Optimal entanglement witnesses for two qutrits

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Abstract

We provide a proof that entanglement witnesses considered recently in [1] are optimal.

In a recent paper [1] we analyzed z class of entanglement witnesses (EW) given by

$$W[a,b,c] = \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a \end{array} \right) , \tag{1}$$

where to make the picture more transparent we replaced zeros by dots (for simplicity we skipped the normalization factor which is not essential). One proves the following result [2]

Theorem 1. *$W[a,b,c]$ defines an entanglement witness if and only if*

1. $0 \leq a < 2$,
2. $a + b + c \geq 2$,
3. *if* $a \leq 1$, *then* $bc \geq (1-a)^2$.

Moreover, being EW it is indecomposable if and only if $bc < (2-a)^2/4$.

In particular we analyzed [1] a subclass of EWs defined by

$$0 \leq a \leq 1 \ , \quad a + b + c = 2 \ , \quad bc = (1-a)^2 \ . \tag{2}$$

The corresponding EWs $W[b,c] := W[2-b-c,b,c]$ belong to the ellipse on bc-plane – see Fig. 1. It was conjectured [1] that $W[b,c]$ are optimal. In the present paper we show that this conjecture is true.

Theorem 2. *EWs $W[b,c]$ defined by (2) are optimal.*

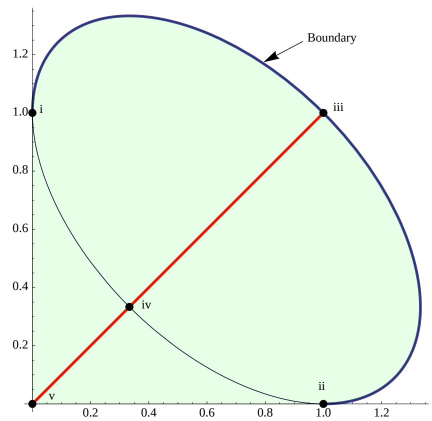

Figure 1: A convex set of EWs $W[b,c]$. A line $b=c$ corresponds to decomposable EW. Special points: (i) and (ii) Choi EWs, (iii) EW corresponding to reduction map, (v) positive operator with $b=c=0$, (iv) decomposable EW with $b=c=1/3$.

Proof: let us define

$$\mathcal{P}_{bc} = \{x \otimes y \in \mathbb{C}^3 \otimes \mathbb{C}^3 \mid \langle x \otimes y | W[b,c] | x \otimes y \rangle = 0\} \ . \tag{3}$$

It is well known [3] that if the set $\mathcal{P}_{bc}$ spans the entire Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$, then $W[b,c]$ is an optimal EW. If we find a set of vectors $y \in \mathbb{C}^3$ such that the 3×3 matrix

$$W_y[b,c] := \mathrm{Tr}_2(W[b,c] \cdot \mathbb{I}_3 \otimes |y\rangle\langle y|) \ , \tag{4}$$

is singular, then for each vector x_y belonging to the kernel of $W_y[b,c]$ the product vector $x_y \otimes y$ belongs to $\mathcal{P}_{bc}$ (Tr_2 denotes a partial trace over the second factor in $\mathbb{C}^3 \otimes \mathbb{C}^3$). The matrix $W_y[b,c]$ is given by the formula

$$W_y[b,c] = \left[\begin{array}{ccc} a|y_1|^2 + b|y_2|^2 + c|y_3|^2 & y_1^* y_2 & y_1^* y_3 \\ y_2^* y_1 & c|y_1|^2 + a|y_2|^2 + b|y_3|^2 & y_2^* y_3 \\ y_3^* y_1 & y_3^* y_2 & b|y_1|^2 + c|y_2|^2 + a|y_3|^2 \end{array} \right]$$

$$= \left[\begin{array}{ccc} (a+1)|y_1|^2 + b|y_2|^2 + c|y_3|^2 & 0 & 0 \\ 0 & c|y_1|^2 + (a+1)|y_2|^2 + b|y_3|^2 & 0 \\ 0 & 0 & b|y_1|^2 + c|y_2|^2 + (a+1)|y_3|^2 \end{array} \right] - |y^*\rangle\langle y^*|$$

Let us observe, that for any a,b,c satisfying Theorem 1 and $y = [e^{i\alpha}, e^{i\beta}, e^{i\gamma}]$ one finds

$$W_y[b,c] = \mathrm{diag}[e^{-i\alpha}, e^{-i\beta}, e^{-i\gamma}] \left[\begin{array}{ccc} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{array} \right] \mathrm{diag}[e^{i\alpha}, e^{i\beta}, e^{i\gamma}] \ .$$

