On the symmetry of the seminal Horodecki state

Dariusz Chruściński and Andrzej Kossakowski
Institute of Physics, Nicolaus Copernicus University,
Grudziądzka 5/7, 87–100 Toruń, Poland

Abstract

It is shown that the seminal Horodecki 2-qutrit state belongs to the class of states displaying symmetry governed by a commutative subgroup of the unitary group $U(3)$. Taking a conjugate subgroup one obtains another classes of symmetric states and one finds equivalent representations of the Horodecki state.

1 Introduction

In a seminal paper [1] Paweł Horodecki provided an example of a density operator living in $\mathbb{C}^3 \otimes \mathbb{C}^3$ which represents entangled state positive under partial transposition (PPT)

$$\rho_a = N_a \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & c \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & c & \cdot & b \end{array} \right) , \tag{1}$$

with

$$N_a = \frac{1}{8a+1} \ , \quad b = \frac{1+a}{2} \ , \quad c = \frac{\sqrt{1-a^2}}{2} \ , \tag{2}$$

and $a \in [0,1]$. The above matrix representation corresponds to the standard computational basis $|ij\rangle = |i\rangle \otimes |j\rangle$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$ and to make the picture more transparent we replaced all zeros by dots. Since the partial transposition $\rho_a^\Gamma = (\mathbb{1} \otimes \mathrm{T})\rho_a \geq 0$ the state is PPT for all $a \in [0,1]$. It is easy to show that for $a = 0$ and $a = 1$ the state is separable and it was shown [1] that for $a \in (0,1)$ the state is entangled (for the recent reviews of quantum entanglement and the methods of its detection see [2] and [3]). Actually, the family (1) provides one of the first examples of bound entanglement. In this Letter we analyze the structure of (1). In particular we study its symmetry group.

2 Symmetry group

Let G be a subgroup of the unitary group $U(d)$ (a group of unitary $d \times d$ matrices). A state ρ living in $\mathbb{C}^d \otimes \mathbb{C}^d$ is $G \otimes \overline{G}$–invariant if

$$U \otimes \overline{U} \rho = \rho \, U \otimes \overline{U} \ , \tag{3}$$

where $U \in G$, and $\overline{U}$ denotes the complex conjugation of the matrix elements with respect to the computational basis $|i\rangle$. It is clear that if ρ is $G \otimes \overline{G}$–invariant then its partial transposition is $G \otimes G$–invariant, that is

$$U \otimes U \rho = \rho \, U \otimes U \ , \tag{4}$$

where $U \in G$. Recall, that if $G = U(d)$, then $G \otimes \overline{G}$–invariant states define a class of isotropic states [4], whereas $G \otimes G$–invariant states define a class of Werner states [5]. Recently [6] we found a class of $G \otimes \overline{G}$–invariant states, where G defines a maximal abelian subgroup of $U(d)$ defined as follows:

$$U_{\mathbf{x}} = \exp\left(i \sum_{k=1}^{d} x_k |k\rangle\langle k| \right) \ , \tag{5}$$

and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. It was shown [6] that states invariant under the maximal abelian subgroup have the following structure

$$\rho = \sum_{i,j=1}^{d} a_{ij} \, |i\rangle\langle j| \otimes |i\rangle\langle j| + \sum_{i \neq j=1}^{d} d_{ij} \, |i\rangle\langle i| \otimes |j\rangle\langle j| \ , \tag{6}$$

where the matrix $||a_{ij}|| \geq 0$, and the numbers $d_{ij} \geq 0$. The normalization condition gives

$$\sum_{i=1}^{d} a_{ii} + \sum_{i \neq j=1}^{d} d_{ij} = 1 \ .$$

The corresponding matrix representation for $d = 3$ reads as follows

$$\rho \ = \left(\begin{array}{ccc|ccc|ccc} a_{11} & \cdot & \cdot & \cdot & a_{12} & \cdot & \cdot & \cdot & a_{13} \\ \cdot & d_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & d_{13} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & d_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{21} & \cdot & \cdot & \cdot & a_{22} & \cdot & \cdot & \cdot & a_{23} \\ \cdot & \cdot & \cdot & \cdot & \cdot & d_{23} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d_{31} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d_{32} & \cdot \\ a_{31} & \cdot & \cdot & \cdot & a_{32} & \cdot & \cdot & \cdot & a_{33} \end{array} \right) \ . \tag{7}$$

