

# On classical and quantum liftings

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## Abstract

We analyze the procedure of lifting in classical stochastic and quantum systems. It enables one to ‘lift’ a state of a system into a state of ‘system+reservoir’. This procedure is important both in quantum information theory and the theory of open systems. We illustrate the general theory of liftings by a particular class related to so called circulant states.

## 1 Introduction

The interest on quantum entanglement has dramatically increased during the last two decades due to the emerging field of quantum information theory [1]. It turns out that quantum entanglement may be used as a basic resource in quantum information processing and communication. The prominent examples are quantum cryptography, quantum teleportation, quantum error correction codes and quantum computation (see the recent review [2]).

The characteristic feature of an entangled state (already stressed by Schrödinger) is that knowing a composed system that is in a pure state one can not say the same thing, in general, for any of its subsystems. Therefore, a full knowledge of a system can guarantee only a partial knowledge of its subsystems. This can’t happen in the classical case i.e. it is a purely quantum feature. In this paper we analyze the converse problem: suppose one knows a state of a subsystem. How one can ‘lift’ this state into a state of the whole system? An appropriate mathematical framework was provided in [3, 4]. This problem is important in many applications of quantum information theory and quantum dynamics.

For example it is of primary importance to be able to construct a state of a composite system knowing only its marginals, i.e. the states of subsystems. An appropriate lifting enables one to define a compound state which takes into account correlations between subsystems [3, 4, 5]. Another important issue arises in the quantum dynamics of open systems [7, 8]. One is interested to extract the dynamics of a quantum system from the unitary dynamics of the ‘system + environment’. Due to the presence of system-environment correlations this problem is highly nontrivial. One usually assumes that initially ‘system + environment’ was described by a product state, i.e. there were no correlations at all. The standard definition of reduced dynamics implies then that the dynamics of the system is completely positive and trace preserving. A product state provides a very special

example of lifting: a system state  $\rho$  is lifted to ‘system + environment’ state  $\rho \otimes \omega$ . Could we use other liftings to define initial ‘system + environment’ state?

One tries to relax the condition upon the ‘product lifting’ and considers dynamics which is only positive but not completely positive (see e.g. [11, 12, 13, 14]). Such positive dynamics enables one to map density matrices into legitimate density matrices. However, one loses the important property that coupling the system to an arbitrary environment will result in positive ‘system + environment’ dynamics. This problem appeared more recently in the context of non-Markovian evolution (see recent papers [15, 16]). An interesting debate upon this problem started already in the 90. of the last century [9, 10].

The problem of positive but not completely positive maps is very much related to the problem of detection of quantum entanglement (see [2] for the review). It is well known that the general structure of positive maps is still unknown and only partial results are available (see [17, 18, 19, 20, 21, 22] for recent examples and constructions). One may try to use appropriate liftings to provide new classes of such maps. Suppose for example that we lift a state  $\rho$  to the compound state [3, 4], perform a positive map and then reduce to the system of interest. It is obvious that the resulting map would be positive and it depends very much upon the applied lifting. An interesting open problem is the classification of ‘lifting assisted maps’ defined this way.

The paper is organized as follows: for pedagogical reason we start with presentation of basic ideas of stochastic classical systems Section 2. In particular we show how to reformulate the classical theory in the quantum framework. Section 3 provides the discussion of classical liftings and Section 4 provides generalization for multipartite case. Then we consider the main topic of this paper in Section 5. Section 6 illustrates the general ideas of liftings by ‘circulant liftings’. These are liftings related to the construction of so called circulant states [23]. In particular we analyze Bell diagonal liftings which play important role in quantum information theory (see [25, 26] and recent paper [27]). Final conclusions are collected in the last Section.

## 2 The quantum framework for classical probability with finite state space

### 2.1 Preliminaries

Let us consider a classical  $n$ -level stochastic system. The primary object for elementary classical probability, i.e. probability on a finite state space, is the space of elementary events  $\Omega = \{\omega_1, \dots, \omega_n\}$ . The corresponding space of states  $S(\Omega)$  consists of probability distributions over  $\Omega$

$$S(\Omega) := \left\{ p = (p_1, \dots, p_n) \mid p_i \geq 0, \sum_{i=1}^n p_i = 1 \right\}. \quad (2.1)$$

It is clear that  $S(\Omega)$  is convex and the corresponding set of extremal points – pure states – reads as follows

$$S_0(\Omega) := \left\{ p \in S(\Omega) \mid p_i^2 = p_i, i = 1, \dots, n \right\}. \quad (2.2)$$

Hence, a state  $p \in S(\Omega)$  is pure iff  $p_i = \delta_{ij}$  for some  $j \in \{1, \dots, n\}$ . It is evident that  $S_0(\Omega)$  consists of the  $n$  vertices of  $(n - 1)$ -dimensional simplex  $S(\Omega)$ . A classical observable is a random variable

$a : \Omega \longrightarrow \mathbb{R}$ , and the corresponding expectation value of  $a$  in the state  $p \in S(\Omega)$  is given by

$$\langle a, p \rangle := \sum_{i=1}^n a_i p_i = \sum_{i=1}^n a(\omega_i) P(\omega_i) , \quad (2.3)$$

where  $a_i = a(\omega_i)$  and  $p_i = P(\omega_i)$  (with  $P : \Omega \rightarrow \mathbb{R}$  being a probability distribution). Let us recall the standard way of translating the above formulation of the classical probability theory using the noncommutative framework of quantum theory. Note, that the space of complex random variables

$$\psi : \Omega \longrightarrow \mathbb{C} , \quad (2.4)$$

defines a Hilbert space

$$\mathcal{H}(\Omega) := \left\{ \psi = (\psi_1, \dots, \psi_n) \mid \psi_i = \psi(\omega_i) \right\} \cong \mathbb{C}^n , \quad (2.5)$$

equipped with the standard inner product

$$\langle \varphi | \psi \rangle = \sum_{i=1}^n \bar{\varphi}_i \psi_i . \quad (2.6)$$

Now, any classical state  $p \in S(\Omega)$  gives rise to a density matrix living in  $\mathcal{H}(\Omega)$

$$p = (p_1, \dots, p_n) \longrightarrow \rho = \sum_{i=1}^n p_i e_{ii} , \quad (2.7)$$

where  $e_{ii} = |i\rangle\langle i|$  and  $e_i \equiv |i\rangle$  denotes an arbitrary (but fixed) orthonormal basis in  $\mathcal{H}(\Omega)$ . Hence classical states are represented by diagonal density matrices with respect to the same fixed basis  $e_i$ . Formula (2.7) defines therefore a decomposition of  $\rho$  which coincides with the spectral decomposition when all the  $p_i$ 's are different. Thus  $S(\Omega)$  defines a commutative subalgebra of

$$S(\Omega) \subset S(\mathbb{C}^n) := \{ \rho : \mathbb{C}^n \longrightarrow \mathbb{C}^n \mid \rho \geq 0 , \text{Tr } \rho = 1 \} , \quad (2.8)$$

consisting of diagonal matrices. Classical pure states

$$S_0(\Omega) \subset S_0(\mathbb{C}^n) := \{ \rho \in S(\mathbb{C}^n) \mid \rho^2 = \rho \} , \quad (2.9)$$

are represented by the rank-1 projectors  $e_{ii}$ . In the same way classical random variables define a commutative subalgebra of  $\mathcal{B}(\mathbb{C}^n)$ :

$$a = (a_1, \dots, a_n) \longrightarrow a = \sum_{i=1}^n a_i e_{ii} , \quad (2.10)$$

that is, one may introduce the ‘‘classical’’ algebra  $\mathcal{B}(\Omega) \subset \mathcal{B}(\mathbb{C}^n)$

$$\mathcal{B}(\Omega) := \left\{ a \in \mathcal{B}(\mathbb{C}^n) \mid a = \sum_{i=1}^n a_i e_{ii} \right\} . \quad (2.11)$$

