Data-driven distributionally robust optimization with Wasserstein metric, moment conditions and robust constraints

Adrián Esteban-Pérez, Juan Miguel Morales

adrianesteban@uma.es

juan.morales@uma.es

Department of Applied Mathematics
University of Malaga
OASYS GROUP
oasys.uma.es

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Distributionally Robust Optimization

Distributionally Robust Optimization (DRO). Introduction

- Stochastic Programming: $\inf_{\mathbf{x} \in X} \mathbb{E}_Q f(\mathbf{x}, \boldsymbol{\xi})$
- Robust Optimization: $\inf_{\mathbf{x} \in X} \sup_{\boldsymbol{\xi} \in \Xi} f(\mathbf{x}, \boldsymbol{\xi})$
- DRO is essentially Stochastic Programming + Robust Optimization.

Data-Driven Distributionally Robust Optimization

Data-Driven Distributionally Robust Optimization (DDRO)

- Input: Training samples: $\widehat{\boldsymbol{\xi}}_1, \ldots, \widehat{\boldsymbol{\xi}}_N$
- Construct a set of probability distributions $\mathcal{Q}_N$ using the traininig samples (ambiguity set)

How to construct an ambiguity set?

- Seek probability distributions *close* to the empirical distribution $\widehat{P}$ based on the training samples.
- How close?
- We use probability metrics. A usually choice is Wasserstein metric.

Data-Driven Distributionally Robust Optimization

The goal is to compute:

$$\inf_{\mathbf{x} \in X} \sup_{Q \in \mathcal{Q}_N} \mathbb{E}_Q f(\mathbf{x}, \boldsymbol{\xi})$$

and the optimal solution $\mathbf{x}^*$.

DDRO

Wasserstein metric of order p

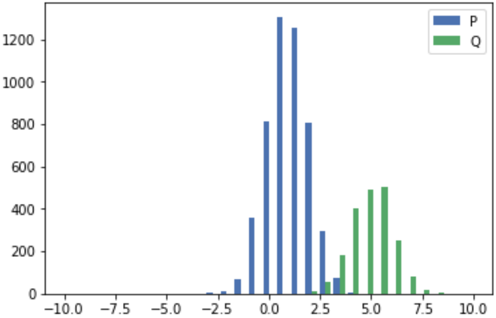
Wasserstein metric (of order p) between two probability distributions P and Q:

$$W_p(P, Q) = \left(\inf_{(X,Y): X \rightsquigarrow P, Y \rightsquigarrow Q} \mathbb{E}(\|X - Y\|^p) \right)^{1/p}$$

DDRO

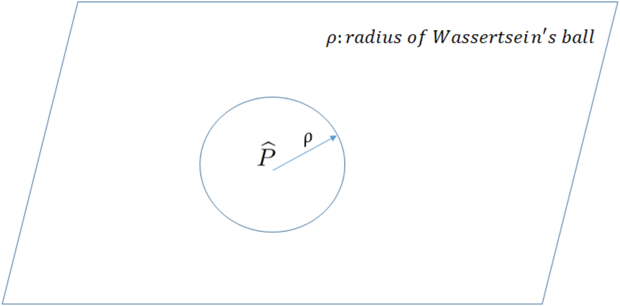
Wasserstein metric

Equivalently, Wasserstein metric between two probability distributions P and Q, $W(P, Q)$, is the minimum cost of moving P to Q. In the discrete case in 1-D:



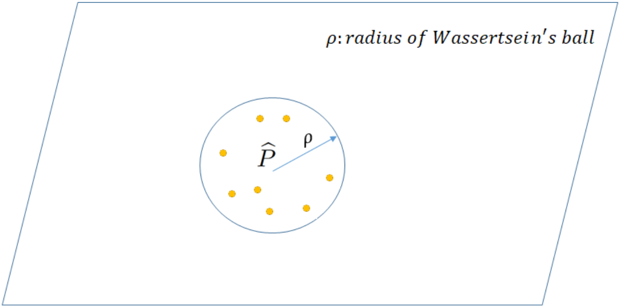
DDRO

- A problem: DDRO with Wasserstein's metric is too conservative:



DDRO

- A problem: DDRO with Wasserstein's metric is too conservative:
- A common approach to solving this problem is to add a priori information!!



