Semi-Global Predefined-Time Stable Vector Systems*

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***Abstract*— In this paper, we expose a control function which allows the semi-global predefined-time stabilization of first-order vector systems. The predefined-time stability is a stronger class of finite-time stability that has as main advantage the settling time as a tunable parameter of the proposed function. To design that stabilizing function, we use the unit control principle jointly to the inverse incomplete gamma function. For the resulting expression, the domain of definition the inverse incomplete gamma function can be made as large as wanted with an appropriate parameter selection, and, as consequence, the attraction domain of the systems. Therefore, we say that the system exhibits semi-global predefined-time stability. As an essential feature, the parameter which defines the settling time bound and those that tune the attraction domain are independent of each other. Finally, the constructed function is used to design predefined-time stabilizing controllers which are robust against vanishing and non-vanishing perturbations.**

I. INTRODUCTION

The high performance and the safety are standard requirements in several problems of control, observation, and optimization. For those cases, fast responses are, commonly, an essential specification of the design. The schemes based on finite-time stability [1]–[5] permit to solve some applications where is necessary to satisfy those hard design constraints. However, this finite time is an unbounded function of the initial conditions of the system. That makes the response times hard or impossible to calculate.

The elimination of this boundlessness present in the conventional finite-time approaches emerges as a desired feature to improve the characteristics of the closed-loop systems. As a response, there is a class of methods related to a stronger form of stability called fixed-time stability, where the settling-time function, is bounded. The references [6]–[10]. investigate the notion of fixed-time stability.

Although the fixed-time stability concept represents a significant advantage over finite-time stability, it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time. To overcome the above, another class of dynamical systems which exhibit the property of predefined-time stability, have been studied in [11]–[15]. For this systems, an upper bound of the fixed stabilization time appears explicitly in their tuning gains.

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Due to its importance, a vector extension of the scalar semi-global predefined-time stabilizing function [16] is proposed in this paper. The constructed function involves the inverse incomplete gamma function, causing this function to be semi-global. Finally, the stabilizing function is used to design predefined-time stabilizing controllers which are robust against vanishing and non-vanishing perturbations. Simulation examples are included.

II. PRELIMINARIES

This paper is a continuation of the research of the authors in predefined-time stability, in particular, it stands as an extension of the reference [16]. As a consequence, since it contains the essential definitions, this section has similar contents to others of the author reports, especially to Section II of the mentioned reference.

A. *On finite-time, fixed-time and predefined-time stability*

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{\rho}), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state, $\boldsymbol{\rho} \in \mathbb{R}^b$ represents the parameters of the system, $\boldsymbol{f} : \mathcal{D} \to \mathbb{R}^n$ is continuous on a neighborhood $\mathcal{D} \subseteq \mathbb{R}^n$ of the origin, and $\boldsymbol{f}(\boldsymbol{0}; \boldsymbol{\rho}) = \boldsymbol{0}$. The initial conditions of this system are $\boldsymbol{x}_0 = \boldsymbol{x}(0) \in \mathcal{D}$.

Definition 2.1 ([4]): The origin is said to be a *finite-time-stable equilibrium* of (1) if it is asymptotically stable and any solution $\boldsymbol{x}(t, \boldsymbol{x}_0) \in \mathcal{D}$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(\boldsymbol{x}_0) : \boldsymbol{x}(t, \boldsymbol{x}_0) = 0$, where $T : \mathcal{N} \to \mathbb{R}_+ \cup \{0\}$, with $\mathcal{N} \subseteq \mathcal{D}$ a neighborhood of the origin, is called the *settling-time function*.

The origin is said to be a *globally finite-time-stable equilibrium* if it is a finite-time-stable equilibrium with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Definition 2.2 ([10]): The origin is said to be a *fixed-time-stable equilibrium* of (1) if it is finite-time-stable and the settling-time function $T(\boldsymbol{x}_0)$ is bounded on $\mathcal{N}$, i.e. $\exists T_{\text{max}} > 0 : \forall \boldsymbol{x}_0 \in \mathcal{N} : T(\boldsymbol{x}_0) \leq T_{\text{max}}$.

The origin is said to be a *globally fixed-time stable equilibrium* if it is a fixed-time-stable equilibrium with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Remark 2.1: Note that there are several possible choices for T_{max}; for example, if $T(\boldsymbol{x}_0) \leq T_{\text{m}}$ for a positive number T_m, also $T(\boldsymbol{x}_0) \leq \lambda T_{\text{m}}$ with $\lambda \geq 1$. This motivates the definition of a set which contains all the bounds of the settling-time function.

