Inventory control for a non-stationary demand perishable product: comparing policies and solution methods

Karin G.J. Pauls-Worm[1], Eligius M.T. Hendrix[2]

[1]Operations Research and Logistics, Wageningen University, orcid 0000-0002-8437-2455, karin.pauls@wur.nl
[2]Computer Architecture, Universidad de Málaga, orcid 0000-0003-1572-1436, eligius@uma.es

Abstract. This paper summarizes our findings with respect to order policies for an inventory control problem for a perishable product with a maximum fixed shelf life in a periodic review system, where chance constraints play a role. A Stochastic Programming (SP) problem is presented which models a practical production planning problem over a finite horizon. Perishability, non-stationary demand, fixed ordering cost and a service level (chance) constraint make this problem complex. Inventory control handles this type of models with so-called order policies.
We compare three different policies: a) production timing is fixed in advance combined with an order-up-to level, b) production timing is fixed in advance and the production quantity takes the age-distribution into account and c) the decision of the order quantity depends on the age-distribution of the items in stock. Several theoretical properties for the optimal solutions of the policies are presented. In this paper, four different solution approaches from earlier studies are used to derive parameter values for the order policies. For policy a), we use MILP approximations and alternatively the so-called Smoothed Monte Carlo method with sampled demand to optimize values. For policy b), we outline a sample based approach to determine the order quantities. The flexible policy c) is derived by SDP. All policies are compared on feasibility regarding the α-service level, computation time and ease of implementation to support management in the choice for an order policy.

Keywords: Inventory · Perishable product · MINLP · Chance constraint · Monte Carlo

1 Introduction

Although in Inventory Control literature, focus is often on infinite horizons with stationary demand, in the reality of retail, demand of perishable food products is typically non-stationary, i.e. uncertain and also fluctuating, partly due to promotions. In this case, we can consider the planning problem as a finite horizon problem with non-stationary demand. The contribution of this paper is to outline a practical inventory control problem of perishable products as a Stochastic Programming (SP) model with a finite horizon and evaluate several solution approaches from earlier studies to handle it.

We focus on perishable products that are processed and get a best-before-date on their package, such as packed cheese, cut and packed lettuce, yoghurt, etc. Those products have a fixed maximum shelf life. Producers and retail organisations have arrangements regarding delivery performance, including the necessary available shelf life for the consumer and the service level. After the maximum shelf life M, the product cannot be used anymore for the intended purpose and is considered waste. An α-service level requirement refers to the probability to be out of stock, i.e. it should be smaller than $1-\alpha$. This implies we are dealing with a chance constraint for each period. For every period (day, week, ...) the producer has to decide whether or not to order and how much, considering a fixed ordering cost, holding cost and disposal cost. This results in replenishment cycle lengths of a varying number of periods. The producer has control over the issuing of products, so in order to minimise waste, a First-in-First-Out (FIFO) policy is used. Excess demand is backlogged. The question is what are the most appropriate inventory policies to handle the inventory control problem in practice.

For products with non-stationary demand, order decisions and production planning will fluctuate, especially for perishable products where smoothing of ordering or production is impossible, because the older items in stock can perish. The fluctuations in demand ask for a specific strategy. Bookbinder and Tan (1988) distinguish three strategies to deal with ordering of non-perishable products with non-stationary demand in a periodic review. The first strategy is called the static uncertainty strategy; the timing and size of the orders are set at the beginning of the time horizon. The replenishment schedule defines when to order beforehand, denoted by $Y_t = 1$ when an order is placed and $Y_t = 0$ if not. This results in replenishment cycles R_t of different length. In case of long lead time, adaptation of the order quantity just before demand is not possible, so also the production quantity is determined at the beginning of the planning horizon. We call this a *YQ* policy. We use the variable *Y* in the policy name instead of *R* to make clear that for every period a decision is made. This policy is appropriate when there is considerable lead time and is investigated in (Pauls-Worm et al., 2016). The second strategy to deal with non-stationary demand is the static-dynamic uncertainty strategy, where timing of the orders is set at the beginning of the time horizon, but the order quantity may be adapted in response of the inventory levels observed during the time horizon. We call this a *YS* policy. In a heuristic approach, Bookbinder and Tan (1988) split the problem in two stages. The first stage determines the timing and the second the quantity. Tarim and Kingsman (2004) considered this approach as a basis for an MILP model formulation for non-stationary stochastic demand for the simultaneous determination of the timing and size of the replenishment orders. The third strategy to deal with non-stationary demand is the dynamic uncertainty strategy where the order quantity is decided at the beginning of every period. We call this a $Q(X)$ policy, where X is the inventory level at the beginning of the period.

A policy according to the dynamic uncertainty strategy is discussed already in the 1960s. Karlin (1960a) shows that a critical number policy is optimal, were the critical numbers are a reorder level s_t, and an order-up-to level S_t, resulting in an (R,s_t,S_t) policy. Karlin (1960b) and Veinott Jr (1963, 1965) developed optimal myopic policies for certain cases. Morton (1978) shows that near myopic bounds are close to optimal, under the assumption of disposal of excess stock. Morton and Pentico (1995) derive near-myopic bounds for the more general case. Zipkin (1989) developed optimal critical number policies for a cyclic demand pattern. He shows that the critical numbers in the optimal policy smooth the fluctuation in the demand data. This “wait-and-see” approach in the critical number policies following the dynamic uncertainty strategy could require an order with setup cost in almost every period. This might be undesirable for the production planning of a company, but in case of large setup cost relative to the holding cost, this is neither optimal (Bookbinder and Tan, 1988).

