neg\varphi_0\urcorner \notin \Gamma$. Therefore, by Lemma 4.5, $\ulcorner\neg\varphi_0\urcorner \notin \Delta_0$, for some Δ_0 such that Δ_0 is PL-maximal and $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \subseteq \Delta_0$. Hence $\Delta_0 \in A_\Gamma$. Thus, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}^+$.

(d) We show that $w_\Gamma \in A_\Gamma$ or $A_\Gamma = \emptyset$. Notice that, by lemmas 4.2.1 and 4.2.6, $\ulcorner\neg\Box(q \wedge \neg q) \supset (\Box\psi \supset \psi)\urcorner \in \Gamma$, for any formula ψ. Suppose that $A_\Gamma \neq \emptyset$. Then '$\Box(q \wedge \neg q)$' $\notin \Gamma$, by Lemma 4.5, since '$q \wedge \neg q$' $\notin \Delta$, for any Δ which is PL-consistent. So, '$\neg\Box(q \wedge \neg q)$' $\in \Gamma$. Therefore $\ulcorner\Box\psi \supset \psi\urcorner \in \Gamma$. Hence $w_\Gamma \in A_\Gamma$, as in (b).

(e) Suppose that $\ulcorner\varphi \equiv \psi\urcorner \in \mathrm{PL}$. Then $\ulcorner\Box\chi[^\varphi/_\psi] \equiv \Box\chi\urcorner \in \Sigma$, since $\mathrm{RE_{PL}} \subseteq \Sigma$. So also $\ulcorner\Box\chi[^\varphi/_\psi] \equiv \Box\chi\urcorner \in \Gamma$, by Lemma 4.2.1. Thus, $V(\Box\chi, w) = V(\Box\chi[^\varphi/_\psi], w)$, by definition of V_Γ. ⊣

Let Σ be a t-regular system, $\Gamma \in \mathrm{Max}_\Sigma$ and $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} = \emptyset$. We say that $\langle w_\Gamma, V_\Gamma\rangle$ is a *canonical model for Σ and Γ* iff it satisfies these conditions:

- $w_\Gamma := \Gamma$,
- $V_\Gamma\colon \mathrm{For} \times \{w_\Gamma\} \to \{0,1\}$ is the valuation such that

$$V_\Gamma(\varphi, w_\Gamma) := \begin{cases} 1 & \text{if } \varphi \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 4.7. *For any t-regular system Σ and any $\Gamma \in \mathrm{Max}_\Sigma$ such that $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} = \emptyset$: $\langle w_\Gamma, V_\Gamma\rangle$ is a queer model.*

PROOF. Thanks to properties of maximal sets in modal systems (see Lemma 4.2), the assignment V_Γ preserves classical conditions for truth-value operators. Moreover, for any $\varphi \in \mathrm{For}$ we have: $\ulcorner\Box\varphi\urcorner \notin \Gamma$. So, $V_\Gamma(\Box\varphi, w_\Gamma) = 0$. ⊣

4.4. Completeness

By lemmas 4.4.2 and 4.6 for very weak t-normal and t-normal rte-systems we obtain

THEOREM 4.8. 1. **S0.5°** *is complete with respect to the class* **nM**.

2. **S0.5°**+(D) *is complete with respect to the class* $\mathbf{nM}^{+}$.
3. **S0.5°**+($\mathrm{T_q}$) *is complete with respect to the class* $\mathbf{nM}^{\mathbf{sa}} \cup \mathbf{nM}^{\varnothing}$.
4. **S0.5** *is complete with respect to the class* $\mathbf{nM}^{\mathbf{sa}}$.
5. $\mathbf{S0.5}^{\circ}_{\mathbf{rte}}$ *is complete with respect to the class* $\mathbf{nM}_{\mathbf{rte}}$.
6. $\mathbf{S0.5}^{\circ}_{\mathbf{rte}}$+(D) *is complete with respect to the class* $\mathbf{nM}^{+}_{\mathbf{rte}}$.
7. $\mathbf{S0.5}^{\circ}_{\mathbf{rte}}$+($\mathrm{T_q}$) *is complete with respect to the class* $\mathbf{nM}^{\mathbf{sa}}_{\mathbf{rte}} \cup \mathbf{nM}^{\varnothing}_{\mathbf{rte}}$.
8. $\mathbf{S0.5}_{\mathbf{rte}}$ *is complete with respect to the class* $\mathbf{nM}^{\mathbf{sa}}_{\mathbf{rte}}$.