An element  $a \in \mathcal{B}(\Omega)$  defines a classical observable iff  $a^* = a$ , i.e.  $\bar{a}_i = a_i$ . Finally, the classical formula (2.3) may be rewritten in a ‘‘quantum fashion’’ as follows

$$\langle a, p \rangle = \text{Tr } a \rho . \quad (2.12)$$

## 2.2 Classical channels

A classical channel is a linear positive map

$$\Lambda : \mathcal{B}(\Omega_1) \longrightarrow \mathcal{B}(\Omega_2) , \quad (2.13)$$

where  $\Omega_1 = \{\omega_1, \dots, \omega_{n_1}\}$  and  $\Omega = \{\varpi_1, \dots, \varpi_{n_2}\}$ . A channel is unital iff  $\Lambda(\mathbb{I}_1) = \mathbb{I}_2$ , where  $\mathbb{I}_k$  denotes an identity element in the  $\mathbb{C}^*$ -algebra  $\mathcal{B}(\Omega_k)$ . Let

$$\mathcal{B}(\Omega_1) := \left\{ a \in \mathcal{B}(\mathbb{C}^{n_1}) \mid a = \sum_{i=1}^{n_1} a_i e_{ii} \right\} , \quad (2.14)$$

and

$$\mathcal{B}(\Omega_2) := \left\{ b \in \mathcal{B}(\mathbb{C}^{n_2}) \mid b = \sum_{i=1}^{n_2} b_i f_{ii} \right\} , \quad (2.15)$$

where  $e_i$  defines an orthonormal basis in  $\mathcal{H}(\Omega_1) \cong \mathbb{C}^{n_1}$ , and  $f_i$  defines an orthonormal basis in  $\mathcal{H}(\Omega_2) \cong \mathbb{C}^{n_2}$ . One obtains

$$\Lambda(a) = \sum_{i=1}^{n_1} a_i \Lambda(e_{ii}) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_i \Lambda_{ij} f_{jj} , \quad (2.16)$$

where

$$\Lambda_{ij} := \text{Tr}(f_{jj} \Lambda(e_{ii})) . \quad (2.17)$$

A linear map  $\Lambda$  is positive if and only if  $\Lambda_{ij} \geq 0$ . A channel  $\Lambda$  transforms  $a = \sum_i a_i e_{ii}$  into  $b = \sum_j b_j f_{jj}$ , with  $b_j = \sum_i a_i \Lambda_{ij}$ . It is represented by the  $n_1 \times n_2$  matrix  $(\Lambda_{ij})$  and it has the following Kraus representation

$$\Lambda(a) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} K_{ij} a K_{ij}^* , \quad (2.18)$$

where the operators  $K_{ij} : \mathbb{C}^{n_1} \longrightarrow \mathbb{C}^{n_2}$  are defined by

$$K_{ij} := \sqrt{\Lambda_{ij}} |f_j\rangle \langle e_i| , \quad (2.19)$$

and  $\sqrt{\Lambda_{ij}}$  is any complex valued square root of  $(\Lambda_{ij})$ . Let us observe that the above representation of the classical channel involves operators which are not diagonal. A channel is unital if

$$\sum_{i=1}^{n_1} \Lambda_{ij} = 1 , \quad j = 1, \dots, n_2 . \quad (2.20)$$

It is therefore clear that a unital channel defines the conditional probability

$$p_{i|j} := \Lambda_{ij} , \quad (2.21)$$

on the space  $\Omega_1 \times \Omega_2$ .

**Example 1 (Unitary channel)** Consider a channel

$$\Lambda : \mathcal{B}(\Omega) \longrightarrow \mathcal{B}(\Omega) , \quad (2.22)$$

defined by

$$\Lambda(a) := UaU^* , \quad (2.23)$$

where  $U$  is an  $n \times n$  unitary matrix corresponding to a permutation  $\pi$  from the symmetric group  $S_n$ , i.e.

$$U_{ij} := \delta_{i\pi(j)} , \quad i, j = 1, \dots, n . \quad (2.24)$$

One easily finds

$$\Lambda(a) = \sum_{i=1}^n a_i e_{\pi(i)\pi(i)} , \quad (2.25)$$

where  $\Lambda$  is represented by the following doubly stochastic matrix

$$\Lambda_{ij} = \text{Tr}(e_{jj}\Lambda(e_{ii})) = \delta_{j\pi(i)} . \quad (2.26)$$

Interestingly one has  $\Lambda = U^T$ .

**Example 2 (Completely depolarizing channel)** Consider the unital channel

$$\Lambda : \mathcal{B}(\Omega_1) \longrightarrow \mathcal{B}(\Omega_2) , \quad (2.27)$$

defined by

$$\Lambda(a) = \frac{1}{n_1} \mathbb{I}_2 \text{Tr} a . \quad (2.28)$$

corresponding to the matrix  $(\Lambda_{ij})$  given by:

$$\Lambda_{ij} = \frac{1}{n_1} . \quad (2.29)$$

Note, that for  $n_1 = n_2$ , the matrix  $(\Lambda_{ij})$  is doubly stochastic. The corresponding representation (2.18) reads as follows

$$\Lambda(a) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} K_{ij} a K_{ij}^* , \quad (2.30)$$

where the  $K_{ij}$  are defined by (2.19).

### 2.3 Composite classical systems

Consider now a composite system consisting of two subsystems with state spaces  $\Omega_1$  and  $\Omega_2$ . The corresponding space of elementary events has the form

$$\Omega_{12} = \Omega_1 \times \Omega_2 , \quad (2.31)$$

hence  $|\Omega_{12}| = n_1 n_2$ . The corresponding space of states consists of the joint probability distributions  $P : \Omega_{12} \longrightarrow \mathbb{R}$

$$S(\Omega_{12}) := \left\{ p_{ij} \mid p_{ij} \geq 0 , \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} p_{ij} = 1 \right\} , \quad (2.32)$$

where  $p_{ij} = P(\omega_i, \varpi_j)$ . Again, one may translate multipartite case into the “quantum” framework by considering the complex random variables  $\psi : \Omega_{12} \rightarrow \mathbb{C}$  which give rise to the following Hilbert space

$$\mathcal{H}(\Omega_{12}) = \mathcal{H}(\Omega_1) \otimes \mathcal{H}(\Omega_2) \cong \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} , \quad (2.33)$$

hence the classical space  $S(\Omega_{12})$  defines a subspace of  $S(\mathcal{H}(\Omega_{12}))$

$$S(\Omega_{12}) \subset S(\mathcal{H}(\Omega_{12})) := \left\{ \rho : \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \rightarrow \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \mid \rho \geq 0 , \text{Tr } \rho = 1 \right\} , \quad (2.34)$$

consisting of diagonal matrices

$$\rho = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} p_{ij} e_{ii} \otimes f_{jj} . \quad (2.35)$$

**Remark 1 (A classical analogue of the maximally entangled state)** Consider a bipartite system with  $\Omega_{12} = \Omega \times \Omega$  and  $|\Omega| = n$ . For any  $U \in U(n)$  one defines a maximally entangled state in  $\mathbb{C}^n \otimes \mathbb{C}^n$

$$P_U := \frac{1}{n} \sum_{i,j=1}^n e_{ij} \otimes U e_{ij} U^* . \quad (2.36)$$

In the classical case the unitaries correspond to permutations from the symmetric group  $S_n$  and one only considers diagonal states. Hence, the classical counterpart of  $P_U$  is given by

$$P_\pi = \frac{1}{n} \sum_{i=1}^n e_{ii} \otimes e_{\pi(i)\pi(i)} . \quad (2.37)$$

Note, that contrarily to  $P_U$ , its classical counterpart  $P_\pi$  is no longer a pure state. One may call  $P_\pi$  a *maximally correlated classical state*.

**Remark 2 (A classical analog of the Choi-Jamiołkowski isomorphism)** There is an evident isomorphism between classical channels and classical states from  $S(\Omega \times \Omega)$ : if  $\Lambda : \mathcal{B}(\Omega) \rightarrow \mathcal{B}(\Omega)$  is a unital channel, then

$$\rho = \frac{1}{n} \sum_{i=1}^n e_{ii} \otimes \Lambda^\#(e_{ii}) , \quad (2.38)$$

where  $\Lambda^\# : \mathcal{S}(\Omega) \rightarrow \mathcal{S}(\Omega)$  is the dual map, defines a classical state of the composite system in  $\Omega \times \Omega$ . Note, that

$$\Lambda^\#(e_{ii}) = \sum_{j=1}^n p_{j|i} e_{jj} , \quad (2.39)$$

implies

$$\rho = \sum_{i,j=1}^n p_{ij} e_{ii} \otimes e_{jj} , \quad (2.40)$$

where  $p_{ij}$  is the joint probability defined by  $p_{ij} = p_{i|j} p_j^{(0)}$ , and  $p^{(0)}$  corresponds to the uniform probability distribution  $p_k^{(0)} = \frac{1}{n}$ .

**Remark 3 (A classical analog of the teleportation protocol)** The state  $P_\pi$  may be used to perform a classical teleportation protocol: suppose that Alice would like to teleport a classical state

$$\rho_A = \sum_{i=1}^n p_i e_{ii} . \quad (2.41)$$

In analogy to the quantum case one defines the joint state  $\rho_A \otimes P_\pi$  living in  $\mathcal{H}(\Omega) \otimes \mathcal{H}(\Omega) \otimes \mathcal{H}(\Omega)$  and performs the joint measurement of  $n^2 P_0 \otimes \mathbb{I}_B$ , where  $P_0$  corresponds to the trivial permutation, i.e.  $P_0 = n^{-1} \sum_i e_{ii} \otimes e_{ii}$ , and  $\mathbb{I}_B$  is an identity operator in the Bob space. One obtains

$$\rho_B = n^2 \text{Tr}_{12}(P_0 \otimes \mathbb{I}_B \cdot \rho_A \otimes P_\pi) = \sum_{i=1}^n p_{\pi^{-1}(i)} e_{ii} . \quad (2.42)$$

It is therefore clear that performing the unitary transformation corresponding to  $\pi$  Bob recovers  $\rho_A$ . Summarizing: a classical teleportation channel consists in simple permutation  $p_i \rightarrow p_{\pi(i)}$ , where the permutation  $\pi$  corresponds to the ‘maximally entangled state’  $P_\pi$  used in the teleportation protocol. It is clear that there is a fundamental difference between classical and quantum teleportation: in the quantum case Bob needs 1 bit of additional classical information from Alice. It is no longer needed in the classical case. Knowing  $\pi$  Bob reconstructs  $\rho_A$  without any additional information.

