Constructing optimal entanglement witnesses. II

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We provide a class of optimal nondecomposable entanglement witnesses for $4N \times 4N$ composite quantum systems or, equivalently, a new construction of nondecomposable positive maps in the algebra of $4N \times 4N$ complex matrices. This construction provides natural generalization of the Robertson map. It is shown that their structural physical approximations give rise to entanglement breaking channels.

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I. INTRODUCTION

Entanglement is one of the essential features of quantum physics and is fundamental in modern quantum technologies [1, 2]. The most general approach to characterize quantum entanglement uses a notion of an entanglement witness (EW) [3, 4]. There is a considerable effort devoted to constructing and analyzing the structure of EWs [5–18]. (see also Ref. [19] for the recent review of entanglement detention). However, the general method of constructing an EW is still not known.

Due to the Choi-Jamiołkowski isomorphism, any EW in $\mathcal{H}_A \otimes \mathcal{H}_B$ corresponds to a linear positive map $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$, where $\mathcal{B}(\mathcal{H})$ denotes the space of bounded operators on the Hilbert space $\mathcal{H}$. Recall that a linear map Λ is said to be positive if it sends a positive operator on $\mathcal{H}_A$ into a positive operator on $\mathcal{H}_B$. It turns out [3] that a state ρ in $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable iff $(\mathbb{1}_A \otimes \Lambda)\rho$ is positive definite for all positive maps $\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$. Unfortunately, in spite of the considerable effort, the structure of positive maps is rather poorly understood (see Refs. [20–22] for the recent research).

In a recent paper we provided a class of nondecomposable positive maps $M_{2K}(\mathbb{C})$ ($M_d(\mathbb{C})$ denotes the algebra of $d \times d$ complex matrices) [23]. For $K = 2$ they are closely related to the Breuer-Hall maps in $M_4(\mathbb{C})$ [16, 17]. It was shown that they provide a class of optimal entanglement witnesses. In the present paper – treated as a second part of Ref. [23] – we provide another construction of a family of positive maps in $M_{4N}(\mathbb{C})$ (see Ref. [23] for all definitions). Our construction provides a natural generalization of the celebrated Robertson map in $M_4(\mathbb{C})$ [24]. We show that proposed maps are nondecomposable (i.e., they are able to detect entangled PPT [positive partial transposed] states) and optimal (i.e., they are able to detect the maximal number of entangled states). As a byproduct we construct new families of PPT entangled states detected by our maps.

The paper is organized as follows: Section II provides the basic construction of a family of positive maps in $M_{4N}(\mathbb{C})$. Then in Section III we study the basic properties of our maps/witnesses (nondecomposability and optimality). In Section IV we discuss the structural physical approximation (SPA) [26–28] of our maps. It is shown that the corresponding SPA gives rise to entanglement breaking channels and hence it supports a recent conjecture [28]. Final conclusions are collected in the last Section.

II. GENERALIZED ROBERTSON MAPS

Our starting point is the reduction map in the matrix algebra $M_2(\mathbb{C})$

$$R_2(X) = \mathbb{I}_2 \mathrm{Tr} X - X \ , \tag{1}$$

and hence its action on a matrix $X = ||x_{ij}||$ reads as follows

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} . \tag{2}$$

It is clear that R_2 is a positive map, since for any rank-1 projector P one finds $R_2(P) = \mathbb{I}_2 - P = P^\perp \geq 0$. There are several ways to generalize formulae (1) and (2) for higher dimensions. An obvious generalization of (1) reads as

$$R_K(X) = \mathbb{I}_K \mathrm{Tr} X - X \ , \tag{3}$$

that is, R_K is the reduction map in $M_K(\mathbb{C})$. The formula (2) may be generalized to $M_{2K}(\mathbb{C})$. Let us observe that $M_{2K}(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_K(\mathbb{C})$ and hence any matrix $X \in M_{2K}(\mathbb{C})$ may be represented as

$$X = \sum_{k,l=1}^{2} |k\rangle\langle l| \otimes X_{kl} \ , \tag{4}$$

where $\{|1\rangle, |2\rangle\}$ denotes the standard basis in $\mathbb{C}^2$ and $X_{kl} \in M_K(\mathbb{C})$. In what follows we shall use the following notation