PROOF. The logics **S0.5°**, **S0.5°**+(D), **S0.5°**+($\mathrm{T_q}$) and **S0.5** are consistent and t-regular. Moreover, for any t-normal logic Λ, if $\Gamma \in \mathrm{Max}_\Lambda$, then $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \neq \emptyset$, since $\Box\mathrm{PL} \subseteq \Lambda \subseteq \Gamma$.

1. Let φ be an arbitrary formula such that $\models_{\mathbf{nM}} \varphi$. Let Γ be an arbitrary **S0.5°**-maximal set. By Lemma 4.6a, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}$. Thus, $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$, by definitions of w_Γ and V_Γ. So, we have shown that φ belongs to all **S0.5°**-maximal sets. Hence $\varphi \in$ **S0.5°**, by Lemma 4.4.2.

2. By Lemma 4.6c, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}^{+}$. The rest as in 1.
3. By Lemma 4.6d, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}^{+} \cup \mathbf{nM}^{\varnothing}$. The rest as in 1.
4. By Lemma 4.6b, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}^{\mathbf{sa}}$. The rest as in 1.
5. By Lemma 4.6e, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}_{\mathbf{rte}}$. The rest as in 1.
6. By Lemma 4.6ce, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}^{+}_{\mathbf{rte}}$. The rest as in 1.
7. By Lemma 4.6de, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}^{\mathbf{sa}}_{\mathbf{rte}} \cup \mathbf{nM}^{\varnothing}_{\mathbf{rte}}$. The rest as in 1.
8. By Lemma 4.6be, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathbf{nM}^{\mathbf{sa}}$. The rest as in 1. ⊣

By lemmas 4.4.2, 4.6 and 4.7 for very weak t-regular and t-regular rte-systems we obtain

Theorem 4.9. 1. **C1** *is complete with respect to the class* rM.

2. **D1** *is complete with respect to the class* $\mathsf{nM}^{+} \cup \mathsf{qM}$.
3. **C1**$+(\mathtt{T_q})$ *is complete with respect to the class* $\mathsf{nM}^{\mathsf{sa}} \cup \mathsf{nM}^{\emptyset} \cup \mathsf{qM}$.
4. **E1** *is complete with respect to the class* $\mathsf{nM}^{\mathsf{sa}} \cup \mathsf{qM}$.
5. $\mathbf{C1}_{\mathsf{rte}}$ *is complete with respect to the class* $\mathsf{rM}_{\mathsf{rte}}$.
6. $\mathbf{D1}_{\mathsf{rte}}$ *is complete with respect to the class* $\mathsf{nM}^{+}_{\mathsf{rte}} \cup \mathsf{qM}$.
7. $\mathbf{E1}_{\mathsf{rte}}+(\mathtt{T_q})$ *is complete with respect to* $\mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}} \cup \mathsf{nM}^{\emptyset}_{\mathsf{rte}} \cup \mathsf{qM}$.
8. $\mathbf{E1}_{\mathsf{rte}}$ *is complete with respect to* $\mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}} \cup \mathsf{qM}$.

Proof. 1. Let φ be an arbitrary formula such that $\models_{\mathsf{rM}} \varphi$. Let Γ be an arbitrary **C1**-maximal set. In both alternative cases from lemmas 4.6 and 4.7, either $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}$ or $\langle w_\Gamma, V_\Gamma\rangle \in \mathsf{qM}$. Thus, in both cases we have $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$, by definitions of w_Γ and V_Γ. So, we have shown that φ belongs to all **C1**-maximal sets. Hence $\varphi \in \mathbf{C1}$, by Lemma 4.4.2.

2. $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}^{+}$ or $\langle w_\Gamma, V_\Gamma\rangle \in \mathsf{qM}$. The rest as in 1.
3. $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}^{\mathsf{sa}} \cup \mathsf{nM}^{\emptyset}$ or $\langle w_\Gamma, V_\Gamma\rangle \in \mathsf{qM}$. The rest as in 1.
4. $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}^{\mathsf{sa}}$ or $\langle w_\Gamma, V_\Gamma\rangle \in \mathsf{qM}$. The rest as in 1.
5. $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}_{\mathsf{rte}}$ or $\langle w_\Gamma, V_\Gamma\rangle \in \mathsf{qM}$. The rest as in 1.
6. $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}^{+}_{\mathsf{rte}}$ or $\langle w_\Gamma, V_\Gamma\rangle \in \mathsf{qM}$. The rest as in 1.
7. $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}} \cup \mathsf{nM}^{\emptyset}_{\mathsf{rte}}$ or $\langle w_\Gamma, V_\Gamma\rangle \in \mathsf{qM}$. The rest as in 1.
8. $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}}$ or $\langle w_\Gamma, V_\Gamma\rangle \in \mathsf{qM}$. The rest as in 1. ⊣

A. Location of very weak modal logics

A.1. Strict implication and strict equivalence

In original Lewis' works (see e.g. [12]) the primitive modal operator is the possibility sign '$\Diamond$'. The necessity sign '$\Box$' is the abbreviation of '$\neg \Diamond \neg$'. Moreover, for the connective of strict implication '⥽' was used $\ulcorner \varphi \mathrel{⥽} \psi \urcorner$ as an abbreviation of a formula $\ulcorner \neg \Diamond(\varphi \wedge \neg\psi)\urcorner$.