DDRO

Wassersein metric

Wasserstein metric paradigm

- *Advantages*: Good theoretical properties: E.g.: Convergence with respect to Wasserstein metric (of order p) is equivalent to the usual weak convergence of measures plus convergence of the first p-th moments, rates of convergence of the empirical distribution to the true distribution.
- *Disadvantages*: We get too conservative distributions.
- Idea behind this approach: *the mass of the empirical distribution is moved to the worst case location points with the worst case mass in such points.*

DDRO

Some remarks about DDRO

A brief scheme

- Formulate the problem and the ambiguity set.
- Reformulate the inner supremum problem in a nice form using duality arguments in order to join it with the infimum outer problem.
- We get a minimization problem with a constraint of the form:

$$\sup_{\boldsymbol{\xi}} F(\mathbf{x}, \boldsymbol{\xi}) \leqslant 0$$

- So, the assumptions of ambiguity set and the objective function are essential!!

DDRO

Remarks and commom assumptions in DDRO

Reformulation of robust constraints in DDRO paradigm

Nowadays, sometimes

$$\sup_{\boldsymbol{\xi} \in \Xi} F(\mathbf{x}, \boldsymbol{\xi}) \leqslant 0$$

can be reformulated in a nice form applying the results existing in:

- *Deriving robust counterparts of nonlinear uncertain inequalities*, Ben-Tal et al. (2015), Mathematical Programming.

Motivation

> *A Wasserstein ball around the empirical distribution includes distributions with different support and allows (in a sense) robustness to unseen data.*

(Sinha et al. (2018), *Certifying Some Distributional Robustness with Principled Adversarial Training*)

- Thus, we consider a Wasserstein's ball.
- Using an ambiguity set, our goal is to find *good* hidden distributions (distributions *closer* to the true distributions with similar features) which reflects the random phenomena of our model.
- How do we do?
- We consider conic constraints in order to add a priori shape information.

Our approach

We split the support set in K regions and we introduce K decision variables (the mass in each region) subject to:

- $\sum_{i=1}^{K} p_i = 1$ and $p_i \geqslant 0$
- The array $(p_i)_{i=1}^{K}$ is in a cone $\mathcal{C}$ which reflects the shape of the distribution.

An example: The Univariate Newsvendor problem

DDRO. Formulation and notation

- x: order quantity (decision variable).
- ξ: demand of the item (random variable).
- h: unit holding cost.
- b: unit backorder cost.

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- ξ: demand of the item (random variable).
- h: unit holding cost.
- b: unit backorder cost.
- $(x-\xi)^+$: quantity in stock, where $z^+ = \max(z, 0)$.
- $(\xi - x)^+$: shortage quantity.

Thus, the problem is the following:

$$\inf_{x \geqslant 0} \sup_{Q \in \mathcal{Q}_N} \mathbb{E}_Q h(x-\xi)^+ + b(\xi - x)^+$$

$\mathcal{Q}_N$ is the ambiguity set which is constructed using a Wasserstein ball (using the Wasserstein metric of order 1) and we consider the conic constraints approach presented before.

Numerical experiments

Data set and assumptions

Parameters of the model: $h = b = 20$.