Let us observe that (7) is PPT if and only if

$$d_{ij} d_{ji} \geq |a_{ij}|^2 \ , \quad i \neq j \ . \tag{8}$$

Surprisingly many well know states considered in the literature belong to this class (see [6] for examples). Note, however, that Horodecki state (1) does not belong to (7) unless $a = 1$. Consider now a subgroup G_0 of the G defined by (5) with $x_1 = x_3$. One finds the following structure of invariant states

$$\rho = \left(\begin{array}{ccc|ccc|ccc} \rho_{11} & \cdot & \rho_{13} & \cdot & \rho_{15} & \cdot & \rho_{17} & \cdot & \rho_{19} \\ \cdot & \rho_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \rho_{28} & \cdot \\ \rho_{31} & \cdot & \rho_{33} & \cdot & \rho_{35} & \cdot & \rho_{37} & \cdot & \rho_{39} \\ \hline \cdot & \cdot & \cdot & \rho_{44} & \cdot & \rho_{46} & \cdot & \cdot & \cdot \\ \rho_{51} & \cdot & \rho_{53} & \cdot & \rho_{55} & \cdot & \rho_{57} & \cdot & \rho_{59} \\ \cdot & \cdot & \cdot & \rho_{64} & \cdot & \rho_{66} & \cdot & \cdot & \cdot \\ \hline \rho_{71} & \cdot & \rho_{73} & \cdot & \rho_{75} & \cdot & \rho_{77} & \cdot & \rho_{79} \\ \cdot & \rho_{82} & \cdot & \cdot & \cdot & \cdot & \cdot & \rho_{88} & \cdot \\ \rho_{91} & \cdot & \rho_{93} & \cdot & \rho_{95} & \cdot & \rho_{97} & \cdot & \rho_{99} \end{array} \right) , \tag{9}$$

and it evidently contains Horodecki state (1). Interestingly, invariant states (9) have almost perfect chessboard structure [7] (see also the recent paper [8]. Note, however, that only a subclass of states considered in [7, 8] are $G_0 \otimes G_0$–invariant. The characteristic feature of (9) is that ρ has a direct sum structure $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$ where the corresponding operators ρ_k are supported on $\mathcal{H}_k$

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathbb{C}}\{\,|11\rangle,\ |13\rangle,\ |22\rangle,\ |31\rangle,\ |33\rangle\,\} , \\ \mathcal{H}_2 &= \mathrm{span}_{\mathbb{C}}\{\,|12\rangle,\ |32\rangle\,\} , \\ \mathcal{H}_3 &= \mathrm{span}_{\mathbb{C}}\{\,|21\rangle,\ |23\rangle\,\} , \end{aligned} \tag{10}$$

giving rise to the direct sum decomposition $\mathbb{C}^3 \otimes \mathbb{C}^3 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Similarly, the partial transposition

$$\rho^\Gamma = \left(\begin{array}{ccc|ccc|ccc} \rho_{11} & \cdot & \rho_{31} & \cdot & \cdot & \cdot & \rho_{17} & \cdot & \rho_{37} \\ \cdot & \rho_{22} & \cdot & \rho_{15} & \cdot & \rho_{35} & \cdot & \rho_{28} & \cdot \\ \rho_{13} & \cdot & \rho_{33} & \cdot & \cdot & \cdot & \rho_{19} & \cdot & \rho_{39} \\ \hline \cdot & \rho_{51} & \cdot & \rho_{44} & \cdot & \rho_{64} & \cdot & \rho_{57} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \rho_{55} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \rho_{53} & \cdot & \rho_{46} & \cdot & \rho_{66} & \cdot & \rho_{59} & \cdot \\ \hline \rho_{71} & \cdot & \rho_{91} & \cdot & \cdot & \cdot & \rho_{77} & \cdot & \rho_{97} \\ \cdot & \rho_{82} & \cdot & \rho_{75} & \cdot & \rho_{95} & \cdot & \rho_{88} & \cdot \\ \rho_{73} & \cdot & \rho_{93} & \cdot & \cdot & \cdot & \rho_{79} & \cdot & \rho_{99} \end{array} \right) \tag{11}$$