Definition 2.3 ([11], [12]): Let the origin be fixed-time-stable for the system (1). The set of all the bounds of the settling-time function is defined as:

$$\mathcal{T} = \{T_{\max} \in \mathbb{R}_+ : T(\boldsymbol{x}_0) \leq T_{\mathrm{m}},\ \forall\, \boldsymbol{x}_0 \in \mathcal{N}\}\,. \qquad (2)$$

In addition, the least upper bound of the settling-time function, denoted by T_f, is defined as

$$T_f = \min \mathcal{T} = \sup_{\boldsymbol{x}_0 \in \mathcal{N}} T(\boldsymbol{x}_0). \qquad (3)$$

Remark 2.2: For several applications it could be desirable for system (1) to stabilize within a time $T_c \in \mathcal{T}$ which can be defined in advance as function of the system parameters, that is $T_c = T_c(\boldsymbol{\rho})$. The cases where this property is present motivate the definition of predefined-time stability. A strong notion of this class of stability is given when $T_c = T_f$, i.e., T_c is the true fixed-time in which the system stabilizes. A weak notion of predefined-time stability is presented when $T_c \geq T_f$, that is, if well it is possible to define an upper bound of the settling-time function in terms of the system parameters, this overestimates the true fixed-time in which the system stabilizes.

Definition 2.4 ([15]): For the system parameters ρ and a constant $T_c(\rho) > 0$, the origin is said to be

(i) *A weakly predefined-time-stable equilibrium* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathcal{N} \to \mathbb{R}$ is such that

$$T(\boldsymbol{x}_0) \leq T_c, \quad \forall \boldsymbol{x}_0 \in \mathcal{N}.$$

In this case, T_c is called a *weak predefined time*.

(ii) *A globally weakly predefined-time-stable equilibrium* for system (1) if it is a weakly predefined-time-stable equilibrium with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

(iii) *A strongly predefined-time-stable equilibrium* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathcal{N} \to \mathbb{R}$ is such that

$$\sup_{\boldsymbol{x}_0 \in \mathcal{N}} T(\boldsymbol{x}_0) = T_c.$$

In this case, T_c is called the *strong predefined time*.

(iv) *A globally strongly predefined-time-stable equilibrium* for system (1) if it is a strongly predefined-time-stable equilibrium with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Theorem 2.1 ([15]): Assume there exist a continuous function $V : \mathcal{D} \to \mathbb{R}$, real numbers $T_c = T_c(\boldsymbol{\rho}) > 0$ and $0 < p < 1$, and a neighborhood $\mathcal{V} \subseteq \mathcal{D}$ of the origin such that:

$$\begin{aligned} &V(\mathbf{0}) = 0 \\ &V(\boldsymbol{x}) > 0, \qquad \forall \boldsymbol{x} \in \mathcal{V} \setminus \{\mathbf{0}\}, \end{aligned}$$

and the derivative of V along the trajectories of the system (1) satisfies

$$\dot{V}(\boldsymbol{x}) \leq -\frac{1}{pT_c} \exp(V(\boldsymbol{x})^p) V(\boldsymbol{x})^{1-p}, \qquad \forall \boldsymbol{x} \in \mathcal{V} \setminus \{\mathbf{0}\}. \qquad (4)$$

Then, the origin is weakly predefined-time-stable for system (1), and a weak predefined time is T_c. If, in addition, $\mathcal{D} = \mathbb{R}^n$, V is radially unbounded, and (4) holds on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, then the origin is a globally weakly predefined-time-stable equilibrium of (1).

Theorem 2.1 characterizes weak predefined-time stability in a very practical way since the Lyapunov condition (4) directly involves a bound on the convergence time. Nevertheless, this condition is not enough to imply strong predefined-time stability. The following theorem provides a Lyapunov characterization for strong predefined-time stability:

