

ON AUSLANDER-REITEN COMPONENTS OF ALGEBRAS WITHOUT EXTERNAL SHORT PATHS

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Dedicated to Daniel Simson on the occasion of his seventieth birthday.

ABSTRACT. We describe the structure of semi-regular Auslander-Reiten components of artin algebras without external short paths in the module category. As an application we give a complete description of self-injective artin algebras whose Auslander-Reiten quiver admits a regular acyclic component without external short paths.

1. INTRODUCTION AND THE MAIN RESULTS

Throughout the paper, by an algebra we mean a basic connected artin algebra over a commutative artin ring K . For an algebra A , we denote by $\text{mod } A$ the category of finitely generated right A -modules and by rad_A the radical of $\text{mod } A$, generated by all non-isomorphisms between indecomposable modules in $\text{mod } A$. Then the infinite radical rad_A^∞ of $\text{mod } A$ is the intersection of all powers rad_A^i , $i \geq 1$, of rad_A . By a result of Auslander [2], $\text{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, there are in $\text{mod } A$ only finitely many indecomposable modules up to isomorphism. Moreover, we denote by Γ_A the Auslander-Reiten quiver of A and by τ_A and τ_A^- the Auslander-Reiten translations $D\text{Tr}$ and $\text{Tr}D$ in $\text{mod } A$, respectively. We do not distinguish between an indecomposable module X in $\text{mod } A$ and the corresponding vertex $\{X\}$ in Γ_A . Moreover, by a component of Γ_A we mean a connected component of the quiver Γ_A .

The Auslander-Reiten quiver Γ_A of an algebra A is an important combinatorial and homological invariant of its module category $\text{mod } A$. Frequently, algebras can be recovered from the graph structure, for example the shape of components, of their Auslander-Reiten quivers. Further, very often the behaviour of components of the Auslander-Reiten quiver Γ_A of an algebra A in the category $\text{mod } A$ leads to essential homological information on A , allowing to determinate A and $\text{mod } A$ completely. Recall that a component \mathcal{C} of an Auslander-Reiten quiver Γ_A is called *regular* if \mathcal{C} contains neither a projective module nor an injective module, and *semi-regular* if \mathcal{C} does not contain both a projective module and an injective module. By general theory (see [26], [27], [62]), every regular component \mathcal{C} of Γ_A is either of the form $\mathbb{Z}\Delta$, for a locally finite acyclic valued quiver Δ , or a *stable tube* $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$. More generally (see [27]), every semi-regular component \mathcal{C} of Γ_A is either a full translation subquiver of such a translation quiver $\mathbb{Z}\Delta$ (acyclic case) or is a *ray tube*, obtained from a stable tube by a finite number (possibly empty)

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of ray insertions, or a *coray tube*, obtained from a stable tube by a finite number (possibly empty) of coray insertions.

A prominent role in the representation theory of algebras is played by the generalized standard Auslander-Reiten components. Following [47] a component \mathcal{C} of an Auslander-Reiten quiver Γ_A is called *generalized standard* if $\text{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in \mathcal{C} . It has been proved in [47] that every generalized standard component \mathcal{C} of Γ_A is *almost periodic*, that is, all but finitely many τ_A -orbits in \mathcal{C} are periodic. Distinguished classes of generalized standard components are formed by the Auslander-Reiten quivers of all algebras of finite representation type, the connecting components of tilted algebras [17] (respectively, double tilted algebras [34], generalized double tilted algebras [35]), the separating families of tubes of quasi-tilted algebras of canonical type [24], [25], or more generally, separating families of almost cyclic coherent components of generalized multicoil algebras [30]. The acyclic generalized standard components have been described completely in [46]. In particular, the regular acyclic generalized standard components are exactly the connecting components of tilted algebras given by regular tilting modules [47]. On the other hand, the description of the support algebras of arbitrary generalized standard components is an exciting but difficult problem (see [53], [55]). Namely, it is shown in [53] and [55] that every algebra Λ over a field K is a factor algebra of an algebra A (even symmetric algebra) with Γ_A having a sincere generalized standard stable tube. Another interesting open problem is to find handy criteria for an almost periodic Auslander-Reiten component to be generalized standard. For stable tubes (respectively, almost acyclic components) such handy criteria have been established in [39], [47], [53] (respectively, [29], [34], [35], [46]).

In this paper we are concerned with the structure of components of the Auslander-Reiten quiver Γ_A of an algebra A having an ordered interaction with other components of Γ_A . Following [33], by an *external short path* of a component \mathcal{C} of Γ_A we mean a sequence $X \rightarrow Y \rightarrow Z$ of non-zero non-isomorphisms between indecomposable modules in $\text{mod } A$ with X and Z in \mathcal{C} but Y not in \mathcal{C} . We mention that every component \mathcal{C} of Γ_A without external short paths has the important property: the additive category $\text{add}(\mathcal{C})$ of \mathcal{C} is closed under extensions in $\text{mod } A$. For a component \mathcal{C} of Γ_A , we denote by $\text{ann}_A(\mathcal{C})$ the *annihilator* of \mathcal{C} in A , that is, the intersection of the annihilators $\text{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all modules X in \mathcal{C} . A component \mathcal{C} of Γ_A with $\text{ann}_A(\mathcal{C}) = 0$ is called *faithful*. We note that $\text{ann}_A(\mathcal{C})$ is an ideal of A , \mathcal{C} is a faithful component of $\Gamma_{A/\text{ann}_A(\mathcal{C})}$, and every simple right $(A/\text{ann}_A(\mathcal{C}))$ -module occurs as a composition factor of a module in \mathcal{C} .

We are now in a position to formulate the main results of the paper.

Theorem 1.1. *Let A be an algebra, \mathcal{C} a component of Γ_A without projective modules and external short paths, and $B = A/\text{ann}_A(\mathcal{C})$. Then one of the following statements holds:*

- (i) *B is a tilted algebra of the form $\text{End}_H(T)$, where H is a hereditary algebra and T is a tilting H -module without non-zero preinjective direct summands, and \mathcal{C} is the connecting component \mathcal{C}_T of Γ_B determined by T .*
- (ii) *B is the opposite algebra of an almost concealed canonical algebra and \mathcal{C} is a faithful coray tube of a separating family of coray tubes of Γ_B .*

Theorem 1.2. *Let A be an algebra, \mathcal{C} a component of Γ_A without injective modules and external short paths, and $B = A/\text{ann}_A(\mathcal{C})$. Then one of the following statements holds:*

- (i) B is a tilted algebra of the form $\text{End}_H(T)$, where H is a hereditary algebra and T is a tilting H -module without non-zero preprojective direct summands, and \mathcal{C} is the connecting component \mathcal{C}_T of Γ_B determined by T .
- (ii) B is an almost concealed canonical algebra and \mathcal{C} is a faithful ray tube of a separating family of ray tubes of Γ_B .

Corollary 1.3. *Let A be an algebra, \mathcal{C} a regular component of Γ_A without external short paths, and $B = A/\text{ann}_A(\mathcal{C})$. Then one of the following statements holds:*

- (i) B is a tilted algebra of the form $\text{End}_H(T)$, where H is a hereditary algebra and T is a regular tilting H -module, and \mathcal{C} is the connecting component \mathcal{C}_T of Γ_B determined by T .
- (ii) B is a concealed canonical algebra and \mathcal{C} is a stable tube of a separating family of stable tubes of Γ_B .

We would like to mention that, by a result of Ringel [41], a hereditary algebra H admits a regular tilting module T if and only if H is neither of Dynkin type nor Euclidean type and has at least three pairwise non-isomorphic simple modules (see also [6], [7], [44, Section XVIII.5] for constructions of regular tilting modules over wild hereditary algebras). We refer to [22] for constructions of regular connecting components of tilted algebras and stable tubes of concealed canonical algebras having all indecomposable modules with every simple module occurring arbitrary many times as a composition factor. Moreover, we refer to [50] for results on the composition factors of modules in generalized standard stable tubes.

As an immediate consequence of Theorems 1.1 and 1.2 we obtain the following fact.

Corollary 1.4. *Let A be an algebra and \mathcal{C} be a semi-regular component of Γ_A without external short paths. Then \mathcal{C} is a generalized standard component of Γ_A .*

We exhibit in Section 2 (Example 2.6) an Auslander-Reiten component without external short paths which is not generalized standard. It would be interesting to know when an Auslander-Reiten component without external short paths is generalized standard. This question can be interpreted in the following way. Following [48], a component quiver Σ_A of an algebra A has the components of Γ_A as the vertices and two components \mathcal{C} and \mathcal{D} of Γ_A are linked in Σ_A by an arrow $\mathcal{C} \rightarrow \mathcal{D}$ if $\text{rad}_A^\infty(X, Y) \neq 0$ for some modules X in \mathcal{C} and Y in \mathcal{D} . In particular, a component \mathcal{C} of Γ_A is generalized standard if and only if Σ_A has no loop at \mathcal{C} . Therefore, we ask when, for a component \mathcal{C} of Γ_A , the absence of short cycles $\mathcal{C} \rightleftarrows \mathcal{D}$ in Σ_A with $\mathcal{C} \neq \mathcal{D}$ forces the absence of loop $\mathcal{C} \circlearrowleft$ at \mathcal{C} in Σ_A .

It has been proved in [47, Theorem 3.6] that the Auslander-Reiten quiver Γ_A of an algebra A has at most finitely many acyclic regular generalized standard components. Hence, we obtain the following immediate consequence of the above results.

Corollary 1.5. *Let A be an algebra. Then all but finitely many components of Γ_A without external short paths are stable tubes.*

A crucial role in our proofs of Theorems 1.1 and 1.2 is played by the following theorem describing the components without external short paths in the Auslander-Reiten quivers of quasi-tilted algebras. Recall that the quasi-tilted algebras are those of the form $\text{End}_{\mathcal{H}}(T)$ for tilting objects T in hereditary abelian Ext-finite

categories \mathcal{H} , or equivalently, the algebras Λ of global dimension at most two and with every indecomposable module in $\text{mod } \Lambda$ of the projective dimension or the injective dimension at most one [16].

Theorem 1.6. *Let A be a quasi-tilted algebra and \mathcal{C} a component of Γ_A . The following statements are equivalent:*

- (i) \mathcal{C} has no external short path.
- (ii) \mathcal{C} is almost periodic.
- (iii) \mathcal{C} is generalized standard.
- (iv) \mathcal{C} is either a preprojective component, a preinjective component, a ray tube, a coray tube, or a connecting component (in case A is a tilted algebra).

We would like to mention that, in general, there are many generalized standard (even faithful) components having external short paths, for example the faithful generalized standard stable tubes over generalized canonical (but not canonical) algebras introduced in [53] (see also [55]). On the other hand, Theorem 1.6 leads to a similar characterization of components without external short paths in the Auslander-Reiten quivers of generalized double tilted algebras investigated in [34], [35]. In particular, it is the case for all algebras A with all but finitely many indecomposable modules in $\text{mod } A$ of the projective dimension (respectively, injective dimension) at most one [49]. We refer also to [54] for a characterization of the class of algebras consisting of the quasi-tilted algebras and generalized double tilted algebras. Finally, we also mention that Theorem 1.6 leads to a similar characterization of components without external short paths in the Auslander-Reiten quivers of algebras having separating families of almost cyclic coherent components, where the connecting components have to be replaced by the generalized multicoils (see [30] for details). On the other hand, it is not clear if every Auslander-Reiten component without external short paths is almost periodic.

From Drozd's Tame and Wild Theorem [10] the class of finite dimensional algebras over an algebraically closed field K may be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d , in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory 'comprises' the representation theories of all finite dimensional algebras over K . Among the tame algebras we may distinguish the classes of algebras of polynomial growth [45] (respectively, domestic [9], [45]) for which there exists a positive integer m such that the indecomposable modules occur, in each dimension d , in a finite number of discrete and at most d^m (respectively, at most m) one-parameter families. Moreover, it has been proved by Crawley-Boevey [8] that, for a tame algebra A , all but finitely many indecomposable A -modules of any fixed dimension d lie in stable tubes of rank one. We refer to [44, Chapter XIX] for precise definitions and properties of tame and wild algebras.

The following result is an immediate consequence of Corollary 1.3, Theorem 1.6, the above theorem of Crawley-Boevey, and the fact that the tilted algebras given by regular tilting modules and the concealed canonical algebras of wild type are wild.

Corollary 1.7. *Let A be a finite dimensional tame algebra over an algebraically closed field K and \mathcal{C} a regular component of Γ_A without external short paths. Then \mathcal{C} is a stable tube and $A/\text{ann}_A(\mathcal{C})$ is a tame concealed algebra or a tubular algebra.*

Moreover, we have also the following consequence of Corollary 1.3, Theorem 1.6, [8], [39], [45, Lemma 3.6] and [52, Theorem A and Corollary B].

Corollary 1.8. *Let A be a finite dimensional tame algebra over an algebraically closed field K such that no component of Γ_A has an external short path. Then the following facts hold.*

- (i) A is of polynomial growth.
- (ii) A is a domestic algebra if and only if all but finitely many components of Γ_A are stable tubes of rank one.

It would be important to know if a finite dimensional algebra A over an algebraically closed field K with every component in Γ_A without external short paths is actually a tame algebra. On the other hand, if for such an algebra A every component in Γ_A is generalized standard, then $\text{rad}_A^\infty(X, X) = 0$ for any indecomposable module X in $\text{mod } A$, and consequently A is a tame algebra (see [51, Proposition 3.3]). Hence we obtain, by Corollary 1.8, the following fact.