has a direct sum structure $\rho^\Gamma = \widetilde{\rho}_1 \oplus \widetilde{\rho}_2 \oplus \widetilde{\rho}_3$ where the corresponding operators $\widetilde{\rho}_k$ are supported on $\widetilde{\mathcal{H}}_k$

$$\begin{aligned} \widetilde{\mathcal{H}}_1 &= \mathrm{span}_{\mathbb{C}}\{\,|11\rangle,\ |13\rangle,\ |31\rangle,\ |33\rangle\,\} , \\ \widetilde{\mathcal{H}}_2 &= \mathrm{span}_{\mathbb{C}}\{\,|12\rangle,\ |21\rangle,\ |23\rangle,\ |32\rangle\,\} , \\ \widetilde{\mathcal{H}}_3 &= \mathrm{span}_{\mathbb{C}}\{\,|22\rangle\,\} , \end{aligned} \tag{12}$$

together with $\mathbb{C}^3 \otimes \mathbb{C}^3 = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2 \oplus \widetilde{\mathcal{H}}_3$. Interestingly one has

$$\widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_3 = \mathcal{H}_1 \ , \quad \mathcal{H}_2 \oplus \mathcal{H}_3 = \widetilde{\mathcal{H}}_2 \ . \tag{13}$$

Hence to check for PPT one needs to check positivity of two 4×4 leading submatrices of (11). Note, that decompositions (10) and (12) remind the characteristic circulant decompositions [9]. There is however important difference: (10) and (12) are governed by the symmetry group G_0 whereas the circulant decompositions are not directly related to any symmetry. For other types of decompositions which simplify PPT conditions see also [10].

3 Another representations of the Horodecki state

Consider now another commutative subgroup G_0' defined by $x_1 = x_2$. It is clear that

$$G_0' = S' G_0 S'^{\dagger} \ , \tag{14}$$

where S' represents permutation $(1, 2, 3) \to (1, 3, 2)$, that is

$$S' = \left(\begin{array}{ccc} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{array} \right) \ . \tag{15}$$

Hence a class of $G_0' \otimes \overline{G}_0'$–invariant states is defined by

$$\rho' = S' \otimes S' \, \rho \, S'^{\dagger} \otimes S'^{\dagger} \ , \tag{16}$$

where ρ is $G_0 \otimes \overline{G}_0$–invariant. The corresponding matrix representation of ρ' has the following form

$$\rho' = \left(\begin{array}{ccc|ccc|ccc} \rho_{11} & \rho_{12} & \cdot & \rho_{14} & \rho_{15} & \cdot & \cdot & \cdot & \rho_{19} \\ \rho_{21} & \rho_{22} & \cdot & \rho_{24} & \rho_{25} & \cdot & \cdot & \cdot & \rho_{29} \\ \cdot & \cdot & \rho_{33} & \cdot & \cdot & \rho_{36} & \cdot & \cdot & \cdot \\ \hline \rho_{41} & \rho_{42} & \cdot & \rho_{44} & \rho_{45} & \cdot & \cdot & \cdot & \rho_{49} \\ \rho_{51} & \rho_{52} & \cdot & \rho_{54} & \rho_{55} & \cdot & \cdot & \cdot & \rho_{59} \\ \cdot & \cdot & \rho_{63} & \cdot & \cdot & \rho_{66} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \rho_{77} & \rho_{78} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \rho_{87} & \rho_{88} & \cdot \\ \rho_{91} & \rho_{92} & \cdot & \rho_{94} & \rho_{95} & \cdot & \cdot & \cdot & \rho_{99} \end{array} \right) \ . \tag{17}$$