Bookbinder and Tan (1988) formulated their strategies for non-perishable products. Perishable products require special attention with respect to order policies taking non-stationary demand into account. Order policies for perishable products with a fixed lifetime are reviewed by Nahmias (1982), Goyal and Giri (2001), and Karaesmen et al. (2011). Almost all papers assume stationary demand. Fries (1975) shows that with a maximum shelf life of $M \geq 2$, neither an (R,S) nor an (R,s,S) policy is optimal. Nahmias (1975) and Fries (1975) observe that in general an optimal order policy for perishable products with a fixed life time should take the age-distribution of the products in stock into account. Even when all perishable items are of the same age, base stock (order-up-to level) policies are not optimal, as argued by Tekin et al. (2001) and Haijema et al. (2007). Some papers, e.g. (Haijema et al., 2007), (Broekmeulen and van Donselaar, 2009), (Minner and Transchel, 2010) assume a cyclic demand pattern, with a weekly demand pattern per day, but stationary expected demand per week. They assume negligible setup cost and follow a dynamic uncertainty strategy, which might not be optimal in case of fixed setup cost.

Food producers often have contracts with their customers, regarding service level. From the above mentioned papers, only Bookbinder and Tan (1988), Tarim and Kingsman (2004) and Minner and Transchel (2010) consider service level constraints. The other papers use penalty cost for each backlogged item. Unfortunately, the size of a penalty to guarantee a desired service level is not straightforward to determine as in general when dealing with chance constraints.

Pauls-Worm et al. (2014) presented an SP inventory model that minimises the expected total costs, including setup cost, unit procurement cost, holding cost and cost of waste, for a perishable product with non-stationary stochastic demand with an α-service level constraint under a FIFO issuing policy. A *YS* policy is convenient in practical planning for a food producer; one knows beforehand in which periods to produce. Using an order-up-to level *S* defines a practical rule to determine the order quantity taking uncertainty into account. In Pauls-Worm et al. (2014), we used an MILP approximation to derive values for a *YS* system, based on the cumulative distribution function (cdf) for the demand during the replenishment cycle and the expected age-distribution of the inventory during the replenishment cycle. However, we know that this approach may fail in meeting service level requirements because the approximated amount of waste is underestimated and the amount of fresh items is overestimated, due to Jensen Inequality. Therefore, we presented in (Hendrix et al., 2015) a computational method based on the so-called Smoothed Monte Carlo method with sampled demand to optimise values for a *YS* system. The resulting MINLP approach uses enumeration, bounding and iterative nonlinear optimisation. The order quantity is determined by order-up-to level *S* minus the total inventory in stock. However, it could be more cost-efficient to consider the age-distribution of all items in stock in determining the order quantity. Let *X* denote the inventory age-distribution at the beginning of a period. Using a sample based *YQ*(*X*) policy, one can take the age-distribution into account. Finally, we can consider a more flexible *Q*(*X*) policy according to a dynamic uncertainty strategy, derived by SDP. A *Q*(*X*) policy determines the order quantity at the start of every period based on the age-distribution of the items in stock. Besides variation in uncertainty strategy, the presented policies vary in ease of implementation in practice. An order-up-to level policy requires only information about the total available inventory and a set of order-up-to levels S_t. When the age-distribution is considered in the policy, information is required about the age-distribution of items in stock, and an order quantity has to be determined applying a table or computation process. With the extra information required, also the calculation time to find a solution may increase.

To summarise, we evaluate three different policies: *YS* policies (production timing is fixed, order-up-to level), a *YQ*(*X*) policy (production timing is fixed, production quantity takes age-distribution into account) and an *Q*(*X*) policy (decides every period on order quantity depending on age-distribution). An overview of investigated strategies and policies is presented in Table 1. We compare the policies for 81 instances and investigate in which situations in practice which policy is most suitable. Section 2 gives the SP model of the practical problem. Section 3 exposes several theoretical properties *YS* and *YQ*(*X*) policy solutions. Section 4 outlines the Smoothed Monte Carlo MINLP approach to determine parameters for a *YS* policy. The *YQ*(*X*) policy is depicted in Section 5 and Section 6 presents the flexible *Q*(*X*) policy. Section 7 compares the policies and Section 8 concludes.

Table 1 Overview of investigated strategies and policies

Uncertainty strategy	Policy	Solution Method	Paper	Instances reported
Static-dynamic	YS	MILP	(Pauls-Worm et al., 2014)	81+5 instances
Static-dynamic	YS	SMC-MINLP	(Hendrix et al., 2015)	1 instance
Static-dynamic	YQ(X)	Sample Based Method	(Hendrix et al., 2015)	1 instance
Dynamic	Q(X)	SDP	(Hendrix et al., 2012)	1 instance

2 Stochastic Programming model

The practical inventory control problem formulated as an SP model in (Pauls-Worm et al., 2014) is described below. Due to stochastic demand d_t, the inventory levels I_{bt} (except the starting inventory I_{b0}) and the order quantities Q_t are random variables. We assume a lead time of zero. The probability in the chance constraints is notated as P(.), and E(.) denotes the expected value operator. We use $(x)^+$ to express max $\{x, 0\}$. Table 2 provides a list of symbols.

Table 2 List of symbols

Indices	
t	period index , $t = 1,..,T$ with T the time horizon
b	age index, $b = 1,..,M$, with M the fixed maximum (internal) shelf life
Data	
d_t	Normally distributed demand with expectation $\mu_t > 0$ and variance (CV x μ_t)^2 where CV is a given coefficient of variation
k	fixed ordering or setup cost, $k > 0$
c	unit procurement cost, $c > 0$
h	unit holding cost, for items that are carried over from one period to the next, $h > 0$
w	unit disposal cost ($w > 0$) or salvage value ($w < 0$) for items becoming waste
α	required service level, $0 < \alpha < 1$
Variables	
Q_t	ordered and delivered quantity at the beginning of period t
I_{bt}	inventory level of items with age b at the end of period t, initial inventory fixed
	$I_{b0} = 0$, $I_{1t} \in \mathbb{R}$, $I_{bt} \geq 0$ for $b = 2,.., M$. Inventory of age M at the end of period t is considered waste

The expected total costs over the time horizon T are minimized.

