Optimal entanglement witnesses from generalized reduction and Robertson maps

Dariusz Chruściński and Justyna Pytel
Institute of Physics, Nicolaus Copernicus University,
Grudziądzka 5/7, 87–100 Toruń, Poland

March 3, 2011

Abstract

We provide a generalization of the reduction and Robertson positive maps in matrix algebras. They give rise to a new class of optimal entanglement witnesses. Their structural physical approximation is analyzed. As a byproduct we provide new examples of PPT (Positive Partial Transpose) entangled states.

1 Introduction

The interest on quantum entanglement has dramatically increased during the last two decades due to the emerging field of quantum information theory [1]. It turns out that quantum entanglement may be used as basic resources in quantum information processing and communication. The prominent examples are quantum cryptography, quantum teleportation, quantum error correction codes and quantum computation.

Since the quantum entanglement is the basic resource for the new quantum information technologies it is therefore clear that there is a considerable interest in efficient theoretical and experimental methods of entanglement detection (see [2] and [3] for the review).

Let us recall that a quantum state represented by the density operator in $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if it can be represented as a convex combination of product states

$$\rho = \sum_\alpha p_\alpha \rho_\alpha^{(A)} \otimes \rho_\alpha^{(B)} \ , \tag{1}$$

where p_α denotes a probability distribution whereas $\rho_\alpha^{(A)}$ and $\rho_\alpha^{(B)}$ are density operators of A and B subsystem, respectively. It is clear that separable states define a convex subset in the space of all density operators in $\mathcal{H}_A \otimes \mathcal{H}_B$ and states which are not separable are called entangled. The most general approach to characterize quantum entanglement uses a notion of an entanglement witness (EW) [4, 5]. A Hermitian operator W defined on a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ is called an entanglement witness if and only if: 1) $\mathrm{Tr}(W\sigma_{\rm sep}) \geq 0$ for all separable states $\sigma_{\rm sep}$, and 2) there exists an entangled state ρ such that $\mathrm{Tr}(W\rho) < 0$ (one says that ρ is detected by W).

It turns out that a state is entangled if and only if it is detected by some EW [4]. In recent years there was a considerable effort in constructing and analyzing the structure of EWs [6]–[18]. In particular several procedures for optimizing EWs for arbitrary states were proposed [7, 19, 20, 21]. Each entangled state ρ may be detected by a specific choice of W. It is therefore clear that each EW provides a new separability test and it may be interpreted as a new type of Bell inequality [10]. There is, however, no general procedure for constructing EWs.

In this paper we provide a new class of EWs. It is well known (see the next section for all details) that each EW is uniquely related to a linear positive map $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$. We provide new classes of linear positive maps by constructing generalization of well known maps, namely reduction map and Robertson map. It is shown that generalized maps and corresponding witnesses are optimal, that is, they detect quantum entanglement in an 'optimal way' (see next section for the precise definition). Optimal EWs are of primary importance since to perform complete classification of quantum states of a bipartite system it is enough to use only optimal EWs. Finally, we discuss how these maps are related to the idea of physical structural approximation (SPA)

[22, 23, 24]. It is shown that there is a strong evidence that these EWs support the conjecture [24] (see also [25]) that physical structural approximation to optimal positive map gives rise to an entanglement breaking channel.

The paper is organized as follows: we recall in Section 2 basic facts about linear positive maps and entanglement witnesses. Section 3 discusses generalization of the reduction map whereas Section 4 discusses generalization of the Robertson map. We show that these maps and the corresponding entanglement witnesses are optimal. Final conclusions are collected in the last Section.

2 Preliminaries and notation

In this paper we consider finite dimensional complex Hilbert spaces. Let $M_n(\mathbb{C})$ denote an algebra (actually, a $\mathbb{C}^*$-algebra) of $n \times n$ complex matrices. A linear map $\Lambda : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is called to be positive if it maps positive elements from $M_n(\mathbb{C})$ into positive elements in $M_m(\mathbb{C})$. It means that for any vectors $|x\rangle \in \mathbb{C}^n$ and $|y\rangle \in \mathbb{C}^m$ one has

$$\mathrm{Tr}(P_y \Lambda(P_x)) \geq 0 \ , \tag{2}$$

where $P_x = |x\rangle\langle x|$ and $P_y = |y\rangle\langle y|$. Equivalently, $\langle y|\Lambda(|x\rangle\langle x|)|y\rangle \geq 0$. Note, that the above condition is in general very hard to check since it does not reduce to any spectral condition. Unfortunately, in spite of the considerable effort, the structure of positive maps is rather poorly understood [28]–[32] (see also the monograph by Paulsen [33]). For some recent works see [34, 35, 36, 37, 17, 18, 38] and for a review paper see [39]. Positive maps play an important role both in physics and mathematics providing generalization of $*$-homomorphisms, Jordan homomorphisms and conditional expectations. Normalized positive maps define affine mappings between sets of states of $\mathbb{C}^*$-algebras. A positive linear map Λ is k-positive if the map

$$\mathbb{1}_k \otimes \Lambda : M_k(M_n(\mathbb{C})) \longrightarrow M_k(M_m(\mathbb{C})), \tag{3}$$

is positive ($M_k(\mathcal{A})$ denotes a set of $k \times k$ complex matrices with entries from the $\mathbb{C}^*$-algebra $\mathcal{A}$). Clearly, a k-positive map is l-positive for all $l < k$. A map which is k-positive for all k is called completely positive. Actually, in the finite dimensional case we consider in this paper Λ is completely positive if and only if it is k positive with $k = \min\{n, m\}$ [30].

Let $\{e_1, \ldots, e_n\}$ be a fixed orthonormal basis in $\mathbb{C}^n$. Denote by $e_{ij} := |e_i\rangle\langle e_j|$ an orthonormal basis in $M_n(\mathbb{C})$. Let $\mathrm{T} : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ denotes transposition map with respect to the fixed basis $\{e_i\}$, that is $\mathrm{T}(e_{ij}) = e_{ji}$. Evidently, 'T' defines linear positive map. Now, a positive map Λ is called decomposable if and only if

$$\Lambda = \Lambda_1 + \Lambda_2 \circ \mathrm{T} \ , \tag{4}$$

where Λ_1 and Λ_2 are completely positive . Maps which are not decomposable are called indecomposable (or nondecomposable).