In this paper—as in [9]—the primitive modal operator is '$\Box$' and $\ulcorner \varphi \mathrel{⥽} \psi \urcorner$ is an abbreviation of $\ulcorner \Box(\varphi \supset \psi)\urcorner$. Moreover, in this paper—as in [12] and [9]—a strict equivalence $\ulcorner \varphi \mathrel{⥼⥽} \psi \urcorner$ is an abbreviation of $\ulcorner (\varphi \mathrel{⥽} \psi) \wedge (\psi \mathrel{⥽} \varphi)\urcorner$.

LEMMA A.1. *For any modal system $\varSigma$ and any $\varphi, \psi \in \mathrm{For}$:*

$$\text{if } \ulcorner \varphi \strictiff \psi \urcorner \in \varSigma, \text{ then } \ulcorner \varphi \strictif \psi \urcorner, \ulcorner \psi \strictif \varphi \urcorner \in \varSigma.$$

PROOF. Let $\ulcorner \varphi \strictiff \psi \urcorner \in \varSigma$, i.e., $\ulcorner \Box(\varphi \supset \psi) \wedge \Box(\psi \supset \varphi) \urcorner \in \varSigma$. Hence $\ulcorner \Box(\varphi \supset \psi) \urcorner, \ulcorner \Box(\psi \supset \varphi) \urcorner \in \varSigma$, by PL, i.e., $\ulcorner \varphi \strictif \psi \urcorner, \ulcorner \psi \strictif \varphi \urcorner \in \varSigma$. ⊣

LEMMA A.2. *For any t-regular system $\varSigma$ and any $\varphi, \psi \in \mathrm{For}$:*

$$\ulcorner \varphi \strictiff \psi) \urcorner \in \varSigma \text{ iff } \ulcorner \Box(\varphi \equiv \psi) \urcorner \in \varSigma.$$

PROOF. If $\ulcorner \Box(\varphi \supset \psi) \wedge \Box(\psi \supset \varphi) \urcorner \in \varSigma$, then $\ulcorner \Box(\varphi \equiv \psi) \urcorner \in \varSigma$, by (MP) and since $\mathrm{R}_{\mathrm{PL}} \subseteq \varSigma$. If $\ulcorner \Box(\varphi \equiv \psi) \urcorner \in \varSigma$, then $\ulcorner \Box(\varphi \supset \psi) \urcorner, \ulcorner \Box(\psi \supset \varphi) \urcorner \in \varSigma$, since $\mathrm{PL}, \mathrm{M}_{\mathrm{PL}} \subseteq \varSigma$. So, $\ulcorner \Box(\varphi \supset \psi) \wedge \Box(\psi \supset \varphi) \urcorner \in \varSigma$, by PL. ⊣

LEMMA A.3 ([4, 9]). *If $\varSigma$ is closed under the following rule*

$$\text{if } \ulcorner \Box \varphi \urcorner \in \varSigma, \text{ then } \varphi \in \varSigma, \tag{RN_*}$$

then $\varSigma$ is closed under the strict version of modus ponens

$$\text{if } \ulcorner \varphi \strictif \psi \urcorner \in \varSigma \text{ and } \varphi \in \varSigma, \text{ then } \psi \in \varSigma. \tag{SMP}$$

Hence, any modal system which contains all instances of (T) is also closed under (RN_*) and (SMP).