Corollary 1.9. *Let A be a finite dimensional algebra over an algebraically closed field K such that the component quiver Σ_A of A has no short cycles $\circ \rightleftarrows \circ$ and no loops $\circ \curvearrowright$. Then A is a tame algebra of polynomial growth.*

As an application of Corollary 1.3 and results established in [12], [37] and [56], [57], [58], we obtain a complete description of all self-injective algebras whose Auslander-Reiten quiver admits a regular acyclic component without external short paths.

Theorem 1.10. *Let A be a self-injective algebra. The following statements are equivalent.*

- (i) Γ_A admits a regular acyclic component \mathcal{C} without external short paths.
- (ii) A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^2)$, where \widehat{B} is the repetitive category of a tilted algebra B of the form $\text{End}_H(T)$, for some hereditary algebra H and a regular tilting H -module T , $\nu_{\widehat{B}}$ is the Nakayama automorphism of \widehat{B} , and φ is a positive automorphism of \widehat{B} .

We refer to [12] for the representation theory of orbit algebras \widehat{B}/G of the repetitive categories \widehat{B} of tilted algebras B of wild type and infinite cyclic automorphism groups G of \widehat{B} . We also note that the problem of describing the self-injective algebras whose Auslander-Reiten quiver admits a stable tube without external short paths is more difficult, because the stable tubes occur in families of quasi-tubes. We refer to [18] for a wide class of self-injective algebras having infinitely many stable tubes without external short paths.

The paper is organized as follows. In Section 2 we prove Theorem 1.6 and recall the related background on tilted algebras and quasi-tilted algebras of canonical type. In Section 3 we provide the proofs of Theorems 1.1 and 1.2, showing that every algebra whose Auslander-Reiten quiver admits a faithful semi-regular component without external short paths is a quasi-tilted algebra, and then applying Theorem 1.6. The final Section 4 is devoted to the proof of Theorem 1.10 and the related background on the orbit algebras of repetitive categories of algebras.

For background on the representation theory we refer to the books [1], [5], [14], [39], [43] and [44].

2. PROOF OF THEOREM 1.6

In the proof of Theorem 1.6 we need translation subquivers of the Auslander-Reiten quivers of special type. Let A be an algebra, \mathcal{C} a component of Γ_A and M an indecomposable module in \mathcal{C} . Then the *left cone* ($\rightarrow M$) of M is the full translation subquiver of \mathcal{C} formed by all predecessors of M in \mathcal{C} and the *right cone* ($M \rightarrow$) of M is the full translation subquiver of \mathcal{C} formed by all successors of M in \mathcal{C} .

It has been proved in [15] that the class of quasi-tilted algebras consists of the tilted algebras [17] (endomorphism algebras of tilting modules over hereditary algebras) and the quasi-tilted algebras of canonical type [25] (endomorphism algebras of tilting objects in hereditary abelian categories whose bounded derived category is equivalent to the bounded derived category of a canonical algebra in the sense of Ringel [39], [42]). Accordingly, we will divide the proof of Theorem 1.6 into two cases: the tilted case and the canonical case.

Let H be a hereditary algebra, Q_H the valued quiver of H , and T a multiplicity-free tilting module in $\text{mod } H$, that is, $\text{Ext}_H^1(T, T) = 0$ and T is a direct sum of n pairwise non-isomorphic indecomposable H -modules with n the rank of the Grothendieck group $K_0(H)$ of H . Consider the associated tilted algebra $B = \text{End}_H(T)$ of type Q_H . Then the tilting module T determines the torsion pair $(\mathcal{F}(T), \mathcal{T}(T))$ in $\text{mod } H$, with the torsion-free part $\mathcal{F}(T) = \{X \in \text{mod } H \mid \text{Hom}_H(T, X) = 0\}$ and the torsion part $\mathcal{T}(T) = \{X \in \text{mod } H \mid \text{Ext}_H^1(T, X) = 0\}$, and the splitting torsion pair $(\mathcal{Y}(T), \mathcal{X}(T))$ in $\text{mod } B$, with the torsion-free part $\mathcal{Y}(T) = \{Y \in \text{mod } B \mid \text{Tor}_1^B(Y, T) = 0\}$ and the torsion part $\mathcal{X}(T) = \{Y \in \text{mod } B \mid Y \otimes_B T = 0\}$. Moreover, by the Brenner-Butler theorem, the functor $\text{Hom}_H(T, -) : \text{mod } H \rightarrow \text{mod } B$ induces an equivalence of $\mathcal{T}(T)$ with $\mathcal{Y}(T)$, and the functor $\text{Ext}_H^1(T, -) : \text{mod } H \rightarrow \text{mod } B$ induces an equivalence of $\mathcal{F}(T)$ with $\mathcal{X}(T)$ (see [1], [17]). Further, the images $\text{Hom}_H(T, I)$ of the indecomposable injective modules I in $\text{mod } H$ via the functor $\text{Hom}_H(T, -)$ belong to one component \mathcal{C}_T of Γ_B , called the *connecting component* of Γ_B determined by T , and form a faithful section $\Delta_T \cong Q_H^{\text{op}}$ of \mathcal{C}_T . The section Δ_T of \mathcal{C}_T has the distinguished property: it connects the torsion-free part $\mathcal{Y}(T)$ with the torsion part $\mathcal{X}(T)$, because every indecomposable predecessor of a module $\text{Hom}_H(T, I)$ from Δ_T in $\text{mod } B$ lies in $\mathcal{Y}(T)$ and every indecomposable successor of a module $\tau_B^- \text{Hom}_H(T, I)$ in $\text{mod } B$ lies in $\mathcal{X}(T)$. We have also the following properties of the connecting component \mathcal{C}_T established in [40]:

- \mathcal{C}_T contains a projective module if and only if T admits a preinjective indecomposable direct summand;
- \mathcal{C}_T contains an injective module if and only if T admits a preprojective indecomposable direct summand;
- \mathcal{C}_T is regular if and only if T is regular.

We also mention that the Auslander-Reiten quiver Γ_H of H has the decomposition

$$\Gamma_H = \mathcal{P}(H) \vee \mathcal{R}(H) \vee \mathcal{Q}(H),$$

where $\mathcal{P}(H)$ is the preprojective component containing all indecomposable projective H -modules, $\mathcal{Q}(H)$ is the preinjective component containing all indecomposable injective H -modules, and $\mathcal{R}(H)$ is the family of all regular components. Moreover, we have

- If Q_H is a Dynkin quiver, then $\mathcal{R}(H)$ is empty and $\mathcal{P}(H) = \mathcal{Q}(H)$.
- If Q_H is a Euclidean quiver, then $\mathcal{P}(H) \cong (-\mathbb{N})Q_H^{\text{op}}$, $\mathcal{Q}(H) \cong \mathbb{N}Q_H^{\text{op}}$ and $\mathcal{R}(H)$ is an infinite family of pairwise orthogonal generalized standard stable tubes [39], [43].
- If Q_H is a wild quiver, then $\mathcal{P}(H) \cong (-\mathbb{N})Q_H^{\text{op}}$, $\mathcal{Q}(H) \cong \mathbb{N}Q_H^{\text{op}}$ and $\mathcal{R}(H)$ is an infinite family of regular components of type $\mathbb{Z}\mathbb{A}_\infty$ [3], [38], [44].

Let H be a hereditary algebra not of Dynkin type and T a multiplicity-free tilting H -module from the additive category $\text{add}(\mathcal{P}(H))$ of the preprojective component $\mathcal{P}(H)$ of Γ_H . Then $B = \text{End}_H(T)$ is called a *concealed algebra* of type Q_H . A concealed algebra $B = \text{End}_H(T)$ is called a *tame concealed algebra* if Q_H is a Euclidean quiver, and a *wild concealed algebra* if Q_H is a wild quiver.

The following fact proved by Baer [6] (see also [44, Theorem XVIII.2.6]) will be important for our considerations.

Lemma 2.1. *Let H be a wild hereditary algebra, and X, Y be two indecomposable modules in $\mathcal{R}(H)$. Then there is a positive integer m such that $\text{Hom}_H(X, \tau_H^r Y) \neq 0$ for all integers $r \geq m$.*

We will prove now Theorem 1.6 in the tilted case.

Proposition 2.2. *Let H be a hereditary algebra, T a multiplicity-free tilting H -module, $B = \text{End}_H(T)$ the associated tilted algebra, and \mathcal{C} a component of Γ_B . The following statements are equivalent:*

- \mathcal{C} has no external short path.
- \mathcal{C} is almost periodic.
- \mathcal{C} is generalized standard.
- \mathcal{C} is either a preprojective component, a preinjective component, a ray tube, a coray tube, or the connecting component \mathcal{C}_T .

Proof. We start with the general view on the module category $\text{mod } B$ due to results established in [19], [20], [21], [28], [61]. Let $\Delta = \Delta_T$ be the canonical section of the connecting component \mathcal{C}_T determined by T . Hence, $\Delta = Q^{\text{op}}$ for $Q = Q_H$. Then \mathcal{C}_T admits a finite (possibly empty) family of pairwise disjoint full translation (valued) subquivers

$$\mathcal{D}_1^{(l)}, \dots, \mathcal{D}_m^{(l)}, \mathcal{D}_1^{(r)}, \dots, \mathcal{D}_n^{(r)}$$

such that the following statements hold:

- For each $i \in \{1, \dots, m\}$, there is an isomorphism of translation quivers $\mathcal{D}_i^{(l)} \cong \mathbb{N}\Delta_i^{(l)}$, where $\Delta_i^{(l)}$ is a connected full valued subquiver of Δ , and $\mathcal{D}_i^{(l)}$ is closed under predecessors in \mathcal{C}_T .
- For each $j \in \{1, \dots, n\}$, there is an isomorphism of translation quivers $\mathcal{D}_j^{(r)} \cong (-\mathbb{N})\Delta_j^{(r)}$, where $\Delta_j^{(r)}$ is a connected full valued subquiver of Δ , and $\mathcal{D}_j^{(r)}$ is closed under successors in \mathcal{C}_T .
- All but finitely many indecomposable modules of \mathcal{C}_T lie in

$$\mathcal{D}_1^{(l)} \cup \dots \cup \mathcal{D}_m^{(l)} \cup \mathcal{D}_1^{(r)} \cup \dots \cup \mathcal{D}_n^{(r)}.$$

- (d) For each $i \in \{1, \dots, m\}$, there exists a tilted algebra $B_i^{(l)} = \text{End}_{H_i^{(l)}}(T_i^{(l)})$, where $H_i^{(l)}$ is a hereditary algebra of type $(\Delta_i^{(l)})^{\text{op}}$ and $T_i^{(l)}$ is a multiplicity-free tilting $H_i^{(l)}$ -module without preinjective indecomposable direct summands such that
- $B_i^{(l)}$ is a quotient algebra of B , and hence there is a fully faithful embedding $\text{mod } B_i^{(l)} \hookrightarrow \text{mod } B$,
 - $\mathcal{D}_i^{(l)}$ coincides with the torsion-free part $\mathcal{Y}(T_i^{(l)}) \cap \mathcal{C}_{T_i^{(l)}}$ of the connecting component $\mathcal{C}_{T_i^{(l)}}$ of $\Gamma_{B_i^{(l)}}$ determined by $T_i^{(l)}$.
- (e) For each $j \in \{1, \dots, n\}$, there exists a tilted algebra $B_j^{(r)} = \text{End}_{H_j^{(r)}}(T_j^{(r)})$, where $H_j^{(r)}$ is a hereditary algebra of type $(\Delta_j^{(r)})^{\text{op}}$ and $T_j^{(r)}$ is a multiplicity-free tilting $H_j^{(r)}$ -module without preprojective indecomposable direct summands such that
- $B_j^{(r)}$ is a quotient algebra of B , and hence there is a fully faithful embedding $\text{mod } B_j^{(r)} \hookrightarrow \text{mod } B$,
 - $\mathcal{D}_j^{(r)}$ coincides with the torsion part $\mathcal{X}(T_j^{(r)}) \cap \mathcal{C}_{T_j^{(r)}}$ of the connecting component $\mathcal{C}_{T_j^{(r)}}$ of $\Gamma_{B_j^{(r)}}$ determined by $T_j^{(r)}$.
- (f) $\mathcal{Y}(T) = \text{add}(\mathcal{Y}(T_1^{(l)}) \cup \dots \cup \mathcal{Y}(T_m^{(l)}) \cup (\mathcal{Y}(T) \cap \mathcal{C}_T))$.
- (g) $\mathcal{X}(T) = \text{add}((\mathcal{X}(T) \cap \mathcal{C}_T) \cup \mathcal{X}(T_1^{(r)}) \cup \dots \cup \mathcal{X}(T_n^{(r)}))$.
- (h) The Auslander-Reiten quiver Γ_B has the disjoint union form

$$\Gamma_B = \left(\bigcup_{i=1}^m \mathcal{Y}\Gamma_{B_i^{(l)}} \right) \cup \mathcal{C}_T \cup \left(\bigcup_{j=1}^n \mathcal{X}\Gamma_{B_j^{(r)}} \right),$$

where

- for each $i \in \{1, \dots, m\}$, $\mathcal{Y}\Gamma_{B_i^{(l)}}$ is the union of all components of $\Gamma_{B_i^{(l)}}$ contained entirely in $\mathcal{Y}(T_i^{(l)})$,
- for each $j \in \{1, \dots, n\}$, $\mathcal{X}\Gamma_{B_j^{(r)}}$ is the union of all components of $\Gamma_{B_j^{(r)}}$ contained entirely in $\mathcal{X}(T_j^{(r)})$.

