On multipartite invariant states I. Unitary symmetry

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We propose a natural generalization of bipartite Werner and isotropic states to multipartite systems consisting of an arbitrary even number of d-dimensional subsystems (qudits). These generalized states are invariant under the action of local unitary operations. We study basic properties of multipartite invariant states: separability criteria and multi-PPT conditions.

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I. INTRODUCTION

Symmetry plays a prominent role in modern physics. In many cases it enables one to simplify the analysis of the corresponding problems and very often it leads to much deeper understanding and the most elegant mathematical formulation of the corresponding physical theory. In Quantum Information Theory [1] the idea of symmetry was first applied by Werner [2] to construct an important family of bipartite $d \otimes d$ quantum states which are invariant under the following local unitary operations

$$\rho \ \longrightarrow \ U \otimes U \, \rho \, (U \otimes U)^\dagger \ , \qquad (1)$$

for any $U \in U(d)$, where $U(d)$ denotes the group of unitary $d \times d$ matrices. Another family of symmetric states (so called isotropic states [3]) is governed by the following invariance rule

$$\rho \ \longrightarrow \ U \otimes \overline{U} \, \rho \, (U \otimes \overline{U})^\dagger \ , \qquad (2)$$

where $\overline{U}$ is the complex conjugate of U in some basis.

In the present paper we propose a natural generalization of these two families of symmetric states to $2K$ partite quantum systems. A generalization is straightforward: instead of 2 d-dimensional systems (qudits), say Alice–Bob pair $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^d$, we introduce $2K$ qudits with the total space $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_{2K} = (\mathbb{C}^d)^{\otimes 2K}$. We may still interpret the total system as a bipartite one with $\mathcal{H}_A = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_K$ and $\mathcal{H}_B = \mathcal{H}_{K+1} \otimes \ldots \otimes \mathcal{H}_{2K}$. Equivalently, we may introduce K Alices and K Bobs with $\mathcal{H}_{A_i} = \mathcal{H}_i$ and $\mathcal{H}_{B_i} = \mathcal{H}_{K+i}$, respectively. Then $\mathcal{H}_A$ and $\mathcal{H}_B$ stand for the composite K Alices' and Bobs' spaces. Now, we call a $2K$ partite quantum state a Werner state state iff it is invariant under (1) in each Alice-Bob pair $A_i \otimes B_i$. Similarly, the defining property of the generalized $2K$ partite isotropic state is that it is invariant under (2) in each Alice-Bob pair $A_i \otimes B_i$. Note, that for $K > 1$ one has much more possibilities: the most general invariant state is invariant under (1) in some pairs, say $A_1 \otimes B_1, \ldots, A_L \otimes B_L$ and it is invariant under (2) in the remaining pairs: $A_{L+1} \otimes B_{L+1}, \ldots, A_K \otimes B_K$. There are exactly 2^K different families of invariant $2K$–partite states and for $K = 1$ they reduce to the family of Werner and isotropic states.

We analyze basic properties of these symmetric families. They are not independent but related by a set of 2^K generalized partial transpositions. Interestingly, each family gives rise to $2^K - 1$–dimensional simplex. We formulate the corresponding multi-separability conditions and derive the generalized PPT criterions.

A generalization of Werner states for four and three partite system was considered in [4] and [5]. Here we solve the problem for even number of parties in full generality.

The symmetric states of bipartite systems proved to be very useful in Quantum Information Theory. In particular The Peres-Horodecki PPT criterion [6, 7] turns out to be the sufficient condition for separability for symmetric states. Moreover, they play crucial role in entanglement distillation [8–10]. It is hoped that multipartite invariant state would play similar role in multipartite composite systems. Recently, there is a considerable effort to explore multipartite entanglement [11–16] and symmetric states may serve as a very useful laboratory.

The paper is organized as follows: in Section II we recall basic properties of symmetric states for bipartite systems. For pedagogical reason we first show in Section III how to generalize symmetric states for 4-partite systems and then in Section IV we construct a general symmetric states for an arbitrary even $2K$ number of parties.

In a forthcoming paper we present new classes of multipartite invariant states by relaxing invariance to certain subgroups of $U(d)$.

II. 2–PARTITE INVARIANT STATES

A. Werner state

Werner states [2] play significant role in quantum information theory. Their characteristic property is that they commute with all unitaries of the form $U \otimes U$, that is, they are invariant under (1):

$$\mathcal{W} = U \otimes U \, \mathcal{W} \, (U \otimes U)^\dagger \ . \tag{3}$$

The space of $U \otimes U$–invariant states is spanned by identity $I^{\otimes 2}$ and the flip (permutation) operator $\mathbf{F}(\psi \otimes \varphi) = \varphi \otimes \psi$ defined by

$$\mathbf{F} = \sum_{i,j=1}^{d} |ij\rangle\langle ji| \ . \tag{4}$$

Hence, any $U \otimes U$–invariant operator may be written as $\alpha I + \beta \mathbf{F}$. Let us introduce two projectors

$$Q^0 = \frac{1}{2}(I^{\otimes 2} + \mathbf{F}) \ , \quad Q^1 = \frac{1}{2}(I^{\otimes 2} - \mathbf{F}) \ , \tag{5}$$

i.e. Q^0 (Q^1) is the projector onto the symmetric (anti-symmetric) subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$. Clearly, Q^α are $U \otimes U$–invariant, $Q^\alpha Q^\beta = \delta_{\alpha\beta} Q^\beta$, and $Q^0 + Q^1 = I^{\otimes 2}$.

Now, the bipartite Werner state may written as follows

$$\mathcal{W}_{\mathbf{q}} = q_0 \, \widetilde{Q}^0 + q_1 \, \widetilde{Q}^1 \ , \tag{6}$$

where $\widetilde{Q}^\alpha = Q^\alpha / \mathrm{Tr} Q^\alpha$ and the corresponding fidelities $\mathbf{q} = (q_0, q_1)$ are given by

$$q_\alpha = \mathrm{Tr}(\mathcal{W}_{\mathbf{q}} Q^\alpha) \ , \tag{7}$$

and satisfy $q_\alpha \geq 0$ together with $q_0 + q_1 = 1$. Werner showed that $\mathcal{W}_{\mathbf{q}}$ is separable iff $q_1 \leq 1/2$.

It is evident that an arbitrary bipartite state ρ may be projected onto the $U \otimes U$–invariant subspace of bipartite Werner state by the following *twirl* operation:

$$\mathcal{D}\rho = \int U \otimes U \, \rho \, U^\dagger \otimes U^\dagger \, dU \ , \tag{8}$$

where dU is an invariant normalized Haar measure on $U(d)$, that is, $\mathcal{D}\rho = \mathcal{W}_{\mathbf{q}}$ with fidelities $q_\alpha = \mathrm{Tr}(\rho Q^\alpha)$.