Consider a classical system living in  $\Omega$  and let  $\rho$  be a classical state from  $S(\Omega)$ . To define a channel let  $\sigma$  be a fixed state of the *ancilla* – again from  $\Omega$  – and define

$$\Lambda^\#(\rho) = \text{Tr}_2(U\rho \otimes \sigma U^*) , \quad (2.43)$$

where  $U$  is an  $n^2 \times n^2$  unitary matrix corresponding to a permutation from the symmetric group  $S_{n^2}$ . The simplest example corresponds to  $n = 2$ : let  $\rho = \sum_i p_i e_{ii}$  and  $\sigma = \sum_i q_i e_{ii}$ . One finds the following  $2^2 = 4$  channels

$$p_i \rightarrow \sum_{j=1}^2 \Lambda_{ij}^{(k)} p_j , \quad (2.44)$$

with

$$\Lambda^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \Lambda^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad (2.45)$$

and

$$\Lambda^{(3)} = \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} = q_1 \Lambda^{(1)} + q_2 \Lambda^{(2)} , \quad \Lambda^{(4)} = \begin{pmatrix} q_2 & q_1 \\ q_1 & q_2 \end{pmatrix} = q_1 \Lambda^{(2)} + q_2 \Lambda^{(1)} . \quad (2.46)$$

Note that all 4 matrices  $[\Lambda_{ij}^{(k)}]$  are doubly stochastic. One easily proves the following general statement.

**Proposition 1** *A classical channel defined via the reduction procedure (2.43) is doubly stochastic.*

We stress that a quantum channel defined by the reduction procedure is trace preserving but not necessarily unital.

## 2.4 Multipartite case

The bipartite case may be easily generalized to many parties. Consider a composite system consisting of  $N$  parties with

$$\Omega_{N\dots 1} = \Omega_N \times \dots \times \Omega_1 , \quad (2.47)$$

and  $|\Omega_k| = n_k$ . It implies  $|\Omega_{1\dots N}| = n_1 \dots n_N$ . The corresponding space of states consists of the joint probability distributions  $P : \Omega_{N\dots 1} \rightarrow \mathbb{R}$

$$S(\Omega_{N\dots 1}) := \left\{ p_{i_N \dots i_1} \mid p_{i_N \dots i_1} \geq 0 , \sum_{i_1 \dots i_N} p_{i_N \dots i_1} = 1 \right\} , \quad (2.48)$$

where  $p_{i_N \dots i_1} = P(\omega_{i_N}^{(N)}, \dots, \omega_{i_1}^{(1)})$ , and  $\omega_i^{(k)}$  is an elementary event from  $\Omega_k$ .

Again, one may translate multipartite case into the quantum framework. Note that complex random variables

$$\psi : \Omega_{N\dots 1} \rightarrow \mathbb{C} , \quad (2.49)$$

give rise to the multipartite Hilbert space

$$\mathcal{H}(\Omega_{N\dots 1}) = \mathcal{H}(\Omega_N) \otimes \dots \otimes \mathcal{H}(\Omega_1) \cong \mathbb{C}^{n_N} \otimes \dots \otimes \mathbb{C}^{n_1} , \quad (2.50)$$

and hence the classical space  $S(\Omega_{N\dots 1})$  defines a subspace of  $S(\mathcal{H}(\Omega_{N\dots 1}))$  consisting of the diagonal density matrices

$$\rho = \sum_{i_1=1}^{n_1} \dots \sum_{i_N=1}^{n_N} p_{i_N \dots i_1} e_{i_N i_N}^{(N)} \otimes \dots \otimes e_{i_1 i_1}^{(1)} , \quad (2.51)$$

where  $\{e_1^{(k)}, \dots, e_{n_k}^{(k)}\}$  defines an orthonormal basis in  $\mathcal{H}(\Omega_k)$ .

## 2.5 Markovian states

Consider now the special case when all subsystems live on the same space  $\Omega$ . A classical  $N$ -partite state on  $\Omega \times \dots \times \Omega$

$$\rho = \sum_{i_N, \dots, i_1=1}^n p_{i_N \dots i_1} e_{i_N i_N} \otimes \dots \otimes e_{i_1 i_1} , \quad (2.52)$$

is Markovian, if the  $N$ -partite joint probabilities  $p_{i_N \dots i_1}$  having the following property

$$p_{i_N \dots i_1} = p_{i_N | i_{N-1}} p_{i_{N-1} | i_{N-2}} \dots p_{i_2 | i_1} p_{i_1} , \quad (2.53)$$

where  $p_{i|j}$  defines the conditional probability on  $\Omega \times \Omega$  and  $p_i$  is a probability distribution on  $\Omega$ . If  $\rho$  is Markovian, then its reduction with respect to the subsystems  $N, N-1, \dots, N-k+1$  gives  $(N-k)$ -partite Markovian state

$$\text{Tr}_N \dots \text{Tr}_{N-k+1} \rho = \sum_{i_{N-k}, \dots, i_1=1}^n p_{i_{N-k} | i_{N-k-1}} \dots p_{i_2 | i_1} p_{i_1} e_{i_{N-k} i_{N-k}} \otimes \dots \otimes e_{i_1 i_1} . \quad (2.54)$$

Let us translate the Markovian property into the quantum noncommutative setting. Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be  $\mathbb{C}^*$ -algebras.



**Definition 1** A linear completely positive map

$$\mathcal{E} : \mathcal{A}_2 \otimes \mathcal{A}_1 \longrightarrow \mathcal{A}_1 , \quad (2.55)$$

is called a transition expectation if

$$\mathcal{E}(\mathbb{1}_2 \otimes \mathbb{1}_1) = \mathbb{1}_1 , \quad (2.56)$$

where  $\mathbb{1}_k$  denotes an identity element in  $\mathcal{A}_k$ .

One has the following

**Proposition 2** An  $N$ -partite state  $\rho$  on  $\Omega^N$  is Markovian if there exist a transition expectation

$$\mathcal{E} : \mathcal{B}(\Omega) \otimes \mathcal{B}(\Omega) \longrightarrow \mathcal{B}(\Omega) , \quad (2.57)$$

such that

$$\text{Tr}(\rho(a_N \otimes \dots \otimes a_1)) = \text{Tr}(\rho_1 \mathcal{E}(a_N \otimes \mathcal{E}(a_{N-1} \otimes \dots \otimes \mathcal{E}(a_1 \otimes \mathbb{1}))) \dots) , \quad (2.58)$$

is satisfied for arbitrary  $a_1, \dots, a_N \in \mathcal{B}(\Omega)$ , where  $\rho_1 = \text{Tr}_1 \rho$  is the reduced density matrix living on the first subsystem.

Indeed, introducing so called Markov operator  $P : \mathcal{B}(\Omega) \longrightarrow \mathcal{B}(\Omega)$

$$P(e_{ii}) = \sum_{k=1}^n p_{i|j} e_{jj} , \quad (2.59)$$

where  $p_{i|j}$  denotes conditional probability, and defining the transition expectation via

$$\mathcal{E}(a_2 \otimes a_1) = P(a_2)a_1 , \quad (2.60)$$

one shows equivalence of (2.58) and (2.53).

### 3 Classical liftings

Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be  $\mathbb{C}^*$ -algebras. One introduces [3] the following

**Definition 2** A lifting from  $\mathcal{A}_1$  to  $\mathcal{A}_2 \otimes \mathcal{A}_1$  is a map

$$\mathcal{E}^\# : \mathcal{S}(\mathcal{A}_1) \longrightarrow \mathcal{S}(\mathcal{A}_2 \otimes \mathcal{A}_1) . \quad (3.1)$$

If the map  $\mathcal{E}^\#$  is affine and its dual  $\mathcal{E} : \mathcal{A}_2 \otimes \mathcal{A}_1 \longrightarrow \mathcal{A}_1$  is completely positive, then one calls  $\mathcal{E}^\#$  a linear lifting. If  $\mathcal{E}^\#$  maps pure states into pure states, one calls it a pure lifting.