$$\text{Min E}(TC) = \text{E}\left(\sum_{t=1}^{T} \left(g(Q_t) + h \sum_{b=1}^{M-1} I_{bt}^{+} + wI_{Mt} \right) \right) = \sum_{t=1}^{T} \text{E}\left(g(Q_t) + h \sum_{b=1}^{M-1} I_{bt}^{+} + wI_{Mt} \right) \tag{1}$$

where the procurement costs are given by the function

$$g(x) = k + cx \text{ if } x > 0 \text{ and } g(0) = 0. \tag{2}$$

The chance constraint requiring the α-service level is expressed by

$$P(I_{1t} \geq 0) \geq \alpha \qquad t = 1,..,T \tag{3}$$

meaning that the probability of not being out-of-stock at the end of period t should be greater than or equal to α. The probability of a stock-out is $1 - \alpha$. Because of a FIFO issuing policy, the inventory levels of the older items are zero in case of a shortage, so only the inventory of the freshest items can be negative. The inventory dynamics for the FIFO issuing is described by

$$I_{bt} = \left(I_{b-1,t-1} - \left(d_t - \sum_{j=b}^{M-1} I_{j,t-1} \right)^{+} \right)^{+} \qquad t = 1,..,T;\ b = 2,..,M \tag{4}$$

and for the freshest items

$$I_{1t} = Q_t - \left(d_t - \sum_{b=1}^{M-1} I_{b,t-1} \right)^{+} \qquad t = 1,..,T \quad . \tag{5}$$

Items of age M become waste at the end of the period and cannot be used in the next period. The inventory of the freshest items can be negative. The unmet demand will be backlogged.

The inventory at the beginning of a period is defined by

$$X = (I_{1,t-1}, \ldots, I_{M-1,t-1}) \quad . \tag{6}$$

As described in Section 1, several order policies can be defined as solutions of the SP model. One way is to define order-up-to levels S_t. The decision maker replenishes in period t the inventory up-to the level S_t, where the order quantity Q_t is defined by

$$Q_t(X) = \left(S_t - \sum_{b=1}^{M-1} X_{bt} \right)^+ \tag{7}$$

In Section 3 we discuss properties of policy solutions according to the static-dynamic uncertainty strategy for this SP model. In Section 4 we develop a *YS* policy and in Section 5 a *YQ*(*X*) policy, both following a static-dynamic uncertainty strategy. In Section 6 we present a more flexible *Q*(*X*) policy according to the dynamic uncertainty strategy to obtain parameters for the SP model.

3 Properties of a solution of the static-dynamic uncertainty strategy

Theoretical properties about the optimal solution of specific cases help to limit their solution space. Therefore, we first focus on the properties of feasible solutions. In Section 3.1, the concept of replenishment cycles and timing are discussed. Section 3.2 shows in which cases the so-called basic order-up-to level is the optimal quantity. For the other order moments we study the mathematical implications of estimating the service level by a Monte Carlo sampling approach in Section 3.3.

3.1 Replenishment cycles and limits on order timing vector *Y*

A replenishment cycle is the number of periods R the order quantity Q_t aims to fulfil. For non-stationary demand, replenishment cycle length R depends on order moment t. When $Y_t = 1$, then $Y_{t+R+1} = 1$ and no orders take place in between. In case of perishable items with maximum shelf life M, the replenishment cycle cannot be longer than the shelf life M, so $R \le M$. So,

Property 1. *Let Y be an order timing vector of the SP model, i.e.* $Y_t = 0 \Rightarrow Q_t = 0$. *Y provides an infeasible solution of the SP model, if it contains more than* $M - 1$ *consecutive zeros.*

Let F_T be the set of all feasible order timing vectors Y of length T. The number of elements $|F_T|$ of horizon T and shelf life of $M < T - 1$ follows recursive rule $|F_{T+1}| = 2|F_T| - |F_{T-M}|$ with the initial terms $|F_t| = 2^{t-1}$ for $t < M + 1$ and $|F_{M+1}| = 2^M - 1$; see (Alcoba et al., 2015). F_T is exponential in the horizon T. However, in practice (Pauls-Worm et al., 2014), it is sufficient to plan ahead for $T = 12$ periods.

3.2 Basic order-up-to level and optimal order quantities

For a certain replenishment cycle length $R = 1,.., M$ we can define a basic order-up-to level $\hat{S}_{Rt}$ as the inventory that should be available at the beginning of period t to cover demand of R periods.

Definition 1. *Let* $d_t + .. + d_{t+R-1}$ *be the stochastic demand during a replenishment cycle of length R with cumulative distribution function (cdf)* G_{Rt}. *The basic order-up-to level* $\hat{S}_{Rt}$ *with probability* α *to fulfil demand is defined by* $G_{Rt}(\hat{S}_{Rt}) = \alpha$ *such that* $\hat{S}_{Rt} = G_{Rt}^{-1}(\alpha)$.

For some replenishment cycles, $\hat{S}_{Rt}$ may be not enough, so $\sum_{b=1}^{M-1} X_{bt} + Q_t \ge \hat{S}_{Rt}$, because products in stock can become waste during the replenishment cycle. Nevertheless, for some replenishment cycles, the basic order-up-to level is sufficient and specifies the optimal order quantity. The following bounds can be derived. For the proofs, see (Hendrix et al., 2015).

Property 2. *Let Y be an order timing vector of the SP model with corresponding cycle length R and X defined by (6). For order moment t having* $Y_{t-M} = 1$*, R = M, the optimal order quantity is* $Q_t = \hat{S}_{Rt}$.

Second, a replenishment cycle may be of just one period; during the cycle no waste occurs.

Property 3. *Let Y be an order timing vector of the SP model and X defined by (6). For an order moment t having* $Y_{t+1} = Y_t = 1$ *the optimal order quantity is* $Q_t = \hat{S}_{1t} - \sum_{b=1}^{M-1} X_{bt}$.

The best order quantity at a negative stock level is of an order-up-to type.

Property 4. *Let Y be an order timing vector of the SP model and X defined by (6),* $Y_t = 1$ *with replenishment cycle length R. If* $X_{1t} \leq 0$*, the optimal order quantity is* $Q_t = \hat{S}_{Rt} - X_{1t}$.