Moreover, we have the following description of the components of Γ_B contained in the parts $\mathcal{Y}\Gamma_{B_i^{(l)}}$ and $\mathcal{X}\Gamma_{B_j^{(r)}}$:

- (1) If $\Delta_i^{(l)}$ is a Euclidean quiver, then $\mathcal{Y}\Gamma_{B_i^{(l)}}$ consists of a unique preprojective component $\mathcal{P}(B_i^{(l)})$ of $\Gamma_{B_i^{(l)}}$ and an infinite family $\mathcal{T}^{B_i^{(l)}}$ of pairwise orthogonal generalized standard ray tubes. Further, $\mathcal{P}(B_i^{(l)})$ coincides with the preprojective component $\mathcal{P}(C_i^{(l)})$ of a tame concealed quotient algebra $C_i^{(l)}$ of $B_i^{(l)}$.
- (2) If $\Delta_i^{(l)}$ is a wild quiver, then $\mathcal{Y}\Gamma_{B_i^{(l)}}$ consists of a unique preprojective component $\mathcal{P}(B_i^{(l)})$ of $\Gamma_{B_i^{(l)}}$ and an infinite family of components obtained from the components of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of ray insertions. Further, $\mathcal{P}(B_i^{(l)})$ coincides with the preprojective component $\mathcal{P}(C_i^{(l)})$ of a wild concealed quotient algebra $C_i^{(l)}$ of $B_i^{(l)}$.

- (3) If $\Delta_j^{(r)}$ is a Euclidean quiver, then $\mathcal{X}\Gamma_{B_j^{(r)}}$ consists of a unique preinjective component $\mathcal{Q}(B_j^{(r)})$ of $\Gamma_{B_j^{(r)}}$ and an infinite family of pairwise orthogonal generalized standard coray tubes. Further, $\mathcal{Q}(B_j^{(r)})$ coincides with the preinjective component $\mathcal{Q}(C_j^{(r)})$ of a tame concealed quotient algebra $C_j^{(r)}$ of $B_j^{(r)}$.
- (4) If $\Delta_j^{(r)}$ is a wild quiver, then $\mathcal{X}\Gamma_{B_j^{(r)}}$ consists of a unique preinjective component $\mathcal{Q}(B_j^{(r)})$ of $\Gamma_{B_j^{(r)}}$ and an infinite family of components obtained from the components of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of coray insertions. Further, $\mathcal{Q}(B_j^{(r)})$ coincides with the preinjective component $\mathcal{Q}(C_j^{(r)})$ of a wild concealed quotient algebra $C_j^{(r)}$ of $B_j^{(r)}$.

It follows from the above facts that the preprojective components, preinjective components, ray tubes and coray tubes of Γ_B are generalized standard, without external short paths, and clearly are almost periodic. On the other hand, the components of Γ_B obtained from the components of the form $\mathbb{Z}\mathbb{A}_\infty$ by ray insertions or coray insertions are not almost periodic, and hence are not generalized standard, by the general result [47, Theorem 2.3]. Therefore, it remains to show that all these components have external short paths. We have two cases to consider.

Assume \mathcal{C} is an acyclic component of Γ_B with infinitely many τ_B -orbits contained in the torsion-free part $\mathcal{Y}(T)$ of $\text{mod } B$. Then it follows from (1) and (2) that there is $i \in \{1, \dots, m\}$ such that $\Delta_i^{(l)}$ is a wild quiver and \mathcal{C} is a component of the Auslander-Reiten quiver $\Gamma_{B_i^{(l)}}$ of the tilted algebra $B_i^{(l)} = \text{End}_{H_i^{(l)}}(T_i^{(l)})$ with $H_i^{(l)}$ a wild hereditary algebra of type $(\Delta_i^{(l)})^{\text{op}}$ and $T_i^{(l)}$ a multiplicity-free tilting $H_i^{(l)}$ -module without preinjective indecomposable direct summands. Since $\mathcal{Y}\Gamma_{B_i^{(l)}}$ contains infinitely many components different from the preprojective component $\mathcal{P}(B_i^{(l)})$, we may choose a regular component \mathcal{D} in $\mathcal{Y}\Gamma_{B_i^{(l)}}$ different from \mathcal{C} . Clearly, \mathcal{D} is of the form $\mathbb{Z}\mathbb{A}_\infty$. Now it follows from [20, Theorem 1] that there exist regular components $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ in $\Gamma_{H_i^{(l)}}$ and indecomposable modules $X \in \mathcal{C}$, $Y \in \mathcal{D}$, $\tilde{X} \in \tilde{\mathcal{C}}$ and $\tilde{Y} \in \tilde{\mathcal{D}}$ such that the functor $F_i^{(l)} = \text{Hom}_{H_i^{(l)}}(T_i^{(l)}, -) : \text{mod } H_i^{(l)} \rightarrow \text{mod } B_i^{(l)}$ induces equivalences of the additive categories of the left cones $F_i^{(l)} : \text{add}(\rightarrow \tilde{X}) \xrightarrow{\sim} \text{add}(\rightarrow X)$ and $F_i^{(l)} : \text{add}(\rightarrow \tilde{Y}) \xrightarrow{\sim} \text{add}(\rightarrow Y)$. Moreover, we have $F_i^{(l)}(\tau_{H_i^{(l)}} M) = \tau_{B_i^{(l)}} F_i^{(l)}(M)$ and $F_i^{(l)}(\tau_{H_i^{(l)}} N) = \tau_{B_i^{(l)}} F_i^{(l)}(N)$ for all modules M in $(\rightarrow \tilde{X})$ and N in $(\rightarrow \tilde{Y})$. Applying now Lemma 2.1, we obtain that there exist positive integers r and s such that $\text{Hom}_{H_i^{(l)}}(\tilde{X}, \tau_{H_i^{(l)}}^r \tilde{Y}) \neq 0$ and $\text{Hom}_{H_i^{(l)}}(\tau_{H_i^{(l)}}^r \tilde{Y}, \tau_{H_i^{(l)}}^s \tilde{X}) \neq 0$. Hence we get $\text{Hom}_{B_i^{(l)}}(X, \tau_{B_i^{(l)}}^r Y) \neq 0$ and $\text{Hom}_{B_i^{(l)}}(\tau_{B_i^{(l)}}^r Y, \tau_{B_i^{(l)}}^s X) \neq 0$, and consequently an external short path $X \rightarrow \tau_{B_i^{(l)}}^r Y \rightarrow \tau_{B_i^{(l)}}^s X$ of \mathcal{C} in $\text{mod } B_i^{(l)}$, and so in $\text{mod } B$, because $B_i^{(l)}$ is a quotient algebra of B and there is a fully faithful embedding $\text{mod } B_i^{(l)} \hookrightarrow \text{mod } B$.

Assume \mathcal{C} is an acyclic component with infinitely many τ_B -orbits contained in the torsion part $\mathcal{X}(T)$ of $\text{mod } B$. Then it follows from (3) and (4) that there

is $j \in \{1, \dots, n\}$ such that $\Delta_j^{(r)}$ is a wild quiver and \mathcal{C} is a component of the Auslander-Reiten quiver $\Gamma_{B_j^{(r)}}$ of the tilted algebra $B_j^{(r)} = \text{End}_{H_j^{(r)}}(T_j^{(r)})$ with $H_j^{(r)}$ a wild hereditary algebra of type $(\Delta_j^{(r)})^{\text{op}}$ and $T_j^{(r)}$ a multiplicity-free tilting $H_j^{(r)}$ -module without preprojective indecomposable direct summands. Since $\mathcal{X}\Gamma_{B_j^{(r)}}$ contains infinitely many components different from the preinjective component $\mathcal{Q}(B_j^{(r)})$, we may choose a regular component \mathcal{D} in $\mathcal{X}\Gamma_{B_j^{(r)}}$ different from \mathcal{C} . Note that \mathcal{D} is of the form $\mathbb{Z}\mathbb{A}_\infty$. We know also that the preinjective component $\mathcal{Q}(B_j^{(r)})$ coincides with the unique preinjective component $\mathcal{Q}(C_j^{(r)})$ of a wild concealed quotient algebra $C_j^{(r)}$ of $B_j^{(r)}$. Then $C_j^{(r)} = \text{End}_{\Lambda_j^{(r)}}(V_j^{(r)})$, where $\Lambda_j^{(r)}$ is a wild hereditary algebra and $V_j^{(r)}$ is a multiplicity-free tilting module from $\text{add}(\mathcal{P}(\Lambda_j^{(r)}))$. In particular, the functor $\text{Hom}_{\Lambda_j^{(r)}}(V_j^{(r)}, -) : \text{mod } \Lambda_j^{(r)} \rightarrow \text{mod } C_j^{(r)}$ induces an equivalence $\text{add}(\mathcal{R}(\Lambda_j^{(r)})) \xrightarrow{\sim} \text{add}(\mathcal{R}(C_j^{(r)}))$ of the categories of regular modules over $\Lambda_j^{(r)}$ and $C_j^{(r)}$. Applying Lemma 2.1, we conclude that for any indecomposable modules M and N in $\mathcal{R}(C_j^{(r)})$ there exists a positive integer p such that $\text{Hom}_{C_j^{(r)}}(M, \tau_{C_j^{(r)}}^s N) \neq 0$ for all integers $s \geq p$. On the other hand, it follows from [20, Theorem 1] that there exist indecomposable modules $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that the left cones $(\rightarrow X)$ of \mathcal{C} and $(\rightarrow Y)$ of \mathcal{D} consist entirely of indecomposable $C_j^{(r)}$ -modules and the restriction of $\tau_{B_j^{(r)}}$ to the left cones $(\rightarrow X)$ and $(\rightarrow Y)$ coincides with $\tau_{C_j^{(r)}}$. Hence, the left cone $(\rightarrow X)$ of \mathcal{C} is the left cone $(\rightarrow \tilde{X})$, with $\tilde{X} = X$, of a component $\tilde{\mathcal{C}}$ of type $\mathbb{Z}\mathbb{A}_\infty$ of $\Gamma_{C_j^{(r)}}$, and the left cone $(\rightarrow Y)$ of \mathcal{D} is the left cone $(\rightarrow \tilde{Y})$, with $\tilde{Y} = Y$, of a component $\tilde{\mathcal{D}}$ of type $\mathbb{Z}\mathbb{A}_\infty$ of $\Gamma_{C_j^{(r)}}$. Observe that $\tilde{\mathcal{C}} \neq \tilde{\mathcal{D}}$ since $\mathcal{C} \neq \mathcal{D}$. Then there exist positive integers p and q such that $\text{Hom}_{B_j^{(r)}}(X, \tau_{B_j^{(r)}}^p Y) = \text{Hom}_{C_j^{(r)}}(X, \tau_{C_j^{(r)}}^p Y) \neq 0$ and $\text{Hom}_{B_j^{(r)}}(\tau_{B_j^{(r)}}^p Y, \tau_{B_j^{(r)}}^q X) = \text{Hom}_{C_j^{(r)}}(\tau_{C_j^{(r)}}^p Y, \tau_{C_j^{(r)}}^q X) \neq 0$. Therefore, we get an external short path $X \rightarrow \tau_{B_j^{(r)}}^p Y \rightarrow \tau_{B_j^{(r)}}^q X$ of \mathcal{C} in $\text{mod } B_j^{(r)}$, and so in $\text{mod } B$, because $B_j^{(r)}$ is a quotient algebra of B and there is a fully faithful embedding $\text{mod } B_j^{(r)} \hookrightarrow \text{mod } B$.

We note that although the proofs in the two considered cases are similar, the applied results concerning the structure of left cones of acyclic components in $\mathcal{Y}(T)$ and $\mathcal{X}(T)$ are different. \square

Let Λ be a canonical algebra in the sense of Ringel [42] (see also [39]). Then the valued quiver Q_Λ of Λ has a unique sink and a unique source. Denote by Q_Λ^* the valued quiver obtained from Q_Λ by removing the unique source of Q_Λ and the arrows attached to it. Then Λ is said to be a *canonical algebra of Euclidean type* (respectively, of *tubular type*, of *wild type*) if Q_Λ^* is a Dynkin quiver (respectively, a Euclidean quiver, a wild quiver). We refer to [60, Theorems 3.1 and 3.2] for the shapes of the valued quivers of canonical algebras of Euclidean and tubular type. The general shape of the Auslander-Reiten quiver Γ_Λ of Λ , described in [42,

Sections 3 and 4], is as follows:

$$\Gamma_\Lambda = \mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda \vee \mathcal{Q}^\Lambda,$$

where \mathcal{P}^Λ is a family of components containing a unique preprojective component $\mathcal{P}(\Lambda)$ and all indecomposable projective Λ -modules, \mathcal{Q}^Λ is a family of components containing a unique preinjective component $\mathcal{Q}(\Lambda)$ and all indecomposable injective Λ -modules, and \mathcal{T}^Λ is an infinite family of pairwise orthogonal generalized standard faithful stable tubes separating \mathcal{P}^Λ from \mathcal{Q}^Λ , and with all but finitely many stable tubes of rank one. An algebra C of the form $\text{End}_\Lambda(T)$, where T is a multiplicity-free tilting module from the additive category $\text{add}(\mathcal{P}^\Lambda)$ of \mathcal{P}^Λ is said a *concealed canonical algebra* of type Λ . More generally, an algebra B of the form $\text{End}_\Lambda(T)$, where T is a multiplicity-free tilting module from the additive category $\text{add}(\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda)$ of $\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda$ is said to be an *almost concealed canonical algebra* of type Λ .