Following [3] let us consider a lifting

$$\mathcal{E}^\# : \mathcal{S}(\Omega_1) \longrightarrow \mathcal{S}(\Omega_2 \times \Omega_1) . \quad (3.2)$$

Note, that any linear lifting has the following form

$$\mathcal{E}^\#(\rho) = \sum_{i,j=1}^{n_1} \sum_{k=1}^{n_2} p_i \mathcal{E}_{ijk} f_{jj} \otimes e_{kk} , \quad (3.3)$$

where

$$\mathcal{E}_{ijk} := \text{Tr} \left( f_{jj} \otimes e_{kk} \mathcal{E}^\#(e_{ii}) \right) = \text{Tr} \left( e_{ii} \mathcal{E}(f_{jj} \otimes e_{kk}) \right) , \quad (3.4)$$

and  $\rho = \sum_i p_i e_{ii}$ . One has

$$\mathcal{E}^\#(\rho) = \sum_{j=1}^{n_2} \sum_{k=1}^{n_1} p_{jk} f_{jj} \otimes e_{kk} , \quad (3.5)$$

and hence

$$p_{jk} = \sum_{i=1}^{n_1} \mathcal{E}_{ijk} p_i . \quad (3.6)$$

**Definition 3** A lifting  $\mathcal{E}^\#$  is called *non-demolishing* for a state  $\rho \in \mathcal{S}(\Omega_1)$  if

$$\text{Tr}_2 \mathcal{E}^\#(\rho) = \rho . \quad (3.7)$$

It is clear that if  $\mathcal{E}^\#$  is linear and non-demolishing for some  $\rho$ , then it is non-demolishing for all states. In this case one has

$$\sum_{k=1}^{n_2} \mathcal{E}_{ijk} = \delta_{ij} . \quad (3.8)$$

**Remark 4** The notion of nondemolition lifting defined here is essentially (i.e., up to minor technicalities) included in the more abstract notion of state extension introduced by Cecchini and Petz [28] (see also [29]).

**Definition 4** A lifting  $\mathcal{E}^\#$  is *Markovian lifting* if

$$\mathcal{E}^\#(e_{ii}) = \sum_{j=1}^{n_2} p_{j|i} f_{jj} \otimes e_{ii} , \quad (3.9)$$

where  $p_{j|i}$  stands for the conditional probability on  $\Omega_2 \times \Omega_1$ .

**Example 3 (Pure lifting)** Let  $s : \{1, \dots, n_1\} \rightarrow \{1, \dots, n_2\}$  be a function and define

$$\mathcal{E}^\#(e_{ii}) = f_{s(i)s(i)} \otimes e_{ii} , \quad (3.10)$$

that is, it lifts a pure state on  $\Omega_1$  into pure states on  $\Omega_2 \times \Omega_1$ . Hence, for  $\rho = \sum_i p_i e_{ii}$  one finds

$$\mathcal{E}^\#(\rho) = \sum_{i=1}^{n_1} p_i f_{s(i)s(i)} \otimes e_{ii} , \quad (3.11)$$

which implies

$$\mathcal{E}_{ijk} = \delta_{ik} \delta_{j,s(i)} . \quad (3.12)$$

**Example 1 (Product lifting)** Let

$$\mathcal{E}_{ijk} = \delta_{ik} q_j , \quad (3.13)$$

where  $\sigma = \sum_k q_k f_{kk}$  is a classical state over  $\Omega_2$ . One obtains

$$\mathcal{E}^\#(\rho) = \sigma \otimes \rho , \quad (3.14)$$

that is, a joint probability is given by a product formula  $p_{jk} = p_j q_k$ .

**Example 2** Let  $\sigma$  be a fixed state on  $\Omega_2$  and

$$\Gamma : \mathcal{B}(\Omega_2 \times \Omega_1) \longrightarrow \mathcal{B}(\Omega_2 \times \Omega_1) , \quad (3.15)$$

be a linear positive map. Define the lifting  $\mathcal{E}^{\#\Gamma, \sigma}$  by

$$\mathcal{E}^{\#\Gamma, \sigma}(\rho) := \Gamma^\#(\sigma \otimes \rho) . \quad (3.16)$$

Note, that if  $\Gamma = \text{id}_2 \otimes \text{id}_1$ , then the above formula recovers a product lifting.

**Example 3 (Classical Ohya lifting)** If

$$\mathcal{E}_{ijk} = \delta_{ik} \delta_{jk} , \quad (3.17)$$

then

$$\mathcal{E}^\#(\rho) = \sum_{i=1}^n p_i e_{ii} \otimes e_{ii} . \quad (3.18)$$

Note, that a characteristic feature of the Ohya lifting is that both reduced states reproduce the original state  $\rho$ , i.e.

$$\text{Tr}_1 \mathcal{E}^\#(\rho) = \text{Tr}_2 \mathcal{E}^\#(\rho) = \rho , \quad (3.19)$$

for an arbitrary classical state  $\rho$  on  $\Omega$ . It shows that Ohya lifting realizes a perfect classical cloning machine [31]. Moreover, since both reduced states are the same Ohya lifting provides at the same time a protocol for a classical broadcasting [32].

## 4 Multipartite classical liftings

It is clear that a lifting (3.1) may be easily generalized to the multipartite case: consider a set  $\mathcal{A}_1, \dots, \mathcal{A}_N$  of  $\mathbb{C}^*$ -algebras.

**Definition 5** *An  $N$ -lifting from  $\mathcal{A}_1$  to  $\mathcal{A}_N \otimes \dots \otimes \mathcal{A}_1$  is a map*

$$\mathcal{E}^\# : \mathcal{S}(\mathcal{A}_1) \longrightarrow \mathcal{S}(\mathcal{A}_N \otimes \dots \otimes \mathcal{A}_1) . \quad (4.1)$$

*If the map  $\mathcal{E}^\#$  is affine and its dual  $\mathcal{E} : \mathcal{A}_N \otimes \dots \otimes \mathcal{A}_1 \longrightarrow \mathcal{A}_1$  is completely positive, then one calls  $\mathcal{E}^\#$  a linear lifting.*

Applying the above definition to the classical commutative case one finds

$$\mathcal{E}_N^\# : \mathcal{S}(\Omega_1) \longrightarrow \mathcal{S}(\Omega_N \times \dots \times \Omega_1) . \quad (4.2)$$

Note, that any linear classical  $N$ -lifting has the following form

$$\mathcal{E}_N^\#(\rho) = \sum_{i=1}^{n_1} \sum_{i_1=1}^{n_1} \dots \sum_{i_N=1}^{n_N} \mathcal{E}_{i_N \dots i_1 i} p_i e_{i_N i_N}^{(N)} \otimes \dots \otimes e_{i_1 i_1}^{(1)} , \quad (4.3)$$

where

$$\mathcal{E}_{i_N \dots i_1 i} := \text{Tr} \left( e_{i_N i_N}^{(N)} \otimes \dots \otimes e_{i_1 i_1}^{(1)} \mathcal{E}_N^\#(e_{ii}^{(1)}) \right) = \text{Tr} \left( \mathcal{E}_N(e_{i_N i_N}^{(N)} \otimes \dots \otimes e_{i_1 i_1}^{(1)}) e_{ii}^{(1)} \right) , \quad (4.4)$$

and  $\rho = \sum_i p_i e_{ii}$ . Note, that  $\mathcal{E}_N^\#(\rho)$  is an  $N$ -partite classical state and hence it corresponds to an  $N$ -partite joint probability distribution  $p_{i_N \dots i_1}$ , that is

$$\mathcal{E}_N^\#(\rho) = \sum_{i=1}^{n_1} \dots \sum_{i_N=1}^{n_N} p_{i_N \dots i_1} e_{i_N i_N}^{(N)} \otimes \dots \otimes e_{i_1 i_1}^{(1)} , \quad (4.5)$$

and hence

$$p_{i_N \dots i_1} = \sum_{i=1}^{n_1} \mathcal{E}_{i_N \dots i_1 i} p_i . \quad (4.6)$$

**Example 4 (Pure  $N$ -lifting)** Let  $s_k : \{1, \dots, n_1\} \longrightarrow \{1, \dots, n_k\}$  ( $k = 2, 3, \dots, N$ ) be a family of functions and define the following linear  $N$ -lifting

$$\mathcal{E}_N^\#(e_{ii}^{(1)}) = e_{s_N(i)s_N(i)}^{(N)} \otimes \dots \otimes e_{s_2(i)s_2(i)}^{(2)} \otimes e_{ii}^{(1)} , \quad (4.7)$$

that is, it lifts a pure state on  $\Omega_1$  into pure states on  $\Omega_N \times \dots \times \Omega_1$ . Hence, for  $\rho = \sum_i p_i e_{ii}^{(1)}$  one finds

$$\mathcal{E}_N^\#(\rho) = \sum_{i=1}^{n_1} p_i e_{s_N(i)s_N(i)}^{(N)} \otimes \dots \otimes e_{s_2(i)s_2(i)}^{(2)} \otimes e_{ii}^{(1)} , \quad (4.8)$$

which implies

$$\mathcal{E}_{i_N \dots i_1 i} = \delta_{i_N, s_N(i)} \dots \delta_{i_2, s_2(i)} \delta_{i_1 i} . \quad (4.9)$$