Using Choi-Jamiołkowski [30, 40] isomorphism each positive map Λ gives rise to entanglement witness W

$$W = (\mathbb{1}_n \otimes \Lambda) P_n^+ \ , \tag{5}$$

where P_n^+ denotes maximally entangled state in $\mathbb{C}^n \otimes \mathbb{C}^n$ and $\mathbb{1}_n$ denotes an identity map acting on $M_n(\mathbb{C})$. Using fixed basis $\{e_i\}$ one has

$$W = \frac{1}{n} \sum_{i,j=1}^{n} e_{ij} \otimes \Lambda(e_{ij}) \ . \tag{6}$$

An entanglement witness W is called (in)decomposable if the corresponding positive map Λ is (in)decomposable. Hence, any decomposable entanglement witness may be represented as follows

$$W = Q_1 + Q_2^\Gamma \ , \tag{7}$$

where $Q_1, Q_2 \geq 0$, and $A^\Gamma := (\mathbb{1}_n \otimes \mathrm{T})A$ denotes partial transposition of A. Let us observe that the positivity of Λ implies that W satisfies

$$\langle x \otimes y|W|x \otimes y\rangle \geq 0 \ , \tag{8}$$

for any vectors $|x\rangle \in \mathbb{C}^n$ and $|y\rangle \in \mathbb{C}^m$. Hermitian operators satisfying (8) are often called block-positive. Note, that if Λ is completely positive then the corresponding W is not only block-positive but even positive.

Let us recall that entanglement witnesses play a key role in the theory of entanglement. A density operator ρ living in $\mathbb{C}^n \otimes \mathbb{C}^m$ is entangled if and only if there exists an entanglement witness W such that

$$\mathrm{Tr}(W\rho) < 0 \ . \tag{9}$$

One says that ρ is detected by W. Recall, that a state represented by a density operator ρ is PPT (Positive Partial Transpose) if $\rho^\Gamma \geq 0$. One has [28, 7]

Proposition 1 *W is an indecomposable entanglement witness if and only if there exists a PPT state ρ detected by W. Equivalently, a PPT state ρ is entangled if and only if there exists an indecomposable entanglement witness which detects ρ.*

Let $\mathcal{D}$ be a subset of density operators of a composite quantum system living in $\mathbb{C}^n \otimes \mathbb{C}^m$ detected by a given entanglement witness W, i.e. $\mathcal{D} = \{\rho \mid \mathrm{Tr}(W\rho) < 0\}$. Given two entanglement witnesses W_1 and W_2 one says that W_2 is finer than W_1 if $\mathcal{D}_1 \subset \mathcal{D}_2$, that is, all states detected by W_1 are also detected by W_2. A witness W is optimal if there is no other entanglement witness which is finer than W. It means that W detects quantum entanglement in the 'optimal way'. It is clear that the knowledge of optimal entanglement witnesses is crucial to classify quantum states of composite systems. One proves [7] the following

Proposition 2 *W is an optimal entanglement witness if and only if $W - Q$ is no longer entanglement witness for arbitrary positive operator Q.*

Authors of Ref. [7] formulated the following criterion for the optimality of W.

Proposition 3 *If the set of product vectors $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^m$ satisfying*

$$\langle x \otimes y | W | x \otimes y \rangle = 0 \ , \tag{10}$$

span the total Hilbert space $\mathbb{C}^n \otimes \mathbb{C}^m$, then W is optimal.

It should be stressed that the converse theorem is not true, i.e. the existence of product vectors which span $\mathbb{C}^n \otimes \mathbb{C}^m$ and satisfy (10) is not necessary for the optimality of W. A well know example is provided by the entanglement witness corresponding to the celebrated Choi indecomposable map [30] which is known to be optimal but does not provide the corresponding collection of $|x \otimes y\rangle$.

Finally, let us comment on an interesting conjecture proposed in [24]: let W be a normalized entanglement witness, i.e. $\mathrm{Tr}\, W = 1$. An operator $\widetilde{W}(p)$ defined by

$$\widetilde{W}(p) = \frac{1-p}{n^2}\, \mathbb{I}_n \otimes \mathbb{I}_n + pW \tag{11}$$

is called structural physical approximation (SPA) of W if $\widetilde{W}(p) \geq 0$. Now, let p_* be a maximal p for which $\widetilde{W}(p)$ defines SPA for W, that is, $\widetilde{W}(p) \geq 0$ for $p \in [0, p_*]$.

Conjecture 1 *If W is an optimal entanglement witness, then $\widetilde{W}(p_*)$ defines a separable state.*

It should be clear that SPA can be equivalently defined for a positive map $\Lambda : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$. Let us recall [26]

Definition 1 *A completely positive map $\Lambda : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is entanglement breaking if and only if $(\mathbb{1}_n \otimes \Lambda)\rho$ defines a separable state for any ρ living in $\mathbb{C}^n \otimes \mathbb{C}^n$.*

Interestingly, any entanglement breaking quantum channel (trace preserving completely positive map) can be represented in the Holevo form [27]

$$\Lambda(\rho) = \sum_i R_i \mathrm{Tr}(F_i \rho) \ , \tag{12}$$

where R_i are density operators in $\mathbb{C}^m$ and F_i are positive operators in $\mathbb{C}^n$ satisfying $\sum_i F_i = \mathbb{I}_n$, i.e. a set $\{F_i\}$ defines a generalized quantum measurement. Now, a positive map Λ is optimal if $\Lambda - \Phi$, with Φ being a completely positive map, is no longer positive. A positive map

$$\widetilde{\Lambda}(p) = (1-p)\mathbb{1}_n + p\Lambda \ , \tag{13}$$

defines a SPA for Λ if $\widetilde{\Lambda}(p)$ is completely positive. The above conjecture may be equivalently formulated as follows: if Λ is an optimal positive map, then $\widetilde{\Lambda}(p_*)$ is entanglement breaking. One proves [17] the following

Theorem 1 *Let $\Lambda : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital map (i.e. $\Lambda(\mathbb{I}_n) = \mathbb{I}_n$) that detects all entangled isotropic states. Then SPA of Λ is an entanglement breaking map.*

Let $\widetilde{W}(p)$ be SPA of W and let $\lambda_{\min}$ be the smallest eigenvalue of W. One easily finds

$$p_* = \frac{1}{1+|\lambda_{\min}|n^2} \ . \tag{14}$$

Now, it follows from Theorem 1 that $p_* = \frac{1}{n+1}$ and hence

Corollary 1 *If $\Lambda : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital map, and the smallest eigenvalue of the corresponding entanglement witness W satisfies*

$$\lambda_{\min} \leq -\frac{1}{n} \ , \tag{15}$$

then SPA of W defines a separable state.

Conjecture 1 is supported by several examples (see [24] and [17, 18]). The present paper provides another family of examples supporting above conjecture.