We note the following statements:

- The class of concealed canonical algebras of Euclidean types coincides with the class of concealed algebras of Euclidean types (tame concealed algebras).
- The class of almost concealed canonical algebras of Euclidean types coincides with the class of tilted algebras of the form $\text{End}_H(T)$, where H is a hereditary algebra of a Euclidean type and T is a multiplicity-free tilting H -module without preinjective indecomposable direct summands.
- The class of the opposite algebras of almost concealed canonical algebras of Euclidean types coincides with the class of tilted algebras of the form $\text{End}_H(T)$, where H is a hereditary algebra of a Euclidean type and T is a multiplicity-free tilting H -module without preprojective indecomposable direct summands.

An almost concealed canonical algebra B of a tubular type is called a *tubular algebra*. It is known that then the opposite algebra B^{op} of B is also a tubular algebra. The shape of the Auslander-Reiten quiver Γ_B of a tubular algebra B , described by Ringel (see [39, Chapter 5], [42, Sections 3 and 4]), is as follows:

$$\Gamma_B = \mathcal{P}(B) \vee \mathcal{T}_0^B \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^B \right) \vee \mathcal{T}_\infty^B \vee \mathcal{Q}(B),$$

where \mathbb{Q}^+ is the set of positive rational numbers, $\mathcal{P}(B)$ is a preprojective component with a Euclidean section, $\mathcal{Q}(B)$ is a preinjective component with a Euclidean section, \mathcal{T}_0^B is an infinite family of pairwise orthogonal generalized standard ray tubes containing at least one indecomposable projective B -module, \mathcal{T}_∞^B is an infinite family of pairwise orthogonal generalized standard coray tubes containing at least one indecomposable injective B -module, and each \mathcal{T}_q^B , for $q \in \mathbb{Q}^+$, is an infinite family of pairwise orthogonal generalized standard faithful stable tubes. Moreover, every component of Γ_B has no external short path in $\text{mod } B$.

We will need also an analogue of Lemma 2.1 for the canonical algebras of wild type.

Let Λ be a canonical algebra of wild type. Then it follows from [16], [23] and [25] that there exist hereditary abelian categories $\mathcal{H}(\Lambda)$ and $\mathcal{T}(\Lambda)$ such that the following statements hold:

- The bounded derived category $D^b(\text{mod } \Lambda)$ of Λ has a decomposition

$$D^b(\text{mod } \Lambda) = \bigvee_{m \in \mathbb{Z}} (\mathcal{H}(\Lambda)[m] \vee \mathcal{T}(\Lambda)[m]) = D^b(\mathcal{H}(\Lambda)^*)$$

with $\mathcal{H}(\Lambda) = \mathcal{H}(\Lambda)[0]$, $\mathcal{T}(\Lambda) = \mathcal{T}(\Lambda)[0]$, and $\mathcal{H}(\Lambda)^* = \text{add}(\mathcal{H}(\Lambda) \vee \mathcal{T}(\Lambda))$.

- $\mathcal{H}(\Lambda)$ is the additive category of infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty$.
- $\mathcal{T}(\Lambda)$ is the additive category of an infinite family of pairwise orthogonal generalized standard stable tubes.
- Every concealed canonical algebra C of type Λ is of the form $\text{End}_{\mathcal{H}(\Lambda)}(T)$ for a tilting object T in $\mathcal{H}(\Lambda)$.
- Every almost concealed canonical algebra B of type Λ is of the form $\text{End}_{\mathcal{H}(\Lambda)^*}(T^*)$ for a tilting object T^* in $\mathcal{H}(\Lambda)^*$.

Then the following lemma is a direct consequence of [23, Theorem 2.7].

Lemma 2.3. *Let Λ be a concealed canonical algebra of wild type, and X, Y be two indecomposable objects in $\mathcal{H}(\Lambda)$. Then there is a positive integer m such that $\text{Hom}_{\mathcal{H}(\Lambda)}(X, \tau_{\mathcal{H}(\Lambda)}^r Y) \neq 0$ for all $r \geq m$.*

We will prove now Theorem 1.6 in the canonical case.

Proposition 2.4. *Let B be a quasi-tilted algebra of canonical type and \mathcal{C} a component of Γ_B . The following statements are equivalent:*

- (i) \mathcal{C} has no external short path.
- (ii) \mathcal{C} is almost periodic.
- (iii) \mathcal{C} is generalized standard.
- (iv) \mathcal{C} is either a preprojective component, a preinjective component, a ray tube, or a coray tube.

Proof. We start with the general view on the module category $\text{mod } B$ due to results established in [25, Sections 3 and 4] and [31]. There are an almost concealed canonical algebra $B^{(r)} = \text{End}_{\Lambda^{(r)}}(T^{(r)})$ and the opposite algebra $B^{(l)}$ of an almost concealed canonical algebra $\text{End}_{\Lambda^{(l)}}(T^{(l)})$, for canonical algebras $\Lambda^{(l)}$ and $\Lambda^{(r)}$ and tilting modules $T^{(l)} \in \text{add}(\mathcal{P}^{\Lambda^{(l)}} \vee \mathcal{T}^{\Lambda^{(l)}})$ and $T^{(r)} \in \text{add}(\mathcal{P}^{\Lambda^{(r)}} \vee \mathcal{T}^{\Lambda^{(r)}})$, such that $B^{(l)}$ and $B^{(r)}$ are quotient algebras of B . Moreover, the Auslander-Reiten quiver Γ_B of B has the disjoint union form

$$\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B,$$

where

- (a) \mathcal{T}^B is a family of pairwise orthogonal generalized standard semi-regular tubes (ray and coray tubes) separating \mathcal{P}^B from \mathcal{Q}^B .
- (b) $\mathcal{P}^B = \mathcal{P}^{B^{(l)}}$ is a family of components consisting entirely of indecomposable $B^{(l)}$ -modules and containing all indecomposable projective B -modules which are not in \mathcal{T}^B .
- (c) \mathcal{P}^B contains a unique preprojective component $\mathcal{P}(B)$ of Γ_B , and $\mathcal{P}(B) = \mathcal{P}(B^{(l)})$ coincides with a unique preprojective component $\mathcal{P}(C^{(l)})$ of the Auslander-Reiten quiver $\Gamma_{C^{(l)}}$ of a connected concealed quotient algebra $C^{(l)}$ of $B^{(l)}$, and hence of B .
- (d) $\mathcal{Q}^B = \mathcal{Q}^{B^{(r)}}$ is a family of components consisting entirely of indecomposable $B^{(r)}$ -modules and containing all indecomposable injective B -modules which are not in \mathcal{T}^B .
- (e) \mathcal{Q}^B contains a unique preinjective component $\mathcal{Q}(B)$ of Γ_B , and $\mathcal{Q}(B) = \mathcal{Q}(B^{(r)})$ coincides with a unique preinjective component $\mathcal{Q}(C^{(r)})$ of the

Auslander-Reiten quiver $\Gamma_{C^{(r)}}$ of a connected concealed quotient algebra $C^{(r)}$ of $B^{(r)}$, and hence of B .

Moreover, we have the following description of components of Γ_B contained in the parts \mathcal{P}^B and \mathcal{Q}^B :

- (1) If $B^{(l)}$ is of Euclidean type, then $\mathcal{P}^B = \mathcal{P}(B)$.
- (2) If $B^{(l)}$ is of tubular type, then

$$\mathcal{P}^B = \mathcal{P}(B^{(l)}) \vee \mathcal{T}_0^{B^{(l)}} \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^{B^{(l)}} \right).$$

- (3) If $B^{(l)}$ is of wild canonical type, then every component of \mathcal{P}^B different from the preprojective component $\mathcal{P}(B) = \mathcal{P}(B^{(l)}) = \mathcal{P}(C^{(l)})$ is obtained from a component of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of ray insertions, and there are infinitely many components of this type in \mathcal{P}^B .
- (4) If $B^{(r)}$ is of Euclidean type, then $\mathcal{Q}^B = \mathcal{Q}(B)$.
- (5) If $B^{(r)}$ is of tubular type, then

$$\mathcal{Q}^B = \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^{B^{(r)}} \right) \vee \mathcal{T}_\infty^{B^{(r)}} \vee \mathcal{Q}(B^{(r)}).$$

- (6) If $B^{(r)}$ is of wild canonical type, then every component of \mathcal{Q}^B different from the preinjective component $\mathcal{Q}(B) = \mathcal{Q}(B^{(r)}) = \mathcal{Q}(C^{(r)})$ is obtained from the component of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of coray insertions, and there are infinitely many components of this type in \mathcal{Q}^B .

It follows from the above facts that the preprojective components, preinjective components, ray tubes, and coray tubes of Γ_B are generalized standard, without external short paths, and clearly are almost periodic. On the other hand, the components of Γ_B obtained from the components of the form $\mathbb{Z}\mathbb{A}_\infty$ by ray insertions or coray insertions are not almost periodic, and hence are not generalized standard, again by [47, Theorem 2.3]. Therefore, it remains to show that all these components have external short paths. We have two cases to consider.

Assume \mathcal{C} is an acyclic component of Γ_B with infinitely many τ_B -orbits contained in the part \mathcal{P}^B . Then, applying (1)-(3), we conclude that $B^{(l)}$ is of wild canonical type and \mathcal{C} is obtained from a component of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of ray insertions. Since \mathcal{P}^B contains infinitely many components we may also choose in \mathcal{P}^B a regular component \mathcal{D} (of the form $\mathbb{Z}\mathbb{A}_\infty$) different from \mathcal{C} . Then it follows from the dual of [31, Theorem 3.4] that there are components $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ of the form $\mathbb{Z}\mathbb{A}_\infty$ in the Auslander-Reiten quiver $\Gamma_{\mathcal{H}(\Lambda^{(l)})}$ of the hereditary abelian category $\mathcal{H}(\Lambda^{(l)})$, associated to the wild canonical algebra $\Lambda^{(l)}$, indecomposable modules $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, indecomposable objects $\tilde{X} \in \tilde{\mathcal{C}}$ and $\tilde{Y} \in \tilde{\mathcal{D}}$, and a functor $F^{(l)} : \mathcal{H}(\Lambda^{(l)}) \rightarrow \text{mod } B^{(l)}$ which induces equivalences of the additive categories of the left cones $F^{(l)} : \text{add}(\rightarrow \tilde{X}) \xrightarrow{\sim} \text{add}(\rightarrow X)$ and $F^{(l)} : \text{add}(\rightarrow \tilde{Y}) \xrightarrow{\sim} \text{add}(\rightarrow Y)$, such that $F^{(l)}(\tau_{\mathcal{H}(\Lambda^{(l)})} M) = \tau_{B^{(l)}} F^{(l)}(M)$ and $F^{(l)}(\tau_{\mathcal{H}(\Lambda^{(l)})} N) = \tau_{B^{(l)}} F^{(l)}(N)$ for all indecomposable objects $M \in (\rightarrow \tilde{X})$ and $N \in (\rightarrow \tilde{Y})$. Applying Lemma 2.3, we obtain that there exist positive integers r and s such that $\text{Hom}_{\mathcal{H}(\Lambda^{(l)})}(\tilde{X}, \tau_{\mathcal{H}(\Lambda^{(l)})}^r \tilde{Y}) \neq 0$ and $\text{Hom}_{\mathcal{H}(\Lambda^{(l)})}(\tau_{\mathcal{H}(\Lambda^{(l)})}^r \tilde{Y}, \tau_{\mathcal{H}(\Lambda^{(l)})}^s \tilde{X}) \neq 0$.

Hence we get $\text{Hom}_{B^{(l)}}(X, \tau_{B^{(l)}}^r Y) \neq 0$ and $\text{Hom}_{B^{(l)}}(\tau_{B^{(l)}}^r Y, \tau_{B^{(l)}}^s X) \neq 0$, and consequently an external short path $X \rightarrow \tau_{B^{(l)}}^r Y \rightarrow \tau_{B^{(l)}}^s X$ of \mathcal{C} in $\text{mod } B^{(l)}$, and hence in $\text{mod } B$, because $B^{(l)}$ is a quotient algebra of B and there is a fully faithful embedding $\text{mod } B^{(l)} \hookrightarrow \text{mod } B$.