Consider now a partial transposition $(\mathbb{1} \otimes \tau)\rho$ (we denote by $\mathbb{1}$ an identity operation acting on M_d = set of $d \times d$ matrices) of a state ρ. Taking into account that

$$(\mathbb{1} \otimes \tau)\mathbf{F} = d \, P_d^+ \ , \tag{9}$$

where P_d^+ is a 1-dimensional projector corresponding to a canonical maximally entangled state $\psi_d^+ = d^{-1/2} \sum_i |ii\rangle$, that is

$$P_d^+ = \frac{1}{d} \sum_{i,j=1}^{d} |ii\rangle\langle jj| \ , \tag{10}$$

and noting that

$$\mathrm{Tr} \, Q^\alpha = \frac{1}{2} d(d + (-1)^\alpha) \ , \tag{11}$$

one easily finds

$$(\mathbb{1} \otimes \tau)\widetilde{Q}^\alpha = \sum_{\beta=0}^{1} \mathbf{X}_{\alpha\beta} \, \widetilde{P}^\beta \ , \tag{12}$$

where we introduced

$$P^1 = P_d^+ \ , \quad P^0 = I^{\otimes 2} - P^1 \ , \tag{13}$$

together with $\widetilde{P}^\alpha = P^\alpha / \mathrm{Tr} P^\alpha$, and the 2×2 matrix $\mathbf{X}$ reads

$$\mathbf{X} = \frac{1}{d} \begin{pmatrix} d-1 & 1 \\ d+1 & -1 \end{pmatrix} \ . \tag{14}$$

Note, that

$$\sum_{\beta=0}^{1} \mathbf{X}_{\alpha\beta} = 1 \ , \tag{15}$$

but $\mathbf{X}_{11} < 0$ which prevents $\mathbf{X}$ to be a stochastic matrix. The partial transposition of $\mathcal{W}_{\mathbf{q}}$ is therefore given by

$$(\mathbb{1} \otimes \tau)\mathcal{W}_{\mathbf{q}} = \sum_{\alpha=0}^{1} p'_\alpha \, \widetilde{P}^\alpha \ , \tag{16}$$

with $q'_\alpha = \sum_\beta q_\beta \mathbf{X}_{\beta\alpha}$. Hence, $\mathcal{W}_{\mathbf{q}}$ is PPT iff $q'_\alpha \geq 0$ which reproduces well known result $q_1 \leq 1/2$, i.e. Werner states $\mathcal{W}_{\mathbf{q}}$ is separable iff it is PPT.

B. Isotropic state

Consider now another class of bipartite states – so called isotropic states [3] – which are invariant under (2), i.e.

$$\mathcal{I} = U \otimes \overline{U} \, \mathcal{I} \, (U \otimes \overline{U})^\dagger \ . \tag{17}$$

Note that

$$U \otimes \overline{U} \, \rho \, (U \otimes \overline{U})^\dagger = (\mathbb{1} \otimes \tau)\Big[(U \otimes U)(\mathbb{1} \otimes \tau)\rho(U \otimes U)^\dagger\Big] \ . \tag{18}$$

Let us observe that the space of $U \otimes \overline{U}$–invariant states is spanned by P^0 and P^1 defined in (13). Moreover, $P^\alpha P^\beta = \delta_{\alpha\beta} P^\beta$ and $P^0 + P^1 = I^{\otimes 2}$. Therefore, an isotropic state may be written as follows:

$$\mathcal{I}_{\mathbf{p}} = \sum_{\alpha=0}^{1} p_\alpha \widetilde{P}^\alpha \ , \tag{19}$$

where the corresponding fidelities

$$p_\alpha = \mathrm{Tr}(\mathcal{I}_{\mathbf{p}} P^\alpha) \ , \tag{20}$$

satisfy $p_\alpha \geq 0$ and $p_0 + p_1 = 1$. An isotropic state is separable iff $p_1 \leq 1/d$.

In analogy to (8) one may define projector into the space of $U \otimes \overline{U}$–invariant states

$$\mathcal{E}\rho = \int U \otimes \overline{U}\, \rho\, (U \otimes \overline{U})^\dagger\, dU \ , \tag{21}$$

such that for any state ρ one has $\mathcal{E}\rho = \mathcal{I}_{\mathbf{p}}$ with $p_\alpha = \mathrm{Tr}(\rho P^\alpha)$. It follows from from (18) that

$$\mathcal{E} = (\mathbb{1} \otimes \tau) \circ \mathcal{D} \circ (\mathbb{1} \otimes \tau) \ . \tag{22}$$

Finally, it is easy to show that the partial transposition $(\mathbb{1} \otimes \tau)\widetilde{P}^\alpha$ is given by

$$(\mathbb{1} \otimes \tau)\widetilde{P}^\alpha = \sum_{\beta=0}^{1} \mathbf{Y}_{\alpha\beta}\, \widetilde{Q}^\beta \ , \tag{23}$$

where the 2×2 matrix $\mathbf{Y}$ reads

$$\mathbf{Y} = \frac{1}{2} \left(\begin{array}{cc} 1 & 1 \\ 1+d & 1-d \end{array} \right) \ . \tag{24}$$

Comparing (12) and (23) it is evident that $\mathbf{Y} = \mathbf{X}^{-1}$. Now, a state $\mathcal{I}_{\mathbf{p}}$ is PPT iff $p'_\alpha = \sum_\beta p_\beta \mathbf{Y}_{\beta\alpha} \geq 0$, that is iff $p_1 \leq 1/d$. Hence, like a Werner state, an isotropic state is separable iff it is PPT.

III. 2×2–PARTITE INVARIANT STATES

A. Werner state

Consider now the following action of the unitary group $U(d) \times U(d)$ on 4-partite state ρ

$$\rho \ \longrightarrow \ \mathbf{U} \otimes \mathbf{U}\, \rho\, \mathbf{U}^\dagger \otimes \mathbf{U}^\dagger \ , \tag{25}$$

where $\mathbf{U} = (U_1, U_2)$, with $U_i \in U(d)$ and

$$\mathbf{U} \otimes \mathbf{U} = U_1 \otimes U_2 \otimes U_1 \otimes U_2 \ .$$

The 4-dimensional space of $\mathbf{U} \otimes \mathbf{U}$–invariant states is spanned by

$$I^{\otimes 4} \ , \quad I_{1|3}^{\otimes 2} \otimes \mathbf{F}_{2|4} \ , \quad \mathbf{F}_{1|3} \otimes I_{2|4}^{\otimes 2} \ , \quad \mathbf{F}_{1|3} \otimes \mathbf{F}_{2|4} \ ,$$

where $L_{i|j}$ denotes a bipartite operator acting on $\mathcal{H}_i \otimes \mathcal{H}_j$. Hence, for example $I_{1|3}^{\otimes 2} \otimes \mathbf{F}_{2|4}$ denotes the following operator in $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_4$:

$$I_{1|3}^{\otimes 2} \otimes \mathbf{F}_{2|4} = \sum_{i,j=1}^{d} I \otimes |i\rangle\langle j| \otimes I \otimes |j\rangle\langle i| \ .$$

Using Alice-Bob terminology the 4-partite operator $I_{1|3}^{\otimes 2} \otimes \mathbf{F}_{2|4}$ represents identity operator on the first pair $A_1 \otimes B_1$ and the operator $\mathbf{F}$ acting on the second pair $A_2 \otimes B_2$.