3 New optimal EWs out of the reduction map

3.1 Reduction map in $M_n(\mathbb{C})$

Let us start with an elementary positive map in $M_n(\mathbb{C})$ called reduction map

$$R_n(X) = \frac{1}{n-1}\Big[\mathbb{I}_n \mathrm{Tr}X - X\Big] \ , \tag{16}$$

for $X \in M_n(\mathbb{C})$. Positivity of R_n follows from the fact that R_n maps rank-1 projectors into projectors. Indeed, for $X = |\psi\rangle\langle\psi|$ with $\langle\psi|\psi\rangle = 1$, one has

$$R_n(|\psi\rangle\langle\psi|) = \frac{1}{n-1}\Big[\mathbb{I}_n - |\psi\rangle\langle\psi|\Big] \ , \tag{17}$$

which is evidently positive, since $\mathbb{I}_n - |\psi\rangle\langle\psi|$ is a projector (of rank '$n-1$') onto the $(n-1)$–dimensional hyperplane orthogonal to $|\psi\rangle$. The corresponding entanglement witness is given by

$$W = \frac{1}{n-1}\Big(\frac{1}{n}\mathbb{I}_n \otimes \mathbb{I}_n - P_n^+\Big) \ . \tag{18}$$

One has for the partial transposition

$$(\mathbb{1} \otimes \mathrm{T})W = \frac{1}{n(n-1)} \sum_{i<j} P_{ij} \ , \tag{19}$$

where

$$P_{ij} = |\psi_{ij}\rangle\langle\psi_{ij}| \ , \tag{20}$$

with

$$|\psi_{ij}\rangle = e_i \otimes e_j - e_j \otimes e_i \ , \tag{21}$$

which shows that $(\mathbb{1}_n \otimes \mathrm{T})W \geq 0$ and hence W defines a decomposable EW. Equivalently, it shows that the map $R_n \circ \mathrm{T}$ is completely positive, i.e. it defines a legitimate quantum channel. Note, that decomposition (19) proves that W is not extremal, since it decomposes into a convex combination of extremal witnesses P_{ij}^{Γ} (it is extremal for $n = 2$ only, due to $W = \frac{1}{2}P_{12}^{-\,\Gamma}$). Interestingly, being not extremal it is still optimal.

Proposition 4 *W is an optimal EW.*

Proof: to show that W is optimal we use Proposition 3. Let us introduce the following set of vectors in $\mathbb{C}^n \otimes \mathbb{C}^n$:

$$f_{kl} = (e_k + e_l) \otimes (e_k + e_l) \ , \quad g_{kl} = (e_k + ie_l) \otimes (e_k - ie_l),$$

for each $1 \leq k < l \leq n$. It is easy to check that n^2 vectors $\{\, e_k \otimes e_k \,, f_{kl} \,, g_{kl} \,\}$ are linearly independent and hence they do span $\mathbb{C}^n \otimes \mathbb{C}^n$. Direct calculation shows that

$$\langle f_{kl}|W|f_{kl}\rangle = 0 \ , \quad \langle g_{kl}|W|g_{kl}\rangle = 0 \ , \quad \langle e_k \otimes e_k|W|e_k \otimes e_k\rangle = 0 \ , \tag{22}$$

which ends the proof. □

Finally, the reduction map R_n supports recent conjecture [24], that is, one has the following

Proposition 5 *The structural physical approximation of R_n is an entanglement breaking map.*

Let as observe that the smallest eigenvalue of W is given by $\lambda_{\min} = -1/n$, and hence, due to Corollary 1, SPA of W is separable. Actually, the above proposition was already proved in [24].

3.2 Generalized reduction map

Let us observe that taking the orthonormal basis e_{ij} in $M_n(\mathbb{C})$ the reduction map R_n may be defined as follows

$$R_n(e_{ii}) = \frac{1}{n-1}\,(\mathbb{I}_n - e_{ii}) \ , \tag{23}$$

$$R_n(e_{ij}) = -\frac{1}{n-1}\,e_{ij} \ , \quad i \neq j \ . \tag{24}$$

Let us take $n(n-1)/2$ complex numbers z_{ij} $(i < j)$ satisfying $|z_{ij}| \leq 1$ and denote by $\mathbf{z}$ the collection $\{z_{12}, \ldots, z_{n-1,n}\}$. Finally, let us define a map

$$R_n^{(\mathbf{z})} \ : \ M_n(\mathbb{C}) \longrightarrow \ M_n(\mathbb{C}) \ , \tag{25}$$

by

$$R_n^{(\mathbf{z})}(e_{ii}) = \frac{1}{n-1}\,(\mathbb{I}_n - e_{ii}) \ , \tag{26}$$

$$R_n^{(\mathbf{z})}(e_{ij}) = -\frac{z_{ij}}{n-1}\,e_{ij} \ , \quad i < j \ , \tag{27}$$

and $z_{ij} = \overline{z_{ji}}$ for $i > j$. It is clear that for $z_{ij} = 1$ one reconstructs the original reduction map R_n.

Proposition 6 *$R_n^{(\mathbf{z})}$ defines a positive decomposable map.*

Proof: let us observe that the corresponding entanglement witness $W_n^{(\mathbf{z})}$ has the following form

$$W^{(\mathbf{z})} = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} e_{ij} \otimes W_{ij}^{(\mathbf{z})} \ , \tag{28}$$

where

$$W_{ii}^{(\mathbf{z})} = \mathbb{I}_n - e_{ii} \ , \quad W_{ij}^{(\mathbf{z})} = -z_{ij}e_{ij} \quad (i < j) \ . \tag{29}$$

To complete the proof observe that $(\mathbb{1}_n \otimes \mathrm{T})W^{(\mathbf{z})}$ is a positive operator. Indeed, one has

$$(\mathbb{1}_n \otimes \mathrm{T})W^{(\mathbf{z})} = \frac{1}{n(n-1)} \sum_{i<j} P_{ij}^{(\mathbf{z})} \ , \tag{30}$$

where the operators $P_{ij}^{(\mathbf{z})}$ are defined by

$$P_{ij}^{(\mathbf{z})} = e_{ii} \otimes e_{jj} + e_{jj} \otimes e_{ii} - z_{ij}e_{ij} \otimes e_{ji} - \overline{z_{ij}}\, e_{ji} \otimes e_{ij} \tag{31}$$

and hence they are positive for $|z_{ij}| \leq 1$. It shows that $(\mathbb{1}_n \otimes \mathrm{T})W^{(\mathbf{z})} \geq 0$ and hence $W^{(\mathbf{z})}$ is a decomposable entanglement witness. □

Note, that if at least one $z_{ij} \neq 0$, then the map $R_n^{(\mathbf{z})}$ is not completely positive. Indeed, the following principal submatrix of $W^{(\mathbf{z})}$

$$\left(\begin{array}{cc} 0 & z_{ij} \\ \overline{z_{ij}} & 0 \end{array} \right) \ ,$$

is not positive definite and hence $W^{(\mathbf{z})} \ngeq 0$. If $|z_{ij}| = 1$, then $P_{ij}^{(\mathbf{z})} = |\psi_{ij}^{(\mathbf{z})}\rangle\langle\psi_{ij}^{(\mathbf{z})}|$, with

$$|\psi_{ij}^{(\mathbf{z})}\rangle = e_i \otimes e_j - \overline{z_{ij}}\, e_j \otimes e_i \ . \tag{32}$$

Proposition 7 *The positive map $R_n^{(\mathbf{z})}$ is optimal if and only if $|z_{ij}| = 1$ for all $i \neq j$.*

Proof: the condition $|z_{ij}| = 1$ is necessary for optimality. Indeed, suppose for example that $|z_{kl}| < 1$ for some pair $k < l$. Then

$$(\mathbb{1}_n \otimes \mathrm{T})W^{(\mathbf{z})} - \frac{1}{n(n-1)} Q_{kl}^{(\mathbf{z})} \ , \tag{33}$$

where

$$Q_{kl}^{(\mathbf{z})} = (1 - |z_{kl}|^2)(e_{kk} \otimes e_{ll} + e_{ll} \otimes e_{kk}) \ , \tag{34}$$

is still a positive operator, and hence

$$W^{(\mathbf{z})} - \frac{1}{n(n-1)} Q_{kl}^{(\mathbf{z})} \ , \tag{35}$$

defines decomposable entanglement witness (note, that $(\mathbb{1}_n \otimes \mathrm{T})Q_{kl}^{(\mathbf{z})} = Q_{kl}^{(\mathbf{z})}$).