In particular one obtains the following representation of the Horodecki state invariant under G_0'

$$\rho_a{}' = S' \otimes S' \, \rho_a \, S'^{\dagger} \otimes S'^{\dagger} \ , \tag{18}$$

or in the matrix form

$$\rho_a{}' = N_a \left(\begin{array}{ccc|ccc|ccc} b & c & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ c & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \end{array} \right) \ . \tag{19}$$

The characteristic feature of (17) is that ρ' has a direct sum structure $\rho' = \rho_1' \oplus \rho_2' \oplus \rho_3'$ where the corresponding operators ρ_k are supported on $\mathcal{H}_k'$

$$\begin{aligned} \mathcal{H}_1' &= (S' \otimes S')\mathcal{H}_1 = \mathrm{span}_{\mathbb{C}}\{\, |11\rangle,\ |12\rangle,\ |21\rangle,\ |22\rangle,\ |33\rangle \,\} \ , \\ \mathcal{H}_2' &= (S' \otimes S')\mathcal{H}_2 = \mathrm{span}_{\mathbb{C}}\{\, |13\rangle,\ |23\rangle \,\} \ , \\ \mathcal{H}_3' &= (S' \otimes S')\mathcal{H}_3 = \mathrm{span}_{\mathbb{C}}\{\, |31\rangle,\ |32\rangle \,\} \ . \end{aligned} \tag{20}$$

One easily finds for the partial transposition

$$\rho'^{\Gamma} = \left(\begin{array}{ccc|ccc|ccc} \rho_{11} & \rho_{21} & \cdot & \rho_{14} & \rho_{24} & \cdot & \cdot & \cdot & \cdot \\ \rho_{12} & \rho_{22} & \cdot & \rho_{15} & \rho_{25} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \rho_{33} & \cdot & \cdot & \rho_{36} & \rho_{19} & \rho_{29} & \cdot \\ \hline \rho_{41} & \rho_{51} & \cdot & \rho_{44} & \rho_{54} & \cdot & \cdot & \cdot & \cdot \\ \rho_{42} & \rho_{52} & \cdot & \rho_{45} & \rho_{55} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \rho_{63} & \cdot & \cdot & \rho_{66} & \rho_{49} & \rho_{59} & \cdot \\ \hline \cdot & \cdot & \rho_{91} & \cdot & \cdot & \rho_{94} & \rho_{77} & \rho_{87} & \cdot \\ \cdot & \cdot & \rho_{92} & \cdot & \cdot & \rho_{95} & \rho_{78} & \rho_{88} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \rho_{99} \end{array}\right) . \tag{21}$$

It is evident that ρ'^{Γ} has a direct sum structure $\rho'^{\Gamma} = \widetilde{\rho}'_1 \oplus \widetilde{\rho}'_2 \oplus \widetilde{\rho}'_3$ where the corresponding operators $\widetilde{\rho}'_k$ are supported on $\mathcal{H}'_k$

$$\begin{aligned} \widetilde{\mathcal{H}}'_1 &= (S' \otimes S')\widetilde{\mathcal{H}}_1 = \mathrm{span}_{\mathbb{C}}\{\, |11\rangle,\ |12\rangle,\ |21\rangle,\ |22\rangle, \,\} , \\ \widetilde{\mathcal{H}}'_2 &= (S' \otimes S')\widetilde{\mathcal{H}}_2 = \mathrm{span}_{\mathbb{C}}\{\, |13\rangle,\ |23\rangle,\ |31\rangle,\ |32\rangle \,\} , \\ \widetilde{\mathcal{H}}'_3 &= (S' \otimes S')\widetilde{\mathcal{H}}_3 = \mathrm{span}_{\mathbb{C}}\{\, |33\rangle \,\} . \end{aligned} \tag{22}$$

Again the analog of the formulae (13) holds, that is

$$\widetilde{\mathcal{H}}'_1 \oplus \widetilde{\mathcal{H}}'_3 = \mathcal{H}'_1 , \quad \mathcal{H}'_2 \oplus \mathcal{H}'_3 = \widetilde{\mathcal{H}}'_2 . \tag{23}$$