LEMMA A.4 ([4]). *Let $\varSigma$ be a rte-system which is closed under (SMP). Then $\varSigma$ is closed under (RN_*).*

PROOF. Let $\ulcorner \Box \varphi \urcorner \in \varSigma$ and $\tau \in \mathrm{PL} \subseteq \varSigma$. Then $\ulcorner \varphi \equiv (\tau \supset \varphi) \urcorner \in \mathrm{PL}$, so $\ulcorner \Box(\tau \supset \varphi) \urcorner \in \varSigma$, by (rte). So $\varphi \in \varSigma$, by (SMP). ⊣

LEMMA A.5. *Let $\varSigma$ be any system which is closed under (SMP) and includes M_{PL}. Then $\varSigma$ is closed under (RN_*).*

PROOF. Let $\ulcorner \Box \varphi \urcorner \in \varSigma$ and $\tau \in \mathrm{PL} \subseteq \varSigma$. Then $\ulcorner \varphi \supset (\tau \supset \varphi) \urcorner \in \mathrm{PL}$, so $\ulcorner \Box \varphi \supset \Box(\tau \supset \varphi) \urcorner \in \varSigma$, since $\mathrm{M}_{\mathrm{PL}} \subseteq \varSigma$. Thus, $\ulcorner \Box(\tau \supset \varphi) \urcorner \in \varSigma$, by (MP), and $\varphi \in \varSigma$, by (SMP). ⊣

A.2. Strict classical modal systems

Imitating [4], we say that a modal system Σ is *strict$_T$ classical* ("*traditionally strict classical*") iff $\Box\mathrm{PL} \subseteq \Sigma$ and Σ is closed under "traditional replacement rule for strict equivalents":

$$\text{if } \ulcorner \varphi \mathrel{\text{⥶}} \psi \urcorner \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi[^{\varphi}/_{\psi}] \in \Sigma. \qquad (\mathrm{RRSE_T})$$

Moreover, a modal system Σ is called *strict classical* iff $\Box\mathrm{PL} \subseteq \Sigma$ and Σ is closed under the following replacement rule:

$$\text{if } \ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi[^{\varphi}/_{\psi}] \in \Sigma. \qquad (\mathrm{RRSE})$$

We obtain that for modal logics which contain (K) and/or (X), the above notions are equivalent (see Lemma A.9).

LEMMA A.6 ([4]). *Let Σ be strict$_T$ or strict classical. Then Σ is also a rte-system.*

PROOF. Suppose that $\ulcorner \varphi \equiv \psi \urcorner \in \mathrm{PL}$ and $\chi \in \Sigma$. Since $\Box\mathrm{PL} \subseteq \Sigma$, so we have that $\ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma$ and $\ulcorner \Box(\varphi \supset \psi) \wedge \Box(\psi \supset \varphi) \urcorner \in \Sigma$, by PL. Hence $\chi[^{\varphi}/_{\psi}] \in \Sigma$ follows by (RRSE) or by (RRSE$_\mathrm{T}$), respectively. ⊣

By definitions we have the following lemma.

LEMMA A.7. *Let Σ be strict$_T$ or strict classical and let Σ contain all instances of* (K). *Then Σ is t-normal.*

Now notice that

LEMMA A.8 ([4, 9]). *Let Σ be strict$_T$ or strict classical and let Σ contain all instances of* (X) *(resp. $\Box$*(X)*). Then Σ contains all instances of* (K) *(resp. $\Box$*(K)*).*

PROOF. Let $\varphi, \psi \in \mathrm{For}$. Since $\Box\mathrm{PL} \subseteq \Sigma$ and $\ulcorner \varphi \equiv (\tau \supset \varphi) \urcorner \in \mathrm{PL}$, for any $\tau \in \mathrm{Taut}$, so we have $\ulcorner \varphi \mathrel{\text{⥶}} (\tau \supset \varphi) \urcorner$, $\ulcorner \Box(\varphi \equiv (\tau \supset \varphi)) \urcorner \in \Sigma$, by PL. Similarly for ψ. Let Σ contain all instances of (X). Then $\ulcorner (\Box(\tau \supset \varphi) \wedge \Box(\varphi \supset \psi)) \supset \Box(\tau \supset \psi) \urcorner \in \Sigma$. Hence $\ulcorner \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi) \urcorner \in \Sigma$, by PL and either (RRSE$_\mathrm{T}$) or (RRSE).

Let Σ contain all instances of $\Box$(X). Then $\ulcorner \Box((\Box(\tau \supset \varphi) \wedge \Box(\varphi \supset \psi)) \supset \Box(\tau \supset \psi)) \urcorner \in \Sigma$. Hence $\ulcorner \Box(\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)) \urcorner \in \Sigma$, by PL and either (RRSE$_\mathrm{T}$) or (RRSE). ⊣

By lemmas 2.4, A.2, A.7 and A.8 we have the following lemma.