$$X = \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) , \qquad (5)$$

to display the block structure of X. Now, one has two maps in $M_{2K}(\mathbb{C})$ that reduce to (2) for $K = 1$:

$$\left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) \longrightarrow \frac{1}{K} \left(\begin{array}{c|c} X_{22} & -X_{12} \\ \hline -X_{21} & X_{11} \end{array}\right) , \qquad (6)$$

and

$$\left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) \longrightarrow \frac{1}{K} \left(\begin{array}{c|c} \mathbb{I}_K \mathrm{Tr} X_{22} & -X_{12} \\ \hline -X_{21} & \mathbb{I}_K \mathrm{Tr} X_{11} \end{array}\right) . \qquad (7)$$

It is easy to show that both maps (6) and (7) are decomposable and hence cannot be used to detect PPT entangled states.

The first example of nondecomposable positive map in $M_{2K}(\mathbb{C})$ was provided by Robertson [24] for $K = 2$:

$$\Phi_4 \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \frac{1}{2} \left(\begin{array}{c|c} \mathbb{I}_2 \mathrm{Tr} X_{22} & -A_{12} \\ \hline -A_{21} & \mathbb{I}_2 \mathrm{Tr} X_{11} \end{array}\right) , \qquad (8)$$

where

$$A_{12} = X_{12} + R_2(X_{21})$$

and

$$A_{21} = X_{21} + R_2(X_{12}) .$$

Recently, Robertson map was generalized to $M_{2K}(\mathbb{C})$ as [23]

$$\Psi_{2K} \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \frac{1}{2K} \left(\begin{array}{c|c} \mathbb{I}_K \mathrm{Tr} X_{22} & -B_{12} \\ \hline -B_{21} & \mathbb{I}_K \mathrm{Tr} X_{11} \end{array}\right) , \qquad (9)$$

where

$$B_{12} = X_{12} + R_N(X_{21})$$

and

$$B_{21} = X_{21} + R_N(X_{12}) ,$$

and it was proved that Ψ_{2K} is nondecomposable. In the present paper we propose another generalization of Φ_4 for $M_{4N}(\mathbb{C})$. Let us observe that

$$R_2(X) = \sigma_y X^{\mathrm{T}} \sigma_y , \qquad (10)$$

where σ_y stands for the y–Pauli matrix. What is important is that σ_y is unitary and anti-symmetric. Essentially (up to a phase factor), it is the only antisymmetric unitary matrix in $M_2(\mathbb{C})$. Now, let us define the following map in $M_{4N}(\mathbb{C})$:

$$\Phi^U_{4N} \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \frac{1}{2N} \left(\begin{array}{c|c} \mathbb{I}_{2N} \mathrm{Tr} X_{22} & -C^U_{12} \\ \hline -C^U_{21} & \mathbb{I}_{2N} \mathrm{Tr} X_{11} \end{array}\right) , \qquad (11)$$

where

$$C^U_{12} = X_{12} + U X_{21}^{\mathrm{T}} U^\dagger$$

and

$$C^U_{21} = X_{21} + U X_{12}^{\mathrm{T}} U^\dagger ,$$

and U is an arbitrary antisymmetric unitary matrix in $M_{2N}(\mathbb{C})$. The above formulae guarantee that Ψ_{2K} and Φ^U_{4N} are unital, i.e.

$$\Psi_{2K}(\mathbb{I}_{2K}) = \mathbb{I}_{2K} , \quad \Phi^U_{4N}(\mathbb{I}_{4N}) = \mathbb{I}_{4N} . \qquad (12)$$

Clearly, Ψ_{2K} and Φ^U_{4N} coincide iff $2K = 4N = 4$. In this case $U = e^{i\lambda}\sigma_y$. However, if $2K = 4N > 4$, they are different. It follows from the fact that for $K > 1$, the reduction map $R_{2K}(X)$ can not be represented as $UX^{\mathrm{T}}U^\dagger$, with a unitary, antisymmetric U. Indeed, one has $R_{2K}(|1\rangle\langle 1|) = \mathbb{I}_{2K} - |1\rangle\langle 1|$, and hence $\mathrm{Tr}[R_{2K}(|1\rangle\langle 1|)] = 2K - 1$. On the other hand $\mathrm{Tr}[U|1\rangle\langle 1|U^\dagger] = 1$. Hence, necessarily $K = 1$.