Finally, let us consider another commutative subgroup G''_0 of G defined by $x_2 = x_3$. It is clear that

$$G''_0 = S'' G_0 S''^{\dagger} , \tag{24}$$

where S'' represents permutation $(1,2,3) \to (2,1,3)$, that is

$$S'' = \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} . \tag{25}$$

Hence a class of $G''_0 \otimes \overline{G}''_0$–invariant states is defined by

$$\rho'' = S'' \otimes S''\, \rho\, S''^{\dagger} \otimes S''^{\dagger} , \tag{26}$$

where ρ is $G_0 \otimes \overline{G}_0$–invariant. The corresponding matrix representation of ρ'' has the following form

$$\rho'' = \left(\begin{array}{ccc|ccc|ccc} \rho_{11} & \cdot & \cdot & \cdot & \rho_{15} & \rho_{16} & \cdot & \rho_{18} & \rho_{19} \\ \cdot & \rho_{22} & \rho_{23} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \rho_{32} & \rho_{33} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \rho_{44} & \cdot & \cdot & \rho_{47} & \cdot & \cdot \\ \rho_{51} & \cdot & \cdot & \cdot & \rho_{55} & \rho_{56} & \cdot & \rho_{58} & \rho_{59} \\ \rho_{61} & \cdot & \cdot & \cdot & \rho_{65} & \rho_{66} & \cdot & \rho_{68} & \rho_{69} \\ \hline \cdot & \cdot & \cdot & \rho_{74} & \cdot & \cdot & \rho_{77} & \cdot & \cdot \\ \rho_{81} & \cdot & \cdot & \cdot & \rho_{85} & \rho_{86} & \cdot & \rho_{88} & \rho_{89} \\ \rho_{91} & \cdot & \cdot & \cdot & \rho_{95} & \rho_{96} & \cdot & \rho_{98} & \rho_{99} \end{array}\right) . \tag{27}$$

In particular one obtains the following representation of the Horodecki state invariant under G_0''

$$\rho_a{}'' = S'' \otimes S'' \, \rho_a \, S''^\dagger \otimes S''^\dagger \ , \tag{28}$$

that is,

$$\rho_a{}'' \ = \ N_a \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & b & c & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot & c & b & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \end{array} \right) \ . \tag{29}$$

Again, the characteristic feature of (27) is that ρ'' has a direct sum structure $\rho'' = \rho_1'' \oplus \rho_2'' \oplus \rho_3''$ where the corresponding operators ρ_k are supported on $\mathcal{H}_k''$

$$\begin{aligned} \mathcal{H}_1'' &= (S'' \otimes S'')\mathcal{H}_1 = \mathrm{span}_{\mathbb{C}}\{ |11\rangle, \ |23\rangle, \ |22\rangle, \ |32\rangle, \ |33\rangle \} \ , \\ \mathcal{H}_2'' &= (S'' \otimes S'')\mathcal{H}_2 = \mathrm{span}_{\mathbb{C}}\{ |21\rangle, \ |31\rangle \} \ , \\ \mathcal{H}_3'' &= (S'' \otimes S'')\mathcal{H}_3 = \mathrm{span}_{\mathbb{C}}\{ |12\rangle, \ |13\rangle \} \ . \end{aligned} \tag{30}$$

One easily finds for the partial transposition

$$\rho''^{\Gamma} = \left(\begin{array}{ccc|ccc|ccc} \rho_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \rho_{22} & \rho_{32} & \rho_{15} & \cdot & \cdot & \rho_{18} & \cdot & \cdot \\ \cdot & \rho_{23} & \rho_{33} & \rho_{16} & \cdot & \cdot & \rho_{19} & \cdot & \cdot \\ \hline \cdot & \rho_{51} & \rho_{61} & \rho_{44} & \cdot & \cdot & \rho_{47} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \rho_{55} & \rho_{65} & \cdot & \rho_{58} & \rho_{68} \\ \cdot & \cdot & \cdot & \cdot & \rho_{56} & \rho_{66} & \cdot & \rho_{59} & \rho_{69} \\ \hline \cdot & \rho_{81} & \rho_{91} & \rho_{74} & \cdot & \cdot & \rho_{77} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \rho_{85} & \rho_{95} & \cdot & \rho_{88} & \rho_{98} \\ \cdot & \cdot & \cdot & \cdot & \rho_{86} & \rho_{96} & \cdot & \rho_{89} & \rho_{99} \end{array} \right) \ , \tag{31}$$