**Example 4 (Product  $N$ -lifting)** Let  $\sigma_N \otimes \dots \otimes \sigma_2$  be a product state on  $\Omega_N \times \dots \times \Omega_2$ . One defines

$$\mathcal{E}_N^\#(\rho) = \sigma_N \otimes \dots \otimes \sigma_2 \otimes \rho , \quad (4.10)$$

that is,

$$\mathcal{E}_{i_N \dots i_1 i} = q_{i_N}^{(N)} \dots q_{i_2}^{(2)} \delta_{i_1 i} , \quad (4.11)$$

where  $\sigma_k = \sum_{i_k} q_{i_k}^{(k)} e_{i_k i_k}^{(k)}$ . It is clear that the corresponding joint probability factorizes as follows

$$p_{i_N \dots i_1} = q_{i_N}^{(N)} \dots q_{i_2}^{(2)} p_{i_1} . \quad (4.12)$$

**Example 5** Let  $\sigma = \sigma_N \otimes \dots \otimes \sigma_2$  be a product state on  $\Omega_N \times \dots \times \Omega_2$  and

$$\Gamma : \mathcal{B}(\Omega_N \times \dots \times \Omega_1) \longrightarrow \mathcal{B}(\Omega_N \times \dots \times \Omega_1) , \quad (4.13)$$

be a linear positive map. Define the  $N$ -lifting  $\mathcal{E}^{\#\Gamma, \sigma}$  by

$$\mathcal{E}_N^{\#\Gamma, \sigma}(\rho) := \Gamma^{\#}(\sigma \otimes \rho) . \quad (4.14)$$

Note, that if  $\Gamma = \text{id}_N \otimes \dots \otimes \text{id}_1$ , then the above formula recovers a product  $N$ -lifting.

**Example 6 (Classical Ohya  $N$ -lifting)** Let  $\Omega_1 = \dots = \Omega_N = \Omega$ . Taking

$$\mathcal{E}_{i_N \dots i_1 i} = \delta_{i_N i} \dots \delta_{i_1 i} , \quad (4.15)$$

one obtains

$$\mathcal{E}_N^{\#}(e_{ii}) = e_{ii} \otimes \dots \otimes e_{ii} , \quad (4.16)$$

and hence for  $\rho = \sum_i p_i e_{ii}$  one finds

$$\mathcal{E}_N^{\#}(\rho) = \sum_{i=1}^n p_i e_{ii} \otimes \dots \otimes e_{ii} . \quad (4.17)$$

Note, that a characteristic feature of the Ohya lifting is that all reduced one-partite states reproduce the original state  $\rho$ , i.e.

$$\text{Tr}_{\check{1}} \mathcal{E}^{\#}(\rho) = \dots = \text{Tr}_{\check{N}} \mathcal{E}^{\#}(\rho) = \rho , \quad (4.18)$$

where  $\text{Tr}_{\check{k}}$  denotes the partial trace with respect all subsystems excluding  $k$ th. It shows that Ohya lifting realizes a perfect classical cloning machine [31]. Moreover, since both reduced states are the same Ohya lifting provides at the same time a protocol for a classical broadcasting [32].

**Remark 1** Let  $\varphi_k : \mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^n)$  be a set of positive unital maps (for  $k = 1, \dots, N$ ). Then the following  $N$ -partite state

$$\rho = \sum_{i=1}^n p_i \varphi_1^{\#}(e_{ii}) \otimes \dots \otimes \varphi_N^{\#}(e_{ii}) , \quad (4.19)$$

is a (quantum)  $N$ -separable state.

Let us observe that in the special case when  $\Omega_2 = \dots = \Omega_N =: \Omega$  there is a simple way to construct an  $N$ -lifting out of the lifting

$$\mathcal{E}^{\#} : \mathcal{S}(\Omega_1) \longrightarrow \mathcal{S}(\Omega_2 \times \Omega_1) , \quad (4.20)$$

Define the  $N$ -lifting

$$\mathcal{E}_N^{\#} : \mathcal{S}(\Omega_1) \longrightarrow \mathcal{S}(\Omega_N \times \dots \times \Omega_1) , \quad (4.21)$$

by the following recurrence formula: for  $N = 3$

$$\mathcal{E}_3^{\#} : \mathcal{S}(\Omega_1) \longrightarrow \mathcal{S}(\Omega_3 \times \Omega_2 \times \Omega_1) , \quad (4.22)$$

one has

$$\mathcal{E}_3^{\#} = (\text{id}_2 \otimes \mathcal{E}^{\#}) \circ \mathcal{E}^{\#} , \quad (4.23)$$

and for  $N > 3$

$$\mathcal{E}_N^{\#} = (\text{id}_{N-1} \otimes \dots \otimes \text{id}_2 \otimes \mathcal{E}^{\#}) \circ \mathcal{E}_{N-1}^{\#} . \quad (4.24)$$

An example of this construction is provided by the Ohya  $N$ -lifting.

## 5 Quantum noncommutative liftings

### 5.1 Linear liftings

To any transition expectation represented by a completely positive linear unital map

$$\mathcal{E} : \mathcal{A}_2 \otimes \mathcal{A}_1 \longrightarrow \mathcal{A}_1 , \quad (5.1)$$

there corresponds a linear lifting

$$\mathcal{E}^\# : S(\mathcal{A}_1) \longrightarrow S(\mathcal{A}_2 \otimes \mathcal{A}_1) . \quad (5.2)$$

Linear liftings play important role in the quantum theory of open systems. It is well known that a linear lifting which is nondemolishing for all states  $\rho$  has the following form

$$\mathcal{E}^\#(\rho) = \omega \otimes \rho , \quad (5.3)$$

for some fixed state  $\omega$  over  $\mathcal{A}_2$ . Let us assume that  $\mathcal{A}_k = \mathcal{B}(\mathcal{H}_k)$ . A product lifting (or assignment maps [9, 7]) defined by (5.3) gives rise to a linear completely positive map

$$\Lambda^\# : S(\mathcal{A}_1) \longrightarrow S(\mathcal{A}_1) ,$$

defined by

$$\Lambda^\#(\rho) := \text{Tr}_2[U(\omega \otimes \rho)U^*] , \quad (5.4)$$

where  $U$  is a unitary operator in  $\mathcal{H}_2 \otimes \mathcal{H}_1$ . It is well known that any linear completely positive map may be obtained this way.

**Remark 5** Let us observe that linear lifting may be used to construct new classes of positive maps which are not completely positive. Let  $\mathcal{E}^\#$  be a product lifting from  $\mathcal{A}_1$  to  $\mathcal{A}_2 \otimes \mathcal{A}_1$  and let

$$\psi : S(\mathcal{A}_2 \otimes \mathcal{A}_1) \longrightarrow S(\mathcal{A}_2 \otimes \mathcal{A}_1) ,$$

be a positive map. Define a new map  $\varphi : S(\mathcal{A}_1) \longrightarrow S(\mathcal{A}_1)$  via

$$\varphi(\rho) := \text{Tr}_2[\psi(\mathcal{E}^\#(\rho))] = \text{Tr}_2[\psi(\omega \otimes \rho)] , \quad (5.5)$$

where  $\omega$  is a fixed state in  $\mathcal{A}_2$ . By construction  $\varphi$  is a linear positive map. Note, however, that if  $\psi$  is only positive but not completely positive then  $\varphi$  needs not be completely positive. As an example consider  $\mathcal{A}_1 = \mathcal{A}_2 = M_2(\mathbb{C})$  and let  $\psi : M_4(\mathbb{C}) \longrightarrow M_4(\mathbb{C})$  be the Robertson map defined as follows [18, 21]

$$\psi \left( \begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array} \right) = \frac{1}{2} \left( \begin{array}{c|c} \mathbb{I}_2 \text{Tr} X_{22} & X_{12} + R_2(X_{21}) \\ \hline X_{21} + R_2(X_{12}) & \mathbb{I}_2 \text{Tr} X_{11} \end{array} \right) , \quad (5.6)$$

where we represented an element  $X \in M_4(\mathbb{C})$  in the block form  $X = \sum_{i,j=1}^2 e_{ij} \otimes X_{ij}$  with  $X_{ij} \in M_2(\mathbb{C})$ . Finally,  $R_2$  denotes the reduction map  $R_2 : M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C})$  defined by

$$R_2(X) = \mathbb{I}_2 \text{Tr} X - X , \quad (5.7)$$

which is known to be positive but not completely positive. Now, if  $\mathcal{E}^\#$  is a product lifting  $\mathcal{E}^\#(\rho) = \omega \otimes \rho$ , then one finds

$$\varphi(\rho) = \frac{1}{2} \begin{pmatrix} 2\rho_{22} & \rho_{12} + \rho_{21} \\ \rho_{12} + \rho_{21} & 2\rho_{11} \end{pmatrix}. \quad (5.8)$$

Interestingly, the positive map  $\varphi$  does not depend upon the state  $\omega$ . To show that  $\varphi$  is not completely positive let us observe that

$$(\text{id} \otimes \varphi)P_2^+ = \frac{1}{4} \left( \begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right), \quad (5.9)$$

is not positively defined ( $P_2^+$  denotes the maximally entangled state in  $\mathbb{C}^2 \otimes \mathbb{C}^2$ , i.e.  $P_2^+ = \frac{1}{2} \sum_{i,j=1}^2 e_{ij} \otimes e_{ij}$ ).