However, the more convenient way to parameterize $\mathbf{U} \otimes \mathbf{U}$–invariant subspace is to introduce the following 4-partite orthogonal projectors:

$$\begin{aligned} \mathbf{Q}^0 &= Q_{1|3}^0 \otimes Q_{2|4}^0 \ , \\ \mathbf{Q}^1 &= Q_{1|3}^0 \otimes Q_{2|4}^1 \ , \\ \mathbf{Q}^2 &= Q_{1|3}^1 \otimes Q_{2|4}^0 \ , \\ \mathbf{Q}^3 &= Q_{1|3}^1 \otimes Q_{2|4}^1 \ , \end{aligned} \tag{26}$$

where Q^α are bipartite projectors defined in (5). It is evident that $\mathbf{Q}^i$ are $U \otimes U$–invariant, $\mathbf{Q}^i\mathbf{Q}^j = \delta_{ij}\mathbf{Q}^j$, and $\sum_{i=0}^{3} \mathbf{Q}^i = I^{\otimes 4}$. Now, let us introduce more compact notation: denote by $\boldsymbol{\alpha}$ a binary 2-dimensional vector, i.e. $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with $\alpha_i \in \{0, 1\}$. Clearly, any binary vector $\boldsymbol{\alpha}$ defines an integer number which can be written in binary notation as $\alpha_1\alpha_2$. Using this notation the family (26) may be rewritten in a compact form as follows:

$$\mathbf{Q}^{\boldsymbol{\alpha}} = Q_{1|3}^{\alpha_1} \otimes Q_{2|4}^{\alpha_2} \ . \tag{27}$$

A 4-partite Werner state is defined by

$$\mathcal{W}_{\mathbf{q}}^{(2)} = \sum_{i=0}^{3} q_i \widetilde{\mathbf{Q}}^i \ \equiv \ \sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha}} \widetilde{\mathbf{Q}}^{\boldsymbol{\alpha}} \ , \tag{28}$$

where $\widetilde{\mathbf{Q}}^{\boldsymbol{\alpha}} = \mathbf{Q}^{\boldsymbol{\alpha}}/\mathrm{Tr}\mathbf{Q}^{\boldsymbol{\alpha}}$, and the corresponding fidelities

$$q_{\boldsymbol{\alpha}} = \mathrm{Tr}(\mathcal{W}_{\mathbf{q}}^{(2)} \mathbf{Q}^{\boldsymbol{\alpha}}) \geq 0 \ , \tag{29}$$

satisfy $\sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha}} = 1$. Note, that

$$\widetilde{\mathbf{Q}}^{\boldsymbol{\alpha}} = \widetilde{Q}_{1|3}^{\alpha_1} \otimes \widetilde{Q}_{2|4}^{\alpha_2} \ , \tag{30}$$

and hence, using (11), one obtains

$$\begin{aligned} \mathrm{Tr}\, \mathbf{Q}^{\boldsymbol{\alpha}} &= \left(\frac{d}{2}\right)^2 (d + (-1)^{\alpha_1})(d + (-1)^{\alpha_2}) \\ &= \left(\frac{d}{2}\right)^2 (d-1)^{|\boldsymbol{\alpha}|}(d+1)^{2-|\boldsymbol{\alpha}|} \ , \end{aligned} \tag{31}$$

where $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 \in \{0, 1, 2\}$.

This way the space of 4-partite-Werner states defines 3–dimensional simplex. The vertices of this simplex correspond to $\widetilde{\mathbf{Q}}^{\boldsymbol{\alpha}}$.

It is evident that an arbitrary 4-partite state ρ may be projected onto the $\mathbf{U} \otimes \mathbf{U}$–invariant subspace of 4-partite Werner state by the following *twirl* operation:

$$\mathcal{D}^{(2)}\rho = \int \mathbf{U} \otimes \mathbf{U}\, \rho\, \mathbf{U}^\dagger \otimes \mathbf{U}^\dagger\, d\mathbf{U} \ , \tag{32}$$

where $d\mathbf{U} = dU_1 dU_2$ is an invariant normalized Haar measure on $U(d)^2$, that is, $\mathcal{D}^{(2)}\rho = \mathcal{W}^{(2)}_{\mathbf{q}}$ with fidelities $q_{\boldsymbol{\alpha}} = \mathrm{Tr}(\rho \mathbf{Q}^{\boldsymbol{\alpha}})$.

To find the corresponding separability criteria note that $\mathcal{W}^{(2)}_{\mathbf{q}}$ is separable iff there exists a separable state ρ such that $\mathcal{D}^{(2)}\rho = \mathcal{W}^{(2)}_{\mathbf{q}}$. Let ρ be an extremal separable state of the form

$$\rho = P_{\psi_1} \otimes P_{\psi_2} \otimes P_{\varphi_1} \otimes P_{\varphi_2} \ , \tag{33}$$

where $P_\psi = |\psi\rangle\langle\psi|$, and ψ_i, φ_i are normalized vectors in $\mathbb{C}^d$. An arbitrary 4-separable state is a convex combination of the extremal states of the form (33). One easily finds for fidelities $\mathrm{Tr}(\rho \mathbf{Q}^{\boldsymbol{\alpha}})$:

$$\begin{aligned} q_0 &= q_{(00)} = \frac{1}{4}(1+a_1)(1+a_2) \ , \\ q_1 &= q_{(01)} = \frac{1}{4}(1+a_1)(1-a_2) \ , \\ q_2 &= q_{(10)} = \frac{1}{4}(1-a_1)(1+a_2) \ , \\ q_3 &= q_{(11)} = \frac{1}{4}(1-a_1)(1-a_2) \ , \end{aligned} \tag{34}$$

with

$$a_1 = |\langle \psi_1 | \varphi_1 \rangle|^2 \ , \qquad a_2 = |\langle \psi_2 | \varphi_2 \rangle|^2 \ . \tag{35}$$

These formulae may be rewritten in a compact form as follows:

$$q_{\boldsymbol{\alpha}} = \frac{1}{4}(1 + (-1)^{\alpha_1} a_1)(1 + (-1)^{\alpha_2} a_2) \ . \tag{36}$$

Now, since $a_i \leq 1$, the projection $\mathcal{D}^{(2)}$ of the convex hull of extremal separable states gives therefore

$$q_{00} \leq 1 \ , \qquad q_{01}, q_{10} \leq \frac{1}{2} \ , \qquad q_{11} \leq \frac{1}{4} \ , \tag{37}$$

together with

$$q_{11} \leq q_{01}, q_{10} \leq q_{00} \ . \tag{38}$$

Note, that using binary notation equations (37) may be compactly rewritten as follows

$$q_{\boldsymbol{\alpha}} \leq \frac{1}{2^{|\boldsymbol{\alpha}|}} \ . \tag{39}$$

B. Isotropic state

Now, in analogy to the bipartite case we may define a 4-partite isotropic state $\mathcal{I}^{(2)}_{\mathbf{p}}$ which is invariant under

$$\rho' = \mathbf{U} \otimes \overline{\mathbf{U}}\, \rho\, (\mathbf{U} \otimes \overline{\mathbf{U}})^\dagger \ , \tag{40}$$

with $\mathbf{U} \otimes \overline{\mathbf{U}} = U_1 \otimes U_2 \otimes \overline{U}_1 \otimes \overline{U}_2$. The recipe is very simple: starting from (26) we may replace both Q's by P's defined in (13). One obtains the following family of orthogonal projectors:

$$\begin{aligned} \mathbf{P}^0 &= P^0_{1|3} \otimes P^0_{2|4} \ , \\ \mathbf{P}^1 &= P^0_{1|3} \otimes P^1_{2|4} \ , \\ \mathbf{P}^2 &= P^1_{1|3} \otimes P^0_{2|4} \ , \\ \mathbf{P}^3 &= P^1_{1|3} \otimes P^1_{2|4} \ . \end{aligned} \tag{41}$$