Suppose now that $|z_{kl}| = 1$. To show that $W^{(\mathbf{z})}$ is optimal we use again the result of Lewenstein et. al. [7]. Let $z_{kl} = e^{i\alpha_{kl}}$. It is easy to check that the following vectors

$$f_{kl} = (e_k + e^{-i\alpha_{kl}/2} e_l) \otimes (e_k + e^{-i\alpha_{kl}/2} e_l) \ , \quad g_{kl} = (e_k + ie^{-i\alpha_{kl}/2} e_l) \otimes (e_k - ie^{-i\alpha_{kl}/2} e_l) \ , \quad e_k \otimes e_k \ ,$$

span the entire Hilbert space $\mathbb{C}^n \otimes \mathbb{C}^n$. Moreover, they satisfy

$$\langle f_{kl}|W^{(\mathbf{z})}|f_{kl}\rangle = \langle g_{kl}|W^{(\mathbf{z})}|g_{kl}\rangle = 0 \tag{36}$$

for $k < l$, and

$$\langle e_k \otimes e_k|W^{(\mathbf{z})}|e_k \otimes e_k\rangle = 0 \ , \tag{37}$$

for $k = 1, \ldots, n$ which proves that $W^{(\mathbf{z})}$ is an optimal entanglement witness. □

Finally, consider the structural physical approximation to $W^{(\mathbf{z})}$

$$\widetilde{W}^{(\mathbf{z})}(p) = \frac{1-p}{n^2}\, \mathbb{I}_n \otimes \mathbb{I}_n + pW^{(\mathbf{z})} \tag{38}$$

and let $\lambda_{\rm min}^{(\mathbf{z})}$ be the smallest eigenvalue of $W^{(\mathbf{z})}$. One has

$$p_*^{(\mathbf{z})} = \frac{1}{1+|\lambda_{\rm min}^{(\mathbf{z})}|n^2} \ . \tag{39}$$

Note, that $\lambda_{\rm min}^{(\mathbf{z})}$ is the smallest eigenvalue to the $n \times n$ Hermitian matrix Z defined by

$$Z_{ii} := 0 \ , \quad Z_{ij} := z_{ij} \ (i<j) \ . \tag{40}$$

Note, that if all $z_{ij} = 1$ (standard reduction map), then

$$\lambda_{\rm min}^{(\mathbf{z})} = -\frac{1}{n} \ , \tag{41}$$

and if all $z_{ij} = -1$, then

$$\lambda_{\rm min}^{(\mathbf{z})} = -\frac{1}{n(n-1)} \ . \tag{42}$$

For a set of arbitrary $z_{ij} = e^{i\alpha_{ij}}$ the analytic formula for $\lambda_{\rm min}^{(\mathbf{z})}$ is not available. However, it is clear that in the general case one has

$$-\frac{1}{n} \le \lambda_{\rm min}^{(\mathbf{z})} \le -\frac{1}{n(n-1)} \ , \tag{43}$$

and hence

$$\frac{1}{n+1} \ge p_*^{(\mathbf{z})} \ge \frac{n-1}{2n+1} \ . \tag{44}$$

We have already shown that for $z_{ij} = 1$ the SPA of $W^{(\mathbf{z})}$ defines a separable state (see Proposition 5).

Proposition 8 *The structural physical approximation $R_n^{(\mathbf{z})}(p_*^{(\mathbf{z})})$, with $|z_{ij}| = 1$, is an entanglement breaking map.*

Proof: one has

$$\widetilde{W}^{(\mathbf{z})}(p_*^{(\mathbf{z})}) = \frac{1-p_*^{(\mathbf{z})}}{n^2}\, \mathbb{I}_n \otimes \mathbb{I}_n + p_*^{(\mathbf{z})}\, W^{(\mathbf{z})} = p_*^{(\mathbf{z})} \left[|\lambda_{\rm min}^{(\mathbf{z})}|\, \mathbb{I}_n \otimes \mathbb{I}_n + W^{(\mathbf{z})} \right] \ , \tag{45}$$

and hence to prove the Proposition one has to show that

$$B^{(\mathbf{z})} = |\lambda_{\rm min}^{(\mathbf{z})}|\, \mathbb{I}_n \otimes \mathbb{I}_n + W^{(\mathbf{z})}$$

defines a separable positive operator.

Lemma 1 *A positive operator*

$$A^{(\mathbf{z})} = \sum_{i,j=1}^n e_{ij} \otimes A_{ij}^{(\mathbf{z})} \ , \tag{46}$$

with

$$A_{ii}^{(\mathbf{z})} = |\lambda_{\rm min}^{(\mathbf{z})}| \mathbb{I}_n \ , \quad A_{ij}^{(\mathbf{z})} = -z_{ij}\, e_{ij} \ , \quad (i<j) \ , \tag{47}$$

is separable.

Proof: consider the following operator living in $\mathbb{C}^n \otimes \mathbb{C}^n$:

$$A^{(\mathbf{z})} = \sum_{i,j=1}^n \widetilde{Z}_{ij}\, e_{ij} \otimes e_{ij} + |\lambda_{\rm min}^{(\mathbf{z})}| \sum_{i \neq j} e_{ii} \otimes e_{jj} \ , \tag{48}$$

where the $n \times n$ matrix $\widetilde{Z}$ is defined as follows

$$\widetilde{Z}_{ii} = |\lambda_{\rm min}^{(\mathbf{z})}| \ , \quad \widetilde{Z}_{ij} = -z_{ij} \ , \ \ (i<j) \ . \tag{49}$$

It is clear that $\widetilde{Z} \geq 0$, and hence $A^{(\mathbf{z})} \geq 0$. Now, let us define the linear map $\Lambda^{(\mathbf{z})} : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ defined as follows

$$\Lambda^{(\mathbf{z})}(X) = \widetilde{Z} \circ X \ , \tag{50}$$

where $\widetilde{Z} \circ X$ denotes the Hadamard product of matrices $X, \widetilde{Z} \in M_n(\mathbb{C})$. Recall, that $[A \circ B]_{ij} := A_{ij} B_{ij}$. It is well known [46] that $\Lambda^{(\mathbf{z})}$ is completely positive due to the positivity of the matrix $\widetilde{Z}$. Observe, that

$$A^{(\mathbf{z})} = (\mathbb{1} \otimes \Lambda^{(\mathbf{z})}) A_0 \ , \tag{51}$$

where

$$A_0 = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ij} + \sum_{i \neq j} e_{ii} \otimes e_{jj} \ . \tag{52}$$

Note, that $A_0 = A^{(\mathbf{z})}$ with $z_{ij} = 1$. Now, it is well known that A_0 defines a separable operator and hence due to (51) the operator $A^{(\mathbf{z})}$ is separable as well. □

It is evident that the separability of $B^{(\mathbf{z})}$ follows from the separability of $A^{(\mathbf{z})}$ which completes the proof of the Proposition. □

Remark 1 *Note, that for $n = 2$ all maps $R_2^{(z)}$ with $|z| = 1$ are unitarily equivalent ($z \equiv z_{12}$)*

$$R_2^{(z)}(X) = V^{(z)} R_2(X) V^{(z)\,\dagger} \ , \tag{53}$$

with

$$V^{(z)} = \begin{pmatrix} 1 & 0 \\ 0 & \overline{z} \end{pmatrix} \ . \tag{54}$$

Clearly, it is not longer true for $n > 2$.