Theorem 2.2 ([15]): Assume there exist a continuous function $V : \mathcal{D} \to \mathbb{R}$, real numbers $T_c = T_c(\boldsymbol{\rho}) > 0$ and $0 < p < 1$, and a neighborhood $\mathcal{V} \subseteq \mathcal{D}$ of the origin such that:

$$\begin{aligned} &V(\mathbf{0}) = 0 \\ &V(\boldsymbol{x}) > 0, \qquad \forall \boldsymbol{x} \in \mathcal{V} \setminus \{\mathbf{0}\}, \\ &\sup_{\boldsymbol{x} \in \mathcal{V}} V(\boldsymbol{x}) = \infty, \end{aligned}$$

and the derivative of V along the trajectories of the system (1) satisfies

$$\dot{V}(\boldsymbol{x}) = -\frac{1}{pT_c} \exp(V(\boldsymbol{x})^p) V(\boldsymbol{x})^{1-p}, \qquad \forall \boldsymbol{x} \in \mathcal{V} \setminus \{\mathbf{0}\}. \qquad (5)$$

Then, the origin is strongly predefined-time-stable for system (1), and the strong predefined time is T_c. If, in addition, $\mathcal{D} = \mathbb{R}^n$, V is radially unbounded, and (5) holds on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, then the origin is a globally strongly predefined-time-stable equilibrium of (1).

B. On the incomplete gamma function inverse

Recall the definition of the Gamma function:

Definition 2.5 ([17]): Let $a > 0$. The *Gamma* function is defined as

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) \mathrm{d}t. \qquad (6)$$

Remark 2.3: The Gamma function satisfies $\Gamma(a+1) = a\Gamma(a)$, which is called the *Functional equation*. Furthermore, note that

$$\Gamma(1) = \int_0^\infty \exp(-t) \mathrm{d}t = 1.$$

Then, for $n \in \mathbb{N}$

$$\Gamma(n+1) = 1 \cdot 2 \cdot \cdots \cdot n = n!.$$

Hence, the Gamma function can be viewed as an extension of the factorial function to positive real numbers.

Splitting the integral (6) at a point $x \geq 0$, two incomplete gamma functions are obtained. These incomplete gamma functions are of great interest in applied mathematics, which motivates the following definition.

Definition 2.6 ([17]): Let $a > 0$ and $x \geq 0$. The *incomplete gamma function* is defined as

$$\gamma(x; a) = \int_0^x t^{a-1} \exp(-t) \mathrm{d}t, \qquad (7)$$

and the *complementary incomplete gamma function* is defined as

$$\Gamma(x; a) = \int_x^\infty t^{a-1} \exp(-t) \mathrm{d}t.$$

Remark 2.4: Some properties concerning the incomplete gamma function (7) are:

(i) Clearly, the following decomposition of the Gamma function (6) is satisfied

$$\Gamma(a) = \gamma(x; a) + \Gamma(x; a).$$

(ii) Since the integrand $t^{a-1}\exp(-t)$ is nonnegative ($t \geq 0$), the incomplete gamma functions are also nonnegative, i.e.,

$$\gamma(x; a) \geq 0 \text{ and } \Gamma(x; a) \geq 0.$$

(iii) From *(i)* and *(ii)*, the incomplete gamma function is bounded above by the Gamma function, i.e.,

$$\gamma(x; a) \leq \Gamma(a).$$

Moreover, $\lim_{x\to\infty} \gamma(x; a) = \Gamma(a)$ (i.e., $y = \Gamma(a)$ is an horizontal asymptote of the function $\gamma(x; a)$).

(iv) Note that $\gamma(x; a) = 0$ if and only if $x = 0$. Furthermore,

$$\frac{\mathrm{d}\gamma(x; a)}{\mathrm{d}x} = x^{a-1}\exp(-x) > 0 \text{ for } x > 0.$$

Then, the function $\gamma(\cdot; a)$ is strictly increasing in $[0, \infty)$, and thus it is injective.

(v) From *(iii)* and *(iv)*, the incomplete gamma function $\gamma(\cdot; a) : [0, \infty) \to [0, \Gamma(a))$ is bijective. Then, there exists the *inverse incomplete gamma function*.

Definition 2.7: Let $a > 0$ and $x \geq 0$. The *incomplete gamma function inverse* $\gamma^{-1}(\cdot; a) : [0, \Gamma(a)) \to [0, \infty)$, is defined as the unique function satisfying $\gamma^{-1}(\gamma(x; a); a) = x$.

Remark 2.5: Some properties of the incomplete gamma function inverse in Definition 2.7 are:

(i) $\lim_{x\to\Gamma(a)^-} \gamma^{-1}(x; a) = \infty$ (i.e., $x = \Gamma(a)$ is a vertical asymptote of the function $\gamma^{-1}(x; a)$).