It is evident that

$$\mathbf{U} \otimes \overline{\mathbf{U}}\, \mathbf{P}^i\, (\mathbf{U} \otimes \overline{\mathbf{U}})^\dagger = \mathbf{P}^i \ . \tag{42}$$

Moreover, one has $\mathbf{P}^i\mathbf{P}^j = \delta_{ij}\mathbf{P}^j$, and $\sum_{i=0}^3 \mathbf{P}^i = I^{\otimes 4}$. Therefore, any $\mathbf{U} \otimes \overline{\mathbf{U}}$–invariant state may be written as follows

$$\mathcal{I}^{(2)}_{\mathbf{p}} = \sum_{i=0}^{3} p_i \widetilde{\mathbf{P}}^i \equiv \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \widetilde{\mathbf{P}}^{\boldsymbol{\alpha}} \ , \tag{43}$$

where as usual $\widetilde{A} = A/\mathrm{Tr}A$, and

$$\mathbf{P}^{\boldsymbol{\alpha}} = P^{\alpha_1}_{1|3} \otimes P^{\alpha_2}_{2|4} \ . \tag{44}$$

One easily finds

$$\mathrm{Tr}\, \mathbf{P}^{\boldsymbol{\alpha}} = (d^2-1)^{2-|\boldsymbol{\alpha}|} \ . \tag{45}$$

The fidelities

$$p_{\boldsymbol{\alpha}} = \mathrm{Tr}(\mathcal{I}^{(2)}_{\mathbf{p}} \mathbf{P}^{\boldsymbol{\alpha}}) \geq 0 \ , \tag{46}$$

satisfy $\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} = 1$.

Denote by $\mathcal{E}^{(2)}$ on orthogonal projector onto the space of $\mathbf{U} \otimes \overline{\mathbf{U}}$–invariant states

$$\mathcal{E}^{(2)}\rho = \int \mathbf{U} \otimes \overline{\mathbf{U}}\, \rho\, \mathbf{U}^\dagger \otimes \overline{\mathbf{U}}^\dagger \, d\mathbf{U} \ . \tag{47}$$

It is evident that

$$\mathcal{E}^{(2)} = (\mathbb{1} \otimes \mathbb{1} \otimes \tau \otimes \tau) \circ \mathcal{D}^{(2)} \circ (\mathbb{1} \otimes \mathbb{1} \otimes \tau \otimes \tau) \ . \tag{48}$$

Now, an isotropic state $\mathcal{I}^{(2)}_{\mathbf{p}}$ is separable iff there exists a separable state ρ such that $\mathcal{E}^{(2)}\rho = \mathcal{I}^{(2)}_{\mathbf{p}}$. Let us consider an extremal separable state $(\mathbb{1} \otimes \mathbb{1} \otimes \tau \otimes \tau)\rho$ with ρ defined in (33), i.e. i.e.

$$(\mathbb{1} \otimes \mathbb{1} \otimes \tau \otimes \tau)\rho = P_{\psi_1} \otimes P_{\psi_2} \otimes P^T_{\varphi_1} \otimes P^T_{\varphi_2} \ , \tag{49}$$

and define the isotropic state $\mathcal{E}^{(2)}(P_{\psi_1} \otimes P_{\psi_2} \otimes P^T_{\varphi_1} \otimes P^T_{\varphi_2})$. One easily finds for fidelities:

$$\begin{aligned} p_0 &= p_{(00)} = (1-b_1)(1-b_2) \ , \\ p_1 &= p_{(01)} = b_1(1-b_2) \ , \\ p_2 &= p_{(10)} = (1-b_1)b_2 \ , \\ p_3 &= p_{(11)} = b_1 b_2 \ , \end{aligned} \tag{50}$$

or equivalently

$$p_{\boldsymbol{\alpha}} = (1-[\alpha_1+(-1)^{\alpha_1}b_1])(1-[\alpha_2+(-1)^{\alpha_2}b_2]) \ , \quad (51)$$

with

$$b_i = \frac{a_i}{d} = \frac{|\langle\psi_i|\varphi_i\rangle|^2}{d} \ . \quad (52)$$

Now, since $b_i \leq 1/d$, the projection $\mathcal{E}^{(2)}$ of the convex hull of extremal separable states gives therefore

$$p_{00} \leq 1 \ , \quad p_{01}, p_{10} \leq \frac{1}{d} \ , \quad p_{11} \leq \frac{1}{d^2} \ , \quad (53)$$

or more compactly in binary notation

$$p_{\boldsymbol{\alpha}} \leq \frac{1}{d^{|\boldsymbol{\alpha}|}} \ , \quad (54)$$

and

$$p_{11} \leq p_{01}, p_{10} \leq p_{00} \ . \quad (55)$$

C. $\boldsymbol{\sigma}$–invariant states

Let us observe that in $\mathcal{H}_A \otimes \mathcal{H}_B$ we may define not only the partial transposition $\mathbb{1}\otimes\mathbb{1}\otimes\tau\otimes\tau$ considered in the previous Section but also the following ones:

$$\begin{aligned} \tau_1 &= (\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\tau) \ , \quad (56)\\ \tau_2 &= (\mathbb{1}\otimes\mathbb{1}\otimes\tau\otimes\mathbb{1}) \ . \quad (57)\end{aligned}$$

All partial transpositions in Alice-Bob system may be conveniently denoted by

$$\tau_{\boldsymbol{\sigma}} = \mathbb{1}\otimes\mathbb{1}\otimes\tau^{\sigma_1}\otimes\tau^{\sigma_2} \ , \quad (58)$$

where

$$\tau^{\alpha} = \left\{ \begin{array}{ll} \mathbb{1} \ , & \alpha = 0 \\ \tau \ , & \alpha = 1 \end{array} \right. \ . \quad (59)$$

Clearly, for $\boldsymbol{\sigma} = (0,0)$ one has trivial operation $\tau_{(00)} = \mathbb{1}^{\otimes 4}$, whereas $\tau_{(01)} = \tau_1$, $\tau_{(10)} = \tau_2$ and $\tau_{(11)}$ reproduces double partial transposition $\mathbb{1}\otimes\mathbb{1}\otimes\tau\otimes\tau$.