LEMMA A.9 ([4]). *For any modal system Σ which contains all instances of* (K) *or* (X)*: Σ is strict$_T$ classical iff Σ is strict classical.*

Moreover, we obtain

LEMMA A.10 ([4]). 1. *If Σ is strict$_T$ classical, then it is also closed under the following "traditional" rule of congruence for strict equivalence*

$$\text{if } \ulcorner \varphi \leftrightarrows \psi \urcorner \in \Sigma, \text{ then } \ulcorner \Box \varphi \leftrightarrows \Box \psi \urcorner \in \Sigma. \tag{RSE$_T$}$$

2. *If Σ is strict classical, then is also closed under the following rule of congruence for strict equivalence*

$$\text{if } \ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma, \text{ then } \ulcorner \Box(\Box \varphi \equiv \Box \psi) \urcorner \in \Sigma. \tag{RSE}$$

PROOF. 1. Since $\Box$PL $\subseteq \Sigma$, we have that $\ulcorner \Box \varphi \leftrightarrows \Box \varphi \urcorner \in \Sigma$, by PL. Hence if $\ulcorner \varphi \leftrightarrows \psi \urcorner \in \Sigma$, then $\ulcorner \Box \varphi \leftrightarrows \Box \psi \urcorner \in \Sigma$, by (RRSE$_T$).

2. Since $\Box$PL $\subseteq \Sigma$, we have that $\ulcorner \Box(\Box \varphi \equiv \Box \varphi) \urcorner \in \Sigma$. Hence if $\ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma$, then $\ulcorner \Box(\Box \varphi \equiv \Box \psi) \urcorner \in \Sigma$, by (RRSE). ⊣

LEMMA A.11 ([4, 9]). *Let Σ be a t-normal system which closed under* (RSE$_T$)*. Then*

1. *Σ is also closed under the following rule of replacement*

$$\text{if } \ulcorner \varphi \leftrightarrows \psi \urcorner \in \Sigma, \text{ then } \ulcorner \chi[{}^{\varphi}/_{\psi}] \leftrightarrows \chi \urcorner \in \Sigma, \tag{RRSE$'_T$}$$

2. *If Σ is also closed under* (SMP)*, then Σ is closed under* (RRSE$_T$)*.*

PROOF. 1. By induction.

2. Let $\ulcorner \varphi \leftrightarrows \psi \urcorner \in \Sigma$ and $\chi \in \Sigma$. Then $\ulcorner \chi[{}^{\varphi}/_{\psi}] \leftrightarrows \chi \urcorner \in \Sigma$, by 1. Hence $\ulcorner \chi \dashv \chi[{}^{\varphi}/_{\psi}] \urcorner \in \Sigma$, by Lemma A.1. So $\chi[{}^{\varphi}/_{\psi}] \in \Sigma$, by (SMP). ⊣

A.3. The logics S0.9, S0.9°, S1 and S1°

In [9] Lemmon provided a simple axiomatization of the Lewis' logic **S1**, where it is the smallest strict$_T$ classical modal logic which contains formulae $\Box$(X), (T) and $\Box$(T). Of course, the logic **S1** contains also (X) and, by Lemma A.8, the formulae (K) and $\Box$(K). So **S1** is strict classical and it is a t-normal rte-logic (see lemmas A.6, A.7 and A.9).

In [9] Lemmon also introduced the logic **S0.9**, where it was meant as the smallest modal logic which included □Taut, contained formulae □(K), (T) and □(T), and is closed under (RSE_T). So **S0.9** contains (K) and is t-normal. Hence, contains (X), since **S0.9** is also t-regular. Moreover, by lemmas A.8 and A.10.1, we obtain that **S0.9** $\subseteq$ **S1**. In [7] it was proved that **S0.9** $\neq$ **S1**, since □(X) $\notin$ **S0.9** (see also [4]).

"The other two systems, **S1°** and **S0.9°**, are often loosely described as **S1** and **S0.9** minus the schema T" [4, p. 12]. In [4] the Feys' logic **S1°** from [5] is described as the smallest strict_T classical modal logic which contains the formulae (X) and □(X), and is closed under (SMP). Thus, **S1°** contains (K) and □(K), by Lemma A.8. So, it is also a strict classical rte-logic.

Moreover, in [4] the logic **S0.9°** is described as the smallest strict_T classical modal logic which contains the formulae (K) and □(K), and is closed under (SMP).

Thus we have the following axiomatizations (of course, in each case PL, (MP) and (US) are added as default items):

- **S0.9**: □Taut, □(K), (T), □(T) and (RSE_T),
- **S0.9°**: □Taut, (K), □(K), ($RRSE_T$) and (SMP),
- **S1**: □Taut, □(X), (T), □(T) and ($RRSE_T$),
- **S1°**: □Taut , (X), □(X), ($RRSE_T$) and (SMP).