3.3 Sample based Monte Carlo estimation of the service level

If theoretical properties of Sections 3.1 and 3.2 do not apply, one can try to find the order quantities fulfilling the chance constraints using samples of the demand series. Let $\boldsymbol{d}$ be the stochastic demand vector ($\boldsymbol{d}_t, \ldots, \boldsymbol{d}_{t+R-1}$) from t, during replenishment cycle length R, X the starting inventory and Q the order quantity. Let $f(Q, X, d) = I_{1,t+R-1}$ define the end inventory of items with an age of one period given a realisation d of $\boldsymbol{d}$ following the inventory dynamics with possible perishing according to (4) and (5). Consider indicator function $\delta : \mathbb{R} \times \mathbb{R}^R \rightarrow \{0, 1\}$

$$\delta(Q,d) = \begin{cases} 1 & \text{if } f(Q,X,d) \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

that translates the service level in constraint (3) for period $t + R - 1$ to

$$a(Q) = P(I_{1,t+R-1} \geq 0) = E_d \delta(Q,d) \tag{9}$$

The idea is that given N sample paths $d_1, .., d_N$ of $\boldsymbol{d}$, the probability (service level) (9) is estimated by

$$\hat{a}(Q) = \frac{1}{N} \sum_{r=1}^{N} \delta(Q, d_r). \tag{10}$$

As is know from handbooks on statistics (e.g. (Lyman Ott and Longnecker, 2001)), given a set of independent random samples $\boldsymbol{d}_r$, (10) is an unbiased estimator of $a(Q)$ with standard deviation

$$\sigma(\hat{a}(Q)) = \sqrt{\tfrac{1}{N}(a(Q) - a(Q)^2)}, \tag{11}$$

which is used in Monte Carlo approaches to determine the number N of samples to reach a desired probabilistic accuracy. A rule of thumb is to have an accuracy of 2σ. Aiming at α = 90%, 95%, 98%, a sample size of N = 5000 gives a rule of thumb accuracy of about 0.005 for the estimator $\hat{a}(Q)$.

4 *YS* policy: sample based SMC-MINLP approach

Consider a *YS* policy, i.e. the decision maker is provided an order timing vector Y, i.e. $Y_t = 0 \Rightarrow Q_t = 0$. The question is to generate values for S_t such that the α-service level constraints are fulfilled for all instances and expected costs are minimized. Properties 2 and 3 can be used to order-up-to level $S_t = \hat{S}_{Rt}$ for specific moments. Sample-based estimation can be used for other service levels. Following the tradition of using scenarios (sample paths), one can write the problem of finding discrete timing Y and continuous order-up-to levels S as a Monte Carlo based Mixed Integer Linear Programming (MC-MILP) model as specified in Section 4.1. Such sample-based model for most instances cannot be solved in reasonable time, e.g. (Rijpkema et al., 2016). Therefore, in Section 4.2 we sketch the use of an equivalent

MINLP model based on the Smoothed Monte Carlo method, see (Hendrix and Olieman, 2008). A specific algorithm is designed that uses enumeration and bounding for the integer part Y of the problem leaving us with iteratively solving an NLP problem in the continuous variables S. This algorithm has been described earlier in (Hendrix et al., 2015).

4.1 Traditional scenario based MC-MILP optimisation of the *YS* policy

The sample-based approach for the *YS* policy can be handled by adding to the SP model a sample (scenario) index $r = 1, .., N$ to the variables I_{btr} and Q_{tr} . The objective (1) is extended towards

$$\text{Min}\frac{1}{N}\sum_{t=1}^{T}\left(kY_t+\sum_{r=1}^{N}\left(cQ_{tr}+h\sum_{b=1}^{M-1}I_{btr}^{+}+wI_{Mtr}\right)\right) \tag{12}$$

with order quantity

$$Q_{tr}=\left(S_t-\sum_{b=1}^{M-1}I_{b,t-1,r}\right)^{+} \qquad r=1,..,N;\; t=1,..,T \tag{13}$$

and the conventional order relation

$$S_t \le \mathcal{M}Y_t \qquad t=1,..,T \tag{14}$$

with a big-$\mathcal{M}$ value. The constraints (4) and (5) are extended to each sample

$$I_{btr}=\left(I_{b-1,t-1,r}-\left(d_{tr}-\sum_{j=b}^{M-1}I_{j,t-1,r}\right)^{+}\right)^{+} \qquad r=1,..,N;\; t=1,..,T;\; b=2,..,M \tag{15}$$

and for the freshest items

$$I_{1tr}=Q_{tr}-\left(d_{tr}-\sum_{b=1}^{M-1}I_{b,t-1,r}\right)^{+} \qquad r=1,..,N;\; t=1,..,T \tag{16}$$

Notice that due to the nonnegativity of the variables, the function $(x)^{+} = \max\{0, x\}$ can be rewritten by additional variables and inequalities in order for (13), (15) and (16) to be linear, as described in (Pauls-Worm et al., 2014). Furthermore, for the chance constraints one adds a binary variable $\delta_{tr} \in \{0, 1\}$ representing the indicator value that specifies whether demand is fulfilled in period t in sample r

$$-I_{1tr}\le m_t(1-\delta_{tr}) \qquad r=1,..,N;\; t=1,..,T \tag{17}$$

with m_t an upper bound on the out of stock $-I_{1t}$. This defines $\hat{a}_t(S):\mathbb{R}^T\rightarrow\{0,\frac{1}{N},\frac{2}{N},..,1\}$ giving the reached service level under the set of samples. The corresponding chance constraints are

$$\hat{a}_t(S):=\frac{1}{N}\sum_{r=1}^{N}\delta_{tr}\ge\alpha \qquad t=1,..,T \tag{18}$$

The variables of the model are Y_t , $\delta_{tr} \in \{0, 1\}$, S_t , I_{btr}, $Q_{tr} \geq 0$. Notice that Y_t and S_t do not depend on sample r and the other variables Q_{rt}, I_{btr} and δ_{tr} that describe the simulation or evaluation part, do so. Solving MC-MILP is in most cases practically impossible due to the large number of binary variables δ and many solutions δ that represent the same obtained service levels $a(S)$. The number of samples N = 5000 mentioned in Section 3.3, implies defining for each period N = 5000 binary variables δ_{tr}.