We call a 4-partite state ρ a $\boldsymbol{\sigma}$–invariant iff $\tau_{\boldsymbol{\sigma}}\rho$ is $\mathbf{U}\otimes\mathbf{U}$–invariant i.e.

$$(\mathbf{U}\otimes\mathbf{U})(\tau_{\boldsymbol{\sigma}}\rho)(\mathbf{U}\otimes\mathbf{U})^\dagger = \tau_{\boldsymbol{\sigma}}\rho \ . \quad (60)$$

To characterize $\boldsymbol{\sigma}$–invariant states let us define the following families of projectors:

$$\begin{aligned} \mathbf{\Pi}^0_{(1)} &= Q^0_{1|3}\otimes P^0_{2|4} \ , \\ \mathbf{\Pi}^1_{(1)} &= Q^0_{1|3}\otimes P^1_{2|4} \ , \\ \mathbf{\Pi}^2_{(1)} &= Q^1_{1|3}\otimes P^0_{2|4} \ , \quad (61)\\ \mathbf{\Pi}^3_{(1)} &= Q^1_{1|3}\otimes P^1_{2|4} \ , \end{aligned}$$

and

$$\begin{aligned} \mathbf{\Pi}^0_{(2)} &= P^0_{1|3}\otimes Q^0_{2|4} \ , \\ \mathbf{\Pi}^1_{(2)} &= P^0_{1|3}\otimes Q^1_{2|4} \ , \\ \mathbf{\Pi}^2_{(2)} &= P^1_{1|3}\otimes Q^0_{2|4} \ , \quad (62)\\ \mathbf{\Pi}^3_{(2)} &= P^1_{1|3}\otimes Q^1_{2|4} \ . \end{aligned}$$

Let us observe that 4 families: $\mathbf{Q}^{\boldsymbol{\alpha}}$, $\mathbf{P}^{\boldsymbol{\alpha}}$, $\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(1)}$ and $\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(2)}$ may be compactly written as

$$\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})} = \Pi^{\alpha_1}_{(\sigma_1)1|3}\otimes\Pi^{\alpha_2}_{(\sigma_2)2|4} \ , \quad (63)$$

where

$$\Pi^{\alpha}_{(\sigma)} = \left\{ \begin{array}{ll} Q^{\alpha} \ , & \sigma = 0 \\ P^{\alpha} \ , & \sigma = 1 \end{array} \right. \ , \quad (64)$$

that is,

$$\begin{aligned} \mathbf{\Pi}^{\boldsymbol{\alpha}}_{(00)} &= \mathbf{Q}^{\boldsymbol{\alpha}} \ , & \mathbf{\Pi}^{\boldsymbol{\alpha}}_{(01)} &= \mathbf{\Pi}^{\boldsymbol{\alpha}}_{(1)} \ , \\ \mathbf{\Pi}^{\boldsymbol{\alpha}}_{(10)} &= \mathbf{\Pi}^{\boldsymbol{\alpha}}_{(2)} \ , & \mathbf{\Pi}^{\boldsymbol{\alpha}}_{(11)} &= \mathbf{P}^{\boldsymbol{\alpha}} \ . \end{aligned}$$

One easily shows that

1. $\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})}$ are $\boldsymbol{\sigma}$–invariant,
2. $\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})}\cdot\mathbf{\Pi}^{\boldsymbol{\beta}}_{(\boldsymbol{\sigma})} = \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}\,\mathbf{\Pi}^{\boldsymbol{\beta}}_{(\boldsymbol{\sigma})}$,
3. $\sum_{\boldsymbol{\alpha}} \mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})} = \mathbb{1}^{\otimes 4}$.

It is therefore clear that any $\boldsymbol{\sigma}$–invariant state may be written as follows:

$$\mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}} = \sum_{\boldsymbol{\alpha}} f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}}\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})} \ , \quad (65)$$

where the corresponding fidelities

$$f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} = \mathrm{Tr}(\mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}}\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})}) \ , \quad (66)$$

satisfy $\sum_{\boldsymbol{\alpha}} f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} = 1$. Clearly, one has $f^{(00)}_{\boldsymbol{\alpha}} = q_{\boldsymbol{\alpha}}$ and $f^{(11)}_{\boldsymbol{\alpha}} = p_{\boldsymbol{\alpha}}$.

Now, to check for separability conditions note that $\mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}}$ is separable iff there exists a separable state ρ such that $\mathcal{D}^{(2)}_{\boldsymbol{\sigma}}\rho$ is separable, where

$$\mathcal{D}^{(2)}_{\boldsymbol{\sigma}} = \tau_{\boldsymbol{\sigma}}\circ\mathcal{D}^{(2)}\circ\tau_{\boldsymbol{\sigma}} \ , \quad (67)$$

denotes the projector onto the subspace of $\boldsymbol{\sigma}$–invariant states. It is evident that $\mathcal{D}^{(2)}_{(00)} = \mathcal{D}^{(2)}$ and $\mathcal{D}^{(2)}_{(11)} = \mathcal{E}^{(2)}$. In analogy to (34) and (50) one easily finds for fidelities corresponding to $\mathcal{D}^{(2)}_{(01)}(\rho)$ with ρ being en extremal separable state (33):

$$\begin{aligned} f^{(01)}_{(00)} &= \frac{1}{2}(1+a_1)(1-b_2) \ , \\ f^{(01)}_{(01)} &= \frac{1}{2}(1+a_1)b_2 \ , \\ f^{(01)}_{(10)} &= \frac{1}{2}(1-a_1)(1-b_2) \ , \quad (68)\\ f^{(01)}_{(11)} &= \frac{1}{2}(1-a_1)b_2 \ , \end{aligned}$$

and similarly for $\mathcal{D}^{(2)}_{(10)}(\rho)$

$$\begin{aligned} f^{(10)}_{(00)} &= \frac{1}{2}\,(1-b_1)(1+a_2)\ , \\ f^{(10)}_{(01)} &= \frac{1}{2}\,(1-b_1)(1-a_2)\ , \\ f^{(10)}_{(10)} &= \frac{1}{2}\,b_1(1+a_2)\ , \\ f^{(10)}_{(11)} &= \frac{1}{2}\,b_1(1+a_2)\ . \end{aligned} \tag{69}$$

The projection $\mathcal{D}^{(2)}_{\boldsymbol{\sigma}}$ of the convex hull of extremal separable states gives therefore

$$f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} \leq \frac{1}{2^{|\boldsymbol{\alpha}|}}\left(\frac{2}{d}\right)^{|\boldsymbol{\sigma\alpha}|}\ , \tag{70}$$

where $\boldsymbol{\sigma\alpha} = (\sigma_1\alpha_1, \sigma_2\alpha_2)$, and

$$f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} \leq f^{(\boldsymbol{\sigma})}_{\boldsymbol{\beta}}\ , \quad \text{for} \quad |\boldsymbol{\alpha}| > |\boldsymbol{\beta}|\ , \tag{71}$$

which generalize (38)–(39) and (54)–(55).

