Memory in a nonlocally damped oscillator

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Abstract

We analyze the new equation of motion for the damped oscillator. It differs from the standard one by a damping term which is nonlocal in time and hence it gives rise to a system with memory. Both classical and quantum analysis is performed. The characteristic feature of this nonlocal system is that it breaks local composition low for the classical Hamiltonian dynamics and the corresponding quantum propagator.

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1 Introduction

The damped harmonic oscillator is one of the simplest quantum systems displaying the dissipation of energy. Moreover, it is of great physical importance and has found many applications especially in quantum optics. For example it plays a central role in the quantum theory of lasers and masers [1, 2, 3]. The standard Hilbert space formulation of Quantum Mechanics gives rise to the unitary evolution with no room for dissipative behavior. The usual approach to include dissipation is the quantum theory of open systems [4, 5, 6, 7, 8]. In this approach the dynamics of a quantum system is no longer unitary but it is defined by a semigroup of completely positive maps in the space of density operators [9].

Another strategy to describe dissipative quantum systems is based on the old idea of Bateman [10]. Bateman has shown that in order to apply the standard canonical formalism of classical mechanics to dissipative and non-Hamiltonian systems, one can double the number of degrees of freedom, so as to deal with an effective isolated classical Hamiltonian system. In this approach the new degrees of freedom may be interpreted as variables describing a reservoir. Applying this idea to a damped harmonic oscillator one obtains a pair of damped oscillators (so called Bateman's dual system): a primary one and its time reversed image. The Bateman dual Hamiltonian has been rediscovered by Morse and Feshbach [11] and Bopp [12] and the detailed quantum mechanical analysis was performed by Feshbach and Tikochinski [13]. The quantum Bateman system was then analyzed by many authors (see for the historical review [14]). Recently it was investigated in connection with quantum field theory and quantum groups [15] (see also [16, 17]). Finally, the detailed analysis of spectral properties of the Bateman system was recently performed in [18, 19] (see also [20]). For other approaches to quantization of damped oscillator see also recent papers [21, 22, 23].

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In the present paper we propose a different approach. Instead of analyzing the standard equation of motion for the damped oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \ , \tag{1}$$

where γ is a positive damping constant, and ω is a frequency of the undamped motion, we shall study a modified equation

$$\ddot{x} + 2\gamma \tanh(\gamma(t-t_0))\dot{x} + \omega^2 x = 0 \ , \tag{2}$$

where t_0 is an arbitrary constant. It is clear that in the limit '$\gamma t \to \infty$', formula (2) reproduces the standard equation (1). Therefore, both dynamics may differ for short times only, that is, for 't' satisfying

$$t \ll 1/\gamma \ .$$

What is the origin of (2)? It turns out that both (1) and (2) give the same frequency of oscillations of the damped motion[1]

$$\Omega = \sqrt{\omega^2 - \gamma^2} \ . \tag{3}$$

Moreover, there is no other linear equation of motion with this property. Interestingly, equation (2) introduces additional parameter t_0. It turns out that t_0 is responsible for an effective memory, that is, in addition to a 'local time t' evolution governed by (2) 'remembers' about its history at t_0. It means that the initial value problem for (2) has to be supplemented by initial data $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$. There is a crucial difference between (1) and (2): the initial data for the standard 'local' equation may be provided for arbitrary initial moment of time. It is no longer true for the modified 'nonlocal' equation: now the initial moment necessarily equals t_0. Changing initial moment t_0 one changes the corresponding equation (2) and hence the whole evolution of our damped system. This property is responsible for a peculiar feature of (2). The composition law for classical dynamics is no longer true, that is, evolution from t_0 to t_1, and then from t_1 to t_2 differs from the direct evolution from t_0 to t_2.

Now, the quantum system corresponding to (2) may be easily constructed. Our damped system turns out to be governed by nonlocal Hamiltonian and hence the standard composition law for quantum propagators

$$K(x_2,t_2;x_0,t_0) = \int dx_1 K(x_2,t_2;x_1,t_1)K(x_1,t_1;x_0,t_0) \ , \tag{4}$$

does not hold. We stress that the local composition law (4) is satisfied by the quantum propagator derived from the standard equation (1) via well known Caldirola-Kanai approach [24, 25]. The violation of (4) defines the basic difference between two descriptions based on (1) and (2), respectively. We hope that our analysis sheds new light on this old problem.