(ii) By the inverse function theorem,

$$\frac{\mathrm{d}\gamma^{-1}(x; a)}{\mathrm{d}x} = \left[\frac{\mathrm{d}\gamma(x; a)}{\mathrm{d}x}\right]^{-1} = \frac{\exp(x)}{x^{a-1}} > 0,$$

for $x \in (0, \Gamma(a))$. Thus, the function $\gamma^{-1}(\cdot; a)$ is strictly increasing in $(0, \Gamma(a))$.

(iii) From *(ii)*, for $a > 1$,

$$\lim_{x\to 0^+} \frac{\mathrm{d}\gamma^{-1}(x; a)}{\mathrm{d}x} = \infty.$$

Example 2.1: For $a = 5$ the plots of the incomplete gamma function and its inverse are shown in Fig. 1 and Fig. 2, respectively.

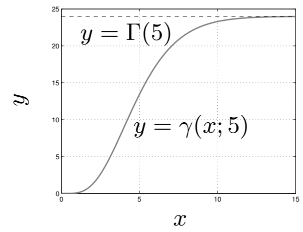

Fig. 1: Incomplete gamma function $\gamma(x; 5)$ (gray solid) and horizontal asymptote $\Gamma(5)$ (black dashed).

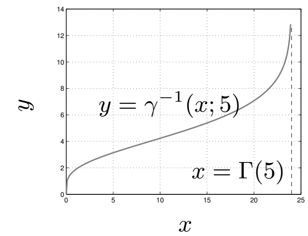

Fig. 2: Incomplete gamma function inverse $\gamma^{-1}(x; 5)$ (gray solid) and vertical asymptote $\Gamma(5)$ (black dashed).

Properties stated on Remarks 2.4 and 2.5 can be observed in Fig. 1 and Fig. 2.

For instance, the incomplete gamma function image is the interval $[0, \Gamma(5)) = [0, 24)$, it strictly is increasing in $[0, \infty)$, it is bijective and $\lim_{x\to\infty} \gamma(x; 5) = \Gamma(5) = 4! = 24$ (see Fig. 1).

On the other hand, the incomplete gamma function inverse domain is the interval $[0, \Gamma(5)) = [0, 24)$, it is strictly increasing in $(0, \Gamma(5))$, $\lim_{x\to\Gamma(5)^-} \gamma^{-1}(x; 5) = \infty$ and $\lim_{x\to 0^+} \frac{\mathrm{d}\gamma^{-1}(x;5)}{\mathrm{d}x} = \infty$ (see Fig. 2).

C. On semi-global predefined-time stable scalar systems

This subsection summarizes the construction on [16].

Definition 2.8 ([16]): Let $m \geq 1$ and $0 < p < 1$. The *semi-global predefined-time stabilizing function* $\phi_{m,p} : \mathcal{D}_{m,p} \subset \mathbb{R} \to \mathbb{R}$ is defined as:

$$\phi_{m,p}(x) := \left[\gamma^{-1}\left(|x|; \frac{1}{mp} + 1\right)\right]^{\frac{1}{mp}} \operatorname{sign}(x), \qquad (8)$$

with domain

$$\mathcal{D}_{m,p} := \left(-\Gamma\left(\frac{1}{mp} + 1\right), \Gamma\left(\frac{1}{mp} + 1\right)\right).$$

Lemma 2.1 ([16]): Let $m \geq 1$ and $0 < p < 1$. The function $\phi_{m,p}(x)$ (8) in Definition 2.8 satisfies the condition

$$\frac{\mathrm{d}\phi_{m,p}(x)}{\mathrm{d}x} = \frac{1}{mp}\exp(|\phi_{m,p}(x)|^{mp})\,|\phi_{m,p}(x)|^{-mp}.$$

Theorem 2.3 ([16]): Let $m \geq 1$, $0 < p < 1$ and $T_c > 0$. If $x_0 \in \mathcal{D}_{m,p}$, the origin of the system

$$\dot{x} = -\frac{1}{T_c}\phi_{m,p}(x)$$

is strongly predefined-time-stable with strong predefined time T_c.

A further result concerning the scalar semi-global predefined-time stabilizing function is obtained below. This will be used in the foregoing.