D. $\boldsymbol{\sigma}$–PPT states

We call a 4-partite state ρ in $\mathcal{H}_A\otimes\mathcal{H}_B = \mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\mathcal{H}_{B_1}\otimes\mathcal{H}_{B_2}$ a $\boldsymbol{\sigma}$–PPT iff

$$\tau_{\boldsymbol{\sigma}}\rho \geq 0\ . \tag{72}$$

Now, if O is $\boldsymbol{\nu}$–invariant operator in $\mathcal{H}_A\otimes\mathcal{H}_B$, then $\tau_{\boldsymbol{\mu}}O$ is $(\boldsymbol{\mu}\oplus\boldsymbol{\nu})$–invariant, where $\oplus$ denotes addition mod 2. Writing O as

$$O = \sum_{\boldsymbol{\alpha}} o_{\boldsymbol{\alpha}} \widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}_{(\boldsymbol{\nu})}\ , \tag{73}$$

one has

$$\tau_{\boldsymbol{\mu}}O = \sum_{\boldsymbol{\alpha}} o_{\boldsymbol{\alpha}}\tau_{\boldsymbol{\mu}} \widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}_{(\boldsymbol{\nu})}\ . \tag{74}$$

One easily computes the $\boldsymbol{\mu}$–partial transposition of $\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}_{(\boldsymbol{\nu})}$:

$$\tau_{\boldsymbol{\mu}}\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}_{(\boldsymbol{\nu})} = \sum_{\boldsymbol{\beta}} \mathbf{Z}^{\boldsymbol{\alpha\beta}}_{(\boldsymbol{\mu}|\boldsymbol{\nu})}\, \widetilde{\mathbf{\Pi}}^{\boldsymbol{\beta}}_{(\boldsymbol{\mu}\oplus\boldsymbol{\nu})}\ , \tag{75}$$

where the 4×4 matrix $\mathbf{Z}_{(\boldsymbol{\mu}|\boldsymbol{\nu})}$ is defined as follows:

$$\mathbf{Z}_{(\boldsymbol{\mu}|\boldsymbol{\nu})} = \mathbf{Z}_{(\mu_1|\nu_1)}\otimes\mathbf{Z}_{(\mu_2|\nu_2)}\ , \tag{76}$$

with

$$\mathbf{Z}_{(\mu|\nu)} = \begin{cases} \mathbf{I}\ , & \mu=0\ ,\quad \nu=0,1 \\ \mathbf{X}\ , & \mu=1\ ,\quad \nu=0 \\ \mathbf{Y}\ , & \mu=1\ ,\quad \nu=1 \end{cases}\ , \tag{77}$$

and $\mathbf{I}$ denotes 2×2 unit matrix. Matrices $\mathbf{X}$ and $\mathbf{Y}$ are defined in (14) and (24), respectively. The corresponding matrix elements are defined in an obvious way

$$(\mathbf{A}\otimes\mathbf{B})^{\boldsymbol{\alpha\beta}} = \mathbf{A}^{\alpha_1\beta_1}\,\mathbf{B}^{\alpha_2\beta_2}\ .$$

The structure of $\mathbf{Z}_{(\mu|\nu)}$ is encoded into the following table:

$\boldsymbol{\mu}\backslash\ \boldsymbol{\nu}$	(00)	(01)	(10)	(11)
(00)	$\mathbf{I}\otimes\mathbf{I}$	$\mathbf{I}\otimes\mathbf{I}$	$\mathbf{I}\otimes\mathbf{I}$	$\mathbf{I}\otimes\mathbf{I}$
(01)	$\mathbf{I}\otimes\mathbf{X}$	$\mathbf{I}\otimes\mathbf{Y}$	$\mathbf{I}\otimes\mathbf{X}$	$\mathbf{I}\otimes\mathbf{Y}$
(10)	$\mathbf{X}\otimes\mathbf{I}$	$\mathbf{X}\otimes\mathbf{I}$	$\mathbf{Y}\otimes\mathbf{I}$	$\mathbf{Y}\otimes\mathbf{I}$
(11)	$\mathbf{X}\otimes\mathbf{X}$	$\mathbf{X}\otimes\mathbf{Y}$	$\mathbf{Y}\otimes\mathbf{X}$	$\mathbf{Y}\otimes\mathbf{Y}$

Now, if $\boldsymbol{\nu}$–invariant operator O is semi-positive, i.e. $o_{\boldsymbol{\alpha}}\geq 0$, then O is $\boldsymbol{\mu}$–PPT iff

$$\sum_{\boldsymbol{\beta}} o_{\boldsymbol{\beta}}\,\mathbf{Z}^{\boldsymbol{\beta\alpha}}_{(\boldsymbol{\mu}|\boldsymbol{\nu})} \geq 0\ , \tag{78}$$

for all binary 2-vectors $\boldsymbol{\alpha}$.

In particular one may look for the $\boldsymbol{\sigma}$–PPT conditions for the 4-partite Werner state. One easily finds that

1. $\mathcal{W}_{\mathbf{q}}$ is (01)–PPT iff

$$q_{00}\geq q_{01}\ ,\quad q_{10}\geq q_{11}\ , \tag{79}$$

2. $\mathcal{W}_{\mathbf{q}}$ is (10)–PPT iff

$$q_{00}\geq q_{10}\ ,\quad q_{01}\geq q_{11}\ , \tag{80}$$

3. $\mathcal{W}_{\mathbf{q}}$ is (11)–PPT iff

$$\begin{aligned} (d-1)(q_{00}-q_{01})+(d+1)(q_{10}-q_{11}) &\geq 0\ , \\ (d-1)(q_{00}-q_{10})+(d+1)(q_{01}-q_{11}) &\geq 0\ , \\ (q_{00}+q_{11})-(q_{01}+q_{10}) &\geq 0\ . \end{aligned} \tag{81}$$

Note that PPT conditions (79)–(81) imply

$$q_{11}\leq q_{01}, q_{10}\leq q_{00}\ , \tag{82}$$

which reproduces (38), together with

$$q_{01}+q_{10}\leq q_{00}+q_{11}\ , \tag{83}$$

which is equivalent to

$$q_{01}+q_{10}\leq \frac{1}{2}\ . \tag{84}$$

Now, (82) and (84) imply

$$2q_{11}\leq q_{01}+q_{10}\leq \frac{1}{2}\ , \tag{85}$$

and hence

$$q_{11} \leq \frac{1}{4} \ , \tag{86}$$

which, together with

$$q_{01}, q_{10} \leq \frac{1}{2} \ , \tag{87}$$

reproduces (39). This shows that 4-partite Werner state is 4-separable iff it is $\boldsymbol{\sigma}$–PPT for all binary vectors $\boldsymbol{\sigma}$. Interestingly, one may prove (see Appendix) that 4-partite Werner state is 12|34 (or $A\otimes B$) bi-separable iff it is (11)–PPT.

One may perform similar analysis for other invariant states. Again, a $\boldsymbol{\mu}$-invariant state is 4-separable iff it is $\boldsymbol{\nu}$–PPT for all binary vectors $\boldsymbol{\nu}$. It is $A\otimes B$ bi-separable iff it is (11)–PPT.

E. Reductions

It is clear that reducing 4-partite invariant state with respect to the pair $A_1 \otimes B_1$ ($A_2\otimes B_2$) one obtains bipartite invariant state of $A_2\otimes B_2$ ($A_1\otimes B_1$). One easily finds

$$\mathrm{Tr}_{13}\, \mathcal{W}^{(2)}_{\mathbf{q}} = \mathcal{W}_{\mathbf{q}'} \ , \tag{88}$$

with

$$q'_\alpha = \sum_\beta q_{(\beta\alpha)} \ . \tag{89}$$

Similarly,

$$\mathrm{Tr}_{24}\, \mathcal{W}^{(2)}_{\mathbf{q}} = \mathcal{W}_{\mathbf{q}''} \ , \tag{90}$$

with

$$q''_\alpha = \sum_\beta q_{(\alpha\beta)} \ . \tag{91}$$

This observation may be easily generalized to an arbitrary 4-partite invariant state $\mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}}$:

$$\mathrm{Tr}_{13}\, \mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}} = \sum_{\alpha_2} f_{\alpha_2}\, \Pi^{\alpha_2}_{(\sigma_2)} \ , \tag{92}$$

where $\Pi^{\alpha}_{(\sigma)}$ is defined in (64) and

$$f_{\alpha_2} = \sum_{\alpha_1} f_{(\alpha_1,\alpha_2)} \ . \tag{93}$$

Finally, let us observe that a reduction with respect to any other pair produces maximally mixed state of the remaining pair, e.g.