2 Hamiltonian formulation

There were many attempts for hamiltonian formulation of Newton equation for systems with one degree of freedom (see e.g. [25, 26, 27, 28, 29]). It is well known that (1) may be derived from the so called Caldirola-Kanai Hamiltonian [24, 25]

$$H(x,p,t) \ = \ \frac{1}{2}m_0\omega^2 x^2\, e^{2\gamma t} + \frac{1}{2m_0}p^2\, e^{-2\gamma t} \ . \tag{5}$$

[1] Throughout the paper we assume $\omega > \gamma$.

It is clear that Caldirola-Kanai Hamiltonian may be considered as the standard Hamiltonian of the harmonic oscillator with time-dependent mass given by

$$m(t) = m_0 e^{2\gamma t} \ . \tag{6}$$

Consider now a family of Hamiltonians with an arbitrary time-dependent mass $m(t)$:

$$H(x,p,t) = \frac{p^2}{2m(t)} + \frac{1}{2} m(t)\omega^2 x^2 \ . \tag{7}$$

The corresponding Newton equation reads as follows

$$\ddot{x} + \frac{\dot{m}(t)}{m(t)}\,\dot{x} + \omega^2 x = 0 \ . \tag{8}$$

Let us assume that in the asymptotic regime '$\gamma t \to \infty$', equation (8) reproduces the standard equation (1), that is

$$\kappa(t) := \frac{1}{2}\,\frac{\dot{m}(t)}{m(t)} \longrightarrow \gamma \ . \tag{9}$$

To solve (8) one represents complexified $x(t)$ as follows

$$x(t) = \lambda(t)\, e^{i\phi(t)} \ , \tag{10}$$

and obtains the following equations for time-dependent real quantities λ, ϕ and κ:

$$\ddot{\lambda} - \lambda\dot{\phi}^2 + 2\kappa\dot{\lambda} + \omega^2\lambda \ = \ 0 \ , \tag{11}$$

$$\ddot{\phi}\lambda + 2\dot{\phi}\dot{\lambda} + 2\kappa\dot{\phi}\lambda \ = \ 0 \, , \tag{12}$$

where we omitted explicit time-dependence. Now, we make our basic assumption that $\dot{\phi} = \Omega$ with Ω defined in (3). Equation (12) reduces then to

$$\dot{\lambda} + \kappa\lambda = 0 \ ,$$

and hence equation (11) implies

$$\Omega^2 = \omega^2 - \kappa^2 - \dot{\kappa} \ ,$$

which together with (3) gives finally

$$\dot{\kappa} + \kappa^2 = \gamma^2 \ .$$

The above equation has the following solutions

$$\kappa(t) \ = \ \gamma \tanh(\gamma(t-t_0)) \quad \text{ or } \quad \kappa(t) \ = \ \pm\gamma \ , \tag{13}$$

with t_0 being an integration constant. Note, however, that due to (9) the solution $\kappa(t) = -\gamma$ is not acceptable. It is clear that $\kappa(t) = \gamma$ corresponds to the Caldirola-Kanai model with $m(t) = e^{2\gamma t}$, whereas $\kappa(t) = \gamma \tanh(\gamma(t-t_0))$ gives rise to our new equation (2) described by a Hamiltonian with time-dependent mass

$$m(t) = m_0 \cosh^2(\gamma(t-t_0)) \ , \tag{14}$$

satisfying $m(t_0) = m_0$. We may therefore interpret (2) as a Newton equation for a particle with constant mass m_0 under the action of the time dependent nonlocal force

$$F(t;t_0) = -2m_0\gamma \tanh(\gamma(t-t_0))\dot{x} - m_0\omega^2 x \ ,$$

or as a Newton equation for a particle with time-dependent nonlocal mass (14) under the action of perfectly local force $-m_0\omega^2 x$.

