Relation-based Galois-connections: towards the residual of a relation

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1 Introduction

Galois connections can be identified in lots of situations, and they have shown to be an interesting tool both for theory and for applications [6, 9]. Recent applications can still be found in different topics, specially in the realm of Formal Concept Analysis and the foundations of Fuzzy Set Theory, see for instance [1,7].

In all the different notions of generalized Galois connection the problem of its construction is of capital importance: specifically, given a mapping $f\colon A \to B$ how can one obtain its residual (aka right adjoint). Freyd's adjoint theorem characterizes when such a residual exists when both A and B have the same structure. One of our recent research topics has been to study what happens if A and B are differently structured, and a number of results have been obtained considering different underlying settings. Namely, in [10] we worked with crisp functions between a poset (resp. preordered set) and an unstructured set; later, in [2] we entered in the fuzzy arena, considering the case in which A is a fuzzy preposet; then, in [3], we extended the previous results by allowing fuzzy equivalence relations as an adequate substitute to equality. In [4] we introduced the notion of relational fuzzy Galois connection, in which the components of the connection are not fuzzy functions but fuzzy relations satisfying certain properties. Before proceeding to further generalizations, it is worth to start again from the crisp case and consider a more adequate notion of Galois connection whose components are crisp relations, and that is the topic of the present paper.

2 Preliminary definitions

A binary relation R between two sets A and B is a subset of the Cartesian product $A \times B$ and it can be also seen as a multivalued function R from the set A to the powerset $\mathcal{P}(B)$. For an element $(a, b) \in R$, it is said that a is related to b and denoted aRb.

Definition 1 *Given a binary relation $R \subseteq A \times B$,*

- *the* afterset aR *of an element $a \in A$ is defined as $\{b \in B \colon aRb\}$*

- *the* foreset Rb *of an element* $b \in B$ *is defined as* $\{a \in A \colon aRb\}$
- *the* domain *of* R *is the set* $\mathrm{Dom}(R) = \{a \in A \colon aR \neq \varnothing\}$
- *the* range *of* R *is the set* $\mathrm{Rng}(R) = \{b \in B \colon Rb \neq \varnothing\}$

Definition 2 *Given a binary relation* $R \subseteq A \times B$ *and a subset* $X \subseteq A$,

- *the* direct image *of* X *under* R *is*

$$R(X) = XR = \{b \in B \colon \textit{ there exists } x \in X \textit{ such that } xRb\} = \bigcup_{x \in X} xR$$

The direct image of a subset X *is the set of those elements of* B *that are related to* at least *one element of* X.

- *the* subdirect image *of* X *under* R *is*

$$R^{\triangleleft}(X) = X^{\triangleleft}R = \{b \in B \colon xRb \textit{ for all } x \in X\} = \bigcap_{x \in X} xR$$

The subdirect image of a subset X *is the set of those elements of* B *that are related to* all *the elements of* X.

Observe that if $X = \{a\}$, the direct image and the subdirect image of X coincides with the afterset, thus aR coincides with $R(a)$ should we interpret R as a multivalued mapping.

In the realm of ordered structures, Ore [12] introduced in 1944 the so-called *Galois connections* as a pair of antitone mappings for which both possible compositions are inflationary. In order to extend this concept to binary relations it is necessary to fix the meaning of antitone and inflationary relation. One possibility is to consider the relation as a multivalued function and extend it naturally, but both the preordered structure in the powerset and the composition of relations admit different approaches.

Definition 3 *Given* P *an arbitrary set and* $\leq$ *a preorder (reflexive and transitive relation) defined over* P, *it is possible to lift the preorder structure to the powerset* $\mathcal{P}(P)$ *by defining*

$$X \ll Y \iff \textit{for all } x \in X \textit{ there exists } y \in Y \textit{ such that } x \leq y \tag{1}$$

$$X \Subset Y \iff \textit{for all } y \in Y \textit{ there exists } x \in X \textit{ such that } x \leq y \tag{2}$$

$$X \ltimes Y \iff \begin{array}{l}\textit{for all } x \in X \textit{ there exists } y \in Y \textit{ such that } x \leq y \textit{ and} \\ \textit{for all } y \in Y \textit{ there exists } x \in X \textit{ such that } x \leq y\end{array} \tag{3}$$

Note that the relations defined above are the preorder relations used in the construction of the, respectively, Hoare, Smyth, and Plotkin powerdomains.