$$\mathrm{Tr}_{12}\, \mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}} = I^{\otimes 2}_{3|4} \ . \tag{94}$$

IV. $2K$–PARTITE INVARIANT STATES

A. General $\boldsymbol{\sigma}$–invariant state

Consider now $2K$–partite system and define the following action of K copies of U(d):

$$\rho' = \mathbf{U}\otimes\mathbf{U}\,\rho\,\mathbf{U}^\dagger\otimes\mathbf{U}^\dagger \ , \tag{95}$$

where $\mathbf{U} = (U_1,\ldots,U_K)$ with $U_i \in U(d)$ and

$$\mathbf{U}\otimes\mathbf{U} = U_1\otimes\ldots U_K\otimes U_1\otimes\ldots U_K \ .$$

A state ρ is $\mathbf{U}\otimes\mathbf{U}$–invariant iff

$$\mathbf{U}\otimes\mathbf{U}\,\rho \ = \ \rho\,\mathbf{U}\otimes\mathbf{U} \ ,$$

for any $\mathbf{U}\in U(d)^K$. Denote by $\mathcal{D}^{(K)}$ the corresponding projector onto the space of $\mathbf{U}\otimes\mathbf{U}$–invariant states

$$\mathcal{D}^{(K)}\rho = \int d\mathbf{U}\, \mathbf{U}\otimes\mathbf{U}\,\rho\,\mathbf{U}^\dagger\otimes\mathbf{U}^\dagger \ , \tag{96}$$

with $d\mathbf{U} = dU_1\ldots dU_K$ being an normalized invariant Haar measure on $U(d)^K$.

Now, let $\boldsymbol{\sigma}$ be a binary K-dimensional vector, i.e. $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_K)$ with $\sigma_j \in \{0,1\}$. For any $\boldsymbol{\sigma}$ one may define $\boldsymbol{\sigma}$–partial transposition on $\mathcal{H}_A\otimes\mathcal{H}_B$ as follows:

$$\tau_{\boldsymbol{\sigma}} = \mathbb{1}^{\otimes K}\otimes\tau^{\sigma_1}\otimes\ldots\otimes\tau^{\sigma_K} \ , \tag{97}$$

where τ^α is defined in (59). We call a state ρ $\boldsymbol{\sigma}$–invariant iff $\tau_{\boldsymbol{\sigma}}\rho$ is $\mathbf{U}\otimes\mathbf{U}$–invariant. The corresponding projector $\mathcal{D}^{(K)}_{\boldsymbol{\sigma}}$ onto the space of $\boldsymbol{\sigma}$–invariant states reads

$$\mathcal{D}^{(K)}_{\boldsymbol{\sigma}} = \tau_{\boldsymbol{\sigma}}\circ\mathcal{D}^{(K)}\circ\tau_{\boldsymbol{\sigma}} \ . \tag{98}$$

To parameterize the space of $\boldsymbol{\sigma}$–invariant states let us introduce the following family of projectors:

$$\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})} = \Pi^{\alpha_1}_{(\sigma_1)1|K+1}\otimes\ldots\otimes\Pi^{\alpha_K}_{(\sigma_K)K|2K} \ , \tag{99}$$

where $\Pi^{\alpha_i}_{(\sigma_i)}$ are defined in (64). It generalizes 4-partite family (63). Note that we have 2^K families parameterized by $\boldsymbol{\sigma}$ each containing 2^K elements.

One easily shows that

1. $\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})}$ are $\boldsymbol{\sigma}$–invariant,
2. $\mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})}\cdot\mathbf{\Pi}^{\boldsymbol{\beta}}_{(\boldsymbol{\sigma})} = \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}\,\mathbf{\Pi}^{\boldsymbol{\beta}}_{(\boldsymbol{\sigma})}$,
3. $\sum_{\boldsymbol{\alpha}} \mathbf{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})} \ = \ \mathbb{1}^{\otimes 2K}$.

It is therefore clear that any $\boldsymbol{\sigma}$–invariant state may be written as follows:

$$\mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}} = \sum_{\boldsymbol{\alpha}} f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}}\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})} \ , \tag{100}$$

where the corresponding fidelities

$$f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} = \mathrm{Tr}(\mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}} \boldsymbol{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})}) \ , \tag{101}$$

satisfy $\sum_{\boldsymbol{\alpha}} f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} = 1$. Hence, the space of $\boldsymbol{\sigma}$–invariant states gives rise to a (2^K-1)–dimensional simplex.

In particular for $\boldsymbol{\sigma} = (0,\ldots,0)$ one obtains a $2K$-partite Werner state

$$\mathcal{W}^{(K)}_{\mathbf{q}} = \sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha}} \widetilde{\mathbf{Q}}^{\boldsymbol{\alpha}} \ , \tag{102}$$

with

$$\widetilde{\mathbf{Q}}^{\boldsymbol{\alpha}} = \widetilde{Q}^{\alpha_1}_{1|K+1} \otimes \ldots \otimes \widetilde{Q}^{\alpha_K}_{K|2K} \ . \tag{103}$$

On the other hand for $\boldsymbol{\sigma} = (1,\ldots,1)$ one obtains $\mathbf{U}\otimes\overline{\mathbf{U}}$–invariant $2K$-partite isotropic state

$$\mathcal{I}^{(K)}_{\mathbf{p}} = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \widetilde{\mathbf{P}}^{\boldsymbol{\alpha}} \ , \tag{104}$$

with

$$\widetilde{\mathbf{P}}^{\boldsymbol{\alpha}} = \widetilde{P}^{\alpha_1}_{1|K+1} \otimes \ldots \otimes \widetilde{P}^{\alpha_K}_{K|2K} \ . \tag{105}$$

B. Separability

To find the corresponding separability conditions for $\boldsymbol{\sigma}$–invariant states let us consider a multi-separable state

$$\rho_{\boldsymbol{\sigma}} = \tau_{\boldsymbol{\sigma}} \rho \ , \tag{106}$$

with ρ being a product state

$$\rho = P_{\psi_1} \otimes \ldots \otimes P_{\psi_K} \otimes P_{\varphi_1} \otimes \ldots \otimes P_{\varphi_K} \ . \tag{107}$$

One easily computes the corresponding fidelities

$$f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} = \mathrm{Tr}(\rho_{\boldsymbol{\sigma}} \boldsymbol{\Pi}^{\boldsymbol{\alpha}}_{(\boldsymbol{\sigma})}) \ , \tag{108}$$

and finds

$$f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} = \frac{1}{2^{K-|\boldsymbol{\sigma}|}} \prod_{i=1}^{K} u_i \ , \tag{109}$$

where

$$u_i = \begin{cases} 1 + (-1)^{\alpha_i}\, a_i \ , & \sigma_i = 0 \\ 1 - [\alpha_i + (-1)^{\alpha_i}\, b_i] \ , & \sigma_i = 1 \end{cases} \ , \tag{110}$$

with

$$a_i = |\langle \psi_i | \varphi_i \rangle|^2 \ , \quad b_i = \frac{a_i}{d} \ . \tag{111}$$