Assume \mathcal{C} is an acyclic component of Γ_B with infinitely many τ_B -orbits contained in the part \mathcal{Q}^B . Then, applying (4)-(6), we conclude that $B^{(r)}$ is of wild canonical type and \mathcal{C} is obtained from a component of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of coray insertions. Since \mathcal{Q}^B contains infinitely many components, we may choose in \mathcal{Q}^B a regular component \mathcal{D} (of the form $\mathbb{Z}\mathbb{A}_\infty$) different from \mathcal{C} . Moreover, by [31, Theorem 6.1], the connected concealed quotient algebra $C^{(r)}$ of $B^{(r)}$ such that $\mathcal{Q}(C^{(r)})$ is a unique preinjective component of Γ_B , is a wild concealed algebra. Hence, $C^{(r)} = \text{End}_{H^{(r)}}(T^{(r)})$ for a wild hereditary algebra $H^{(r)}$ and a multiplicity-free tilting $H^{(r)}$ -module from the additive category $\text{add}(\mathcal{Q}(H^{(r)}))$ of the preinjective component $\mathcal{Q}(H^{(r)})$ of $\Gamma_{H^{(r)}}$. In particular, the functor $\text{Ext}_{H^{(r)}}^1(T^{(r)}, -) : \text{mod } H^{(r)} \rightarrow \text{mod } C^{(r)}$ induces an equivalence $\text{add}(\mathcal{R}(H^{(r)})) \xrightarrow{\sim} \text{add}(\mathcal{R}(C^{(r)}))$ of the categories of regular modules over $H^{(r)}$ and $C^{(r)}$. Applying Lemma 2.1, we conclude that for any indecomposable modules M and N in $\mathcal{R}(C^{(r)})$ there exists a positive integer p such that $\text{Hom}_{C^{(r)}}(M, \tau_{C^{(r)}}^s N) \neq 0$ for all integers $s \geq p$. On the other hand, it follows from [31, Theorem 6.4] that there exist indecomposable modules $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that the left cones $(\rightarrow X)$ of \mathcal{C} and $(\rightarrow Y)$ of \mathcal{D} consist entirely of indecomposable $C^{(r)}$ -modules and the restriction of $\tau_{B^{(r)}}$ to the left cones $(\rightarrow X)$ and $(\rightarrow Y)$ coincides with $\tau_{C^{(r)}}$. Hence, the left cone $(\rightarrow X)$ of \mathcal{C} is the left cone $(\rightarrow \tilde{X})$, with $X = \tilde{X}$, of a component $\tilde{\mathcal{C}}$ of type $\mathbb{Z}\mathbb{A}_\infty$ of $\Gamma_{C^{(r)}}$, and the left cone $(\rightarrow Y)$ of \mathcal{D} is the left cone $(\rightarrow \tilde{Y})$, with $Y = \tilde{Y}$, of a component $\tilde{\mathcal{D}}$ of the form $\mathbb{Z}\mathbb{A}_\infty$ of $\Gamma_{C^{(r)}}$. Note that $\tilde{\mathcal{C}} \neq \tilde{\mathcal{D}}$ since $\mathcal{C} \neq \mathcal{D}$. Then there exist positive integers p and q such that $\text{Hom}_{B^{(r)}}(X, \tau_{B^{(r)}}^p Y) = \text{Hom}_{C^{(r)}}(X, \tau_{C^{(r)}}^p Y) \neq 0$ and $\text{Hom}_{B^{(r)}}(\tau_{B^{(r)}}^p Y, \tau_{B^{(r)}}^q X) = \text{Hom}_{C^{(r)}}(\tau_{C^{(r)}}^p Y, \tau_{C^{(r)}}^q X) \neq 0$. Therefore, we obtain an external short path $X \rightarrow \tau_{B^{(r)}}^p Y \rightarrow \tau_{B^{(r)}}^q X$ of \mathcal{C} in $\text{mod } B^{(r)}$, and so in $\text{mod } B$, because $B^{(r)}$ is a quotient algebra of B and there is a fully faithful embedding $\text{mod } B^{(r)} \hookrightarrow \text{mod } B$.

We note also that although the proofs in the two considered cases are similar, the applied results concerning the structure of left cones of acyclic components in \mathcal{P}^B and \mathcal{Q}^B are different. \square

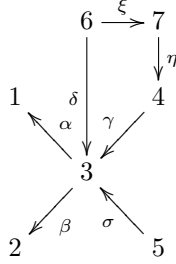
The following corollary is a direct consequence of Propositions 2.2 and 2.4.

Corollary 2.5. *Let B be a quasi-tilted algebra. The following statements are equivalent.*

- (i) *No component of Γ_B has an external short path.*
- (ii) *The component quiver Σ_B is acyclic.*
- (iii) *Every component of Γ_B is generalized standard.*
- (iv) *Γ_B is almost periodic.*

We end this section with an example showing that an Auslander-Reiten component without external short paths is not necessarily generalized standard.

Example 2.6. Let K be an algebraically closed field, Q the quiver



I the ideal in the path algebra KQ of Q generated by the elements $\delta\alpha$, $\delta\beta$, $\xi\eta$ and $\eta\gamma$, and $A = KQ/I$ the associated bound quiver algebra. We denote by H the path algebra $K\Delta$ of the full subquiver Δ of Q given by the vertices 1, 2, 3, 4, 5 and the arrows α , β , γ , σ . Further, let Ω be the quiver obtained from Q by removing the arrow η , J the ideal of the path algebra $K\Omega$ of Ω generated by $\delta\alpha$ and $\delta\beta$, and $B = K\Omega/J$ the associated bound quiver algebra. Then H is a hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_4$ whose Auslander-Reiten quiver Γ_H consists of a preprojective component $\mathcal{P}(H)$, a preinjective component $\mathcal{Q}(H)$, and a $\mathbb{P}_1(K)$ -family $\mathcal{T}^H = (\mathcal{T}_\lambda^H)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes, the three tubes \mathcal{T}_∞^H , \mathcal{T}_0^H , \mathcal{T}_1^H of rank 2, and the remaining tubes \mathcal{T}_λ^H , $\lambda \in \mathbb{P}_1(K) \setminus \{0, 1, \infty\}$, of rank 1 (we refer to [43, Theorem XIII.2.9] for a detailed description of the stable tubes of Γ_H). In particular, the simple H -module S_3 at the vertex 3 lies on the mouth of a stable tube, say \mathcal{T}_1^H of rank 2. Further, B is a tubular (branch) extension (see [39, (4.7)] or [44, Chapters XV-XVII]) of H , involving the simple module S_3 , which is a tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_6$ such that the Auslander-Reiten quiver Γ_B of B has disjoint union form

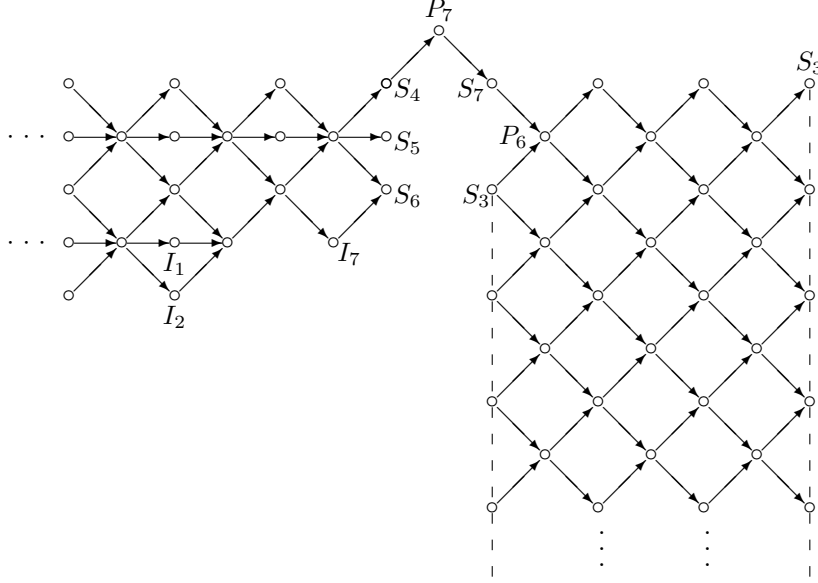
$$\Gamma_B = \mathcal{P}(B) \vee \mathcal{T}^B \vee \mathcal{Q}(B),$$

where $\mathcal{P}(B) = \mathcal{P}(H)$ is a unique preprojective component, $\mathcal{Q}(B)$ is a unique preinjective component containing all indecomposable injective B -modules and $\mathcal{T}^B = (\mathcal{T}_\lambda^B)_{\lambda \in \mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$ -family of pairwise orthogonal generalized standard ray tubes with $\mathcal{T}_\lambda^B = \mathcal{T}_\lambda^H$ for $\lambda \in \mathbb{P}_1(K) \setminus \{1\}$, and \mathcal{T}_1^B is a ray tube obtained from the stable tube \mathcal{T}_1^H by insertion of two rays, containing the indecomposable projective B -module P_6 and P_7 at the vertices 6 and 7. Moreover, we have $\text{Hom}_B(\mathcal{T}^B, \mathcal{P}(B)) = 0$, $\text{Hom}_B(\mathcal{Q}(B), \mathcal{T}^B) = 0$, $\text{Hom}_B(\mathcal{Q}(B), \mathcal{P}(B)) = 0$, and $\text{Hom}_B(\mathcal{P}(B), \mathcal{T}_\lambda^B) \neq 0$, $\text{Hom}_B(\mathcal{T}_\lambda^B, \mathcal{Q}(B)) \neq 0$ for all $\lambda \in \mathbb{P}_1(K)$. Observe also that B is the quotient algebra of A by the ideal generated by the coset $\eta + I$ of the arrow η , and hence we have the fully faithful embedding $\text{mod } B \hookrightarrow \text{mod } A$. Then using the canonical equivalences $\text{mod } A \xrightarrow{\sim} \text{rep}_K(Q, I)$ and $\text{mod } B \xrightarrow{\sim} \text{rep}_K(\Omega, J)$ (see [1, Theorem III.1.6]) we easily infer that there is only one indecomposable module in $\text{mod } A$ which is not in $\text{mod } B$, namely the 2-dimensional projective-injective module $P_7 = I_4$ whose socle is the simple module S_4 at the vertex 4 and whose top is the simple module S_7 at the vertex 7. Therefore, the Auslander-Reiten quiver Γ_A of A has the disjoint union form

$$\Gamma_A = \mathcal{P}(A) \vee \mathcal{T}^A \vee \mathcal{C},$$

where $\mathcal{P}(A) = \mathcal{P}(B) = \mathcal{P}(H)$ is a preprojective component, $\mathcal{T}^A = (\mathcal{T}_\lambda^A)_{\lambda \in \mathbb{P}_1(K) \setminus \{1\}} = (\mathcal{T}_\lambda^B)_{\lambda \in \mathbb{P}_1(K) \setminus \{1\}}$ is a family of pairwise orthogonal generalized standard stable

tubes, and \mathcal{C} is the component of the form below, obtained by gluing the ray tube \mathcal{T}_1^B and the preinjective component $\mathcal{Q}(B)$ by the projective-injective module $P_7 = I_4$,



where the modules along the vertical dashed lines have to be identified. We note that $\text{Hom}_A(\mathcal{C}, \mathcal{D}) = 0$ for any component \mathcal{D} of Γ_A different from \mathcal{C} , and hence \mathcal{C} has no external short path in $\text{mod } A$. On the other hand, the canonical monomorphism from the simple A -module S_7 to its injective envelope I_7 in $\text{mod } A$ belongs to $\text{rad}_A^\infty(S_7, I_7)$, and so the component \mathcal{C} is not generalized standard.

3. PROOFS OF THEOREMS 1.1 AND 1.2

The aim of this section is to provide the proof of Theorem 1.1. Observe that Theorem 1.2 follows from Theorem 1.1 applied to the opposite algebra A^{op} of A .

Let A be an algebra. Following [4] a module M in $\text{mod } A$ is said to be the *middle of a short chain* if there is some indecomposable module X in $\text{mod } A$ with $\text{Hom}_A(X, M) \neq 0$ and $\text{Hom}_A(M, \tau_A X) \neq 0$. We note that if M and N are indecomposable modules in $\text{mod } A$ with the same composition factors and M is not the middle of a short chain, then M and N are isomorphic (see [4], [36]).

The following lemma follows from [33, Lemma 1.3] (see also [36, Theorem 1.6]).

Lemma 3.1. *Let A be an artin algebra, M a module in $\text{mod } A$ and X an indecomposable module in $\text{mod } A$ which is not isomorphic to a direct summand of M . Assume that $\text{Hom}_A(X, M) \neq 0$ and $\text{Hom}_A(M, \tau_A X) \neq 0$. Then there is a short path $Y \rightarrow V \rightarrow Z$ in $\text{mod } A$, where Y and Z are indecomposable direct summands of M , and $V = X$ or V is an indecomposable direct summand of the middle term E of an Auslander-Reiten sequence $0 \rightarrow \tau_A X \rightarrow E \rightarrow X \rightarrow 0$ in $\text{mod } A$.*

The next proposition will reduce the proof of Theorem 1.1 to Theorem 1.6.

Proposition 3.2. *Let A be an algebra, \mathcal{C} a semi-regular component of Γ_A without external short paths, and $B = A/\text{ann}_A(\mathcal{C})$. Then B is a quasi-tilted algebra.*

Proof. We may assume (by duality) that \mathcal{C} is without projective modules. Since \mathcal{C} is a component of Γ_B with $\text{ann}_B(\mathcal{C}) = 0$, we may also assume that $\text{ann}_A(\mathcal{C}) = 0$. We choose a module M in the additive category $\text{add}(\mathcal{C})$ of \mathcal{C} such that $\text{ann}_A(M) = \text{ann}_A(\mathcal{C}) = 0$ (see [33, Lemma 1.1]). Then there are a monomorphism $A \rightarrow M^r$ and an epimorphism $M^s \rightarrow D(A)$ in $\text{mod } A$, for some positive integers r and s (see [1, Lemma VI.2.2]).