Proposition 1 *Φ^U_{4N} defines a linear positive map in $M_{4N}(\mathbb{C})$.*

Proof: to prove that Φ^U_{4N} defines a positive map it is enough to show that each rank-1 projector $P \in M_4(\mathbb{C})$ is mapped via Φ^U_{4N} into a positive element in $M_4(\mathbb{C})$, that is, $\Phi^U_{4N}(P) \geq 0$. Let $P = |\psi\rangle\langle\psi|$ with arbitrary ψ from $\mathbb{C}^{4N}$ satisfying $\langle\psi|\psi\rangle = 1$. Now, since $\mathbb{C}^{4N} = \mathbb{C}^{2N} \oplus \mathbb{C}^{2N}$ one has

$$\psi = \sqrt{a}\, \psi_1 \oplus \sqrt{1-a}\, \psi_2 , \qquad (13)$$

with normalized $\psi_1, \psi_2 \in \mathbb{C}^{2N}$ and $a \in [0, 1]$. One has

$$P = \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \left(\begin{array}{c|c} a\, |\psi_1\rangle\langle\psi_1| & b\, |\psi_1\rangle\langle\psi_2| \\ \hline b\, |\psi_2\rangle\langle\psi_1| & (1-a)\, |\psi_2\rangle\langle\psi_2| \end{array}\right) ,$$

where $b = \sqrt{a(1-a)}$. Therefore

$$\Phi^U_{4N}(P) = \frac{1}{2N} \left(\begin{array}{c|c} (1-a)\, \mathbb{I}_{2N} & -b\, M \\ \hline -b\, M^\dagger & a\, \mathbb{I}_{2N} \end{array}\right) , \qquad (14)$$

where

$$M = |\psi_1\rangle\langle\psi_2| + U(|\psi_2\rangle\langle\psi_1|)^{\mathrm{T}} U^\dagger . \qquad (15)$$

It is clear that if $a = 0$, then

$$\Phi^U_{4N}(P) = \frac{1}{2N} \left(\begin{array}{c|c} \mathbb{I}_{2N} & \mathbb{O}_{2N} \\ \hline \mathbb{O}_{2N} & \mathbb{O}_{2N} \end{array}\right) \geq 0 . \qquad (16)$$

Similarly, for $a = 1$ one finds

$$\Phi^U_{4N}(P) = \frac{1}{2N} \left(\begin{array}{c|c} \mathbb{O}_{2N} & \mathbb{O}_{2N} \\ \hline \mathbb{O}_{2N} & \mathbb{I}_{2N} \end{array} \right) \geq 0 \ . \qquad (17)$$

Assume now that $0 < a < 1$. Let us recall [25] that a Hermitian matrix $X \in M_{2K}(\mathbb{C})$,

$$X = \left(\begin{array}{c|c} A & M \\ \hline M^\dagger & B \end{array} \right)$$

with strictly positive matrices $A, B \in M_K(\mathbb{C})$, is positive if and only if

$$A \geq M B^{-1} M^\dagger \ . \qquad (18)$$

Hence, to show that $\Phi^U_{4N}(P) \geq 0$ one has to prove

$$\mathbb{I}_{2N} \geq M M^\dagger \ . \qquad (19)$$

Taking into account that $(|\psi_2\rangle\langle\psi_1|)^{\rm T} = |\psi_1^*\rangle\langle\psi_2^*|$, and $\langle\psi|U|\psi^*\rangle = 0$ for any unitary anti-symmetric matrix U, one obtains

$$M M^\dagger = Q + Q^U \ , \qquad (20)$$

where $Q = |\psi_1\rangle\langle\psi_1|$ and $Q^U = U Q^{\rm T} U^\dagger$. Clearly, Q and Q^U are mutually orthogonal rank-1 projectors and hence $Q + Q^U \leq \mathbb{I}_{2N}$, which proves the positivity of Φ^U_{4N}.

Remark 1 One may replace the antisymmetric unitary matrix U by any antisymmetric matrix satisfying $UU^\dagger \leq \mathbb{I}_{4N}$. In particular, if $U = \mathbb{O}_{4N}$, one reproduces (7).