Let $m \geq 1$ and $0 < p < 1$ as in Definition 2.8. Note that the absolute value of the function $\phi_{m,p}(x)$ is

$$|\phi_{m,p}(x)| = \left[\gamma^{-1}\left(|x|; \frac{1}{mp} + 1\right)\right]^{\frac{1}{mp}}.$$

Then, using the chain rule the derivative of the above function is:

$$\begin{aligned}\frac{d\left|\phi_{m,p}(x)\right|}{dx} &= \operatorname{sign}(\phi_{m,p}(x))\frac{d\phi_{m,p}(x)}{dx} \\ &= \operatorname{sign}(x)\frac{d\phi_{m,p}(x)}{dx}.\end{aligned} \tag{9}$$

Now, consider the following function

$$\kappa_{m,p}(u) := \left[\gamma^{-1}\left(u;\frac{1}{mp}+1\right)\right]^{\frac{1}{mp}}, \tag{10}$$

and notice that $|\phi_{m,p}(x)| = \kappa_{m,p}(|x|)$.

Using the chain rule again, another expression of the derivative of the function $|\phi_{m,p}(x)|$ is

$$\frac{d\left|\phi_{m,p}(x)\right|}{dx} = \left.\frac{d\kappa_{m,p}(u)}{du}\right|_{u=|x|}\operatorname{sign}(x). \tag{11}$$

Hence, equating (9) and (11), it yields

$$\begin{aligned}\left.\frac{d\kappa_{m,p}(u)}{du}\right|_{u=|x|} &= \frac{d\phi_{m,p}(x)}{dx} \\ &= \frac{1}{mp}\exp(|\phi_{m,p}(x)|^{mp})\,|\phi_{m,p}(x)|^{-mp} \\ &= \frac{1}{mp}\exp\left(\gamma^{-1}\left(|x|\,;\frac{1}{mp}+1\right)\right)\times \\ &\qquad \left[\gamma^{-1}\left(|x|\,;\frac{1}{mp}+1\right)\right]^{-1}.\end{aligned}$$

The above analysis is summarized in the following definition and lemma.

Definition 2.9: Let $m \geq 1$ and $0 < p < 1$. The function $\kappa_{m,p} : \left[0, \Gamma(\frac{1}{mp}+1)\right) \to \mathbb{R}$ is defined as (10).

Lemma 2.2: Let $m \geq 1$ and $0 < p < 1$. The derivative of the function $\kappa_{m,p}$ in Definition 2.9 is

$$\begin{aligned}\frac{d\kappa_{m,p}(u)}{du} = \frac{1}{mp}\exp\left(\gamma^{-1}\left(u;\frac{1}{mp}+1\right)\right)\times \\ \left[\gamma^{-1}\left(u;\frac{1}{mp}+1\right)\right]^{-1}.\end{aligned} \tag{12}$$

III. Semi-Global Predefined-Time Stabilizing Vector Function

As an extension of the scalar semi-global predefined-time stabilizing function and based on the unit control algorithm [3], [18], we define the *vector semi-global predefined-time stabilizing function* as

Definition 3.1: Let $m \geq 1$ and $0 < p < 1$. The *semi-global predefined-time stabilizing function* $\boldsymbol{\phi}_{m,p} : \mathcal{D}_{m,p} \subset \mathbb{R}^n \to \mathbb{R}^n$ is defined as:

$$\boldsymbol{\phi}_{m,p}(\boldsymbol{x}) := \left[\gamma^{-1}\left(||\boldsymbol{x}||\,;\frac{1}{mp}+1\right)\right]^{\frac{1}{mp}}\frac{\boldsymbol{x}}{||\boldsymbol{x}||}, \tag{13}$$

with domain

$$\mathcal{D}_{m,p} := \left\{\boldsymbol{x} : ||\boldsymbol{x}|| < \Gamma\left(\frac{1}{mp}+1\right)\right\}. \tag{14}$$

Here, $||\cdot||$ stands for the euclidean 2-norm in $\mathbb{R}^n$.

Now, the main result of this paper is stated and proved in the following theorem.

Theorem 3.1: Let $m \geq 1$, $0 < p < 1$ and $T_c > 0$. If $\boldsymbol{x}_0 \in \mathcal{D}_{m,p} \subset \mathbb{R}^n$, the origin of the system

$$\dot{\boldsymbol{x}} = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x}) \tag{15}$$

is strongly predefined-time-stable with strong predefined time T_c.