The concept of antitone multivalued function between two preordered sets depends on the preorder relation considered over the powerset. We explore three possibilities, being one approach more restrictive than the others:

Definition 4 *Let $\mathbb{A} = (A, \le_A), \mathbb{B} = (B, \le_B)$ be two preordered sets and $R \subseteq A \times B$ a binary relation between A and B.*

- *R is said to be* h-antitone *if $a_1 \le_A a_2$ implies $a_2R \ll a_1R$, for all $a_1, a_2 \in A$.*
- *R is said to be* s-antitone *if $a_1 \le_A a_2$ implies $a_2R \Subset a_1R$, for all $a_1, a_2 \in A$.*
- *R is said to be* p-antitone *if $a_1 \le_A a_2$ implies $a_2R \ltimes a_1R$, for all $a_1, a_2 \in A$, equivalently, if it is h-antitone and s-antitone.*

Analogously, the definition of inflationary multivalued function also admits three possibilities.

Definition 5 *Let $\mathbb{A} = (A, \le_A)$ be a preordered set and $R \subseteq A \times A$ a binary relation on A.*

- *R is said to be* h-inflationary *if $\{a\} \ll aR$, for all $a \in A$, that is, there exists (at least) $x \in aR$ such that $a \le x$.*
- *R is said to be* s-inflationary *if $\{a\} \Subset aR$, for all $a \in A$, that is, $a \le x$ for all $x \in aR$.*
- *R is said to be* p-inflationary *if it is h-inflationary and s-inflationary.*

Remark 1 *Notice that if a relation is s-inflationary, then it is also h-inflationary, therefore p-inflationary and s-inflationary are equivalent notions.*

The condition of the composition of two multivalued mappings being inflationary in some sense requires also to fix which definition of composition will be used.

Definition 6 *Let R be a binary relation between A and B and S be a binary relation between B and C.*

- *The* classical *composition of R and S is defined as follows*

$$\begin{aligned} R \circ S &= \{(x, z) \in A \times C\colon \text{ there exists } b \in B \text{ such that } xRb \text{ and } bSz\} \\ &= \{(x, z) \in A \times C\colon xR \cap Sz \neq \varnothing\} \end{aligned}$$

- *The $\triangleleft$ composition of R and S is defined as follows*

$$\begin{aligned} R \triangleleft S &= \{(x, z) \in A \times C\colon \text{ for all } b \in B \text{ such that } xRb \text{ it holds that } bSz\} \\ &= \{(x, z) \in A \times C\colon xR \subseteq Sz\} \end{aligned}$$

Observe that for an element $a \in A$, the afterset $a[R \circ S]$ coincides with the direct image of the afterset aR under S, that is

$$a[R \circ S] = (aR)S = \bigcup_{b \in aR} bS$$

Analogously, for an element $a \in A$, the afterset $a[R \triangleleft S]$ coincides with the subdirect image of the afterset aR under S, that is

$$a[R \triangleleft S] = (aR)^{\triangleleft}S = \bigcap_{b \in aR} bS$$

3 Two types of relational Galois connections

Definition 7 *Let $\mathbb{A} = (A, \leq_A), \mathbb{B} = (B, \leq_B)$ be two posets, $R \subseteq A \times B$ a binary relation between A and B and $S \subseteq B \times A$ a binary relation between B and A.*

The pair (R, S) is said to be an s-Galois connection *between $\mathbb{A}$ and $\mathbb{B}$ if*

i) R and S are s-antitone.

ii) $R \circ S$ and $S \circ R$ are s-inflationary.

Proposition 1 *Let $\mathbb{A} = (A, \leq_A), \mathbb{B} = (B, \leq_B)$ be two posets and (R, S) be an* s-Galois connection *between $\mathbb{A}$ and $\mathbb{B}$. Then if $b \in \operatorname{Rng}(R)$ then bS is at most a singleton, so, the restriction of S to $\operatorname{Rng}(R)$ is a (partial) single-valued function.*

Proof: If $b \in \operatorname{Rng}(R) \smallsetminus \operatorname{Dom}(S)$, then there is nothing to prove; therefore, let us assume that $b \in \operatorname{Rng}(R) \cap \operatorname{Dom}(S)$.

1. As $b \in \operatorname{Rng}(R)$, there exists $a \in A$ such that $b \in aR$ and, as $b \in \operatorname{Dom}(S)$, we have that bS is nonempty. We will now see that, $b \in xR$ for all $x \in bS$.