Remark 2 Note that

$$U_0 = \sigma_y \oplus \ldots \oplus \sigma_y \in M_{2N}(\mathbb{C}) \qquad (21)$$

is evidently antisymmetric and unitary. One may call Φ^0_{4N} corresponding to $U = U_0$ the canonical generalization of the Robertson map. Note that if $V \in M_{2N}(\mathbb{R})$ is orthogonal, i.e. $VV^{\rm T} = \mathbb{I}_{2N}$, then $U = VU_0V^{\rm T}$ is antisymmetric and unitary.

Remark 3 Let us recall that Breuer-Hall maps

$$\Lambda^U_{2K}(X) = R_{2K}(X) - UX^{\rm T}U^\dagger \ , \qquad (22)$$

with U antisymmetric unitary matrix in $M_{2K}(\mathbb{C})$, provide another generalization of the Robertson map. One has $\Phi_4 = \Lambda^0_4$, where again Λ^0_4 corresponds to $U = U_0$. We stress, however, that for $K > 2$, Breuer-Hall maps Λ^U_{2K} differ both form Ψ_{2K} and Φ^U_{4N}.

III. ENTANGLEMENT WITNESSES

To show that a positive map Φ^U_{4N} can be used to detect quantum entanglement one has to show that it is not completely positive. It means that the corresponding Choi matrix

$$W^U_{4N} = (\mathbb{1} \otimes \Phi^U_{4N}) P^+_{4N} \ , \qquad (23)$$

where P^+_d stands for the maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$, is not positive, i.e., it possess a strictly negative eigenvalue. Direct calculation shows that the spectrum of W reads as follows:

$$\frac{1}{4N} \times \left\{ \begin{array}{ll} -1 & \text{single} \\ 0 & (12N^2 - 2)\text{–fold} \\ \frac{1}{N} & 4N^2\text{–fold} \\ 1 & \text{single} \end{array} \right. .$$

It proves that W is indeed an entanglement witness.

Proposition 2 *W is a nondecomposable entanglement witness.*

Proof: to prove nondecomposability of W one has to show that there exists a PPT state ρ such that ${\rm Tr}(W\rho) < 0$. Let us construct the following density matrix

$$\rho = \mathcal{N} \sum_{i,j=1}^{4N} |i\rangle\langle j| \otimes \rho_{ij} \ , \qquad (24)$$

where the blocks $\rho_{ij} \in M_{4N}(\mathbb{C})$ are defined as follows: the diagonal blocks

$$\rho_{ii} = \left(\begin{array}{c|c} 4N \cdot \mathbb{I}_{2N} & \mathbb{O}_{2N} \\ \hline \mathbb{O}_{2N} & \mathbb{I}_{2N} \end{array} \right) \ , \qquad (25)$$

for $i = 1, \ldots, 2N$, and

$$\rho_{ii} = \left(\begin{array}{c|c} \mathbb{I}_{2N} & \mathbb{O}_{2N} \\ \hline \mathbb{O}_{2N} & 4N \cdot \mathbb{I}_{2N} \end{array} \right) \ , \qquad (26)$$

for $i = 2N+1, \ldots, 4N$. The off-diagonal blocks

$$\rho_{i,i+2N} = -8N^2 \cdot {\rm W}_{i,i+2N} \ , \qquad (27)$$

for $i = 1, \ldots, 2N$. Finally, for any $i = 1, \ldots, 2N$ and $j = 2N+1, \ldots, 4N$, provided that $j \neq i + 2N$ one defines

$$\rho_{ij} = |i\rangle\langle j| \ . \qquad (28)$$

All the remaining elements do vanish, i.e. $\rho_{ij} = \mathbb{O}_{2N}$. One finds for the normalization factor

$$\mathcal{N} = \frac{1}{8N^2(1+4N)} \ . \qquad (29)$$

Direct calculation shows that $\rho \geq 0$ and $\rho^\Gamma \geq 0$, i.e., ρ is PPT. Finally, one easily finds for the trace

$$\mathrm{Tr}(W\rho) = -\frac{\mathcal{N}}{8N^2} \ , \qquad (30)$$

which proves nondecomposability of W.