This matrix has rank 2 and its 1-dim. kernel is spanned by the vector $x_y = [e^{-i\alpha}, e^{-i\beta}, e^{-i\gamma}]$. Hence we have the following continuous family of product vectors

$$x_y \otimes y = [1, e^{i(\beta-\alpha)}, e^{i(\gamma-\alpha)}, e^{i(\alpha-\beta)}, 1, e^{i(\gamma-\beta)}, e^{i(\alpha-\gamma)}, e^{i(\beta-\gamma)}, 1] \tag{5}$$

Note that this family spans at most 7-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$. To show, that this subspace is exactly 7-dimensional, it suffices to consider the following set of (α, β, γ)

$$(0,0,0), \quad (0,0,\pi), \quad (0,\pi,0), \quad (0,\pi,\pi), \quad (0,0,\pi/2), \quad (0,\pi/2,0), \quad (0,\pi/2,-\pi/2)\ . \tag{6}$$

Consider now $y = (0, y_2, y_3)$. One has

$$W_y[b,c] = \begin{bmatrix} b|y_2|^2 + c|y_3|^2 & 0 & 0 \\ 0 & a|y_2|^2 + b|y_3|^2 & -y_2^* y_3 \\ 0 & -y_3^* y_2 & c|y_2|^2 + a|y_3|^2 \end{bmatrix} .$$

Its determinant is given by the formula:

$$\det W_y[b,c] = (b|y_2|^2 + c|y_3|^2)(ab|y_2|^4 + (a^2 + ac - 1)|y_2|^2|y_3^2| + bc|y_3|^4)\ .$$

We are looking for $y \in \mathbb{C}^3$, that the determinant vanishes.

Case 1: $b, c \neq 0$.

Now, the first term is always positive and so the second term has to vanish. Taking $||y|| = 1$, one can replace $|y_3|^2$ by $1 - |y_2|^2$. The second term reads as follows

$$a\,(4-3a)|y_2|^4 + 2\,a\,(a-b-1)|y_2|^2 + ab = 0\ . \tag{7}$$

We use here relations $bc = (a-1)^2$ and $a = 2 - b + c$. One also assume that $b < c$ (the case $c < b$ may be treated in the same way using a symmetry $b \longleftrightarrow c$ [1]). One obtains the following formulae for b and c

$$b = \frac{1}{2}(2 - a - \sqrt{4a - 3a^2}\,)\ , \quad c = \frac{1}{2}(2 - a + \sqrt{4a - 3a^2}\,)$$

The discriminant of the quadratic equation (for $|y_2|^2$) vanishes (it can not be positive due to the fact that $W[b,c]$ is an EW) and one easily solves (7) to get

$$|y_2|^2 = \frac{1 + b - a}{4 - 3a}\ .$$

The vector y is then equal (after calculating $|y_3|^2$, we drop the normalization):

$$y = [0, \sqrt{1 + b - a}, \sqrt{3 - b - 2a}e^{i\phi}] =: [0, p, qe^{i\phi_1}]\ . \tag{8}$$

For such y, the kernel of $W_y[b,c]$ is spanned by the vector

$$x_y = [0, y_2^* \cdot y_3, a|y_2|^2 + b|y_3|^2] =: [0, re^{i\phi_1}, s] \tag{9}$$

The numbers p, q, r, s are nonzero and depend only on parameters a, b, c. Let

$$\Psi^{(1)} := x_y \otimes y = [0,0,0,0, pre^{i\phi_1}, ps, 0, qre^{2i\phi_1}, qse^{i\phi_1}]\ .$$

Because of the cyclic symmetry of the problem, one can find the similar product vectors for $y_2 = 0$ and $y_3 = 0$:

$$\Psi^{(2)} = [qse^{i\phi_2}, 0, qre^{2i\phi_2}, 0, 0, 0, ps, 0, pre^{i\phi_2}] \ ,$$
$$\Psi^{(3)} = [pre^{i\phi_3}, ps, 0, qre^{2i\phi_3}, qse^{i\phi_3}, 0, 0, 0, 0] \ .$$