which is supported the direct product of three subspaces

$$\begin{aligned} \widetilde{\mathcal{H}}_1'' &= (S'' \otimes S'')\widetilde{\mathcal{H}}_1 = \mathrm{span}_{\mathbb{C}}\{ |21\rangle, \ |23\rangle, \ |32\rangle, \ |33\rangle \} \ , \\ \widetilde{\mathcal{H}}_2'' &= (S'' \otimes S'')\widetilde{\mathcal{H}}_2 = \mathrm{span}_{\mathbb{C}}\{ |12\rangle, \ |21\rangle, \ |13\rangle, \ |31\rangle \} \ , \\ \widetilde{\mathcal{H}}_3'' &= (S'' \otimes S'')\widetilde{\mathcal{H}}_3 = \mathrm{span}_{\mathbb{C}}\{ |11\rangle \} \ . \end{aligned} \tag{32}$$

It is evident that the analog of (13) is satisfied for $\mathcal{H}_k''$ and $\widetilde{\mathcal{H}}_k''$.

4 Conlcusions

We shown that the celebrated Horodecki state [1] belongs to a class of states invariant under a commutative subgroup G_0 of $U(3)$. Taking conjugate subgroups G_0' and G_0'' we provided another classes of invariant states. In particular we found equivalent representations of the Horodecki state invariant under G_0' and G_0'', respectively (cf. formulae (19) and (29)). Interestingly, known entanglement witnesses detecting PPT entangled state (1) display G_0-invariance (see [11, 12]). It should be clear that our discussion can be immediately generalized from $3 \otimes 3$ to $d \otimes d$ (d arbitrary but finite). Now, the maximal commutative subgroup of $U(d)$ defined by (5) gives rise to a number of subgroups corresponding to $x_{k_1} = \ldots = x_{k_l}$. In particular using a subgroup defined by $x_1 = x_d$ one may introduce the generalized Horodecki state in $d \otimes d$. We believe that our discussion opens new perspectives to study symmetric states of composite quantum systems. It would be interesting to generalize our analysis to multipartite case [13, 14].

Acknowledgments

This work was partially supported by the Polish Ministry of Science and Higher Education Grant No 3004/B/H03/2007/33.

References

[1] P. Horodecki, Phys. Lett. A **223**, 1 (1996).

[2] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[3] O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).

[4] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).

[5] R.F. Werner, Phys. Rev. A **40**, 4277 (1989).

[6] D. Chruściński and A Kossakowski, Phys. Rev. A **74**, 022308 (2006).

[7] D. Bruss and A. Peres, Phys. Rev. A **61**, 030301(R) (2000).

[8] D.Z. Djokovic, *The checkerboard family of entangled states of two qutrits*, arXiv:0911.2797.

[9] D. Chruściński and A. Kossakowski, Phys. Rev. A **76**, 032308 (2007).

[10] F.E.S. Steinhoff and M.C. de Oliveira, *Families of bipartite states classifiable by the positive partial transposition criterion*, arXiv:0906.1297.

[11] S. Yu and N. Liu, Phys. Rev. Lett, **95**, 150504 (2005).

[12] N. Ganguly and S. Adhikari, Phys. Rev. A **80**, 032331 (2009).

[13] K.G.H. Vollbrecht and R.F.Werner, Phys. Rev. A **64**, 062307 (2001).

[14] D. Chruściński, A. Kossakowski, Phys. Rev. A **73**, 062313 (2006); Phys. Rev. A **73**, 062314 (2006).