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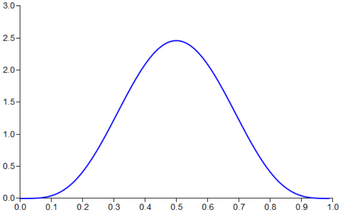
- The probability distribution is a Beta distribution $B(5,5)$



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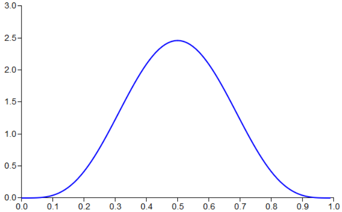

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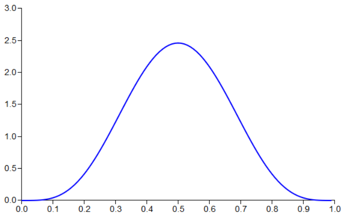

- The probability distribution is *unimodal*.

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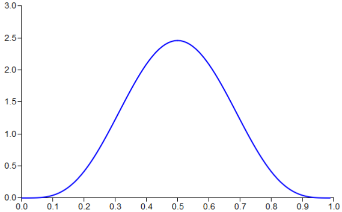


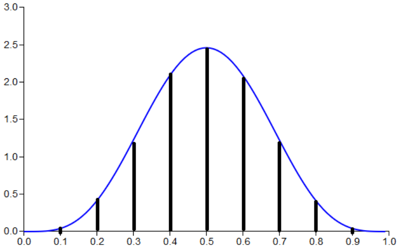
The decision-maker knows:

- The probability distribution is *unimodal*.
- The support set is the interval $[0, 1]$.

Numerical experiments

Data set

We construct K regions over the support set of the demand:

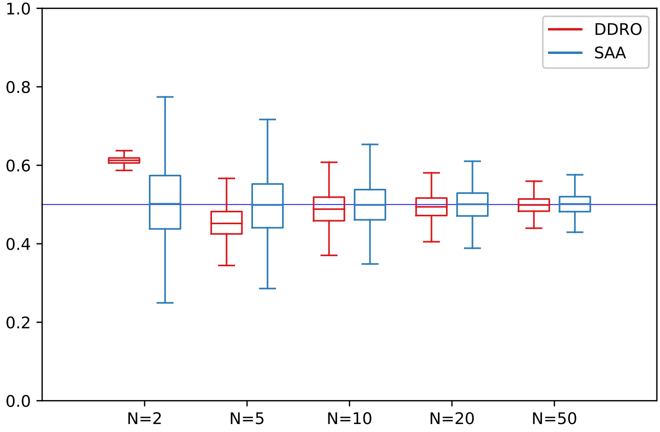


For each $i = 1, \ldots, K$, we assign a probability mass p_i to the i-th region. We consider the cone

$$\mathcal{C} = \{p \in \mathbb{R}^K : p_1 \leqslant p_2 \leqslant \ldots \leqslant p_m \geqslant p_{m+1} \geqslant \ldots \geqslant p_K\}$$

Numerical experiments

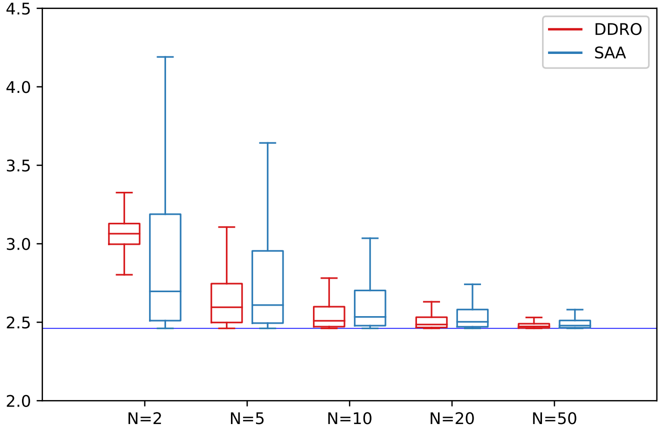
Order Quantity



Our DDRO approach obtains less variance

Numerical experiments

Actual Expected Cost



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Conclusions

- Shape information helps us to get better solutions than SAA method.

- Shape information is added in an easy way using conic constraints which becomes linear!!

Questions?

Thanks for the attention!

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