3 Classical dynamics

Consider now classical Hamiltonian dynamics generated by (8) with $m(t)$ defined in (14). It is given by

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \Lambda^{\rm new}(t;t_0) \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} , \tag{15}$$

where the 2×2 matrix

$$\Lambda^{\rm new}(t;t_0) \;=\; \begin{pmatrix} \Lambda^{\rm new}_{xx} & \Lambda^{\rm new}_{xp} \\ \Lambda^{\rm new}_{px} & \Lambda^{\rm new}_{pp} \end{pmatrix} ,$$

reads as follows

$$\Lambda^{\rm new}_{xx} = \frac{\cos(\Omega(t-t_0))}{\cosh(\gamma(t-t_0))} , \qquad \Lambda^{\rm new}_{xp} = \frac{1}{m_0\Omega}\frac{\sin(\Omega(t-t_0))}{\cosh(\gamma(t-t_0))} , \tag{16}$$

and

$$\Lambda^{\rm new}_{px} \;=\; -m_0\Big[\Omega\cosh(\gamma(t-t_0))\sin(\Omega(t-t_0)) + \gamma\sinh(\gamma(t-t_0))\cos(\Omega(t-t_0))\Big] , \tag{17}$$

$$\Lambda^{\rm new}_{pp} \;=\; \Omega\cosh(\gamma(t-t_0))\cos(\Omega(t-t_0)) - \frac{\gamma}{\Omega}\sinh(\gamma(t-t_0))\sin(\Omega(t-t_0)) . \tag{18}$$

The crucial feature of (15) is that it breaks the local composition law

$$\Lambda(t_2;t_0) = \Lambda(t_2;t_1)\circ\Lambda(t_1;t_0) , \tag{19}$$

for $t_0 \le t_1 \le t_2$, that is, formula (19) does not hold for $\Lambda = \Lambda^{\rm new}$.

Let us compare our new evolution (15) with the classical dynamics of the Caldirola-Kanai system. Note, that we may trivially change the formula (6) for time-dependent mass

$$m(t) = m_0 e^{2\gamma(t-t_0)} , \tag{20}$$

without affecting the equation of motion. Now, these two models contain additional parameter t_0. One easily finds for the corresponding Caldirola-Kanai dynamics $\Lambda^{\rm CK}(t;t_0)$:

$$\Lambda^{\rm CK}_{xx} \;=\; \Big[\cos(\Omega(t-t_0)) + \frac{\gamma}{\Omega}\sin(\Omega(t-t_0))\Big]e^{-\gamma(t-t_0)}, \tag{21}$$

$$\Lambda^{\rm CK}_{xp} \;=\; \frac{1}{m_0\Omega}\sin(\Omega(t-t_0))e^{-\gamma(t-t_0)}, \tag{22}$$

$$\Lambda^{\rm CK}_{px} \;=\; -m_0\Omega\Big(1+\frac{\gamma^2}{\Omega^2}\Big)\sin(\Omega(t-t_0))e^{\gamma(t-t_0)}, \tag{23}$$

$$\Lambda^{\rm CK}_{pp} \;=\; \Big[\cos(\Omega(t-t_0)) - \frac{\gamma}{\Omega}\sin(\Omega(t-t_0))\Big]e^{\gamma(t-t_0)}. \tag{24}$$

One may check that $\Lambda^{\rm CK}$ satisfies

$$\Lambda^{\rm CK}(t_2;t_0) = \Lambda^{\rm CK}(t_2;t_1)\circ\Lambda^{\rm CK}(t_1;t_0) , \tag{25}$$

for $t_0 \le t_1 \le t_2$. Note, that computing $\Lambda^{\rm CK}(t_2,t_1)$ one applies (21)–(24) with t_0 replaced by t_1, and m_0 replaced by $m_1 := m(t_1) = m_0 e^{2\gamma(t_1-t_0)}$. Then formula (25) easily follows as a simple consequence of $e^{a+b} = e^a e^b$. It is no longer true if we replace (20) by (14).