4 New optimal EWs out of the Robertson map

4.1 Robertson map in $M_{2k}(\mathbb{C})$

Robertson provided [41] the following linear map $\Phi_4 : M_4(\mathbb{C}) \longrightarrow M_4(\mathbb{C})$

$$\Phi_4 \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array} \right) = \frac{1}{2} \left(\begin{array}{c|c} \mathbb{I}_2 \, \mathrm{Tr} X_{22} & -[X_{12} + R_2(X_{21})] \\ \hline -[X_{21} + R_2(X_{12})] & \mathbb{I}_2 \, \mathrm{Tr} X_{11} \end{array} \right) \ , \tag{55}$$

where $X_{kl} \in M_2(\mathbb{C})$. It turns out [41] that Φ_4 defines a unital positive indecomposable map. Moreover, Φ_4 is extremal and hence optimal. Interestingly, Robertson map supports the SPA conjecture [24].

Recently, [43, 44] (see also discussion in [17, 18, 42]) Robertson map was generalized to a linear map $\Phi_{2k} : M_{2k}(\mathbb{C}) \longrightarrow M_{2k}(\mathbb{C})$

$$\Phi_{2k} \left(\begin{array}{c|c|c|c} X_{11} & X_{12} & \cdots & X_{1k} \\ \hline X_{21} & X_{22} & \cdots & X_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline X_{k1} & X_{k2} & \cdots & X_{kk} \end{array} \right) = \frac{1}{2(k-1)} \left(\begin{array}{c|c|c|c} A_1 & -B_{12} & \cdots & -B_{1k} \\ \hline -B_{21} & A_2 & \cdots & -B_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -B_{k1} & -B_{k2} & \cdots & A_k \end{array} \right) \ , \tag{56}$$

where

$$A_k = \mathbb{I}_2 (\mathrm{Tr} X - \mathrm{Tr} X_{kk}) \ , \tag{57}$$

and

$$B_{kl} = X_{kl} - R_2(X_{lk}) \ . \tag{58}$$

It was shown [43] that Φ_{2k} defines an indecomposable optimal positive map. Analyzing the spectrum of the corresponding entanglement witness $W = (\mathbb{1}_{2k} \otimes \Phi_{2k}) P^+_{2k}$ one finds single negative eigenvalue '$-1/2k$', one strictly positive eigenvalue '$1/[2k(k-1)]$' with multiplicity $2k^2 - (k+1)$, and $k(2k+1)$ zero-modes. Therefore, due to the Corollary 1 the SPA of Φ_{2k} defines an entanglement breaking map and hence supports conjecture of [24].

Remark 2 *Note, that Φ_{2k} defines a special example of the Breuer-Hall map [43, 44]*

$$\Phi^U_{2k}(X) = \frac{1}{2(k-1)}\left(R_{2k}(X) - UX^{\mathrm{T}}U^\dagger\right) \ , \tag{59}$$

where U is a unitary antisymmetric $2k \times 2k$ matrix. It corresponds to

$$U = \mathbb{I}_k \otimes \sigma_y \ . \tag{60}$$

It was shown [43] that for any U the map Φ^U_{2k} is indecomposable and optimal. The special form of Φ^U_{2k} resembling the original Robertson map in $M_4(\mathbb{C})$ was proposed in [17].

4.2 Generalized Robertson map in $M_{2k}(\mathbb{C})$

In analogy to the reduction map discussed in the previous section we propose the following generalization of the Robertson map Φ_{2k}: for any collection of $k(k-1)/2$ complex numbers z_{ij}, with $i<j$, satisfying $|z_{ij}| \leq 1$ we define $\Phi^{(\mathbf{z})}_{2k} : M_{2k}(\mathbb{C}) \longrightarrow M_{2k}(\mathbb{C})$ by

$$\Phi^{(\mathbf{z})}_{2k}\left(\begin{array}{c|c|c|c} X_{11} & X_{12} & \cdots & X_{1k} \\ \hline X_{21} & X_{22} & \cdots & X_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline X_{k1} & X_{k2} & \cdots & X_{kk} \end{array}\right) = \frac{1}{2(k-1)}\left(\begin{array}{c|c|c|c} A_1 & -z_{12}B_{12} & \cdots & -z_{1k}B_{1k} \\ \hline -\overline{z}_{21}B_{21} & A_2 & \cdots & -z_{2k}B_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -\overline{z}_{k1}B_{k1} & -\overline{z}_{k2}B_{k2} & \cdots & A_k \end{array}\right) . \tag{61}$$

The main result of this section consists in the following

Theorem 2 *$\Phi^{(\mathbf{z})}_{2k}$ defines a positive map.*

Proof: to prove the positivity of $\Phi^{(\mathbf{z})}_{2k}$ one has to show that for any rank-1 projector $P_{2k} = |\psi\rangle\langle\psi|$, one has

$$\Phi^{(\mathbf{z})}_{2k}(P_{2k}) \geq 0 \ , \tag{62}$$

where $\psi \in \mathbb{C}^{2k}$ and $\langle\psi|\psi\rangle = 1$. Now, any normalized $|\psi\rangle \in \mathbb{C}^{2k}$ may be considered as a direct sum

$$|\psi\rangle = \sqrt{\alpha_1}\,|\psi_1\rangle \oplus \ldots \oplus \sqrt{\alpha_k}\,|\psi_k\rangle \ , \tag{63}$$

where $|\psi_i\rangle \in \mathbb{C}^2$, such that $\langle\psi_i|\psi_i\rangle = 1$, and $\alpha_1, \ldots, \alpha_k \geq 0$ satisfy normalization condition

$$\alpha_1 + \ldots + \alpha_k = 1 \ . \tag{64}$$

Using such representation the projector $P_{2k} = |\psi\rangle\langle\psi|$ has the following form