4.2 Smoothed Monte Carlo MINLP approach to the *YS* policy

Considering MC-MILP from the point of view of an NLP problem in the continuous variables S given Y, the function $\hat{a}_t(S)$: $\mathbb{R}^T\rightarrow\{0,\frac{1}{N},\frac{2}{N},..,1\}$ in (18) is piecewise constant, i.e. changing the values of S a bit may not change the evaluated value of $\hat{a}_t(S)$. According to Hendrix and Olieman (2008), the reached service level can be made practically a continuous function by following the MC smoothing approach. Let $z_{rt} = \sum_{b=1}^{M} I_{btr}$ be the total amount of product left over at the end of period t in sample r. Measuring how close $\hat{a}_t(S)$ is to change value using the least nonnegative total inventory $p_t^{[in]}(S) = \min_r \{z_{rt} \mid z_{rt} \geq 0\}$ and the least negative inventory $p_t^{[out]}(S) = \min_r \{-z_{rt} \mid z_{rt} < 0\}$. The suggested smoothing function $o_t(S)$ is

$$o_t(S) = \frac{1}{2N}\left(\frac{2p_t^{[in]}(S)}{p_t^{[in]}(S) + p_t^{[out]}(S)} - 1\right). \quad (19)$$

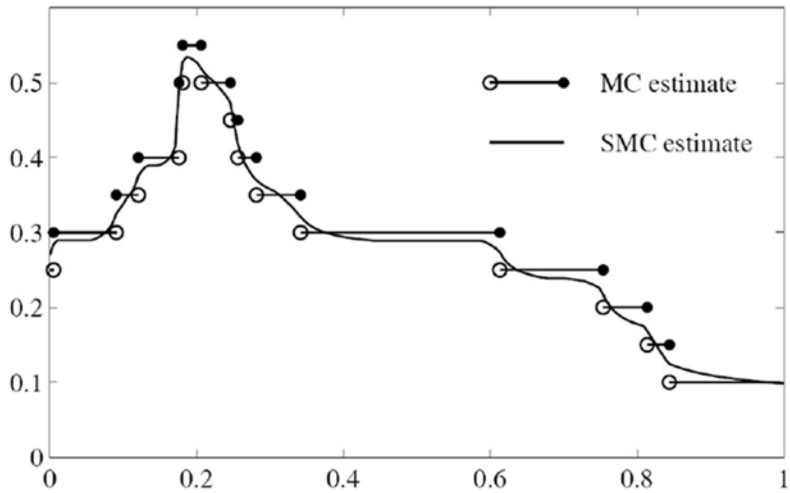

Fig. 1. Illustration of Smoothed MC from (Hendrix and Olieman, 2008), where the estimated probability on the y-axis depends on varying one parameter on the x-axis, in our case the order-up-to level S

Hendrix and Olieman (2008) show that $\hat{a}_t(S) + o_t(S)$ is continuous in interesting values of S, as illustrated in Figure 1 and the function $\hat{a}_t(S) + o_t(S)$ deviates at most $\frac{1}{2N}$ from the reached service level $\hat{a}_t(S)$ which is much smaller than the possible estimation error. Problem $NLPS(Y)$ is defined as MC-MILP replacing constraint (18) by

$$\hat{a}_t(S) + o_t(S) \geq \alpha \qquad t = 1,..,T \quad (20)$$

as a smooth optimization problem that in principle can be solved by a nonlinear optimisation routine. This can be used in enumeration routine Algorithm 1.

Algorithm 1. *YS*smooth in: samples d_{tr}, cost data, α, $\hat{S}_{Rt}$, out: Y^*, S^*

Set the best function value $C^U := \infty$
Generate the set of feasible order timing Y
<u>for</u> all Y
  <u>if</u> for the lower bound on cost $LBc(Y) < C^U$
    solve $NLPS(Y)$ using $\hat{S}_{Rt}$ values → S and cost C
    <u>if</u> $C < C^U$
      save the best found values $C^U := C$, $S^* := S$, $Y^* = Y$

5 *YQ*(*X*) policy: sample based algorithm

In policy *YQ*(*X*), the decision maker is provided a replenishment schedule *Y*, i.e. $Y_t = 0 \Rightarrow Q_t = 0$. The order quantity depends on the age distribution *X* of the items in stock. Practically, this is more complex than an order-up-to level as the decision maker requires an information system advising the order quantity given the actual age composition of inventory. The results of Section 3 also hold for the *YQ*(*X*) policy. Quantities of order moments fulfilling Properties 2, 3 or 4, can be set to basic order-up-to level $\hat{S}_{Rt}$. Otherwise, for a positive inventory *X*, the sample-based estimation of Section 3.3 can be used. At an order moment, i.e. $Y_t = 1$ where the inventory position is positive and $R > 1$, the order quantity may be larger than the basic order-up-to level

$$Q_t(X) \geq \left(\hat{S}_{Rt} - \sum_{b=1}^{M-1} X_{bt} \right)^+ \tag{21}$$

due to the occurrence of expected waste during the replenishment cycle. To compute the optimal order quantity for this case, we focus on the total inventory at the end of the replenishment cycle as function of the starting inventory *X*, the order quantity *Q* and the demand $d_t,.., d_{t+R-1}$.

Definition 2. *The function* $Z: \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^R \to \mathbb{R}$ *is defined as the transformation* $z = Z(Q, X, d)$ *giving the total inventory* $z = \sum_{b=1}^{M} I_{b,t+R-1}$ *following the dynamics (4), (5) with starting inventory X, order quantity Q and demand vector* $(d_1,.., d_{t+R-1})$.

Actually, we are looking for the minimum value Q_t for which the chance constraint holds; this is the value of Q_t for which $P(Z(Q_t, X_t, d_t,.., d_{t+R-1}) \geq 0) = \alpha$. We can use the following property.

Property 5. *Let function Z be defined by Definition 2,* $R \leq M$, *starting inventory X* and $z = Z(0, X, d_t,.., d_{t+R-1})$ *with cdf* Γ. *The optimal order quantity in period t is* $Q_t = (-\Gamma^{-1}(1-\alpha))^+$.