By Lemma A.10 the logic **S0.9°** is also closed under the rules (RSE_T) and (RSE_T). So **S0.9°** $\subsetneq$ **S0.9**, since **S0.9** is also closed under (SMP) and (T), □(T) $\notin$ **S0.9°**. Hence, by Lemma A.8, we have that **S0.9°** $\subsetneq$ **S1°**, since □(X) $\notin$ **S0.9**. Moreover, since **S1** is also closed under (SMP) and (T), □(T) $\notin$ **S1°**. We have that **S1°** $\subsetneq$ **S1**.

By Lemma A.6, the logics **S0.9°**, **S1** and **S1°** are a t-normal rte-logic. Moreover, by lemmas A.3, A.11, A.9 and A.6, we have:

COROLLARY A.12 ([4]). **S0.9** *is* $\textit{strict}_T$ *and strict classical, and it is a t-normal rte-logic.*

Notice that using lemmas given in sections A.1 and A.2 as well as Lemma 1.1 we obtain the following facts.

FACT A.13 ([4]). 1. **S0.9** *is the smallest rte-logic which is closed under* (RN_*) *and* (RRSE) *(resp.* ($RRSE_T$)*), and contains the formulae* (N), □(T) *and* □(K)*.*

2. $\mathbf{S0.9}^{\circ}$ *is the smallest rte-logic which is closed under* (RN$_*$) *and* (RRSE) (*resp.* (RRSE$_{\mathrm{T}}$)), *and contains the formulae* (N) *and* $\Box$(K).
3. $\mathbf{S1}$ *is the smallest rte-logic which is closed under* (RN$_*$) *and* (RRSE) (*resp.* (RRSE$_{\mathrm{T}}$)), *and contains the formulae* (N), $\Box$(T) *and* $\Box$(X).
4. $\mathbf{S1}^{\circ}$ *is the smallest rte-logic which is closed under* (RN$_*$) *and* (RRSE) (*resp.* (RRSE$_{\mathrm{T}}$)), *and contains the formulae* (N) *and* $\Box$(X).

Fact A.14. 1. $\mathbf{S0.9}$ *is the smallest strict* (*resp. strict*$_{\mathrm{T}}$) *classical logic which is closed under* (RN$_*$), *and contains the formulae* $\Box$(T) *and* $\Box$(K).

2. $\mathbf{S0.9}^{\circ}$ *is the smallest strict* (*resp. strict*$_{\mathrm{T}}$) *classical logic which is closed under* (RN$_*$), *and contains the formula* $\Box$(K).
3. $\mathbf{S1}$ *is the smallest strict* (*resp. strict*$_{\mathrm{T}}$) *classical logic which is closed under* (RN$_*$), *and contains the formulae* $\Box$(T) *and* $\Box$(X).
4. $\mathbf{S1}^{\circ}$ *is the smallest strict* (*resp. strict*$_{\mathrm{T}}$) *classical logic which is closed under* (RN$_*$), *and contains the formula* $\Box$(X).

A.4. The logics S2, S2°, S3, S3.5, S4 and S5

We say the a modal logic Λ is closed under *Becker's rule* iff

$$\text{if } \ulcorner \varphi \dashv\!\!3\, \psi \urcorner \in \Lambda, \text{ then } \ulcorner \Box\,\varphi \dashv\!\!3\, \Box\,\psi \urcorner \in \Lambda. \tag{RB}$$

In [9] (see also [1]) the logic $\mathbf{S2}$ is described as the smallest modal logic which includes $\Box$Taut, contains the formulae (T), $\Box$(T), and $\Box$(K), and is closed under (RB). Of course, $\mathbf{S2}$ includes $\Box$PL, contains (K) and, by Lemma A.3, it is closed under (RN$_*$) and (SMP).

Moreover, in [1] the logic $\mathbf{S2}^{\circ}$ is described as the smallest modal logic which includes $\Box$Taut, contains $\Box$(K), and is closed under (RB) and (RN$_*$). Of course, $\mathbf{S2}^{\circ}$ includes $\Box$PL, contains (K) and, by Lemma A.3, it is closed under (SMP). So $\mathbf{S2}^{\circ} \subsetneq \mathbf{S2}$. For example (T), $\Box$(T) $\notin \mathbf{S2}^{\circ}$.

Moreover, by (RB) and PL, the logics $\mathbf{S2}$ and $\mathbf{S2}^{\circ}$ are closed under (RSE$_{\mathrm{T}}$). Thus, by lemmas A.3, A.11 and A.9, the logics $\mathbf{S2}$ and $\mathbf{S2}^{\circ}$ are strict$_{\mathrm{T}}$ and strict classical, but they are not congruential.