Finally, let us compare both approaches in the simplified situation corresponding to pure dissipation, that is, $\omega = 0$. In this case the corresponding formula for $\Lambda^{\rm new}(t;t_0)$ considerably simplifies

$$\Lambda_0^{\rm new}(t;t_0) \;=\; \left(\begin{array}{cc} 1 & (m_0\gamma)^{-1}\tanh(\gamma(t-t_0)) \\ 0 & 1 \end{array} \right) . \tag{26}$$

However, $\Lambda_0^{\rm new}$ still breaks (19). In the case of Caldirola-Kanai dynamics one finds

$$\Lambda_0^{\rm CK}(t;t_0) \;=\; \left(\begin{array}{cc} 1 & (m_0\gamma)^{-1}\sinh(\gamma(t-t_0))\, e^{-\gamma(t-t_0)} \\ 0 & 1 \end{array} \right) . \tag{27}$$

Note, that in a asymptotic regime $\gamma t \longrightarrow \infty$ one obtains

$$\Lambda_{\rm asympt}^{\rm new} \;=\; \left(\begin{array}{cc} 1 & (m_0\gamma)^{-1} \\ 0 & 1 \end{array} \right) , \qquad \Lambda_{\rm asympt}^{\rm CK} \;=\; \left(\begin{array}{cc} 1 & (2m_0\gamma)^{-1} \\ 0 & 1 \end{array} \right) , \tag{28}$$

that is, both dynamics have different asymptotic states: $(x_0 + p_0/(m_0\gamma), p_0)$ and $(x_0 + p_0/(2m_0\gamma), p_0)$ for the Caldirola-Kanai system.

4 Quantum propagator

Consider now a quantum propagator corresponding to Hamiltonian (7) with $m(t)$ given by (14). Since the Hamiltonian is quadratic in (x,p) one easily finds for the propagator (see e.g. [32])

$$K(x,t;x_0,t_0) = N(t-t_0)\, A(x,t;x_0,t_0)\cdot B(x,t;t_0) \;, \tag{29}$$

where

$$\begin{aligned} A(x,t;x_0,t_0) \;&=\; \exp\left\{ \frac{i}{2\hbar}\frac{m_0\Omega}{\sin(\Omega(t-t_0))}\Big[(x_0^2+x^2)\cos(\Omega(t-t_0)) - 2x_0x\cosh(\gamma(t-t_0))\Big]\right\} \;, \\ B(x,t;t_0) \;&=\; \exp\left\{ \frac{i}{2\hbar}\, m_0x^2\, \frac{\sinh(\gamma(t-t_0))}{\sin(\Omega(t-t_0))}\Big[\Omega\cos(\Omega(t-t_0))\sinh(\gamma(t-t_0)) \right. \\ &\qquad \left. - \gamma\sin(\Omega(t-t_0))\cosh(\gamma(t-t_0))\Big]\right\} \;, \end{aligned} \tag{30}$$

and the normalization factor is given by

$$N(t-t_0) = \sqrt{\frac{m_0\Omega\cosh(\gamma(t-t_0))}{2\pi i\hbar\sin(\Omega(t-t_0))}} \;. \tag{31}$$

Note, that when $\gamma = 0$ the factor B equals to 1 and one recovers standard propagator for the harmonic oscillator [30]. On the other hand when $\omega = 0$, that is, in a pure damping case, formula (29) reduces to

$$K_0(x,t;x_0,t_0) = N_0(t-t_0)\exp\left\{ \frac{i\gamma}{2\hbar}\frac{m_0(x-x_0)^2}{\tanh(\gamma(t-t_0))}\right\} , \tag{32}$$

with

$$N_0(t-t_0) = \sqrt{\frac{\gamma m_0}{2\pi i\hbar\tanh(\gamma(t-t_0))}} \;. \tag{33}$$

For $\gamma = 0$ formula (32) reproduces quantum propagator for a free particle [30]. Interestingly, the formula (32) for a quantum propagator for a particle undergoing linear damping was recently derived by Kochan [31] by using purely geometrical considerations. Here we showed that it may be derived via standard methods by taking suitable nonlocal Hamiltonian with memory. In [31] the formula for a full propagator of the damped oscillator was also presented which however differs from (29). Actually, Kochan propagator is given by

$$K_{\rm Kochan}(x,t;x_0,t_0) = N(t-t_0)\, A(x,t;x_0,t_0)\ , \tag{34}$$

and therefore differs from (29) only by a factor $B(x,t;t_0)$. Both propagators K and $K_{\rm Kochan}$ produce the same limits for $\gamma = 0$ (harmonic oscillator) and for ($\gamma = 0$ & $\omega = 0$) (free particle). Observe that $B(x,t;t_0)$ produces only an additional phase factor at the final position 'x'. Therefore both formulae give the same results for a mean position of a particle: starting from the initial wave function $\psi(x_0,t_0)$ one finds