We prove first that $\text{gl.dim } A \leq 2$. Take an indecomposable projective module P in $\text{mod } A$ and an indecomposable direct summand X of the radical $\text{rad } P$ of P . Observe that X is not in \mathcal{C} since there is an irreducible homomorphism $X \rightarrow P$ and P is not in \mathcal{C} . On the other hand, we have $\text{Hom}_A(X, M) \neq 0$ because there is a monomorphism $A \rightarrow M^r$ and P is a direct summand of A . We claim that $\text{pd}_A X \leq 1$. Assume $\text{pd}_A X \geq 2$. Then we have $\text{Hom}_A(D(A), \tau_A X) \neq 0$ (see [1, Lemma IV.2.7] or [39, (2.4)]), and hence $\text{Hom}_A(M, \tau_A X) \neq 0$, because there is an epimorphism $M^s \rightarrow D(A)$ in $\text{mod } A$. Thus M is the middle of a short chain $X \rightarrow M \rightarrow \tau_A X$ with M in $\text{add}(\mathcal{C})$ and $X, \tau_A X$ not in \mathcal{C} . Applying Lemma 3.1, we conclude that there is a short path $Y \rightarrow V \rightarrow Z$, where Y and Z are indecomposable direct summands of M , and $V = X$ or V is an indecomposable direct summand of the middle term E of an Auslander-Reiten sequence

$$0 \rightarrow \tau_A X \rightarrow E \rightarrow X \rightarrow 0$$

in $\text{mod } A$. Since X is not in \mathcal{C} , V is also not in \mathcal{C} , and so $Y \rightarrow V \rightarrow Z$ is an external short path of \mathcal{C} , a contradiction. Hence, indeed $\text{pd}_A X \leq 1$. This shows that $\text{pd}_A \text{rad } P \leq 1$, and consequently $\text{gl.dim } A \leq 2$.

Let N be an indecomposable module in $\text{mod } A$. We claim that $\text{pd}_A N \leq 1$ or $\text{id}_A N \leq 1$. We have two cases to consider. Assume first that N belongs to \mathcal{C} . We prove that then $\text{id}_A N \leq 1$. Suppose $\text{id}_A N \geq 2$. Then $\text{Hom}_A(\tau_A^- N, A) \neq 0$ (see again [1, Lemma IV.2.7] or [39, (2.4)]), and so $\text{Hom}_A(\tau_A^- N, P') \neq 0$ for an indecomposable projective right A -module P' . Clearly, we have also $\text{Hom}_A(P', Z) \neq 0$ for an indecomposable direct summand Z of M , because there is a monomorphism $A \rightarrow M^r$. Therefore, since \mathcal{C} is without projective modules, we obtain a short path $\tau_A^- N \rightarrow P' \rightarrow Z$ in $\text{mod } A$ with $\tau_A^- N$ and Z in \mathcal{C} and P' not in \mathcal{C} , and so an external short path of \mathcal{C} , a contradiction. Hence, indeed $\text{id}_A N \leq 1$.

Assume now that N is not in \mathcal{C} and $\text{id}_A N \geq 2$. We claim that then $\text{pd}_A N \leq 1$. We show first that $\text{pd}_A \tau_A^- N \leq 1$. Assume $\text{pd}_A \tau_A^- N \geq 2$. Then we have $\text{Hom}_A(D(A), N) = \text{Hom}_A(D(A), \tau_A(\tau_A^- N)) \neq 0$, and consequently $\text{Hom}_A(M, N) \neq 0$, because there is an epimorphism $M^s \rightarrow D(A)$ in $\text{mod } A$. On the other hand, the assumption $\text{id}_A N \geq 2$ gives $\text{Hom}_A(\tau_A^- N, A) \neq 0$. This implies $\text{Hom}_A(\tau_A^- N, M) \neq 0$, because there is a monomorphism $A \rightarrow M^r$ in $\text{mod } A$. Observe also that N and $\tau_A^- N$ are not isomorphic to a direct summand of M , since N and $\tau_A^- N$ are not in \mathcal{C} . Therefore, applying Lemma 3.1 to the short chain $\tau_A^- N \rightarrow M \rightarrow N$, we conclude that there is in $\text{mod } A$ an external short path $U \rightarrow V \rightarrow W$ of \mathcal{C} , where U and W are indecomposable direct summands of M , and $V = \tau_A^- N$ or V is an indecomposable direct summand of the middle term F of an Auslander-Reiten sequence

$$0 \rightarrow N \rightarrow F \rightarrow \tau_A^- N \rightarrow 0$$

in $\text{mod } A$, a contradiction. Hence, indeed $\text{pd}_A \tau_A^- N \leq 1$. Take now an indecomposable direct summand L of the middle term F of the above Auslander-Reiten sequence. We claim that $\text{pd}_A L \leq 1$. Choose an irreducible homomorphism

$f : L \rightarrow \tau_A^- N$ in $\text{mod } A$. By general theory we know that f is either a proper monomorphism or a proper epimorphism. Hence we have two cases to consider.

Assume f is a monomorphism. Then we have in $\text{mod } A$ a short exact sequence

$$0 \rightarrow L \rightarrow \tau_A^- N \rightarrow R \rightarrow 0,$$

and hence an exact sequence of functors

$$\text{Ext}_A^2(\tau_A^- N, -) \rightarrow \text{Ext}_A^2(L, -) \rightarrow \text{Ext}_A^3(R, -)$$

on $\text{mod } A$. Since $\text{pd}_A \tau_A^- N \leq 1$ and $\text{gl.dim } A \leq 2$, we have $\text{Ext}_A^2(\tau_A^- N, -) = 0$ and $\text{Ext}_A^3(R, -) = 0$, which leads to $\text{Ext}_A^2(L, -) = 0$, or equivalently, to $\text{pd}_A L \leq 1$.

Assume f is an epimorphism. Then $\text{Hom}_A(\tau_A^- N, M) \neq 0$ forces $\text{Hom}_A(L, M) \neq 0$. Assume $\text{pd}_A L \geq 2$. Then $\text{Hom}_A(D(A), \tau_A L) \neq 0$, and hence $\text{Hom}_A(M, \tau_A L) \neq 0$, because there is an epimorphism $M^s \rightarrow D(A)$ in $\text{mod } A$. Therefore, M is the middle term of a short chain $L \rightarrow M \rightarrow \tau_A L$ with L and $\tau_A L$ not in \mathcal{C} , since N is not in \mathcal{C} . Applying Lemma 3.1, we conclude that there is an external short path of \mathcal{C} of the form $S \rightarrow V \rightarrow T$, where S and T are indecomposable direct summands of M , and $V = L$ or V is an indecomposable direct summand of the middle term G of an Auslander-Reiten sequence

$$0 \rightarrow \tau_A L \rightarrow G \rightarrow L \rightarrow 0$$

in $\text{mod } A$, a contradiction. Hence we obtain that $\text{pd}_A L \leq 1$. Summing up, we have in $\text{mod } A$ a short exact sequence

$$0 \rightarrow N \rightarrow F \rightarrow \tau_A^- N \rightarrow 0$$

with $\text{pd}_A F \leq 1$. Since $\text{gl.dim } A \leq 2$, we have an epimorphism of functors $\text{Ext}_A^2(F, -) \rightarrow \text{Ext}_A^2(N, -)$, and hence $\text{Ext}_A^2(F, -) = 0$ forces $\text{Ext}_A^2(N, -) = 0$. This proves that $\text{pd}_A N \leq 1$.

Therefore, A is a quasi-tilted algebra. \square

We complete now the proof of Theorem 1.1.

Let A be an algebra, \mathcal{C} a component of Γ_A without projective modules and external short paths, and $B = A/\text{ann}_A(\mathcal{C})$. It follows from Proposition 3.2 that then B is a quasi-tilted algebra. Moreover, \mathcal{C} is a faithful component of Γ_B . We have two cases to consider.

Assume \mathcal{C} is acyclic. Then it follows from Propositions 2.2 and 2.4 (and their proofs) that B is a tilted algebra $\text{End}_H(T)$, for a hereditary algebra H and a multiplicity-free tilting H -module T , and \mathcal{C} is the connecting component \mathcal{C}_T of Γ_B determined by T . Further, since $\mathcal{C} = \mathcal{C}_T$ is without projective modules, we conclude also that T has no non-zero preinjective direct summands. We note that $\mathcal{C} = \mathcal{C}_T$ is a preinjective component if and only if T is a preprojective tilting H -module, or equivalently, B is a concealed algebra. Clearly, the component $\mathcal{C} = \mathcal{C}_T$ is regular if and only if T is a regular tilting H -module.

Assume \mathcal{C} contains an oriented cycle. Since \mathcal{C} is without projective modules, it follows from the general result of Liu [27, Theorem 2.5] that \mathcal{C} is a coray tube. Hence B is a quasi-tilted algebra with a faithful coray tube \mathcal{C} . Applying Propositions 2.2 and 2.4 (and their proofs) and [25, Theorem 3.4] we infer that B is the opposite algebra of an almost concealed canonical algebra and \mathcal{C} is a coray tube of a separating family \mathcal{T}^B of coray tubes of Γ_B . We note that, if \mathcal{C} contains an injective module, then the remaining coray tubes of \mathcal{T}^B are stable tubes. On the

other hand, \mathcal{C} is a regular component if and only if the algebra $B = A/\text{ann}_A(\mathcal{C})$ is a concealed canonical algebra.

4. PROOF OF THEOREM 1.10

Let B be an algebra and $1_B = e_1 + \dots + e_n$ a decomposition of the identity 1_B of B into a sum of pairwise orthogonal primitive idempotents. The *repetitive category* of B is the self-injective locally bounded K -category \widehat{B} with the objects $e_{m,i}$, $m \in \mathbb{Z}$, $i \in \{1, \dots, n\}$, the morphism K -modules defined as follows

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_j B e_i, & r = m \\ D(e_i B e_j), & r = m + 1 \\ 0, & \text{otherwise,} \end{cases}$$

and the composition of morphisms in \widehat{B} is given by the B - B -bimodule structures on B and $D(B)$. We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of \widehat{B} defined by $\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$ for all $m \in \mathbb{Z}$, $i \in \{1, \dots, n\}$. An automorphism φ of the K -category \widehat{B} is said to be:

- *positive* if, for each pair $(m, i) \in \mathbb{Z} \times \{1, \dots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and some $j \in \{1, \dots, n\}$;
- *rigid* if, for each pair $(m, i) \in \mathbb{Z} \times \{1, \dots, n\}$, we have $\varphi(e_{m,i}) = e_{m,j}$ for some $j \in \{1, \dots, n\}$;
- *strictly positive* if it is positive but not rigid.

A group G of automorphisms of the K -category \widehat{B} is said to be *admissible* if G acts freely on the set of objects of \widehat{B} and has finitely many orbits. Following Gabriel [13] we may then consider the orbit bounded K -category \widehat{B}/G , where the objects are the G -orbits of objects of \widehat{B} , and hence the basic connected self-injective artin algebra $\bigoplus(\widehat{B}/G)$ given by the direct sum of all morphism K -modules in \widehat{B}/G . We will identify \widehat{B}/G with $\bigoplus(\widehat{B}/G)$ and call the *orbit algebra* of \widehat{B} with respect to G . We note that the infinite cyclic group $(\nu_{\widehat{B}})$ generated by the Nakayama automorphism $\nu_{\widehat{B}}$ of \widehat{B} is admissible and the orbit algebra $\widehat{B}/(\nu_{\widehat{B}})$ is isomorphic to the trivial extension $B \rtimes D(B)$ of B by $D(B)$. We refer to [59, Theorem 5.3] for a criterion on a self-injective algebra A to be of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$ with B an algebra and φ a positive automorphism of \widehat{B} .

The implication (ii) \Rightarrow (i) in Theorem 1.10 follows from the part (iii) of the following proposition.

Proposition 4.1. *Let B be a tilted algebra of the form $\text{End}_H(T)$ for a hereditary algebra H and a regular tilting H -module T , \mathcal{C}_T the connecting component of Γ_B determined by T , ψ a positive automorphism of \widehat{B} and $A = \widehat{B}/(\psi\nu_{\widehat{B}})$ the orbit algebra of \widehat{B} with respect to the infinite cyclic group generated by $\psi\nu_{\widehat{B}}$. Then the following statements hold.*

- (i) B is a quotient algebra of A .
- (ii) \mathcal{C}_T is a generalized standard regular component of Γ_A .
- (iii) \mathcal{C}_T has no external short path in $\text{mod } A$ if and only if $\psi = \varphi\nu_{\widehat{B}}$ for a positive automorphism φ of \widehat{B} .

Proof. The statement (i) follows from the definition of the orbit algebra $\widehat{B}/(\psi\nu_{\widehat{B}})$ and the positivity of ψ . In order to show the statements (ii) and (iii), we present

the structure of $\Gamma_{\widehat{B}}$ and $\text{mod } \widehat{B}$ established in [12, Section 3] (see also [37, Section 2]).