   Since $\{a\} \Subset a[R \circ S]$, taking into account that $b \in aR$, we have $a \leq x$ for all $x \in bS$. Now, as R is s-antitone, there exists $b' \in xR$ such that $b' \leq b$. The other inequality $b \leq b'$ follows because of $\{b\} \Subset b[S \circ R]$. As a result, $b = b' \in xR$.

2. Consider two elements $x, x^\star \in bS$, by definition of composition and the previous item, we have $x^\star \in x[R \circ S]$; since, by hypothesis, we have $\{x\} \Subset x[R \circ S]$, it turns out that $x \leq x^\star$. Applying that R is s-antitone, there exists $b^\star \in x^\star R$ such that $b^\star \leq b$. Again, the hypothesis $\{b\} \Subset b[S \circ R]$ implies $b \leq b^\star$, and we obtain $b = b^\star \in x^\star R$. This, together with $x \in bS$, proves that $x \in x^\star[R \circ S]$; finally, applying once again $x^\star \Subset x^\star[R \circ S]$, we obtain $x^\star \leq x$ and, therefore $x^\star = x$ and bS is a singleton. $\square$

Notice that the previous result shows that the definition of s-Galois connection necessarily collapses the relations R and S to be (partial) functions in the case of posets. If we drop the antisymmetry and consider the more general case of preordered sets, we obtain a similar result in that the images of the relations are clusters.

As a result, it seems more convenient to consider alternative approaches either by changing the ordering between subsets and/or slightly modifying the notions of antitone or inflationary relation.

A promising definition seems to be the following:

Definition 8 *Let $\mathbb{A} = (A, \leq_A), \mathbb{B} = (B, \leq_B)$ be two preordered sets, $R \subseteq A \times B$ a binary relation between A and B and $S \subseteq B \times A$ a binary relation between B and A. The pair (R, S) is said to be an* h-Galois connection *between $\mathbb{A}$ and $\mathbb{B}$ if*

i) R and S are h-antitone.

ii) For all $a \in A$ and all $b \in B$ the following conditions hold:

$$\{a\} \ll yS \text{ for all } y \in aR \quad \text{and} \quad \{b\} \ll xR \text{ for all } x \in bS. \tag{4}$$

It is not difficult to check that condition (4) above is a consequence of the property of $R \lhd S$ and $S \lhd R$ being h-inflationary but, in general, are not equivalent.

The following result shows a necessary condition for a pair (R, S) to be an h-Galois connection.

Lemma 1 *If the pair (R, S) is an h-Galois connection between $\mathbb{A}$ and $\mathbb{B}$, then the following inclusions hold:* $\mathrm{Rng}(R) \subseteq \mathrm{Dom}(S)$ *and* $\mathrm{Rng}(S) \subseteq \mathrm{Dom}(R)$.

Proof: Given $b \in \mathrm{Rng}(R)$, there exists $a \in A$ such that $b \in aR$. Now, by condition (4) above, we obtain that $bS \neq \varnothing$ and, therefore, $b \in \mathrm{Dom}(S)$.

The other inclusion can be proved similarly. □

Notice that, in fact, the proof of the previous lemma does not use the antitonicity of either R or S.

With the condition of Lemma 1 in mind, we obtain an equivalence with the usual notion of Galois connection, as stated below:

Theorem 1 *Let $\mathbb{A} = (A, \leq_A), \mathbb{B} = (B, \leq_B)$ be two preordered sets, $R \subseteq A \times B$ a binary relation between A and B and $S \subseteq B \times A$ a binary relation between B and A. The pair (R, S) is an h-Galois connection between $\mathbb{A}$ and $\mathbb{B}$ if and only if the following holds:*

$$\mathrm{Rng}(R) \subseteq \mathrm{Dom}(S) \quad \text{and} \quad \mathrm{Rng}(S) \subseteq \mathrm{Dom}(R) \tag{5}$$

$$\{a\} \ll bS \iff \{b\} \ll aR \tag{6}$$

Proof: Given (R, S) an h-Galois connection between $\mathbb{A}$ and $\mathbb{B}$, condition (5) follows by Lemma 1. Now, for condition (6), assume that $\{a\} \ll bS$. Then, there exists $x \in bS$ such that $a \leq x$ and, by R h-antitone, we obtain $xR \ll aR$. On the other hand, by condition (4) we have $\{b\} \ll xR$. Now, by transitivity of $\ll$, we obtain that $\{b\} \ll aR$. The proof that $\{b\} \ll aR$ implies $\{a\} \ll bS$ is similar.