Hence, a $\boldsymbol{\sigma}$–invariant state $\mathcal{I}^{(\boldsymbol{\sigma})}_{\mathbf{f}}$ is multi-separable iff

$$f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} \leq \frac{1}{2^{|\boldsymbol{\alpha}|}} \left(\frac{2}{d}\right)^{|\boldsymbol{\sigma}\boldsymbol{\alpha}|} \ , \tag{112}$$

where $\boldsymbol{\sigma}\boldsymbol{\alpha} = (\sigma_1\alpha_1,\ldots,\sigma_K\alpha_K)$, and

$$f^{(\boldsymbol{\sigma})}_{\boldsymbol{\alpha}} \leq f^{(\boldsymbol{\sigma})}_{\boldsymbol{\beta}} \ , \quad \text{for} \quad |\boldsymbol{\alpha}| > |\boldsymbol{\beta}| \ . \tag{113}$$

In particular for $2K$-partite Werner state, i.e. $\boldsymbol{\sigma} = (0,\ldots,0)$ one has

$$q_{\boldsymbol{\alpha}} \leq \frac{1}{2^{|\boldsymbol{\alpha}|}} \ , \tag{114}$$

whereas for $2K$-partite isotropic state, i.e. $\boldsymbol{\sigma} = (1,\ldots,1)$, one finds

$$p_{\boldsymbol{\alpha}} \leq \frac{1}{d^{|\boldsymbol{\alpha}|}} \ . \tag{115}$$

Finally, one may prove that a general $2K$–partite $\boldsymbol{\mu}$-invariant state is $2K$-separable iff it is $\boldsymbol{\nu}$–PPT for all binary vectors $\boldsymbol{\nu}$ and it is $A\otimes B$ bi-separable iff it is $(1\ldots1)$–PPT.

C. Reductions

It is evident that reducing the $2K$ partite $\boldsymbol{\sigma}$–invariant state with respect to $A_i \otimes B_i$ pair one obtains $2(K-1)$–partite $\boldsymbol{\sigma}_{(i)}$–invariant state with

$$\boldsymbol{\sigma}_{(i)} = (\sigma_1,\ldots,\check{\sigma}_i,\ldots,\sigma_K) \ , \tag{116}$$

where $\check{\sigma}_i$ denotes the omitting of σ_i. The reduced state lives in

$$\mathcal{H}_1 \otimes \ldots \check{\mathcal{H}}_i \otimes \ldots \otimes \check{\mathcal{H}}_{i+K} \otimes \ldots \otimes \mathcal{H}_{2K} \ . \tag{117}$$

The corresponding fidelities are given by

$$f^{(\boldsymbol{\sigma}_{(i)})}_{(\alpha_1\ldots\alpha_{K-1})} = \sum_{\beta} f^{(\boldsymbol{\sigma})}_{(\alpha_1\ldots\alpha_{i-1}\beta\alpha_i\ldots\alpha_{K-1})} \ . \tag{118}$$

Note, that reduction with respect to a 'mixed' pair, say $A_i \otimes B_j$ with $i \neq j$, is equivalent to two 'natural' reductions with respect to $A_i \otimes B_i$ and $A_j \otimes B_j$ and hence it gives rise to $2(K-2)$–partite invariant state. This procedure establishes a natural hierarchy of multipartite invariant states.

Appendix

The 4-partite Werner state $\mathcal{W}^{(2)}_{\mathbf{q}}$ is $12|34$ (or $A \otimes B$) separable iff there exists a bi-separable state ϱ such that $\mathcal{W}^{(2)}_{\mathbf{q}} = \mathcal{D}^{(2)}\varrho$. Consider an extremal $A|B$ separable state $\varrho = P_A \otimes P_B$ where P_A and P_B are bipartite projectors living in $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}_1 \otimes \mathcal{H}_2 \equiv (\mathbb{C}^d)^{\otimes 2}$. Simple calculations give rise to the corresponding fidelities $q_{\boldsymbol{\alpha}} = \mathrm{Tr}(\varrho\, \mathbf{Q}^{\boldsymbol{\alpha}})$:

$$
\begin{aligned}
q_{00} &= \frac{1}{4}\Big\{1 + \mathrm{Tr}_2\left(\mathrm{Tr}_1 P_A \cdot \mathrm{Tr}_1 P_B\right) + \mathrm{Tr}_1\left(\mathrm{Tr}_2 P_A \cdot \mathrm{Tr}_2 P_B\right) + \mathrm{Tr}_{12}(P_A \cdot P_B)\Big\} \ , \\
q_{01} &= \frac{1}{4}\Big\{1 - \mathrm{Tr}_2\left(\mathrm{Tr}_1 P_A \cdot \mathrm{Tr}_1 P_B\right) + \mathrm{Tr}_1\left(\mathrm{Tr}_2 P_A \cdot \mathrm{Tr}_2 P_B\right) - \mathrm{Tr}_{12}(P_A \cdot P_B)\Big\} \ , \\
q_{10} &= \frac{1}{4}\Big\{1 + \mathrm{Tr}_2\left(\mathrm{Tr}_1 P_A \cdot \mathrm{Tr}_1 P_B\right) - \mathrm{Tr}_1\left(\mathrm{Tr}_2 P_A \cdot \mathrm{Tr}_2 P_B\right) - \mathrm{Tr}_{12}(P_A \cdot P_B)\Big\} \ , \\
q_{11} &= \frac{1}{4}\Big\{1 - \mathrm{Tr}_2\left(\mathrm{Tr}_1 P_A \cdot \mathrm{Tr}_1 P_B\right) - \mathrm{Tr}_1\left(\mathrm{Tr}_2 P_A \cdot \mathrm{Tr}_2 P_B\right) + \mathrm{Tr}_{12}(P_A \cdot P_B)\Big\} \ ,
\end{aligned}
\tag{A.1}
$$

where Tr_1 denotes a partial trace in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Therefore, for a general $A|B$ separable state (convex hull of extremal product states) one obtains from (A.1):

$$
q_{01}, q_{10}, q_{11} \leq q_{00} \ , \tag{A.2}
$$

and

$$
q_{01} + q_{10} \leq \frac{1}{2} \ . \tag{A.3}
$$

Note, that above conditions are equivalent to the condition (81) for (11)–PPT. The third equation in (81) implies (A.3) whereas the first (second) and third gives $q_{00} \geq q_{01}$ ($q_{00} \geq q_{10}$). Note, that 4-separable Werner state is necessarily bi-separable but the converse is not true. Taking $\varrho = P_A \otimes P_B$ such that

$$
\mathrm{Tr}_2\left(\mathrm{Tr}_1 P_A \cdot \mathrm{Tr}_1 P_B\right) = \mathrm{Tr}_1\left(\mathrm{Tr}_2 P_A \cdot \mathrm{Tr}_2 P_B\right) \ , \tag{A.4}
$$

and $\mathrm{Tr}_{12}(P_A \cdot P_B) \neq 0$ one obtains a bi-separable Werner state $\mathcal{D}^{(2)}(\varrho)$ with

$$
q_{01} = q_{10} < q_{11} \ , \tag{A.5}
$$

which contradicts 4-separability.

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