Let $Q = Q_H$ be the valued quiver of H and $\Delta = Q^{\text{op}}$. Then Δ is a wild quiver with at least 3 vertices, since we have the regular tilting H -module T . Applying [12, Theorem 3.5] we conclude that there exist tilted convex subcategories $B^+ = \text{End}_H(T^+)$ and $B^- = \text{End}_H(T^-)$ of \widehat{B} , for a tilting H -module T^+ without non-zero preinjective direct summands and a tilting H -module T^- without non-zero preprojective direct summands, such that for the shifts $B_{2q}^+ = B_{2q}^- = \nu_{\widehat{B}}^q(B)$, $B_{2q+1}^+ = \nu_{\widehat{B}}^q(B^+)$, $B_{2q+1}^- = \nu_{\widehat{B}}^q(B^-)$, $q \in \mathbb{Z}$, of B , B^+ and B^- inside \widehat{B} , the following statements hold:

- (a) The Auslander-Reiten quiver $\Gamma_{\widehat{B}}$ of \widehat{B} has the disjoint union form

$$\Gamma_{\widehat{B}} = \bigsqcup_{q \in \mathbb{Z}} (\mathcal{C}_q \vee \mathcal{R}_q),$$

where $\nu_{\widehat{B}}(\mathcal{C}_q) = \mathcal{C}_{q+2}$, $\nu_{\widehat{B}}(\mathcal{R}_q) = \mathcal{R}_{q+2}$, $\text{Hom}_{\widehat{B}}(\mathcal{R}_q, \mathcal{C}_q) = 0$, for any $q \in \mathbb{Z}$, and $\text{Hom}_{\widehat{B}}(\mathcal{C}_p \vee \mathcal{R}_p, \mathcal{C}_q \vee \mathcal{R}_q) = 0$ for all $p > q$. Moreover, each \mathcal{C}_q separates $\bigvee_{p < q} (\mathcal{C}_p \vee \mathcal{R}_p)$ from $\mathcal{R}_q \vee (\bigvee_{p > q} (\mathcal{C}_p \vee \mathcal{R}_p))$ and each \mathcal{R}_q separates $\bigvee_{p < q} (\mathcal{C}_p \vee \mathcal{R}_p) \vee \mathcal{C}_q$ from $\bigvee_{p > q} (\mathcal{C}_p \vee \mathcal{R}_p)$.

- (b) For each $q \in \mathbb{Z}$, \mathcal{C}_{2q} is the regular connecting component \mathcal{C}_T of $\Gamma_{B_{2q}^+}$ (respectively, $\Gamma_{B_{2q}^-}$) of the form $\mathbb{Z}\Delta$ determined by the regular tilting H -module T .
- (c) For each $q \in \mathbb{Z}$, \mathcal{C}_{2q+1} is an acyclic component with the stable part \mathcal{C}_{2q+1}^s of the form $\mathbb{Z}\Delta$, the torsion-free part $\mathcal{Y}(T^+) \cap \mathcal{C}_{T^+}$ of the connecting component \mathcal{C}_{T^+} of $\Gamma_{B_{2q+1}^+} = \Gamma_{B^+}$ determined by T^+ is a full translation subquiver of \mathcal{C}_{2q+1} which is closed under predecessors, and the torsion part $\mathcal{X}(T^-) \cap \mathcal{C}_{T^-}$ of the connecting component \mathcal{C}_{T^-} of $\Gamma_{B_{2q+1}^-} = \Gamma_{B^-}$ determined by T^- is a full translation subquiver of \mathcal{C}_{2q+1} which is closed under successors.
- (d) For each $q \in \mathbb{Z}$, \mathcal{R}_q is an infinite family of components whose stable parts are of the form $\mathbb{Z}\mathbb{A}_\infty$, and the simple composition factors of modules in \mathcal{R}_q are simple B_q^- -modules or simple B_{q+1}^+ -modules.
- (e) For each $m \in \mathbb{Z}$, the indecomposable projective-injective \widehat{B} -modules $P_{m+1,i} = e_{m+1,i}\widehat{B} = \nu_{\widehat{B}}(e_{m,i}\widehat{B}) = \nu_{\widehat{B}}(P_{m,i})$, $i \in \{1, \dots, n\}$, lie in $\mathcal{R}_{2m} \vee \mathcal{C}_{2m+1} \vee \mathcal{R}_{2m+1}$.

Moreover, we claim that the following fact holds.

- (f) For each $q \in \mathbb{Z}$, the family of components \mathcal{R}_q contains at least one indecomposable projective-injective \widehat{B} -module.

Assume that \mathcal{R}_q , for some $q \in \mathbb{Z}$, consists entirely of regular components of the form $\mathbb{Z}\mathbb{A}_\infty$. Then it follows from the above description of $\Gamma_{\widehat{B}}$ that in the Auslander-Reiten quiver Γ_B of the tilted algebra $B = \text{End}_H(T)$ either all indecomposable projective B -modules lie in the unique preprojective component $\mathcal{P}(B)$ or all indecomposable injective B -modules lie in the unique preinjective component $\mathcal{Q}(B)$. Hence, Γ_B admits at least two

components having a section of type Δ , one of them the regular connecting component \mathcal{C}_T . On the other hand, by a result due to Ringel [40, Lectures 2 and 3], if the Auslander-Reiten quiver Γ_B contains at least two components with complete sections then B is a concealed algebra, a contradiction.

We note the following obvious consequence of the description of the simple composition factors of indecomposable \widehat{B} -modules presented above.

- (g) For each $q \in \mathbb{Z}$, we have $\text{Hom}_{\widehat{B}}(\mathcal{C}_{2q}, \mathcal{C}_p \vee \mathcal{R}_p) = 0$ for all $p \geq 2q + 2$. In particular, we obtain that $\text{Hom}_{\widehat{B}}(\mathcal{C}_0, \mathcal{C}_p \vee \mathcal{R}_p) = 0$ for all $p \geq 2$.

We prove now that:

- (h) For each $q \in \mathbb{Z}$ and all but finitely many indecomposable modules X in \mathcal{C}_{2q} , there exists a short path in $\text{mod } \widehat{B}$ of the form $X \rightarrow P \rightarrow Z$ with an indecomposable projective-injective module P in \mathcal{R}_{2q+1} and Z in \mathcal{C}_{2q+3} .

Since $\mathcal{C}_{p+2} = \nu_{\widehat{B}}(\mathcal{C}_p)$ and $\mathcal{R}_{p+2} = \nu_{\widehat{B}}(\mathcal{R}_p)$ for each $p \in \mathbb{Z}$, we may assume that $q = 0$. It is known (see [22, Lemma 4.1] and [37, Lemma 1.8]) that all but finitely many indecomposable modules in the regular connecting component $\mathcal{C}_0 = \mathcal{C}_T$ of the tilted algebra $B = \text{End}_H(T)$ are sincere B -modules. Further, the simple B -modules are exactly the socles of the indecomposable projective \widehat{B} -modules $P_{1,i} = e_{1,i}\widehat{B}$, $i \in \{1, \dots, n\}$, which are exactly the indecomposable projective-injective \widehat{B} -modules located in $\mathcal{R}_0 \vee \mathcal{C}_1 \vee \mathcal{R}_1$. Applying (f) we conclude that there is $i \in \{1, \dots, n\}$ such that $P_{1,i}$ belongs to \mathcal{R}_1 . Then, for an arbitrary sincere indecomposable module X in \mathcal{C}_0 , we have $\text{Hom}_{\widehat{B}}(X, P_{1,i}) \neq 0$, since the socle of $P_{1,i}$ is a composition factor of X . Moreover, we have a canonical non-zero homomorphism $f : P_{1,i} \rightarrow P_{2,i} = \nu_{\widehat{B}}(P_{1,i})$, since the top of $P_{1,i}$ is the socle of $P_{2,i}$. Observe that $P_{2,i}$ lies in $\mathcal{R}_3 = \nu_{\widehat{B}}(\mathcal{R}_1)$. Finally, since the component \mathcal{C}_3 separates $\bigvee_{p < 3} (\mathcal{C}_p \vee \mathcal{R}_p)$ from $\mathcal{R}_3 \vee (\bigvee_{p > 3} (\mathcal{C}_p \vee \mathcal{R}_p))$ in $\text{mod } \widehat{B}$, we conclude that the homomorphism f factors through a module M from the additive category $\text{add}(\mathcal{C}_3)$ of \mathcal{C}_3 . In particular, we obtain a short path $X \rightarrow P \rightarrow Z$ with $P = P_{1,i}$ in \mathcal{R}_1 and Z an indecomposable module in \mathcal{C}_3 .

Consider now the push-down functor [13] $F_\lambda : \text{mod } \widehat{B} \rightarrow \text{mod } A$ associated to the Galois covering $F : \widehat{B} \rightarrow \widehat{B}/(\psi\nu_{\widehat{B}})$. Since, by the properties (a) - (d), the repetitive category \widehat{B} is locally support-finite, we conclude by the density theorem of [11] that the push-down functor F_λ is dense. In particular, by [13, Theorem 3.6], the Auslander-Reiten quiver Γ_A of A is the orbit quiver $\Gamma_{\widehat{B}}/(\psi\nu_{\widehat{B}})$ of $\Gamma_{\widehat{B}}$ with respect to the induced action of $(\psi\nu_{\widehat{B}})$ on $\Gamma_{\widehat{B}}$. In fact, for $g = \psi\nu_{\widehat{B}}$, $G = (g)$, and the induced action of G on $\text{mod } \widehat{B}$ (see [13]), the push-down functor $F_\lambda : \text{mod } \widehat{B} \rightarrow \text{mod } A$ induces isomorphisms of K -modules

$$\bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\widehat{B}}(U, g^r V) \xrightarrow{\sim} \text{Hom}_{\widehat{B}}(F_\lambda(U), F_\lambda(V))$$

for all indecomposable modules U and V in $\text{mod } \widehat{B}$. Since $g = \psi\nu_{\widehat{B}}$ with ψ a positive automorphism of \widehat{B} and $\nu_{\widehat{B}}(\mathcal{C}_0) = \mathcal{C}_2$, we conclude that $\mathcal{C}_T = \mathcal{C}_0 = F_\lambda(\mathcal{C}_0)$

is a regular component of Γ_A consisting entirely of indecomposable B -modules. Then \mathcal{C}_T is a generalized standard component of Γ_A , because \mathcal{C}_T is a generalized standard component of Γ_B , and $B = A/\text{ann}_A(\mathcal{C})$. This shows the required property (ii).

For (iii), assume first that $\psi = \varphi\nu_{\widehat{B}}$ for a positive automorphism φ of \widehat{B} . Then $g = \varphi\nu_{\widehat{B}}^2$ and $g(\mathcal{C}_T) = g(\mathcal{C}_0) = \mathcal{C}_p$ for some $p \geq 4$. Hence, applying (a) and (g), we conclude that there is no short path in $\text{mod } \widehat{B}$ of the form $X \rightarrow Y \rightarrow Z$ with X in \mathcal{C}_0 , Z in $g^r\mathcal{C}_0$ and Y not in $g^r\mathcal{C}_0$, for any $r \in \mathbb{Z}$. This shows that $\mathcal{C}_T = F_\lambda(\mathcal{C}_0)$ has no external short path in $\text{mod } B$. On the other hand, if $g = \psi\nu_{\widehat{B}}$ is not of the form $\varphi\nu_{\widehat{B}}^2$ with φ a positive automorphism of \widehat{B} , then $g(\mathcal{C}_0) = \mathcal{C}_2$ or $g(\mathcal{C}_0) = \mathcal{C}_3$. In the first case, $\mathcal{C}_T = F_\lambda(\mathcal{C}_0)$ is a sincere regular component in $\text{mod } A$, and then we have an external short path $X \rightarrow Q \rightarrow X$ of \mathcal{C}_T for an arbitrary sincere indecomposable module X in \mathcal{C}_T and Q an arbitrary indecomposable projective-injective A -module. For $g = \psi\nu_{\widehat{B}}$ with $g(\mathcal{C}_0) = \mathcal{C}_3$, we conclude from (h) that there is an external short path $F_\lambda(X) \rightarrow F_\lambda(P) \rightarrow F_\lambda(Z)$ of $\mathcal{C}_T = F_\lambda(\mathcal{C}_0) = F_\lambda(\mathcal{C}_3)$ for some modules $X \in \mathcal{C}_0$, $Z \in \mathcal{C}_3$ and an indecomposable projective-injective A -module $F_\lambda(P) \in F_\lambda(\mathcal{R}_1)$. Therefore, the equivalence (iii) holds. \square

We complete now the proof of Theorem 1.10 by showing that the implication (i) \Rightarrow (ii) also holds.