$$P_{2k} = \left(\begin{array}{c|c|c|c} \alpha_1|\psi_1\rangle\langle\psi_1| & \sqrt{\alpha_1\alpha_2}|\psi_1\rangle\langle\psi_2| & \cdots & \sqrt{\alpha_1\alpha_k}|\psi_1\rangle\langle\psi_k| \\ \hline \sqrt{\alpha_2\alpha_1}|\psi_2\rangle\langle\psi_1| & \alpha_2|\psi_2\rangle\langle\psi_2| & \cdots & \sqrt{\alpha_2\alpha_k}|\psi_2\rangle\langle\psi_k| \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \sqrt{\alpha_k\alpha_1}|\psi_k\rangle\langle\psi_1| & \sqrt{\alpha_k\alpha_2}|\psi_k\rangle\langle\psi_2| & \cdots & \alpha_k|\psi_k\rangle\langle\psi_k| \end{array}\right) , \tag{65}$$

and hence

$$\Phi^{(\mathbf{z})}_{2k}(P_{2k}) = \frac{1}{2(k-1)}\left(\begin{array}{c|c|c|c} (1-\alpha_1)\mathbb{I}_2 & -z_{12}M_{12} & \cdots & -z_{1k}M_{1k} \\ \hline -\overline{z}_{12}M_{21} & (1-\alpha_2)\mathbb{I}_2 & \cdots & -z_{2k}M_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -\overline{z}_{1k}M_{k1} & -\overline{z}_{2k}M_{k2} & \cdots & (1-\alpha_k)\mathbb{I}_2 \end{array}\right) , \tag{66}$$

where the 2×2 matrices M_{ij} are defined as follows

$$M_{ij} = \sqrt{\alpha_i\alpha_j}\left[|\psi_i\rangle\langle\psi_j| + \sigma_y\overline{|\psi_i\rangle\langle\psi_j|}\sigma_y\right] . \tag{67}$$

Lemma 2 *Matrices M_{ij} satisfy the following properties:*

1. $M_{ij}M_{ji} = \alpha_i\alpha_j\,\mathbb{I}_2$,
2. $M_{ij}M_{jk} = \alpha_j M_{ik}$.

One proves this lemma by direct calculation. To prove (62) we perform the induction with respect to k. For $k = 2$ any $\Phi_4^{(\mathbf{z})}$ is unitarily equivalent to the Robertson map Φ_4. Suppose now that the theorem is true for $k = n-1$. To prove that it holds for $k = n$ we use the following well known

Lemma 3 (Bhatia [46]) *A block matrix*

$$\left(\begin{array}{cc} A & X \\ X^\dagger & B \end{array} \right) ,$$

with $A \geq 0$ and $B > 0$, is positive if and only if

$$A \geq XB^{-1}X^\dagger \ . \tag{68}$$

Hence

$$2(k-1)\,\Phi_{2k}^{(\mathbf{z})}(P_{2k}) \ = \ \left(\begin{array}{c|c|c|c} (1-\alpha_1)\mathbb{I}_2 & -z_{12}M_{12} & \cdots & -z_{1n}M_{1n} \\ \hline -\overline{z}_{12}M_{21} & (1-\alpha_2)\mathbb{I}_2 & \cdots & -z_{2n}M_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -\overline{z}_{1n}M_{n1} & -\overline{z}_{2n}M_{n2} & \cdots & (1-\alpha_n)\mathbb{I}_2 \end{array} \right) \geq 0 \ , \tag{69}$$

if and only iff

$$\left(\begin{array}{c|c|c|c} (1-\alpha_1)\mathbb{I}_2 & -z_{12}M_{12} & \cdots & -z_{1,n-1}M_{1,n-1} \\ \hline \overline{z}_{12}M_{21} & (1-\alpha_2)\mathbb{I}_2 & \cdots & -z_{2,n-1}M_{2,n-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -\overline{z}_{1,n-1}M_{n-1,1} & -\overline{z}_{2,n-1}M_{n-1,2} & \cdots & (1-\alpha_{n-1})\mathbb{I}_2 \end{array} \right) \geq$$
$$\frac{\alpha_n}{1-\alpha_n} \left(\begin{array}{c|c|c|c} \alpha_1\mathbb{I}_2 & z_{1n}\overline{z}_{2n}M_{12} & \cdots & z_{1n}\overline{z}_{n-1,n}M_{1,n-1} \\ \hline \overline{z}_{1n}z_{2n}M_{21} & \alpha_2\mathbb{I}_2 & \cdots & z_{2n}\overline{z}_{n-1,n}M_{2,n-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \overline{z}_{1n}z_{n-1,n}M_{n-1,1} & \overline{z}_{2n}z_{n-1,n}M_{n-1,2} & \cdots & \alpha_{n-1}\mathbb{I}_2 \end{array} \right) . \tag{70}$$

Now let us define a new set of positive numbers

$$\alpha_i' := \frac{\alpha_i}{1-\alpha_n} \ , \quad i = 1,\ldots,n-1 \ , \tag{71}$$

and new set of matrices M_{ij}'

$$M_{ij}' := \sqrt{\frac{\alpha_i'\alpha_j'}{\alpha_i\alpha_j}}\, M_{ij} \ , \tag{72}$$

for $i,j = 1,\ldots,n-1$. It is clear that

$$\alpha_1' + \ldots + \alpha_{n-1}' = 1 \ , \tag{73}$$

and the matrices M_{ij}' satisfy Lemma 2 with α_i replaced by α_i'. Using these new quantities and the condition $|z_{ij}| \leq 1$ the inequality (70) may be rewritten as follows

$$\left(\begin{array}{c|c|c|c} (1-\alpha_1')\mathbb{I}_2 & -z_{12}'M_{12}' & \cdots & -z_{1,n-1}'M_{1,n-1}' \\ \hline -\overline{z}_{12}'M_{21}' & (1-\alpha_2')\mathbb{I}_2 & \cdots & -z_{2,n-1}'M_{2,n-1}' \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -\overline{z}_{1,n-1}'M_{n-1,1}' & -\overline{z}_{2,n-1}'M_{n-1,2}' & \cdots & (1-\alpha_{n-1}')\mathbb{I}_2 \end{array} \right) \geq 0 \ , \tag{74}$$

where

$$z'_{ij} := (1-\alpha_n)z_{ij} + \alpha_n z_{in}\overline{z}_{jn} \ . \tag{75}$$

Note, that

$$|z'_{ij}| \leq (1-\alpha_n)|z_{ij}| + \alpha_n|z_{in}\overline{z}_{jn}| \leq 1 \ , \tag{76}$$

due to $|z_{ij}| \leq 1$. Hence inequality (74) is equivalent to

$$\Phi^{(\mathbf{z}')}_{2(n-1)}(P_{2(n-1)}) \geq 0 \ , \tag{77}$$

which is true due to our original assumption that the theorem holds for $k = n-1$. □

It should be stressed that $\Phi^{(\mathbf{z})}_{2k}$ does not in general correspond to the Breuer-Hall map [43, 44]. One has

Proposition 9 *A map $\Phi^{(\mathbf{z})}_{2k}$ is equivalent to the Breuer-Hall map iff $z_{ij} = z_i\overline{z_j}$, where $(z_1,\ldots,z_{2k})$ are defined by $z_k = e^{i\alpha_k}$.*

Proof: indeed, any such vector gives rise to the unitary matrix $U^{(\mathbf{z})}$ via

$$U^{(\mathbf{z})}_{kl} = \delta_{kl} z_l \ . \tag{78}$$

One has

$$\Phi^{(\mathbf{z})}_{2k}(X) = U^{(\mathbf{z})}\Phi_{2k}(X)U^{(\mathbf{z})} \ , \tag{79}$$

and hence $\Phi^{(\mathbf{z})}_{2k}$ is unitary equivalent to the Breuer-Hall map. If $z_{ij} \neq z_i\overline{z_j}$, then the corresponding entanglement witness $W^{(\mathbf{z})}$ has different spectrum and hence cannot be equivalent to the entanglement witness corresponding to the Breuer-Hall map.