The cases where $Q_t = \hat{S}_{Rt} - \sum_{b=1}^{M-1} X_{bt}$ is the optimal solution are given by Properties 2 and 3. The value may deviate in other cases due to the waste during the replenishment cycle. Taking this value as benchmark provides a corollary which follows directly from the former properties.

Corollary 1. *Let Z be given by Definition 2,* $R \leq M$, $z = Z(\hat{S}_{Rt} - \sum_{b=1}^{M-1} X_{bt}, X, d_t,.., d_{t+R-1})$ *with cdf* Γ *and starting inventory X. The optimal order quantity is* $Q_t = \left(\hat{S}_{Rt} - \sum_{b=1}^{M-1} X_{bt} - \Gamma^{-1}(1-\alpha)\right)$.

This means that for cases following Properties 2 and 3 and if $X_{1t} \leq 0$ (no waste occurs during the cycle), the optimal choice is $Q_t = \hat{S}_{Rt} - \sum_{b=1}^{M-1} X_{bt}$ as $\Gamma^{-1}(1-\alpha) = 0$. In other cases, waste can be generated and $\Gamma^{-1}(1-\alpha) < 0$. To estimate the quantile $\Gamma^{-1}(1-\alpha)$, Monte Carlo simulation can be used as discussed in Section 3.3. Let *D* be an $N \times T$ matrix with samples d_{tr}. For a starting inventory *X*, giving the order quantity $Q_t = \hat{S}_{Rt} - \sum_{b=1}^{M-1} X_{bt}$, one can evaluate $z_r = Z(Q, X, d_{tr},.., d_{t+R-1,r})$ being the total inventory of sample *r* at the end of the cycle. The adjusting amount $-\Gamma^{-1}(1-\alpha)$ is estimated by

$$A_t(X) = (-\text{quantile } (\{z_r, r = 1,.., N\}, 1-\alpha))^+ , \tag{22}$$

where quantile $(\{\}, \alpha)$ is the α sample quantile of set $\{\}$. So the order quantity for any starting inventory according to Corollary 1, can be approximated by

$$Q_t(X) = \hat{S}_{Rt} - \sum_{b=1}^{M-1} X_{bt} + A_t(X)\,. \tag{23}$$

It should be noted that the order quantity is in fact based on a conditional chance constraint and may lead to unnecessarily high service levels, i.e. given the age distribution, (23) can only focus on fulfilling the chance constraint in the future, no matter how likely the current age distribution X is, see (Rossi et al., 2008). To provide some intuition, consider a discrete demand and inventory age-distribution X_t at the beginning of the period and I_t the inventory age-distribution at the end of the period, or alternatively replenishment cycle. Then we have $P(I_{1t} \geq 0) = \Sigma_X P(X_t = X) \cdot P(I_{1t} \geq 0 \mid X)$. Having a policy that assures for all possible situations X the conditional probability $P(I_{1t} \geq 0 \mid X) \geq \alpha$ is sufficient to imply $P(I_{1t} \geq 0) \geq \alpha$, but not necessary.

The order quantities for the *YQ*(*X*) policy are now defined either by the theoretical results, or by the sample-based estimation in (22) and (23). The next question is to generate the best advice for the order timing *Y*. Algorithm (2) enumerates the possible replenishment schedules replenishment schedules *Y*. Property 1 can be used to leave out those with too large periods between two orders. For each vector *Y*, the average cost is evaluated for a large simulation run that uses different random numbers than the ones in matrix *D* used to determine the order quantities by (22) and (23).

Algorithm 2. *YQ* in: samples d_{tr}, cost data, α, $\hat{S}_{Rt}$, out: Y^*

Set the best function value $C^U := \infty$
Generate the set of feasible order timing Y
for all Y
    if for the lower bound on cost $LBc(Y) < C^U$
        Determine C by simulating N sample paths
        During the simulation
        if $Y_t = 1$
            if starting inventory X not positive or $R_t = 1$ take $Q_t = \hat{S}_{Rt} - \sum_{b=1}^{M-1} X_{bt}$
            else simulate the replenishment cycle with N paths from X
            Determine the order quantity Q_t from (23)
        if $C < C^U$
            save the best found values $C^U := C$, $Y^* = Y$

The *YQ*(*X*) policy takes the age-distribution into account. For the decision maker the required use of sampling and possibly interpolation is more inconvenient than using a simple order-up-to strategy with a list of order-up-to levels of the *YS* policy. The question arises in which cases the policy *YQ*(*X*) performs significantly better than the *YS* policy.

6 Flexible *Q*(*X*) policy according to the dynamic uncertainty strategy

In this policy, at the beginning of every period *t*, the order quantity is determined, based on the age-distribution of the available inventory. We obtain this policy by Stochastic Dynamic Programming (SDP), an appropriate technique to solve the SP model, as it is clearly separable in *t*. The SDP approach to solve the SP model has been described in (Hendrix et al., 2012). The state values are given by *X*, the transition is provided by equations (4) and (5), so we have transition function Φ:

$$I_t = \Phi(X_t, Q_t, d_t) \qquad t = 1,..,T \tag{24}$$

The chance constraints can be written as

$$Q_t \geq \left(\Gamma_t^{-1}(\alpha) - \sum_{b=1}^{M-1} X_{bt} \right)^+ \qquad t = 1,..,T \tag{25}$$

The waste I_{Mt} at the end of period t is a function of the inventory at the beginning of the period and the demand: $I_{Mt} = f\,(X_t,d_t)$. We can write the expected contribution to the objective function in period t as function of state X_t and decision Q_t:

$$EC\,(X_t,Q_t) = g(Q_t) + E\{wf\,(X_t,d_t) + h1^T \Phi\,(X_t,Q_t,d_t)\} \tag{26}$$

where 1 is the all-ones vector. The SDP objective function can be written down via the Bellmann equation using a value function V:

$$V_t\,(X) = \min_Q \left(EC(Q,X) + E[V_{t+1}(\Phi\,(X,Q,d_t))] \right) \tag{27}$$

subject to fulling (25). The argmin of (27) represents the optimal policy $Q(X)$. To implement this policy, at the beginning of every period, the decision maker needs an information system with the optimal strategy in tables, advising on the order quantity given the actual age composition of the inventory.