## 5.2 Nonlinear liftings

Let  $\Lambda$  be a linear completely positive unital map

$$\Lambda : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}). \quad (5.10)$$

**Definition 6** ([34]) *A positive operator*

$$\pi := \sum_{i,j=1}^d e_{ij} \otimes \Lambda(e_{ij}), \quad (5.11)$$

living in  $\mathcal{H}_2 \otimes \mathcal{H}_1$  ( $\mathcal{H}_k = \mathbb{C}^d$ ) is called the quantum conditional probability (QCP) operator.

QCP generalizes classical conditional probability for the quantum noncommutative case.<sup>1</sup> Note, that in the classical case one may define

$$\pi := \sum_{i=1}^d e_{ii} \otimes \Lambda(e_{ii}), \quad (5.12)$$

with

$$\Lambda(e_{ii}) = \sum_{j=1}^d p_{i|j} e_{jj}. \quad (5.13)$$

Then

$$\sum_i p_{i|j} = 1, \quad (5.14)$$

is equivalent to

$$\text{Tr}_2 \pi = \mathbb{I}, \quad (5.15)$$

which follows from  $\Lambda(\mathbb{I}) = \mathbb{I}$ . Following [34] let us define a nonlinear lifting by

$$\mathcal{E}^\#(\rho) := (\mathbb{I} \otimes \rho^{\frac{1}{2}}) \pi (\mathbb{I} \otimes \rho^{\frac{1}{2}}). \quad (5.16)$$

---

<sup>1</sup>For slightly different approach to quantum conditional probability see [35] and [36].

**Proposition 3** A lifting  $\mathcal{E}^\#$  satisfies

$$\mathrm{Tr}_2 \mathcal{E}^\#(\rho) = \rho , \quad (5.17)$$

and

$$\mathrm{Tr}_1 \mathcal{E}^\#(\rho) = \Lambda^\#(\rho)^\mathrm{T} . \quad (5.18)$$

**Remark 2** Note, that in the special case if

$$\Lambda(x) = \mathbb{I} \mathrm{Tr} \omega^T x , \quad (5.19)$$

for some density operator  $\omega$  in  $\mathcal{H}_2$ , then (5.16) defines a linear product lifting

$$\mathcal{E}^\#(\rho) = \omega \otimes \rho . \quad (5.20)$$

**Example 5** Let  $\Lambda(a) = UaU^*$ , with unitary  $U$ . One has for the lifting

$$\begin{aligned} \mathcal{E}^\#(\rho) &= (\mathbb{I} \otimes \rho^{\frac{1}{2}}) \sum_{i,j=1}^n e_{ij} \otimes U e_{ij} U^* (\mathbb{I} \otimes \rho^{\frac{1}{2}}) \\ &= \sum_{i,j=1}^n e_{ij} \otimes \rho^{\frac{1}{2}} U e_{ij} U^* \rho^{\frac{1}{2}} , \end{aligned} \quad (5.21)$$

that is

$$\mathcal{E}^\#(\rho) = \sum_{i,j=1}^n e_{ij} \otimes \phi(e_{ij}) , \quad (5.22)$$

where  $\phi$  is the  $\rho$ -dependent CP map

$$\phi(a) = KaK^* ,$$

with the  $\rho$ -dependent operator given by  $K = \rho^{\frac{1}{2}} U$ .

**Example 7** Let  $\Lambda(a) = \sum_i \mathrm{Tr}(ae_{ii}) e_{ii}$ . It leads to the following lifting

$$\mathcal{E}^\#(\rho) = \sum_{i=1}^n e_{ii} \otimes \rho^{\frac{1}{2}} e_{ii} \rho^{\frac{1}{2}} . \quad (5.23)$$

Note, that if  $e_i$  defines eigen-basis for  $\rho$ , then

$$\mathcal{E}^\#(\rho) = \sum_{i=1}^n \rho_i e_{ii} \otimes e_{ii} , \quad (5.24)$$

with  $\rho_i$  being an eigenvalue of  $\rho$ , i.e.  $\rho e_i = \rho_i e_i$ . The above formula reproduces the nonlinear Ohya lifting. Recall, that the nonlinear Ohya lifting  $\mathcal{E}^\# : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{A})$  is defined as follows [3]

$$\mathcal{E}^\#(\rho) = \sum_k p_k E_k \otimes E_k , \quad (5.25)$$

where

$$\rho = \sum_k p_k E_k , \quad (5.26)$$

stands for the spectral decomposition of  $\rho$ . We stress that  $\mathcal{E}^\#$  is nonlinear since both  $p_k$  and  $E_k$  are  $\rho$ -dependent.



**Remark 3** Note, that a characteristic feature of the Ohya lifting is that both reduced states reproduce the original state  $\rho$ , i.e.

$$\mathrm{Tr}_1 \mathcal{E}^\#(\rho) = \mathrm{Tr}_2 \mathcal{E}^\#(\rho) = \rho , \quad (5.27)$$

for an arbitrary classical state  $\rho$  on  $\Omega$ . It shows that Ohya lifting realizes a perfect classical cloning machine [31]. Moreover, since both reduced states are the same Ohya lifting provides at the same time a protocol for a classical broadcasting [32].

**Remark 4** Let  $\theta$  be a compound state in  $\mathcal{H}_2 \otimes \mathcal{H}_1$  with marginals  $\rho$  and  $\sigma$ . If  $\phi$  is a CP map such that

$$\theta = \sum_{i,j=1}^n e_{ij} \otimes \phi(e_{ij}) , \quad (5.28)$$

then one can rewrite  $\theta$  as follows

$$\theta = \sum_{i,j=1}^n e_{ij} \otimes \rho^{\frac{1}{2}} \Lambda(e_{ij}) \rho^{\frac{1}{2}} , \quad (5.29)$$

where  $\Lambda$  is a unital CP map defined by

$$\Lambda(a) = \rho^{-\frac{1}{2}} \phi(a) \rho^{-\frac{1}{2}} , \quad (5.30)$$

where we assume that  $\rho$  is faithful state, i.e.  $\rho > 0$ .

Suppose now that we have two QCP operators  $\pi_1$  and  $\pi_2$ . Let us define a 3-partite operator  $\pi_1 \circ \pi_2$

$$\pi_1 \circ \pi_2 : \mathcal{H}_3 \otimes \mathcal{H}_2 \otimes \mathcal{H}_1 \longrightarrow \mathcal{H}_3 \otimes \mathcal{H}_2 \otimes \mathcal{H}_1 , \quad (5.31)$$

with  $\mathcal{H}_k = \mathbb{C}^d$ , by

$$\pi_1 \circ \pi_2 := (\mathbb{I} \otimes \pi_1^{\frac{1}{2}}) (\pi_2 \otimes \mathbb{I}) (\mathbb{I} \otimes \pi_1^{\frac{1}{2}}) . \quad (5.32)$$

**Proposition 4** *The operator  $\pi_1 \circ \pi_2$  is positive and it satisfies the following basic properties:*

$$\mathrm{Tr}_3 (\pi_1 \circ \pi_2) = \pi_1 , \quad (5.33)$$

$$\mathrm{Tr}_{32} (\pi_1 \circ \pi_2) = \mathbb{I} . \quad (5.34)$$

It is clear that for  $\pi_1 = \pi_2 = \pi$ , the operator  $\pi \circ \pi$  may be used to define a lifting from  $M_d^+$  into  $M_d^+ \otimes M_d^+ \otimes M_d^+$ . One defines

$$\mathcal{E}_{3|1}^\#(\rho) = (\mathbb{I} \otimes \mathbb{I} \otimes \rho^{\frac{1}{2}}) (\pi \circ \pi) (\mathbb{I} \otimes \mathbb{I} \otimes \rho^{\frac{1}{2}}) . \quad (5.35)$$