Let A be a self-injective algebra such that Γ_A admits a regular acyclic component \mathcal{C} without external short paths in $\text{mod } A$. Then it follows from Corollary 1.4 that \mathcal{C} is a generalized standard component of Γ_A . Let I be the annihilator $\text{ann}_A(\mathcal{C})$ of \mathcal{C} in A and $B = A/I$. We may choose a set e_1, \dots, e_n of pairwise orthogonal primitive idempotents of A such that $1_A = e_1 + \dots + e_n$ and $\{e_i | 1 \leq i \leq m\}$, for some $m \leq n$, is the set of all idempotents in $\{e_i | 1 \leq i \leq n\}$ which are not contained in I . Then $e = e_1 + \dots + e_m$ is an idempotent of A such that $e + I$ is the identity 1_B of B , uniquely determined by I up to an inner automorphism of A , called the *residual identity* of $B = A/I$. Moreover, $B \cong eAe/eIe$. Further, it follows from Corollary 1.3 that B is a tilted algebra of the form $\text{End}_H(T)$, for a (wild) hereditary algebra H and a regular tilting H -module T , and \mathcal{C} is the connecting component \mathcal{C}_T of Γ_B determined by T . In particular, the valued quiver Q_B of B is acyclic. Applying [56, Proposition 2.3 and Theorem 5.1], we obtain also that $Ie = l_A(I)$ and $eI = r_A(I)$, where $l_A(I)$ and $r_A(I)$ denote the left annihilator and the right annihilator of I in A , respectively, and $\text{soc}(A)$ is contained in I . We note that by a theorem of Nakayama [32] the left socle $\text{soc}({}_A A)$ and right socle $\text{soc}(A_A)$ of A coincide, and are denoted by $\text{soc}(A)$, which is an ideal of A . Observe that $IeI = 0$, and so Ie is a right B -module and eI is a left B -module. In fact, by [56, Proposition 2.3], Ie is an injective cogenerator in $\text{mod } B$ and eI is an injective cogenerator in $\text{mod } B^{\text{op}} = B - \text{mod}$ (the category of finitely generated left B -modules). We have also $l_{eAe}(I) = eIe = r_{eAe}(I)$, and so I is a deforming ideal in the sense of [56, (2.1)]. The canonical isomorphism of algebras $eAe/eIe \rightarrow A/I$ allows to consider I as an $(eAe/eIe) - (eAe/eIe)$ -bimodule. Denote by $A[I]$ the direct sum $(eAe/eIe) \oplus I$ of K -modules with the multiplication

$$(b, x) \cdot (c, y) = (bc, by + xc + xy)$$

for $b, c \in eAe/eIe$ and $x, y \in I$. Then, by [56, Theorem 4.1], $A[I]$ is a basic connected self-injective artin algebra with the identity $(e + eIe, 1_A - e)$, and the same

Nakayama permutation as A . We note that $\{(e_i + eIe, 0) | 1 \leq i \leq m\} \cup \{(0, e_j) | m < j \leq n\}$ is a complete set of pairwise orthogonal idempotents of $A[I]$ whose sum is the identity $1_{A[I]}$ of $A[I]$. Further, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider I as an ideal of $A[I]$. Then $e = (e + eIe, 0)$ is a residual identity of $A[I]/I = eAe/eIe \xrightarrow{\sim} A/I$, $eA[I]e = (eAe/eIe) \oplus eIe$ and the canonical algebra epimorphism $eA[I]e \rightarrow eA[I]e/eIe$ is a retraction. Finally, $IeI = (0, IeI) = 0$ and $Ie = l_{A[I]}(I)$, $eI = r_{A[I]}(I)$. Applying now [57, Theorem 3.8] we conclude that $A[I]$ is isomorphic to the orbit algebra $\widehat{B}/(\psi\nu_{\widehat{B}})$ for a positive automorphism ψ of \widehat{B} .

It follows also from [56, Theorem 4.1] that there is a canonical algebra isomorphism $\phi : A[I]/\text{soc}(A[I]) \rightarrow A/\text{soc}(A)$. Observe that $\text{soc}(A[I]) = \text{soc}(I) = \text{soc}(A)$, since $\text{soc}(A)$ and $\text{soc}(A[I])$ are contained in I . In particular, we obtain the isomorphism $\phi^* : \text{mod}(A[I]/\text{soc}(A[I])) \rightarrow \text{mod}(A/\text{soc}(A))$ of module categories, induced by ϕ . Note that the regular component $\mathcal{C} = \mathcal{C}_T$ of Γ_A consists entirely of B -modules, and hence is a component of the Auslander-Reiten quiver $\Gamma_{A/\text{soc}(A)}$ of $A/\text{soc}(A)$. Clearly, \mathcal{C} is a regular acyclic component of $\Gamma_{A/\text{soc}(A)}$ without external short paths in $\text{mod}(A/\text{soc}(A))$. Denote by \mathcal{C}' the regular acyclic component of the Auslander-Reiten quiver $\Gamma_{A[I]/\text{soc}(A[I])}$ of $A[I]/\text{soc}(A[I])$ such that $\phi^*(\mathcal{C}') = \mathcal{C}$. In fact, we have $I = \text{ann}_{A[I]}(\mathcal{C}')$ and $A[I]/I \cong eAe/eIe \xrightarrow{\sim} A/I = B$ canonically, so \mathcal{C}' is the regular connecting component of the tilted algebra eAe/eIe , isomorphic to B . We claim that \mathcal{C}' has no external short path in $\text{mod } A[I]$. Suppose there is a short path $X \rightarrow Y \rightarrow Z$ in $\text{mod } A[I]$ with X and Z in \mathcal{C}' but Y not in \mathcal{C}' . Obviously, if Y is not a projective module in $A[I]$, then Y is a module in $\text{mod}(A[I]/\text{soc}(A[I]))$, and we obtain an external short path $\phi^*(X) \rightarrow \phi^*(Y) \rightarrow \phi^*(Z)$ of \mathcal{C} in $\text{mod } A$, a contradiction. Assume $Y = P$ is an indecomposable projective, hence injective, module in $\text{mod } A[I]$. Then there is an indecomposable projective-injective module Q in $\text{mod } A$ such that $\phi^*(P/\text{soc}(P)) = Q/\text{soc}(Q)$. Since the Nakayama permutations of the algebras $A[I]$ and A coincide, we conclude also that $\phi^*(\text{soc}(P)) = \text{soc}(Q)$ and $\phi^*(\text{top}(P)) = \text{top}(Q)$.

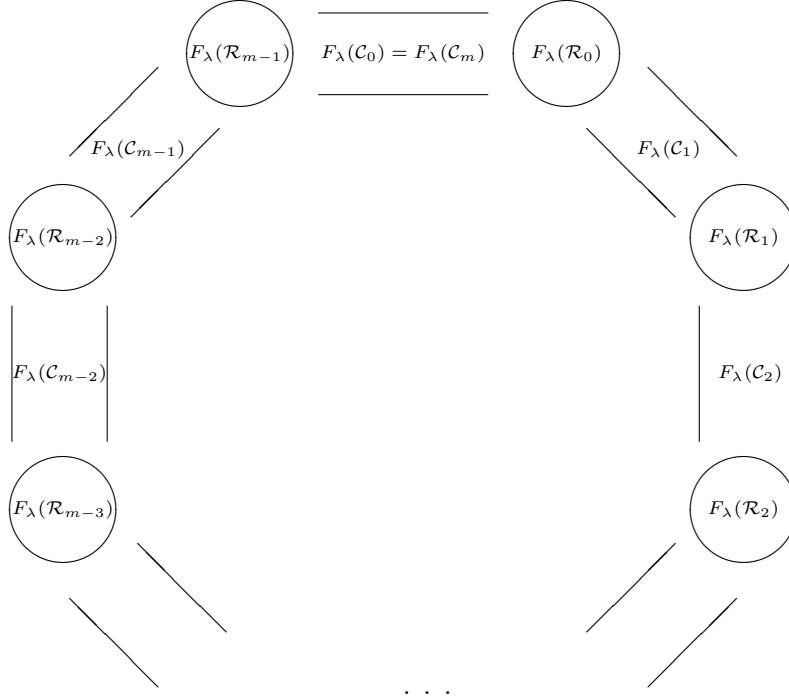
Further, since there are non-zero homomorphisms $X \rightarrow P$ and $P \rightarrow Y$ in $\text{mod } A[I]$, we conclude that $\text{soc}(P)$ is a simple composition factor of X and $\text{top}(P)$ is a simple composition factor of Y . But then $\text{soc}(Q)$ is a simple composition factor of $\phi^*(X)$ and $\text{top}(Q)$ is a simple composition factor of $\phi^*(Z)$. Hence we obtain a short path $\phi^*(X) \rightarrow Q \rightarrow \phi^*(Z)$ in $\text{mod } A$, with $\phi^*(X)$ and $\phi^*(Z)$ in $\mathcal{C} = \mathcal{C}_T$ and Q not in \mathcal{C} , because Q is projective and \mathcal{C} is regular. Thus \mathcal{C} admits an external short path in $\text{mod } A$. Summing up, \mathcal{C}' is a regular acyclic component of $\Gamma_{A[I]/\text{soc}(A[I])}$ without external short paths in $\text{mod}(A[I]/\text{soc}(A[I]))$.

Therefore, applying Proposition 4.1 to $\widehat{B}/(\psi\nu_{\widehat{B}})$, isomorphic to $A[I]$, we conclude that $A[I]$ is isomorphic to $\widehat{B}/(\varphi\nu_{\widehat{B}}^2)$ for a positive automorphism φ of \widehat{B} . Invoking now the structure of the Auslander-Reiten quiver of $\widehat{B}/(\psi\nu_{\widehat{B}}) = \widehat{B}/(\varphi\nu_{\widehat{B}}^2)$ described in the proof of Proposition 4.1, we conclude that, for any indecomposable projective-injective module P in $\text{mod } A[I]$, the top $\text{top}(P)$ of P is not isomorphic to the socle $\text{soc}(P)$ of P . Since the Nakayama permutations of A and $A[I]$ coincide, we conclude that for any indecomposable projective-injective module Q in $\text{mod } A$, we have that the $\text{top}(Q)$ of Q is not isomorphic to the socle $\text{soc}(Q)$ of Q . In particular, if ν is the Nakayama permutation of A (so of the set $\{1, \dots, n\}$), and e_i is a primitive summand of e (so $i \in \{1, \dots, m\}$) then $e_i \neq e_{\nu(i)}$. Applying [58, Proposition 3.2] we then conclude that the algebras A and $A[I]$ are isomorphic. Therefore, A is

isomorphic to $\widehat{B}/(\varphi\nu_{\widehat{B}}^2)$. This proves that the implication (i) \Rightarrow (ii) in Theorem 1.10 holds.

We end this section with comments concerning the structure of the Auslander-Reiten quivers of self-injective algebras having regular acyclic components without external short paths.

Let H be a wild hereditary algebra, T a multiplicity-free regular tilting H -module, $B = \text{End}_H(T)$ the associated tilted algebra, φ a positive automorphism of \widehat{B} , and $A = \widehat{B}/(\varphi\nu_{\widehat{B}}^2)$. Moreover, let $Q = Q_H$ be the valued quiver of H and $\Delta = Q^{\text{op}}$. We use the notation introduced in the proof of Proposition 4.1 for description the structure of the Auslander-Reiten quiver $\Gamma_{\widehat{B}}$ of the repetitive category \widehat{B} of B . Since the push-down functor $F_\lambda : \text{mod } \widehat{B} \rightarrow \text{mod } A$, associated to the Galois covering $F : \widehat{B} \rightarrow \widehat{B}/(\varphi\nu_{\widehat{B}}^2)$, is dense and preserves the Auslander-Reiten sequences, we conclude that $\Gamma_A = \Gamma_{\widehat{B}/(\varphi\nu_{\widehat{B}}^2)} = \Gamma_{\widehat{B}}/(\varphi\nu_{\widehat{B}}^2)$ and may be visualised as follows



for a positive integer $m \geq 4$, where, for each $i \in \{0, 1, \dots, m-1\}$, $F_\lambda(C_i)$ is an acyclic component with the stable part of the form $\mathbb{Z}\Delta$ and $F_\lambda(\mathcal{R}_i)$ is an infinite family of components with the stable parts of the form $\mathbb{Z}\mathbb{A}_\infty$, containing at least one indecomposable projective module and a simple module. Concerning the distribution of simple and projective modules in the components $F_\lambda(C_i)$, $i \in \{0, 1, \dots, m-1\}$, we have two cases (see [12, Section 5]).

Case 1. The connecting component \mathcal{C}_T of Γ_B contains a simple B -module. Then $m = 2r$, for $r \geq 2$, $\varphi\nu_{\widehat{B}}^2 = \varrho\nu_{\widehat{B}}^r$ for a rigid automorphism ϱ of \widehat{B} , the components $F_\lambda(\mathcal{C}_{2i})$, $i \in \{0, 1, \dots, r-1\}$, are regular and contain a simple A -module, while the components $F_\lambda(\mathcal{C}_{2i+1})$, $i \in \{0, 1, \dots, r-1\}$, contain a projective module but not a simple module.

We also mention there is always a regular tilting H -module T^* such that the connecting component \mathcal{C}_{T^*} of the tilted algebra $B^* = \text{End}_H(T^*)$ determined by T^* contains a simple B^* -module (see [12, Proposition 2.5]). For a concrete example of such tilted algebra we refer to [22, Example 6.2].

Case 2. The connecting component \mathcal{C}_T of Γ_B does not contain a simple B -module. Then all components $F_\lambda(\mathcal{C}_i)$, $i \in \{0, 1, \dots, m-1\}$, are regular and without simple modules, and consequently all simple and projective modules are located in the parts $F_\lambda(\mathcal{R}_i)$, $i \in \{0, 1, \dots, m-1\}$. We note that it is an open problem if in this case m is even, or equivalently, $\varphi\nu_{\widehat{B}} = \varrho\nu_{\widehat{B}}^r$ for some positive integer $r \geq 2$ and a rigid automorphism ϱ of \widehat{B} .

We also mention that, by [22, Corollary 4], there are infinitely many pairwise non-isomorphic tilted algebras $B^* = \text{End}_H(T^*)$ given by regular tilting H -modules T^* such that the connecting component \mathcal{C}_{T^*} of Γ_{B^*} determined by T^* does not contain simple modules.

Finally, we note that all regular components of the form $\mathbb{Z}\Delta$ in Γ_A have no external short paths. Similarly, for $m \geq 5$, all non-regular components with the stable parts $\mathbb{Z}\Delta$ in Γ_A have no external short paths. On the other hand, for $m = 4$ and \mathcal{C}_T containing a simple module, the non-regular components $F_\lambda(\mathcal{C}_1)$ and $F_\lambda(\mathcal{C}_3)$ have external short paths in $\text{mod } A$.

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