7 Comparison of policies

To compare the different policies, we use the erratic demand pattern and the design of experiments used in (Pauls-Worm et al., 2014). The demand pattern is depicted in Table 3. In 81 experiments, we systematically vary fixed setup cost k = 1500, 500, 2000, disposal cost w = -0.5, 0, 0.5, α-service level = 90%, 95%, 98%, CV = 0.1, 0.25, 0.333. The other values are kept constant: the product has a shelf life of M = 3, unit production cost c = 2 and unit holding cost h = 0.5. Negative disposal cost means the product has a salvage value, which is usually much less than the unit production cost c, zero disposal cost means that only the unit production cost is lost in case of waste, and positive disposal cost means that there is a cost to discard the wasted items.

Table 3 Erratic demand pattern

	t	1	2	3	4	5	6	7	8	9	10	11	12
Data	$E(d_t)$	800	950	200	900	800	150	650	800	900	300	150	600

We compare the policies of Table 1, i.e. the YS_{MILP} policy with parameters generated by MILP (Pauls-Worm et al., 2014), a $YS_{\text{SMC-MINLP}}$ policy with parameters generated by the smoothed Monte Carlo MINLP approach described in Section 4, a $YQ(X)$ policy according to the sample based algorithm presented in Section 5, and the flexible $Q(X)$ policy generated by SDP as discussed in Section 6. The inventory system is simulated for all policies using the same (pseudo) random number series of 10,000 runs and compared on expected total costs with the reference value for the $YS_{\text{SMC-MINLP}}$ policy set to 100, as reported in Table 4.
The calculation time to generate parameter values for the policies differs significantly. Values for the YS_{MILP} policy are calculated within a second, while values for the $YS_{\text{SMC-MINLP}}$ policy take about 100 seconds. Calculations for the $YQ(X)$ policy use about 274 seconds, and for the $Q(X)$ policy about 150 seconds in Matlab on an Intel 7 processor. The calculation time is determined based on instances with a setup cost k = 1500, where the replenishment cycles vary in length, resulting in the longest computation time.

Table 4 shows that in most instances the expected total costs are very close. The YS_{MILP} policy provides often the lowest costs, but does not always meet the service level requirements. However, this is mostly in the last period T = 12 due to end-of-horizon effects. Simulation shows that in 96.4% of the periods the service level requirements are met, with an error tolerance of 1%. The other policies are designed such that they always meet the service level requirements. The $YS_{\text{MC-MINLP}}$ policy is in general the policy that meets the service level best at lowest costs. The shaded cells show the instances where a more complex policy is preferred regarding the costs.

Table 4. Simulated cost results of four order policies relative to $YS_{\text{SMC-MINLP}}$ costs = 100, shaded cells highlight the instances where a more complex policy is preferred regarding the costs

	CV = 0.10				CV = 0.25				CV = 0.33			
k = 1500	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$
Service level 90%												
w = -0.5	1	100.03	100.12	102.56	10	99.86	100.28	103.01	19	99.13	100.34	102.83
w = 0	2	100.04	100.14	102.12	11	99.51	99.86	102.18	20	98.80	99.57	101.82
w = 0.5	3	99.74	100.16	101.59	12	99.31	99.99	101.66	21	98.49	99.35	101.24
Service level 95%												
w = -0.5	4	100.02	100.13	101.61	13	99.24	99.68	101.22	22	98.76	97.20	100.77
w = 0	5	100.02	100.15	100.96	14	99.12	99.90	100.82	23	98.23	98.88	99.76
w = 0.5	6	99.40	99.83	100.27	15	98.81	99.39	100.41	24	97.13	97.84	98.40
Service level 98%												
w = -0.5	7	100.04	100.18	100.61	16	98.79	96.98	99.51	25	98.70	98.64	98.66
w = 0	8	99.93	100.21	99.72	17	98.23	98.89	99.11	26	98.51	98.41	98.02
w = 0.5	9	99.55	99.80	99.03	18	97.99	98.72	98.72	27	98.44	98.29	97.43

	CV = 0.10				CV = 0.25				CV = 0.33			
k = 500	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$
Service level 90%												
w = -0.5	28	99.95	99.99	100.95	37	99.66	99.91	100.17	46	99.33	99.88	99.28
w = 0	29	99.55	99.99	100.76	38	99.65	99.90	100.15	47	99.32	99.86	99.20
w = 0.5	30	99.63	99.98	100.90	39	99.65	99.89	100.11	48	99.30	99.84	99.10
Service level 95%												
w = -0.5	31	99.56	100.00	100.30	40	99.56	99.94	99.27	49	99.18	99.88	98.39
w = 0	32	99.67	99.98	100.46	41	99.56	99.94	99.23	50	99.17	99.87	98.27
w = 0.5	33	99.77	99.98	100.59	42	99.55	99.92	99.18	51	99.16	99.86	98.15
Service level 98%												
w = -0.5	34	99.96	100.02	99.89	43	99.50	100.03	98.43	52	99.14	100.04	97.60
w = 0	35	99.96	100.01	100.06	44	99.49	100.02	98.36	53	99.13	100.03	97.44
w = 0.5	36	99.96	100.01	100.21	45	99.48	100.01	98.29	54	99.13	100.02	97.29