Conversely, assume that equivalence (6) holds, and let us prove that (R, S) is an h-Galois connection.

Firstly, we will show condition (4): Given $a \in A$ and $y \in aR$, since $y \leq y$, then $\{y\} \ll aR$ which by (6) implies that $\{a\} \ll yS$ for all $y \in aR$. The other part is similar.

Now, consider $a_1 \leq a_2$ in A. Then, since $\{a_2\} \ll yS$ for all $y \in a_2R$, it also holds that $\{a_1\} \ll yS$ for all $y \in a_2R$. Hence, by (6), we have $\{y\} \ll a_1R$, for all $y \in a_2R$ which means that $a_2R \ll a_1R$. The antitonicity of S follows analogously. □

4 Conclusions and further work

The problem of considering relations within the notion of Galois connection is not new, since it can be dated back to [8], nor outdated, since one can still find recent references dealing with different aspects of the integration of relations and Galois connections, see for instance [5, 11, 13].

We have obtained some prospective results on the notion of relational-based Galois connection, in which the components of the connection are relations between posets. There are several possibilities depending both on the (pre-)order relation between subsets in the underlying powerdomain and the chosen type of relational composition. We have just scratched the surface of the problem, and shown that one of the most reasonable approaches collapses in that the involved relations R and S should actually be functions. The second proposed definition uses a different approach in that, apart from considering an alternative ordering in the underlying powerdomain for the definition of antitonicity, it also generalizes the notion of being inflationary. This way, we have obtained a promising result in the form of Theorem 1.

As future work, we are planning to continue the line initiated in [2, 3] and attempt the construction of the residual, in the sense of relation-based (fuzzy) Galois connections, to a given mapping between differently structured domain and codomain, as stated in the introduction.

References

[1] L. Antoni, S. Krajči, and O. Krídlo, Representation of fuzzy subsets by Galois connections. *Fuzzy Sets and Systems*, 2017. `https://doi.org/10.1016/j.fss.2017.05.020`.

[2] I.P. Cabrera, P. Cordero, F. García-Pardo, M. Ojeda-Aciego, and B. De Baets. On the construction of adjunctions between a fuzzy preposet and an unstructured set, *Fuzzy Sets and Systems*, 320:81–92, 2017.

[3] ——, Adjunctions between a fuzzy preposet and an unstructured set with underlying fuzzy equivalence relations, 2017, submitted.

[4] I.P. Cabrera, P. Cordero, and M. Ojeda-Aciego. Relational fuzzy Galois connections. In *Proc. of the 17th World Congress of Intl Fuzzy Systems Association (IFSA-SCIS'17)*, 2017. To appear.

[5] M. Couceiro. Galois connections for generalized functions and relational constraints. In *Contributions to General Algebra*, volume 16, pages 35–54. Verlag Johannes Heyn, 2005.

[6] K. Denecke, M. Erné, and S. Wismath. *Galois connections and applications*, ser. Mathematics and its Applications. Springer, 2004, vol. 565.

[7] J.T. Denniston, A. Melton, S.E. Rodabaugh. Formal Contexts, Formal Concept Analysis, and Galois Connections. *Electr. Proc. on Theoretical Computer Science* 129: 105–120, 2013.

[8] I. Fleischer and I. Rosenberg. The Galois connection between partial functions and relations. *Pacific Journal of Mathematics*, 79(1):93–97, 1978.

[9] F. García-Pardo, I.P. Cabrera, P. Cordero, and M. Ojeda Aciego. On Galois connections and soft computing, *Lect. Notes in Computer Science*, vol. 7903, pp. 224–235, 2013.

[10] F. García-Pardo, I.P. Cabrera, P. Cordero, M. Ojeda Aciego, and F. Rodríguez. On the definition of suitable orderings to generate adjunctions over an unstructured codomain, *Information Sciences*, vol. 286, pp. 173–187, 2014.

[11] E. Jeřábek. Galois connection for multiple-output operations. `https://arxiv.org/abs/1612.04353v1, dec 2016`.

[12] O. Ore. Galois Connexions. *Transactions of the American Mathematical Society*, 55(3):493–513, 1944.

[13] Á. Száz. A particular Galois connection between relations and set functions. *Acta Univ. Sapientiae, Mathematica*, 6(1):73–91, 2014.