Note, that due to (5.33) one has

$$\mathrm{Tr}_3 \mathcal{E}_{3|1}^\#(\rho) = \mathcal{E}_{2|1}^\#(\rho) , \quad (5.36)$$

where  $\mathcal{E}_{2|1}^\#(\rho) := \mathcal{E}^\#(\rho)$  is defined in (5.16). This procedure may be immediately generalized for arbitrary  $N$ . One has the following recurrence formula for the  $N$ -partite operator

$$\pi_1 \circ \dots \circ \pi_{N-1} : \mathcal{H}_N \otimes \dots \otimes \mathcal{H}_1 \longrightarrow \mathcal{H}_N \otimes \dots \otimes \mathcal{H}_1 , \quad (5.37)$$

$$\pi_1 \circ \dots \circ \pi_{N-1} := (\mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \pi_1^{\frac{1}{2}}) (\pi_2 \circ \dots \circ \pi_{N-1} \otimes \mathbb{I}) (\mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \pi_1^{\frac{1}{2}}) . \quad (5.38)$$

One easily proves

$$\mathrm{Tr}_N (\pi_1 \circ \dots \circ \pi_{N-1}) = \pi_1 \circ \dots \circ \pi_{N-2} , \quad (5.39)$$

$$\mathrm{Tr}_{N,N-1} (\pi_1 \circ \dots \circ \pi_{N-1}) = \pi_1 \circ \dots \circ \pi_{N-3} , \quad (5.40)$$

⋮

$$\mathrm{Tr}_{N,\dots,2} (\pi_1 \circ \dots \circ \pi_{N-1}) = \mathbb{I} . \quad (5.41)$$

Again, when  $\pi_1 = \dots = \pi_{N-1} = \pi$ , then  $N$ -partite  $\pi \circ \dots \circ \pi$  is positive and one defines

$$\mathcal{E}_{N|1}^\#(\rho) = (\mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \rho^{\frac{1}{2}}) (\pi \circ \dots \circ \pi) (\mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \rho^{\frac{1}{2}}) . \quad (5.42)$$

It is clear that

$$\mathrm{Tr}_N \mathcal{E}_{N|1}^\#(\rho) = \mathcal{E}_{N-1|1}^\#(\rho) . \quad (5.43)$$

Note, that in the classical case defined one finds

$$\mathcal{E}_{N|1}^\#(\rho) = \sum_{i_1, \dots, i_N=1}^d p_{i_N|i_{N-1}} \dots p_{i_2|i_1} p_{i_1} e_{i_N i_N} \otimes \dots \otimes e_{i_1 i_1} , \quad (5.44)$$

where  $\rho = \sum_i p_i e_{ii}$ . Hence, the lifted  $N$ -partite state  $\mathcal{E}^\#(\rho)$  generalizes classical Markovian state (for entanglement properties of a particular class of pure Markovian multipartite state see [30]).

## 6 Circulant liftings

In this section we analyze a particular class of liftings defined in terms of circulant states [23] (see also [24]). Circulant states play important role in Quantum Information Theory since majority of states of composite quantum systems considered in the literature turn out to be circulant states. The most important example of circulant states is provided by generalized Bell states. Let us mention also the class of isotropic states, Werner states, Bell diagonal states and many others (see [23] for more examples). Therefore, it is interesting to investigate the class of liftings directly related to the class of circulant states. This section provides also the illustration of the theoretical concepts introduced so far. It is organized as follows: we start with the notion of circulant decomposition of any Hilbert space being the tensor product  $\mathbb{C}^d \otimes \mathbb{C}^d$ . It turns out that it may be represented as a direct product of  $d$  subspaces each of dimension  $d$ . This decomposition (we call it *cyclic decomposition*) provides starting point in the construction of a circulant state. Having defined circulant state we analyze the class of liftings such that  $\mathcal{E}^\#(\rho)$  is a circulant state for any  $\rho$  living in  $\mathbb{C}^d$  and illustrate the construction by the special class of Bell diagonal liftings.

### 6.1 Circulant decompositions

The basic idea is to decompose the total Hilbert space  $\mathbb{C}^d \otimes \mathbb{C}^d$  into a direct sum of  $d$  orthogonal  $d$ -dimensional subspaces related by a certain cyclic property. Let us introduce a  $d$ -dimensional subspace

$$\Sigma_0 = \mathrm{span} \{ e_0 \otimes e_0 , e_1 \otimes e_1 , \dots , e_{d-1} \otimes e_{d-1} \} , \quad (6.1)$$

where  $\{e_0, \dots, e_{d-1}\}$  denote the standard computational basis in  $\mathbb{C}^d$ , together with the shift operator  $S : \mathbb{C}^d \rightarrow \mathbb{C}^d$  defined by

$$S e_k = e_{k+1} \ , \quad \text{mod } d \ . \quad (6.2)$$

Now, for any  $\alpha = 0, \dots, d-1$ , we define

$$\Sigma_\alpha = (\mathbb{1} \otimes S^\alpha) \Sigma_0 \ . \quad (6.3)$$

It is clear that  $\Sigma_\alpha$  and  $\Sigma_\beta$  are mutually orthogonal (for  $\alpha \neq \beta$ ). Moreover,

$$\Sigma_0 \oplus \dots \oplus \Sigma_{d-1} = \mathbb{C}^d \otimes \mathbb{C}^d \ . \quad (6.4)$$

Due to the cyclic property of the shift operator (6.2), we call (6.4) a circulant decomposition [23]. For example in the case of two qubits ( $d = 2$ ) one has

$$\begin{aligned} \Sigma_0 &= \text{span} \{e_0 \otimes e_0, e_1 \otimes e_1\} \ , \\ \Sigma_1 &= \text{span} \{e_0 \otimes e_1, e_1 \otimes e_0\} \ , \end{aligned} \quad (6.5)$$

whereas for two qutrits ( $d = 3$ ) one obtains

$$\begin{aligned} \Sigma_0 &= \text{span} \{e_0 \otimes e_0, e_1 \otimes e_1, e_2 \otimes e_2\} \ , \\ \Sigma_1 &= \text{span} \{e_0 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_0\} \ , \\ \Sigma_2 &= \text{span} \{e_0 \otimes e_2, e_1 \otimes e_0, e_2 \otimes e_1\} \ . \end{aligned} \quad (6.6)$$

## 6.2 Circulant states

The circulant decomposition (6.4) gives rise to the following construction of a state in  $\mathbb{C}^d \otimes \mathbb{C}^d$ : let  $a^{(0)}, \dots, a^{(d-1)}$  be a set of  $d$  positive  $d \times d$  matrices. Let us observe that

$$\rho^{(\alpha)} = \sum_{i,j=0}^{d-1} a_{ij}^{(\alpha)} e_{ij} \otimes S^\alpha e_{ij} S^{*\alpha} \ , \quad \alpha = 0, \dots, d-1 \ , \quad (6.7)$$

is a positive operator in  $\mathbb{C}^d \otimes \mathbb{C}^d$  supported on  $\Sigma_\alpha$ . Hence, the following sum

$$\rho = \rho^{(0)} + \dots + \rho^{(d-1)} \ , \quad (6.8)$$

gives rise to a positive operator in  $\mathbb{C}^d \otimes \mathbb{C}^d$ . It defines a legitimate state iff  $\text{Tr} \rho = 1$ , which is equivalent to

$$\text{Tr} \left[ a^{(0)} + \dots + a^{(d-1)} \right] = 1 \ . \quad (6.9)$$

For obvious reason we call (6.8) a circulant state. This simple construction recovers many well known bipartite states from the literature (see [23]).

Consider now a partial transposition of the circulant state (6.8). It turns out that  $\rho^\Gamma = (\mathbb{1} \otimes \tau) \rho$  is again circulant but it corresponds to another cyclic decomposition of the original Hilbert space  $\mathbb{C}^d \otimes \mathbb{C}^d$ . Recall, that  $\rho$  is PPT (Positive Partial Transpose) if  $\rho^\Gamma \geq 0$ . Let us introduce the following permutation  $\pi$  from the symmetric group  $S_d$ :

$$\pi(0) = 0 \ , \quad \pi(i) = d - i \ , \quad i = 1, 2, \dots, d-1 \ . \quad (6.10)$$

We use  $\pi$  to introduce

$$\tilde{\Sigma}_0 = \text{span} \{e_0 \otimes e_{\pi(0)}, e_1 \otimes e_{\pi(1)}, \dots, e_{d-1} \otimes e_{\pi(d-1)}\} , \quad (6.11)$$

and

$$\tilde{\Sigma}_\alpha = (\mathbf{1} \otimes S^\alpha) \tilde{\Sigma}_0 . \quad (6.12)$$