Proposition 10 *$\Phi^{(\mathbf{z})}_{2k}$, with $|z_{ij}| = 1$, defines an indecomposable map.*

Proof: let us consider the following state ρ living in $\mathbb{C}^{2k}\otimes\mathbb{C}^{2k}$:

$$\rho^{(\mathbf{z})} = \mathcal{N}\sum_{i,j=1}^{2k} e_{ij}\otimes\rho^{(\mathbf{z})}_{ij} \ , \tag{80}$$

where $\rho^{(\mathbf{z})}_{ij} \in M_{2k}(\mathbb{C})$ are defined as follows: if $i+j = 2\ell$, then

$$\rho^{(\mathbf{z})}_{ij} = -W^{(\mathbf{z})}_{ij} \ . \tag{81}$$

If $i+j = 2\ell+1$, one has either

$$\rho^{(\mathbf{z})}_{ij} = \mathbb{O}_{2k} \ , \tag{82}$$

for $(i,j) = (2m-1,2m)$ and $m = 1,\ldots,k$, or

$$\rho^{(\mathbf{z})}_{ij} = \frac{z_{ij}}{4k(k-1)}\, e_{ij} \ , \tag{83}$$

for $(i,j) \neq (2m-1,2m)$. Finally, the normalization constant reads $\mathcal{N} = 1/3$. One easily checks that $\rho^{(\mathbf{z})}$ defines a PPT state. Now direct calculation shows that

$$\mathrm{Tr}(W^{(\mathbf{z})}\rho^{(\mathbf{z})}) = -\frac{1}{24k(k-1)} < 0 \ , \tag{84}$$

which proves that $W^{(\mathbf{z})}$ is an indecomposable entanglement witness. □

Corollary 2 *The formula (80) defines a new class of PPT entangled states in $\mathbb{C}^{2k}\otimes\mathbb{C}^{2k}$.*

4.3 Optimality and SPA

Finally, let us analyze the problem of optimality of $\Phi^{(\mathbf{z})}_{2k}$. One has the following

Proposition 11 $\Phi^{(\mathbf{z})}_{2k}$ *is optimal if and only if* $|z_{ij}| = 1$.

Proof: the necessity of $|z_{ij}| = 1$ is obvious (compare the proof of Proposition 7). Now, to prove that this condition is also sufficient we use again the result of Lewenstein et. al. [7] (cf. Proposition 3). Let $z_{kl} = e^{i\alpha_{kl}}$, as before. It is easy to check that the following vectors

$$f_{kl} = (e_k + e^{-i\alpha_{kl}/2}e_l) \otimes (e_k + e^{-i\alpha_{kl}/2}e_l) \ , \quad g_{kl} = (e_k + ie^{-i\alpha_{kl}/2}e_l) \otimes (e_k - ie^{-i\alpha_{kl}/2}e_l) \ , \quad e_k \otimes e_k \ ,$$

span the whole Hilbert space $\mathbb{C}^{2k} \otimes \mathbb{C}^{2k}$ and that they satisfy condition:

$$\langle f_{kl}|W^{(\mathbf{z})}|f_{kl}\rangle = \langle g_{kl}|W^{(\mathbf{z})}|g_{kl}\rangle = 0 \ , \ \ \langle e_k \otimes e_k|W^{(\mathbf{z})}|e_k \otimes e_k\rangle = 0. \tag{85}$$

Thus, $W^{(\mathbf{z})} = (\mathbb{1} \otimes \Phi^{(\mathbf{z})}_{2k})P^+_{2k}$ is an optimal entanglement witness. □

Concerning SPA we have the following

Proposition 12 *SPA for* $\Phi^{(\mathbf{z})}_6$ *and* $z_{ij} = -1$ *is entanglement breaking.*

Proof: consider the following class of states living in $\mathbb{C}^d \otimes \mathbb{C}^d$

$$\rho = \sum_{k,l=1}^{d} a_{ij} e_{ij} \otimes e_{ij} + \sum_{i \neq j} b_{ij} e_{ii} \otimes e_{jj} \ , \tag{86}$$

where the $d \times d$ complex matrix a_{ij} is positive semidefinite. It was shown [45] that ρ is invariant under the maximal abelian subgroup of $U(d)$

$$U_{\mathbf{x}} \otimes \overline{U}_{\mathbf{x}}\, \rho = \rho\, U_{\mathbf{x}} \otimes \overline{U}_{\mathbf{x}} \ , \tag{87}$$

where

$$U_{\mathbf{x}} = \exp\left(i \sum_{k=0}^{d-1} x_k e_{kk} \right) \ , \tag{88}$$

and $\mathbf{x} = (x_1, \ldots, x_d) \in [0, 2\pi) \times \ldots \times [0, 2\pi)$. Let $\mathcal{P}$ denotes the following projector

$$\mathcal{P}(\rho) := \frac{1}{(2\pi)^d} \int_0^{2\pi} dx_1 \ldots \int_0^{2\pi} dx_d \ U_{\mathbf{x}} \otimes \overline{U}_{\mathbf{x}}\, \rho (U_{\mathbf{x}} \otimes \overline{U}_{\mathbf{x}})^\dagger \ , \tag{89}$$

that is, $\mathcal{P}(\rho)$ performs symmetrization of ρ with respect to $U_{\mathbf{x}}$. It is clear that $\mathcal{P}$ maps separable states into separable states. Now, observe that

$$W^{(\mathbf{z})} = \mathcal{P}(V_1) + \mathcal{P}[(\mathbb{1}_n \otimes \sigma_x) V_2 (\mathbb{1}_n \otimes \sigma_x)] + D \ , \tag{90}$$

where

$$V_1 = \sum_{i=1}^{4} |\psi_i \otimes \psi_i\rangle\langle\psi_i \otimes \psi_i| \ , \qquad V_2 = \sum_{i=1}^{4} |\psi_i \otimes \phi_i\rangle\langle\psi_i \otimes \phi_i| \ , \tag{91}$$

with

$$\psi_1 = [1\,0\,1\,0\,1\,0] \ , \quad \psi_2 = [1\,0\,0\,1\,0\,1] \ , \quad \psi_3 = [0\,1\,1\,0\,0\,1] \ , \quad \psi_4 = [0\,1\,0\,1\,1\,0] \ ,$$

and

$$\phi_1 = [1\,0\,1\,0\,1\,0] \ , \quad \phi_2 = [1\,0\,0\ -1\,0\ -1] \ , \quad \phi_3 = [0\,1\ -1\,0\,0\,1] \ , \quad \phi_4 = [0\,1\,0\,1\ -1\,0] \ .$$