$$\psi(x,t;t_0) = \int\limits_{-\infty}^{\infty} K(x,t;x_0,t_0)\, \psi(x_0,t_0) dx_0\ , \tag{35}$$

and

$$\psi_{\rm Kochan}(x,t;t_0) \ = \ \int\limits_{-\infty}^{\infty} K_{\rm Kochan}(x,t;x_0,t_0)\, \psi(x_0,t_0) dx_0 \ = \ \frac{1}{B(x,t;t_0)}\, \psi(x,t;t_0)\ , \tag{36}$$

and hence

$$\langle x(t;t_0)\rangle \ = \ \int\limits_{-\infty}^{\infty} x\, |\psi(x,t;t_0)|^2 dx \ = \ \int\limits_{-\infty}^{\infty} x\, |\psi_{\rm Kochan}(x,t;t_0)|^2 dx\ . \tag{37}$$

It is easy to show that $\langle x(t;t_0)\rangle$ satisfies our modified nonlocal equation (2):

$$\frac{d^2}{dt^2}\langle x(t;t_0)\rangle \ + \ 2\gamma \tanh(\gamma(t-t_0))\frac{d}{dt}\langle x(t;t_0)\rangle \ + \ \omega^2\langle x(t;t_0)\rangle = 0\ . \tag{38}$$

In the Appendix we show that this additional phase factor $B(x,t;t_0)$ may be eliminated by a suitable canonical transformation. The price we pay for a simpler formula for a propagator is the much more complex form of the Hamiltonian which apart from the standard quadratic terms x^2 and p^2 contains also a mixed term xp (see Appendix).

Let us compare the above formulae with the propagator for the Caldirola-Kanai system. One easily finds the CK propagator corresponding to $m(t)$ given by (20)

$$\begin{aligned} K_{\rm CK}(x,t;x_0,t_0) \ &= \ N_{\rm CK}(t-t_0)\exp\Big\{\frac{i}{2\hbar}\, m_0\Big[\gamma\Big(x_0^2 e^{2\gamma t_0} - x^2 e^{2\gamma t}\Big) \\ &+ \ \Omega\cot(\Omega(t-t_0))\Big(x_0^2 e^{2\gamma t_0} + x^2 e^{2\gamma t}\Big) - \ 2\frac{\Omega}{\sin(\Omega(t-t_0))} x x_0 e^{\gamma(t+t_0)}\Big]\Big\}, \end{aligned} \tag{39}$$

where the normalization factor

$$N_{\rm CK}(t-t_0) = \sqrt{\frac{m_0\Omega\, e^{\gamma(t-t_0)}}{2\pi i\hbar \sin(\Omega(t-t_0))}}\,. \tag{40}$$

Again (39) correctly reproduces both the propagator for the harmonic oscillator (for $\gamma = 0$) and the propagator for the free particle ($\gamma = 0$ and $\omega = 0$). However, for pure damping both propagators are different.

There is a crucial difference between K and $K_{\rm CK}$. The Caldirola-Kanai propagator satisfies local composition law

$$K_{\rm CK}(x_2, t_2; x_0, t_0) = \int dx_1 K_{\rm CK}(x_2, t_2; x_1, t_1) K_{\rm CK}(x_1, t_1; x_0, t_0) \ . \tag{41}$$

It is no longer true for K defined in (29). Therefore, as in the classical case memory term encoded into dynamical equation is responsible for the breaking of the local composition law for the quantum evolution.