Proposition 3 *W is an optimal entanglement witness.*

Proof: to show that W^U_{4N} is optimal we use the following result of Lewenstein et al. [7]: if the family of product vectors $\psi \otimes \phi \in \mathbb{C}^{4N} \otimes \mathbb{C}^{4N}$ satisfying

$$\langle \psi \otimes \phi | W | \psi \otimes \phi \rangle = 0 \ , \qquad (31)$$

span the total Hilbert space $\mathbb{C}^{4N} \otimes \mathbb{C}^{4N}$, then W is optimal. Let us introduce the following sets of vectors:

$$f_{mn} = e_m + e_n \ , \quad g_{mn} = e_m + ie_n \ ,$$

for each $1 \leq m < n \leq 4N$. It is easy to check that $(4N)^2$ vectors $\psi_\alpha \otimes \psi^*_\alpha$ with ψ_α belonging to the set $\{\, e_l \, , f_{mn} \, , g_{mn} \,\}$, are linearly independent and hence they do span $\mathbb{C}^{4N} \otimes \mathbb{C}^{4N}$. Direct calculation shows that

$$\langle \psi_\alpha \otimes \psi^*_\alpha | W^U_{4N} | \psi_\alpha \otimes \psi^*_\alpha \rangle = 0 \ , \qquad (32)$$

which proves that W^U_{4N} is an optimal EW.

Remark 4 Actually, W^U_{4N} is not only an optimal EW but even nd-optimal. An EW W is optimal if $W - A$ is not EW for any $A \geq 0$, that is, subtracting from W any positive operator one destroys block-positivity of W. Now, W is nd-optimal if $W - D$ is not EW for any decomposable operator D (D is decomposable if $D = A + B^\Gamma$, with $A, B \geq 0$). Clearly, any nd-optimal EW is optimal and hence nd-optimal EWs define a proper subset of optimal witnesses. Recall, that a nondecomposable EW W is nd–optimal if and only if both W and W^Γ are optimal. Note that $(W^U_{4N})^\Gamma = VW^U_{4N}V^\dagger$, where the unitary matrix V is defined as follows

$$V = |1\rangle\langle 1| \otimes U^\dagger + |2\rangle\langle 2| \otimes U \ , \qquad (33)$$

and hence the optimality of $(W^U_{4N})^\Gamma$ easily follows from the optimality of W^U_{4N}.

Remark 5 Let us observe that for any unitarities $V_1, V_2 : \mathbb{C}^{4N} \to \mathbb{C}^{4N}$ a new map

$$\Phi^{U,V_1,V_2}_{4N}(X) := V_1^\dagger \Big[\Phi^U_{4N}(V_2 X V_2^\dagger) \Big] V_1 \ , \qquad (34)$$

is again positive (unital) and nondecomposable. Indeed, positivity is clear, and indecomposability follows from the following observation: if Φ^U_{4N} detects a PPT entangled state ρ , i.e., $(\mathbb{1} \otimes \Phi^U_{4N})\rho \ngeq 0$, then Φ^{U,V_1,V_2}_{4N} detects a PPT state $\widetilde{\rho} = (\mathbb{I}_{4N} \otimes V_2^\dagger)\rho(\mathbb{I}_{4N} \otimes V_2)$.

The corresponding entanglement witness W^{U,V_1,V_2}_{4N} reads as follows

$$\begin{aligned} W^{U,V_1,V_2}_{4N} &= (\mathbb{1} \otimes \Phi^{U,V_1,V_2}_{4N}) P^+_{4N} \qquad (35) \\ &= \frac{1}{4N} \sum_{k,l=1}^{4N} |k\rangle\langle l| \otimes V_1^\dagger \Big[\Phi^U_{4N}(V_2 |k\rangle\langle l| V_2^\dagger) \Big] V_1 \ , \end{aligned}$$

that is,

$$W^{U,V_1,V_2}_{4N} = (\mathbb{I}_{4N} \otimes V_1^\dagger) \Big[(\mathbb{1} \otimes \Phi^U_{4N}) \widetilde{P}^+_{4N} \Big] (\mathbb{I}_{4n} \otimes V_1) \ ,$$

where

$$\widetilde{P}^+_{4N} = (\mathbb{I}_{4N} \otimes V_2) P^+_{4N} (\mathbb{I}_{4n} \otimes V_2) \ . \qquad (36)$$