In [4] the *Lewis version* $\mathbf{Lew}(\Lambda)$ of a logic Λ understood as the smallest modal logic which includes Λ and contains the formula (N). We have: $\mathbf{S2}^{\circ} = \mathbf{Lew}(\mathbf{C2})$ and $\mathbf{S2} = \mathbf{Lew}(\mathbf{E2})$. Moreover, for every $\varphi \in$ For: $\varphi \in \mathbf{C2}$ iff $\ulcorner \Box\varphi \urcorner \in \mathbf{S2}^{\circ}$; $\varphi \in \mathbf{E2}$ iff $\ulcorner \Box\varphi \urcorner \in \mathbf{S2}$ (see e.g. [4, 8]).

In [9] Lemmon proved that $\Box(\mathtt{X}) \in \mathbf{S2}$. His proof shows that also $\Box(\mathtt{X}) \in \mathbf{S2}^\circ$. We have that $\mathbf{S1}^\circ \subsetneq \mathbf{S2}^\circ$ and $\mathbf{S1} \subsetneq \mathbf{S2}$. For example, the formulae '$\Box(p \wedge q) \prec (\Box p \wedge \Box q)$', '$(\Box p \wedge \Box q) \prec \Box(p \wedge q)$' and '$\Diamond(p \wedge q) \prec \Diamond p$' belong to $\mathbf{S2}^\circ$, but they are not members of $\mathbf{S1}$.

In [9] the logic **S3** is described as the smallest modal logic which includes $\Box$Taut and contains the formulae ($\mathtt{T}$), $\Box(\mathtt{T})$ and $\Box(\mathtt{sK})$. Of course, **S3** contains ($\mathtt{sK}$) and ($\mathtt{K}$). Moreover, it contains also $\Box(\mathtt{K})$.[6] So **S3** is also closed under (RB), ($\mathrm{RSE_T}$), (RSE), and it is strict$_\mathrm{T}$ and strict classical. We have $\mathbf{S2} \subsetneq \mathbf{S3}$. For example $(\mathtt{sK}), \Box(\mathtt{sK}) \notin \mathbf{S2}$. We have: $\mathbf{S3} = \mathbf{Lew}(\mathbf{E3})$. Moreover, for every $\varphi \in$ For: $\varphi \in \mathbf{E3}$ iff $\ulcorner\Box\varphi\urcorner \in \mathbf{S3}$ (see e.g. [8]).

Åqvist's logic **S3.5** is obtained by adding

$$\Diamond p \supset \Box\Diamond p \tag{5}$$

or equivalently

$$p \supset \Box\Diamond p \tag{B}$$

to Lewis' logic **S3** (see e.g. [6, p. 208]). We have that $\mathbf{S3} \subsetneq \mathbf{S3.5}$. For example $(5), (\mathtt{B}) \notin \mathbf{S3}$.

In [9] the logic **S4** is described as the smallest modal logic which contains the formulae ($\mathtt{T}$) and ($\mathtt{sK}$), and is closed under (RN). Of course, **S4** contains ($\mathtt{K}$), $\Box(\mathtt{K})$, ($\mathtt{sK}$) and $\Box(\mathtt{sK})$. It is closed under (RB), ($\mathrm{RSE_T}$) and is strict$_\mathrm{T}$ and strict classical. It is known (see e.g. [9]) that **S4** is the smallest normal logic which contains the formulae ($\mathtt{T}$) and

$$\Box p \supset \Box\Box p \tag{4}$$

We have that $\mathbf{S3} \subsetneq \mathbf{S4}$. For example $(4) \notin \mathbf{S3}$.

Finally, **S5** is the smallest normal logic which contains ($\mathtt{T}$) and (5). Moreover, **S5** is the smallest normal logic which contains ($\mathtt{T}$), ($\mathtt{B}$) and (4); resp. ($\mathtt{D}$), ($\mathtt{B}$) and (4); resp. ($\mathtt{D}$), ($\mathtt{B}$) and (5); resp. ($\mathtt{D}$) (5) and ($\mathtt{T_q}$). It is known that $\mathbf{S3.5} \subsetneq \mathbf{S5}$ and $\mathbf{S4} \subsetneq \mathbf{S5}$. For example $\Box(5) \notin \mathbf{S3.5}$ and $(5) \notin \mathbf{S4}$. Note that $\Box(5)$ strengthens **S3** to **S5** (see e.g. [6, p. 208]).