Now, it turns out that 7 vectors from the family (5) generated by a set (6) plus two arbitrary vectors from the family $(\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)})$ defines a basis in $\mathbb{C}^3 \otimes \mathbb{C}^3$. Indeed, taking 7 vectors from (5) and $\Psi^{(1)}$, $\Psi^{(2)}$ one obtains the following 9×9 matrix:

$$\left[\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & i & 1 & 1 & i & -i & -i & 1 \\ 1 & i & 1 & -i & 1 & -i & 1 & i & 1 \\ 1 & i & -i & -i & 1 & -1 & i & i & 1 \\ 0 & 0 & 0 & 0 & pre^{i\phi_1} & ps & 0 & qre^{2i\phi_1} & qse^{i\phi_1} \\ qse^{i\phi_2} & 0 & qre^{2i\phi_2} & 0 & 0 & 0 & ps & 0 & pre^{i\phi_2} \end{array} \right] . \qquad (10)$$

Its determinant reads

$$(-32 + 160i)e^{i(\phi_1+\phi_2)}[\,(qs)^2 + (pr)^2 - qspr\,] \ ,$$

and is different from zero except $qs = pr = 0$. Note, however, that for $b, c \neq 0$ one has $qs, pr \neq 0$.

Case 2: $b = 0, c = 1$.

Now, the determinant reads

$$\det W_y[b, c] = |y_1|^2|y_2|^4 + |y_2|^2|y_3|^4 + |y_3|^2|y_1|^4 - 3|y_1|^2|y_2|^2|y_3|^2 \ .$$

If one of coordinates, say y_1 is zero, then the determinant is equal $|y_2|^2|y_3|^4$ and vanishes only if y_2 or y_3 vanishes, so the only vectors y with at least one zero coordinate for which $W_y[b, c]$ vanishes are

$$\Phi^{(1)} := [1, 0, 0] \otimes [0, 0, 1] \ , \quad \Phi^{(2)} := [0, 1, 0] \otimes [1, 0, 0] \ , \quad \Phi^{(3)} := [0, 0, 1] \otimes [0, 1, 0] \ . \qquad (11)$$

Now we will look for the remaining vectors and we assume that all coordinates are non-zero. Dividing the determinant by $|y_1|^2|y_2|^2|y_3|^2$ and gets the following equation

$$\frac{|y_2|}{|y_3|} + \frac{|y_3|}{|y_1|} + \frac{|y_1|}{|y_2|} - 3 = 0 \ .$$

Its LHS is nonnegative and vanishes only for $|y_1| = |y_2| = |y_3|$, and hence

$$y = [e^{i\alpha}, e^{i\beta}, e^{i\gamma}] \ , \quad x_y = [e^{-i\alpha}, e^{-i\beta}, e^{-i\gamma}] \ ,$$

and one gets again the 7-dimensional family of vectors (5). However, vectors $\Phi^{(k)}$ are not linearly independent from (5). Therefore, $\mathcal{P}_{01}$ spans only 7-dim. subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Actually, one obtains $\Phi^{(k)}$ from $\Psi^{(k)}$ in the limit $b \to 0$. Let us recall that the determinant of (10) vanishes only when $qs = pr = 0$. Now, $p = s = 0$ when $b = 0$ and $c = 1$, whereas $q = r = 0$ when $b = 1$ and $c = 0$. Hence, apart from two witnesses corresponding to Choi maps $W[1,0]$ and $W[0,1]$, the remaining EWs have spanning property, i.e. $\mathcal{P}_{bc}$ spans $\mathbb{C}^3 \otimes \mathbb{C}^3$, and hence they are optimal. $\Box$

As this paper was completed we were informed by professors Kil-Chan Ha and Seung-Hyeok Kye that they provided an independent proof of optimality [4]. Moreover, they proved [5] that all witnesses $W[b,c]$ are exposed (and hence extremal) except $W[1,1]$, $W[1,0]$ and $W[0,1]$.

References

[1] D. Chruściński and F.A. Wudarski, *Geometry of entanglement witnesses for two qutrits*, arXiv:1105.4821.

[2] S. J. Cho, S.-H. Kye, and S. G. Lee, Linear Algebr. Appl. **171**, 213 (1992).

[3] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A **62**, 052310 (2000).

[4] K.-C. Ha and S.-H. Kye, *One parameter family of indecomposable optimal entanglement witnesses arising from generalized Choi maps*, arXiv:1107.2720.

[5] K.-C. Ha and S.-H. Kye, *Entanglement witnesses arising from exposed positive linear maps*, arXiv:1108.0130.