Using the fact that P^+_{4N} is $V \otimes \overline{V}$–invariant, one obtains

$$W^{U,V_1,V_2}_{4N} = (\overline{V}_2^\dagger \otimes V_1^\dagger) W^U_{4N} (\overline{V}_2 \otimes V_1) \ . \qquad (37)$$

Hence, if $\langle \phi_k \otimes \psi_k | W^U_{4N} | \phi_k \otimes \psi_k \rangle = 0$ and $\phi_k \otimes \psi_k$ do span $\mathbb{C}^{4N} \otimes \mathbb{C}^{4N}$, then $\langle \widetilde{\phi}_k \otimes \widetilde{\psi}_k | W^U_{4N} | \widetilde{\phi}_k \otimes \widetilde{\psi}_k \rangle = 0$, with

$$\widetilde{\phi}_k \otimes \widetilde{\psi}_k = (\overline{V}_2^\dagger \otimes V_1^\dagger)(\phi_k \otimes \psi_k) \ .$$

Clearly, $\widetilde{\phi}_k \otimes \widetilde{\psi}_k$ do span $\mathbb{C}^{4N} \otimes \mathbb{C}^{4N}$. Hence, it proves that W^{U,V_1,V_2}_{4N} defines an optimal entanglement witness.

IV. STRUCTURAL PHYSICAL APPROXIMATION

The idea of the structural physical approximation (SPA) [26, 27] consists of mixing a positive map Λ with some completely positive map making the mixture $\widetilde{\Lambda}$ completely positive. In the recent paper Ref. [28], the authors analyze the SPA to a positive map $\Lambda : \mathcal{B}(\mathcal{H_A}) \to \mathcal{B}(\mathcal{H_B})$ obtained through minimal admixing of white noise

$$\widetilde{\Lambda}(\rho) = p \frac{\mathbb{I}_B}{d_B} \mathrm{Tr}(\rho) + (1-p)\Lambda(\rho) \ . \qquad (38)$$

The minimal means that the positive mixing parameter $0 < p < 1$ is the smallest one for which the resulting map $\widetilde{\Lambda}$ is completely positive, i.e., it defines a quantum channel. Equivalently, one may introduce the SPA of an entanglement witness W:

$$\widetilde{W} = \frac{p}{d_A d_B} \mathbb{I}_A \otimes \mathbb{I}_B + (1-p)W \ , \qquad (39)$$

where p is the smallest parameter for which $\widetilde{W}$ is a positive operator in $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e. it defines a (possibly unnormalized) state.

It was conjectured that the SPA to optimal positive maps correspond to entanglement breaking maps (quantum channels) [28]. Equivalently, the SPA to optimal entanglement witnesses corresponds to separable (unnormalized) states. We show that the family of optimal maps/witnesses constructed in this paper supports this conjecture.

The corresponding SPA of W^U_{4N} is therefore given by

$$\widetilde{W}^U_{4N} = \frac{p}{(4N)^2}\,\mathbb{I}_{4N} \otimes \mathbb{I}_{4N} + (1-p)\,W^U_{4N} \ . \qquad (40)$$

The above definition guarantees that $\mathrm{Tr}\widetilde{W}^U_{4N} = 1$. Using the fact that the negative eigenvalue of W^U_{4N} equals "$-1/4N$" one easily finds the following condition for the positivity of $\widetilde{W}^U_{4N}$

$$p \geq \frac{4N}{4N+1} \ . \qquad (41)$$

To show that the SPA of Φ^U_{4N} is entanglement breaking we use the following result [23]: let $\Lambda : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be a positive unital map. Then the SPA of Λ is entanglement breaking if Λ detects all entangled isotropic states in $\mathbb{C}^d \otimes \mathbb{C}^d$. If, in addition, Λ is self-dual, i.e.,

$$\mathrm{Tr}(X \cdot \Lambda(Y)) = \mathrm{Tr}(\Lambda(X) \cdot Y) \ , \qquad (42)$$

for all $A, B \in M_d(\mathbb{C})$, then it is enough to check whether all entangled isotropic states are detected by the corresponding witness $W_\Lambda = (\mathbb{1} \otimes \Lambda)P^+_d$.