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[6] Notice the formula '$\Box(p \supset \Box q) \supset \Box(p \supset q)$' belongs to **S2** and **S3**. By the substitution $p/\Box(p \supset q)$ and $q/\Box p \supset \Box q$ we have $\ulcorner\Box(\mathtt{sK}) \supset \Box(\mathtt{K})\urcorner$.

A.5. Location

Using semantics result we will to situate the logics **C1**, **D1**, **C1**+($\mathtt{T_q}$), **E1**, **S0.5°**, **S0.5°**+(D), **S0.5°**+($\mathtt{T_q}$), **S0.5**, $\mathbf{C1_{rte}}$, $\mathbf{D1_{rte}}$, $\mathbf{E1_{rte}}$+($\mathtt{T_q}$), $\mathbf{E1_{rte}}$, $\mathbf{S0.5^{\circ}_{rte}}$, $\mathbf{S0.5^{\circ}_{rte}}$+(D), $\mathbf{S0.5^{\circ}_{rte}}$+($\mathtt{T_q}$) and $\mathbf{S0.5_{rte}}$ among other logics (see Fig. 1; see also diagrams in [1, p. 3], [3, p. 132], [4, p. 21], [9, p. 186], [10, p. 48] and [11, p. 58]).

Using names of formulae, to simplify notation of normal logics we write the *Lemmon code* $\mathbf{KA}_1 \dots \mathbf{A}_n$ to denote the smallest normal logic containing the formulae ($\mathtt{A}_1$), ..., ($\mathtt{A}_n$) (see [2, 3]). Thus, for example, **KT4** is the smallest normal modal logic which contains (T) and (4). We standardly put **T** := **KT** and **D** := **KD**. We have **S4** = **KT4**, **KT** = $\mathbf{KDT_q}$, **KB4** = **KB5** = $\mathbf{K5T_q}$ and **S5** = **KT5** = **KTB4** = **KDB4** = **KDB5** = $\mathbf{KD5T_q}$ (see e.g. [9, 10, 4]).

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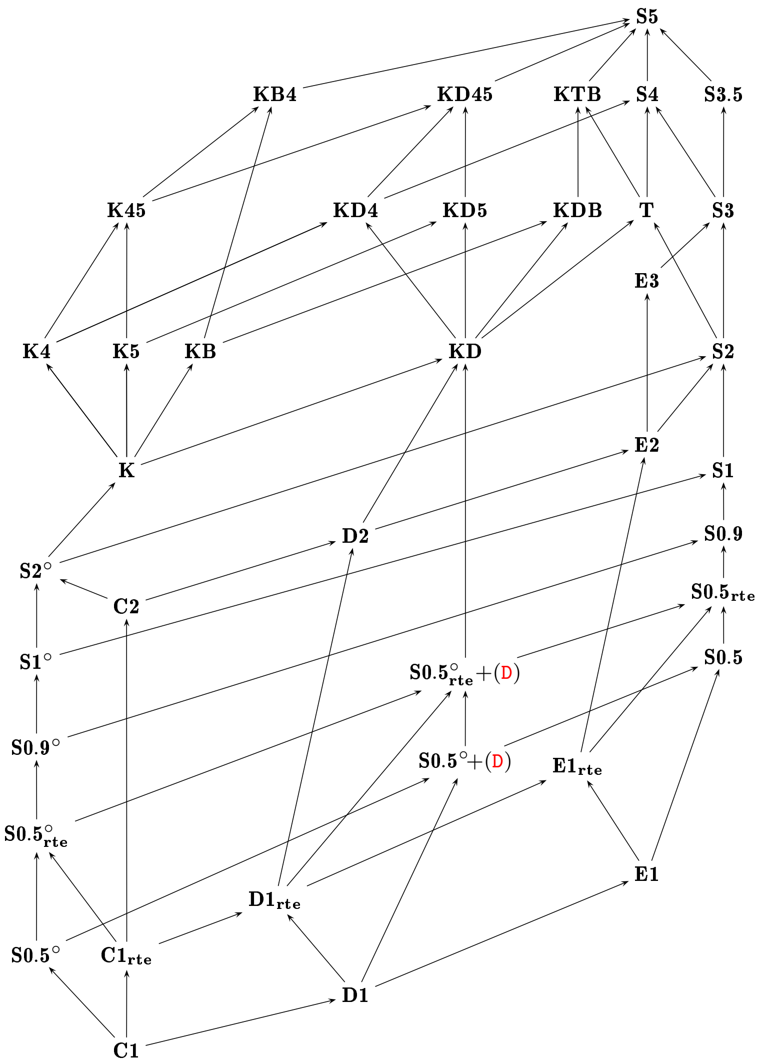

Figure 1. Some t-regular modal logics

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