Finally, D is diagonal. It is clear, that V_1 and V_2 are separable. Hence, $W^{(\mathbf{z})}$ is separable being the convex combination of symmetrized separable operators and diagonal D. □

Remark 3 *Clearly the above proposition is trivially satisfied for* $\Phi^{(\mathbf{z})}_4$ *and* $z_{12} = -1$. *Actually, there is a strong numerical evidence that SPA for* $\Phi^{(\mathbf{z})}_4$ *with* $|z_{12}| = 1$ *is entanglement breaking.*

5 Conclusions

We provided a generalization of the well known linear positive maps: reduction map in $M_n(\mathbb{C})$ and Robertson map in $M_{2k}(\mathbb{C})$: $R_n^{(\mathbf{z})}$ and $\Phi_{2k}^{(\mathbf{z})}$, respectively. We showed that for each collection z_{ij} $(i < j)$ satisfying $|z_{ij}| \leq 1$ these maps are positive. Hence, each collection of points from the unit disc in the complex plane $\mathbb{C}$ gives rise to a positive map. Interestingly, points from the boundary, i.e. satisfying $|z_{ij}| = 1$, generate optimal maps: decomposable in the case of reduction map and indecomposable in the case of Robertson map.

Our construction gives rise to the new classes of entanglement witnesses: decomposable entanglement witnesses corresponding to $R_n^{(\mathbf{z})}$, and indecomposable entanglement witnesses corresponding to $\Phi_{2k}^{(\mathbf{z})}$. As a byproduct we provided new examples of PPT entangled states in $\mathbb{C}^{2k} \otimes \mathbb{C}^{2k}$ detected by indecomposable entanglement witnesses. Our analysis supports recent conjecture [24, 25] that structural physical approximation to an optimal positive map defines entanglement breaking completely positive map. Actually, we were able to prove it for generalized reduction map. Concerning generalized Robertson map Proposition 12 provides evidence that it supports conjecture [24, 25] as well.

Acknowledgement

This work was partially supported by the Nicolaus Copernicus Grant 383-F.

References

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, (Cambridge University Press, Cambridge, England, 2000).

[2] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[3] O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).

[4] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[5] B.M. Terhal, Phys. Lett. A **271**, 319 (2000).

[6] B. M. Terhal, Theor. Comput. Sci. **287**, 313 (2002).

[7] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A **62**, 052310 (2000).

[8] M. Lewenstein, B. Kraus, P. Horodecki, and J. I. Cirac, Phys. Rev. A **63**, 044304 (2001).

[9] B. Kraus, M. Lewenstein, and J. I. Cirac, Phys. Rev. A **65**, 042327 (2002).

[10] P. Hyllus, O. Gühne, D. Bruss, and M. Lewenstein Phys. Rev. A **72**, 012321 (2005).

[11] D. Bruß, J. Math. Phys. **43**, 4237 (2002).

[12] G. Tóth and O. Gühne, Phys. Rev. Lett. **94**, 060501 (2005)

[13] R. A. Bertlmann, H. Narnhofer and W. Thirring, Phys. Rev. A **66**, 032319 (2002).

[14] F.G.S.L. Brandão, Phys. Rev. A **72**, 022310 (2005).

[15] J. Eisert, P. Hyllus, O. Gühne, and M. Curty, Phys. Rev. A **70**, 062317 (2004);

[16] D. Chruściński and A. Kossakowski, J. Phys. A: Math. Theor. **41**, 145301 (2008).

[17] D. Chruściński, J. Pytel and G. Sarbicki, Phys. Rev. A **80** (2009) 062314.

[18] D. Chruściński and J. Pytel, Phys. Rev. A **82** 052310 (2010).

[19] A.C. Doherty, P.A. Parrilo and F.M. Spedalieri, Phys. Rev. Lett. **88**, 187904 (2002).

[20] F.G.S.L. Brandão and R.O. Vianna, Phys. Rev. Lett. **93**, 220503 (2004).

[21] J. Eisert, P. Hyllus, O. Gühne, and M. Curty, Phys. Rev. A **70**, 062317 (2004);

[22] P. Horodecki, Phys. Rev. A **68**, 052101 (2003).

[23] P. Horodecki and A. Ekkert, Phys. Rev. Lett. **89**, 127902 (2002).

[24] J.K. Korbicz, M.L. Almeida, J. Bae, M. Lewenstein and A. Acin, Phys. Rev. A **78**, 062105 (2008).

[25] R. Augusiak, J. Bae, Ł. Czekaj, and M. Lewenstein, *On structural physical approximations and entanglement breaking maps*, arXiv:1008.5056.

[26] M. Horodecki, P.W. Shor, and M. Ruskai, Rev. Mat. Phys. **15**, 629 (2003).

[27] A. Holevo, Russ. Math. Surv. **53**, 1295 (1990).

[28] E. Størmer, Acta Math. **110**, 233 (1963); Proc. Am. Math. Soc. **86**, 402 (1982).

[29] W. Arveson, Acta Math. **123**, 141 (1969).

[30] M.-D. Choi, Lin. Alg. Appl. **10**, 285 (1975); *ibid* **12**, 95 (1975).

[31] S.L. Woronowicz, Rep. Math. Phys. **10**, 165 (1976).

[32] S.L. Woronowicz, Comm. Math. Phys. **51**, 243 (1976).

[33] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, 2003.

[34] Ł. Skowronek and K. Życzkowski, J. Phys. A: Math. Theor. **42**, 325302 (2009).

[35] Ł. Skowronek, E. Størmer, and K. Życzkowski, J. Math. Phys. **50**, 062106 (2009)

[36] D. Chruściński and A. Kossakowski, Comm. Math. Phys. **290**, 1051 (2009).

[37] D. Chruściński and A. Kossakowski, Phys. Lett. A **373**, 2301 (2009).

[38] D. Chruściński, A. Kossakowski, and G. Sarbicki, Phys. Rev A **80**, 042314 (2009).

[39] D. Chruściński and A. Kossakowski, Open Systems and Inf. Dynamics, **14**, 275 (2007).

[40] A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972).

[41] A.G. Robertson, J. London Math. Soc. (2) **32**, 133 (1985).

[42] D. Chruściński and A. Kossakowski, J. Phys. A: Math. Theor. **41**, 215201 (2008).

[43] H.-P. Breuer, Phys. Rev. Lett. **97**, 0805001 (2006).

[44] W. Hall, J. Phys. A: Math. Gen. **39**, (2006) 14119.

[45] D. Chruściński and A Kossakowski, Phys. Rev. A **74**, 022308 (2006).

[46] R. Bhatia, *Positive Definite Matrices*, (Princeton University Press, 2006).