It is clear that  $\tilde{\Sigma}_\alpha$  and  $\tilde{\Sigma}_\beta$  are mutually orthogonal (for  $\alpha \neq \beta$ ). Moreover,

$$\tilde{\Sigma}_0 \oplus \dots \oplus \tilde{\Sigma}_{d-1} = \mathbb{C}^d \otimes \mathbb{C}^d , \quad (6.13)$$

and hence it defines another circulant decomposition. Note, that for  $d = 2$  one has  $\tilde{\Sigma}_\alpha = \Sigma_\alpha$ . It is no longer true for  $d \geq 3$ . For  $d = 3$  one finds

$$\begin{aligned} \tilde{\Sigma}_0 &= \text{span} \{e_0 \otimes e_0, e_1 \otimes e_2, e_2 \otimes e_1\} , \\ \tilde{\Sigma}_1 &= \text{span} \{e_0 \otimes e_1, e_1 \otimes e_0, e_2 \otimes e_2\} , \\ \tilde{\Sigma}_2 &= \text{span} \{e_0 \otimes e_2, e_1 \otimes e_1, e_2 \otimes e_0\} . \end{aligned} \quad (6.14)$$

Now, the partially transformed state  $\rho^\tau$  has again a circulant structure but with respect to the new decomposition (6.13):

$$\rho^\Gamma = \tilde{\rho}^{(0)} + \dots + \tilde{\rho}^{(d-1)} , \quad (6.15)$$

where

$$\tilde{\rho}^{(\alpha)} = \sum_{i,j=0}^{d-1} \tilde{a}_{ij}^{(\alpha)} e_{ij} \otimes S^\alpha e_{\pi(i)\pi(j)} S^{\dagger\alpha} , \quad \alpha = 0, \dots, d-1 , \quad (6.16)$$

and the new  $d \times d$  matrices  $[\tilde{a}_{ij}^{(\alpha)}]$  are given by the following formulae:

$$\tilde{a}^{(\alpha)} = \sum_{\beta=0}^{d-1} a^{(\alpha+\beta)} \circ (\Pi S^\beta) , \quad \text{mod } d , \quad (6.17)$$

where “ $\circ$ ” denotes the Hadamard product,<sup>2</sup> and  $\Pi$  being a  $d \times d$  permutation matrix corresponding to  $\pi$ , i.e.  $\Pi_{ij} := \delta_{i,\pi(j)}$ . It is therefore clear that our original circulant state (6.8) is PPT iff all  $d$  matrices  $\tilde{a}^{(\alpha)}$  satisfy

$$\tilde{a}^{(\alpha)} \geq 0 , \quad \alpha = 0, \dots, d-1 . \quad (6.18)$$

### 6.3 Circulant liftings

Circulant states provide interesting example of a linear lifting. Denote by  $M_d$  a  $\mathbb{C}^*$ -algebra of  $d \times d$  complex matrices and consider the following lifting

$$\mathcal{E}^\# : S(M_d) \longrightarrow S(M_d \otimes M_d) , \quad (6.19)$$

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<sup>2</sup>A Hadamard (or Schur) product of two  $n \times n$  matrices  $A = [A_{ij}]$  and  $B = [B_{ij}]$  is defined by

$$(A \circ B)_{ij} = A_{ij} B_{ij} .$$

defined by

$$\mathcal{E}^\#(\rho) = \sum_{\alpha=0}^{d-1} c_{ij}^{(\alpha)} e_{ij} \otimes V_{i\alpha} \rho V_{i\alpha}^* , \quad (6.20)$$

where  $[c_{ij}^{(\alpha)}]$  are  $d \times d$  positive matrices for  $\alpha = 0, 1, \dots, d-1$  such that  $\text{Tr } c^{(\alpha)} = 1$ , and

$$V_{i\alpha} = |e_i + e_\alpha\rangle \langle e_\alpha| . \quad (6.21)$$

Note that

$$\mathcal{E}^\#(\rho) = \sum_{\alpha=0}^{d-1} a_{ij}^{(\alpha)} e_{ij} \otimes S^\alpha e_{ij} S^{\alpha*} , \quad (6.22)$$

where

$$a_{ij}^{(\alpha)} = p_\alpha c_{ij}^{(\alpha)} , \quad (6.23)$$

and

$$p_\alpha = \rho_{\alpha\alpha} . \quad (6.24)$$

It shows that for any  $\rho$  its lifting  $\mathcal{E}^\#(\rho)$  defines a circulant state. Now, if  $c^{(\alpha)}$  are rank-1 projectors, i.e.  $c_{mn}^{(\alpha)} = c_m^{(\alpha)} \bar{c}_n^{(\alpha)}$  are the Gram matrices for the  $d$  complex  $d$ -vectors  $c^{(\alpha)}$ , the above formula simplifies to

$$\mathcal{E}^\#(\rho) := V \mathcal{D}(\rho) V^* , \quad (6.25)$$

where

$$\mathcal{D}(\rho) := \sum_{i=0}^{d-1} e_{ii} \rho e_{ii} , \quad (6.26)$$

is the projection onto the diagonal part of  $\rho$ , and

$$V : \mathbb{C}^d \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^d \quad (6.27)$$

is defined by

$$V e_\alpha = \sum_{j=0}^{d-1} c_j^{(\alpha)} e_j \otimes e_{j+\alpha} \quad (6.28)$$

Note, that due the trace condition  $\text{Tr } c^{(\alpha)} = 1$  the linear operator  $V$  defines an isometry

$$V^* V = \mathbb{I} . \quad (6.29)$$

It should be stressed that the above circulant lifting is never pure.

## 6.4 Bell diagonal lifting

Consider a simplex of Bell diagonal states [25, 26] defined by

$$\rho = \sum_{m,n=0}^{d-1} p_{mn} P_{mn} , \quad (6.30)$$

where  $p_{mn} \geq 0$ ,  $\sum_{m,n} p_{mn} = 1$  and

$$P_{mn} = (\mathbb{I} \otimes U_{mn}) P_d^+ (\mathbb{I} \otimes U_{mn}^\dagger) , \quad (6.31)$$

with  $U_{mn}$  being the collection of  $d^2$  unitary matrices defined as follows

$$U_{mn} e_k = \lambda^{mk} S^n e_k = \lambda^{mk} e_{k+n} , \quad (6.32)$$

with

$$\lambda = e^{2\pi i/d} . \quad (6.33)$$

The matrices  $U_{mn}$  define an orthonormal basis in the space  $M_d(\mathbb{C})$  of complex  $d \times d$  matrices. One easily shows

$$\text{Tr}(U_{mn} U_{rs}^\dagger) = d \delta_{mr} \delta_{ns} . \quad (6.34)$$

It is clear [27] that Bell diagonal states define a subclass of circulant states.

**Definition 7** *A circulant lifting  $\mathcal{E}^\#$  is Bell diagonal if  $\mathcal{E}^\#(\rho)$  defines Bell diagonal state for any  $\rho$ .*

Suppose now that  $c^{(0)} = \dots = c^{(d-1)} =: c$ , where the positive matrix  $c$  is defined by

$$c_{kl} = \frac{1}{d} \sum_{m=0}^{d-1} p_m \lambda^{m(k-l)} . \quad (6.35)$$

The above formula defines a circulant matrix<sup>3</sup>

$$c_{kl} = f_{k-l} , \quad (6.36)$$

where the  $d$ -vector  $f_n$  is defined via the discrete Fourier transform of the probability  $d$ -vector  $p_m$ . One finds for the lifted state  $\rho$

$$\mathcal{E}^\#(\rho) = \sum_{m,n=0}^{d-1} p_{mn}(\rho) P_{mn} , \quad (6.37)$$

where

$$p_{mn}(\rho) = p_m \langle n | \rho | n \rangle . \quad (6.38)$$

Hence, the joint distribution  $p_{mn}$  is the product of  $p_m$  and the classical probability distribution defined by the diagonal elements  $\rho_{nn}$  of  $\rho$ . For more detailed analysis of Bell diagonal states we refer to the recent paper [27].

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<sup>3</sup>Recall, that a  $d \times d$  matrix  $c_{ij}$  is circulant if  $c_{ij}$  depends only upon the difference  $i - j \pmod{d}$ . For example

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} , \quad \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} ,$$

are circulant matrices.

## 7 Conclusions

We analyzed the procedure of lifting in classical stochastic and quantum systems. Lifting enables one to ‘lift’ a state of a classical/quantum system into a state of ‘system+reservoir’. This procedure is important both in quantum information theory and in the theory of open systems. It turns out that it is very much related to the problem of completely positive maps are the workhorse in these fields. We discussed both linear and nonlinear liftings and provided instructive illustration of the general theory by a particular class related to so called circulant states. In particular it is shown how to lift a state of a subsystem to the Bell diagonal state of the composed system. The theory of liftings may provide new constructions of classical/quantum channels. Moreover, it may be used to construct new classes of linear maps which are positive but not completely positive. It is well known that such maps define a basic tool for detection quantum entanglement. We therefore conclude that the theory of liftings might provide an interesting insight both in quantum information theory and **in** the intricate mathematics of (completely) positive maps. Both problems deserve further studies.

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