	CV = 0.10				CV = 0.25				CV = 0.33			
k = 2000	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$	Instance	YS_{MILP}	$YQ(X)$	$Q(X)_{SDP}$
Service level 90%												
w = -0.5	55	100.03	100.11	103.34	64	100.02	100.26	105.04	73	100.01	100.32	105.15
w = 0	56	100.03	100.13	103.49	65	100.03	100.30	104.16	74	99.92	100.37	103.93
w = 0.5	57	100.04	100.15	103.45	66	100.21	100.52	103.34	75	99.09	100.42	102.55
Service level 95%												
w = -0.5	58	100.02	100.12	102.46	67	100.02	100.27	103.41	76	100.01	100.34	103.14
w = 0	59	100.02	100.14	102.46	68	100.02	100.31	102.15	77	99.41	99.95	101.54
w = 0.5	60	100.02	100.16	102.23	69	99.19	100.35	100.83	78	98.76	99.10	100.05
Service level 98%												
w = -0.5	61	100.04	100.16	101.66	70	99.97	100.32	101.78	79	100.02	100.44	101.22
w = 0	62	100.04	100.20	101.50	71	99.31	100.37	100.08	80	99.31	98.71	98.78
w = 0.5	63	100.05	100.23	100.86	72	98.68	99.11	99.04	81	98.55	98.42	97.99

In case of low setup cost (k = 500), one orders almost every period. The expected total costs of the different order policies are almost equal. Only with increasing uncertainty and higher service levels, the $Q(X)_{\text{SDP}}$ might significantly save costs. However, the dynamic uncertainty strategy $Q(X)_{\text{SDP}}$ policy may raise the expense on planning, which is not part of this model. In case of high setup cost (k = 2000), the static-dynamic uncertainty YS and $YQ(X)$ policies have mostly the same production plans based on the derived basic order-up-to levels and therefore, the costs show no significant differences. The $Q(X)_{SDP}$ policy has in most cases significantly higher costs. This is due to overachievement of the service level, as an SDP policy meets a conditional service level requirement as is illustrated in (Rossi, 2013) and (Pauls-Worm and Hendrix, 2015). Moreover, the $Q(X)_{SDP}$ policy is allowed to order every period, which is very costly in case of high setup cost. The $YQ(X)$ policy meets a conditional service level requirement, but here the production moments are fixed, which results in a less "nervous" system (Tunc et al., 2013). The so-called setup-oriented system nervousness can be prevented against a minor cost increase in case of a non-perishable product (Tunc et al., 2013). The results in this paper give a similar indication for a perishable product. Due to this behaviour, in instances 80 and 81 the $YQ(X)$ policy has lower costs, comparable to the costs of $Q(X)_{SDP}$. This means the $YQ(X)$ policy might be preferred in practice as it fixes the order timing.

As also shown in (Pauls-Worm et al., 2014), an intermediate level of setup cost is the most interesting situation. The replenishment cycles are varying in length, resulting in waste during the replenishment cycles. Finding the optimal order timing is more difficult as discussed in Section 3, and considering the age-distribution of the items in stock becomes more important. This is confirmed by the costs of the different order policies, in the instances with higher uncertainty and higher service levels (16 − 18, 22 − 27). The $YQ(X)$ policy, taking the age-distribution of the items in stock into account, gives clearly lower costs, as expected.

8 Conclusion

We studied a practical inventory control problem with non-stationary demand for a perishable product, investigating the question which order policies might be appropriate for this problem. We evaluated two different strategies to deal with the uncertainty in the practical problem, resulting in three different policies.

The inventory control can be handled according to a static-dynamic uncertainty strategy, where we distinguish a YS policy and a $YQ(X)$ policy. In inventory literature of perishable products this strategy is little studied. We first looked at possible bounds of the solution space discussing properties of the policies. Next we sketched several solution methods for the control policies. First a computational method based on the Smoothed Monte Carlo method with sampled demand, called the $YS_{SMC\text{-}MINLP}$ policy has been outlined and a sample based method to calculate values for the $YQ(X)$ policy. These policies were compared to an YS_{MILP} policy and a more flexible $Q(X)$ policy generated by SDP, according to a dynamic uncertainty strategy. The experimental evaluation comprises 81 instances with the same erratic demand pattern, but with varying setup cost, service level, cost of waste and uncertainty measured in the Coefficient of Variation. Table 5 presents an overview of the compared policies and solution methods.

For most instances, the expected total costs of the policies are very close, and a YS policy gives a cost efficient and easy to implement solution. From a production planning perspective, the static-dynamic uncertainty strategy is the most convenient strategy to follow. In situations of relatively low setup cost, (production takes place every period), or high setup cost (each M periods a production takes place), MILP generates appropriate parameter values. In situations of intermediate setup cost, were the replenishment cycles are highly varying, the optimized SMC-MINLP parameters might be more suitable, although more calculation time is needed; about 100 seconds compared to less than 1 second.

Table 5. Comparing policies and solution methods for 81 instances with horizon $T = 12$

Uncertainty Strategy	**Policy**	**Solution Method**	**Minimal α-service level**		**Comp Time**	**At Implementation**		**Average tot. costs (indexed)**
			Type	**Feasibility**		**Pre-det. ordering**	**Info needed**	
Static-dynamic	*YS*	MILP	Expected α-SL	96.4%	<1s	Yes	Order-up-to level Total Inventory	99.45
Static-dynamic	*YS*	SMC-MINLP	Expected α-SL	100%	≈100s*	Yes	Order-up-to level Total Inventory	100
Static-dynamic	*YQ*(*X*)	Sample Based Method	Conditional α-SL	100%	≈274s*	Yes	Inventory distr. Computer program to determine *Q* based on sampling	99.79
Dynamic	*Q*(*X*)	SDP	Conditional α-SL	100%	≈150s*	No	Inventory distr. Find *Q* in table based on inventory distribution	100.64

* In Matlab on an Intel 7 processor

When also the required service level is high or the CV is 0.25 or more, a *YQ*(*X*) policy, where the age-distribution of the inventory is taken into account, might have less inventory costs. However, for the decision maker the required use of samples and possibly interpolation is more inconvenient than using a simple order-up-to strategy with a list of order-up-to levels of the *YS* policy. Also the computation time of about 274 seconds might be a disadvantage. The dynamic uncertainty strategy *Q*(*X*) policy is only appropriate in situations with relatively low setup cost, a CV of 0.33 and a high service level of 98%. The same implementation drawback as for the *YQ*(*X*) policy applies, with a computation time of around 150 seconds. Based on the findings of this paper, it is up to management to decide which uncertainty strategy and which order policy is most appropriate.

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