Finally, let us consider a damped (without oscillations) evolution of the initial Gaussian wave packet

$$\psi_0(x; t_0) = C\, e^{-(x-x_0)^2/\sigma + i p_0 x/\hbar} \ . \tag{42}$$

One easily finds for the probability density $\rho(x, t; t_0) = |\psi(x, t; t_0)|^2$:

$$\rho(x, t; t_0) = \frac{|C|^2}{\sqrt{\sigma(t-t_0)/\sigma}} \exp\left[-2\frac{(x - x(t))^2}{\sigma(t-t_0)}\right] \ , \tag{43}$$

where $x(t)$ is given by the classical formula

$$x(t) = x_0 + p_0 \frac{\tanh(\gamma(t-t_0))}{m_0\gamma} \ , \tag{44}$$

and

$$\sigma(t-t_0) = \sigma\left[1 + \left(\frac{\hbar \tanh(\gamma(t-t_0))}{m_0\sigma\gamma}\right)^2\right] \ . \tag{45}$$

It shows that the presence of the damping term modifies the way the initial way function spreads in time. Note, that in the asymptotic regime one obtains

$$\rho_{\rm asympt}(x; t_0) = \frac{|C|^2}{\sqrt{\sigma_{\rm asympt}/\sigma}} \exp\left[-2\frac{(x - x_{\rm asympt})^2}{\sigma_{\rm asympt}}\right] \ , \tag{46}$$

with the classical formula for the asymptotic position

$$x_{\rm asympt} = x_0 + \frac{p_0}{m_0\gamma} \ , \tag{47}$$

and asymptotic dispersion

$$\sigma_{\rm asympt} = \sigma\left[1 + \left(\frac{\hbar}{m_0\gamma\sigma}\right)^2\right] \ . \tag{48}$$

It is therefore clear that the damping prevents the wave function from the total spreading. An initial Gaussian probability density $\rho_0(x; t_0)$ cannot 'spread more' than $\rho_{\rm asympt}(x; t_0)$. A similar phenomenon appears for Caldirola-Kanai dynamics with different asymptotic state

$$x^{\rm CK}_{\rm asympt} = x_0 + \frac{p_0}{2m_0\gamma} \ , \tag{49}$$

but with the same asymptotic dispersion $\sigma_{\rm asympt}$ given by (48).

5 Conclusions

We performed an analysis of the new equation of motion for the damped oscillator (2). It differs from the standard one by a damping term – $2\gamma \tanh(\gamma(t-t_0))\dot{x}$ – which is non-local in time and nonlinear in the damping constant γ. The new parameter t_0 introduces effective memory. For long time behavior $t \gg 1/\gamma$ one recovers standard equation with a damping term – $2\gamma\dot{x}$. Both classical and quantum analysis is performed. The characteristic feature of this nonlocal system is that it breaks local composition law for the classical Hamiltonian dynamics and the corresponding quantum propagator. Interestingly, the same propagator was recently derived in [31]. Without referring to Hamiltonian formulation and using purely geometric methods the author of [31] derived the corresponding propagator starting from the standard equation (1). We have shown that the corresponding classical limit leads to the new damping equation (2). Therefore, we conclude that the quantum nonlocal propagator derived in [31] does originate in nonlocal equation (2). Finally, it was shown that the purely damped behavior modifies well known property of the free quantum dynamics leading to the perfect spreading of the initial wave function. It is no longer true when the dissipation is present. Now, instead of the perfect spreading there is an asymptotic state giving rise to the asymptotic probability density (46).

Appendix

Using a general formula for a propagator corresponding to quadratic Hamiltonian (see e.g. [32]) one easily finds that

$$H(x,p,t) \;=\; \frac{1}{2}\mu(t)p^2 + \nu(t)xp + \frac{1}{2}\lambda(t)x^2, \tag{A.1}$$

with

$$\mu(t) = \frac{1}{m_0 \cosh^2(\gamma(t-t_0))} \tag{A.2}$$

and

$$\nu(t) \;=\; \tanh(\gamma(t-t_0))\Big(\Omega\frac{\tanh(\gamma(t-t_0))}{\tan(\Omega(t-t_0))} - \gamma\Big), \tag{A.3}$$

$$\lambda(t) \;=\; m_0\Omega^2\left\{\frac{2\gamma}{\Omega}\,\frac{\tanh(\gamma(t-t_0))}{\tan(\Omega(t-t_0))} + \frac{1-[1+\tanh^2(\gamma(t-t_0))]\cos^2(\Omega(t-t_0))}{\sin^2(\Omega(t-t_0))}\right\}, \tag{A.4}$$

produces (34), that is, it annihilates additional phase factor $B(x,t;t_0)$ out of (29).

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