Lemma 1 *Φ^U_{4N} is self-dual.*

Using the definition of Φ^U_{4N} one obtains

$$\mathrm{Tr}[X \cdot \Phi^U_{4N}(Y)] = a - b \ ,$$

where

$$a = \mathrm{Tr}[X_{11}Y_{11} - X_{12}Y_{21} - X_{21}Y_{12} + X_{22}Y_{22}] \ ,$$

and

$$b = \mathrm{Tr}[X_{12}UY^{\mathrm{T}}_{12}U^\dagger + X_{21}UY^{\mathrm{T}}_{21}U^\dagger] \ .$$

On the other hand,

$$\mathrm{Tr}[\Phi^U_{4N}(X) \cdot Y] = a - b' \ ,$$

where

$$b' = \mathrm{Tr}[UX^{\mathrm{T}}_{12}U^\dagger Y_{12} + UX^{\mathrm{T}}_{21}U^\dagger Y_{21}] \ .$$

Now, using $\mathrm{Tr}X^{\mathrm{T}} = \mathrm{Tr}X$, and $U^{\mathrm{T}} = -U$, one proves that $b = b'$ and hence Φ^U_{4N} is self-dual.

Let

$$\rho_\lambda = \frac{\lambda}{(4N)^2}\,\mathbb{I}_d \otimes \mathbb{I}_d + (1-\lambda)P^+_{4N} \ , \qquad (43)$$

be an isotropic state which is known to be entangled iff

$$\lambda < \frac{4N}{4N+1} \ . \qquad (44)$$

Lemma 2 *If ρ_λ is entangled, then* $\mathrm{Tr}(W^U_{4N} \cdot \rho_\lambda) < 0$.

One has

$$\mathrm{Tr}(W^U_{4N} \cdot \rho_\lambda) = \frac{\lambda}{(4N)^2} + (1-\lambda)\,\mathrm{Tr}(W^U_{4N} \cdot P^+_{4N}) \ , \qquad (45)$$

where we have used $\mathrm{Tr}W^U_{4N} = 1$. Moreover,

$$\mathrm{Tr}(W^U_{4N} \cdot P^+_{4N}) = \frac{1}{(4N)^2} \sum_{k,l=1}^{4N} \langle k|\, \Phi^U_{4N}(|l\rangle\langle k|)\,|l\rangle \ .$$

Finally, direct calculation shows that

$$\sum_{k,l=1}^{4N} \langle k|\, \Phi^U_{4N}(|l\rangle\langle k|)\,|l\rangle = -4N \ , \qquad (46)$$

and hence

$$\mathrm{Tr}(W^U_{4N} \cdot \rho_\lambda) = \frac{1}{4N}\left(\frac{\lambda}{4N} + \lambda - 1\right) \ . \qquad (47)$$

Therefore, if $\lambda < 4N/(4N+1)$, then $\mathrm{Tr}(W^U_{4N}\rho_\lambda) < 0$, which shows that W^U_{4N} detects all entangled isotropic states.

Remark 6 One easily shows that the SPA for Φ^{U,V_1,V_2}_{4N} provides again an entanglement breaking channel.

V. CONCLUSIONS

We have provided a new construction of EWs in $\mathbb{C}^{4N} \otimes \mathbb{C}^{4N}$. It was shown that these EWs are nondecomposable, i.e., they are able to detect PPT entangled states. The crucial property of witnesses W^U_{4N} is optimality. Equivalently, our construction gives rise to the new class of positive maps in algebras of $4N \times 4N$

complex matrices. For $N = 1$ this construction reproduces the Robertson map [24] and hence it defines the special case of Brauer-Hall maps [16, 17].

Interestingly, a class of EWs W^U_{4N} is nd-optimal, i.e., both W^U_{4N} and its partial transposition $(W^U_{4N})^\Gamma$ are optimal EWs and hence provide the best "detectors" of PPT entangled states. We have shown that the structural physical approximation for our new class of positive maps gives rise to entanglement breaking channels and hence it supports the conjecture of Ref. [28].

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