Proof: Consider the positive definite Lyapunov function candidate $V(x) = ||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||^m$; its derivative along the trajectories of system (15) is

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= m\,||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||^{m-1}\frac{\partial\,||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||}{\partial\boldsymbol{x}}\left[-\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x})\right] \\ &= -\frac{m}{T_c}\,||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||^{m-1}\frac{\partial\,||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||}{\partial\boldsymbol{x}}\boldsymbol{\phi}_{m,p}(\boldsymbol{x}).\end{aligned} \tag{16}$$

Note that

$$||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})|| = \left[\gamma^{-1}\left(||\boldsymbol{x}||\,;\frac{1}{mp}+1\right)\right]^{\frac{1}{mp}} = \kappa_{m,p}(||\boldsymbol{x}||).$$

Hence, using the chain rule, the derivative $\frac{\partial||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||}{\partial\boldsymbol{x}}$ can be calculated as

$$\frac{\partial\,||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||}{\partial\boldsymbol{x}} = \left.\frac{\mathrm{d}\kappa_{m,p}(u)}{\mathrm{d}u}\right|_{u=||\boldsymbol{x}||}\frac{\boldsymbol{x}^T}{||\boldsymbol{x}||}. \tag{17}$$

Now, by Lemma 2.2

$$\begin{aligned}\left.\frac{\mathrm{d}\kappa_{m,p}(u)}{\mathrm{d}u}\right|_{u=||\boldsymbol{x}||} &= \frac{1}{mp}\exp\left(\gamma^{-1}\left(||\boldsymbol{x}||\,;\frac{1}{mp}+1\right)\right)\times \\ &\qquad \left[\gamma^{-1}\left(||\boldsymbol{x}||\,;\frac{1}{mp}+1\right)\right]^{-1} \\ &= \frac{1}{mp}\exp(||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||^{mp})\,||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||^{-mp} \\ &= \frac{1}{mp}\exp(V(\boldsymbol{x})^p)V(\boldsymbol{x})^{-p}.\end{aligned} \tag{18}$$

Finally, replacing (17) and (18) into (16), and noticing that $\frac{\boldsymbol{x}^T}{||\boldsymbol{x}||}\boldsymbol{\phi}_{m,p}(\boldsymbol{x}) = ||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||$, it yields

$$\dot{V}(\boldsymbol{x}) = -\frac{1}{T_c p}\exp(V(\boldsymbol{x})^p)V(\boldsymbol{x})^{1-p}.$$

From the above and Theorem 2.2, the origin is strongly predefined-time stable for system (15), with T_c as the strong predefined time. ■

Remark 3.1: The semi-global property refers to the fact that even though the region $\mathcal{D}_{m,p}$ (14) is a proper subset of $\mathbb{R}^n$, it can be made as large as wanted with an appropriate selection of the parameters m and p. For instance, for a given $m \geq 1$, select $p = \frac{1}{rm}$ with $r > 1$. Thus, with this selection $\mathcal{D}_{m,\frac{1}{rm}} = \{\boldsymbol{x} : ||\boldsymbol{x}|| < \Gamma(r+1)\}$. Since the Gamma function (6) grows very fast (even faster than exponential), so does this region. Furthermore, in the limit $r \to \infty$, $\mathcal{D}_{m,\frac{1}{rm}}$ becomes $\mathbb{R}^n$.

Remark 3.2: Note that the time parameter T_c is completely independent of the parameters m and p.

Remark 3.3: The function $||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||$ takes arbitrarily large values for $\boldsymbol{x}$ near the boundary of the region $\mathcal{D}_{m,p}$ (see

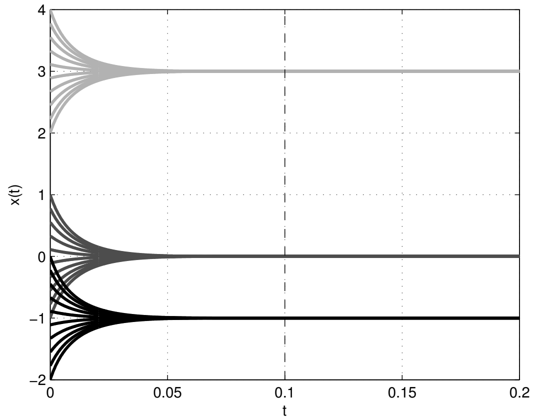

Fig. 3: Response of the system $\dot{\boldsymbol{x}} = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x}-\boldsymbol{x}_{ss})$ to several initial conditions.

Remark 2.5, part *(i)*). Hence, the parameters m and p should be selected carefully to ensure that $\boldsymbol{x}$ remains far enough of the points $\left\{\boldsymbol{x} : ||\boldsymbol{x}|| = \Gamma\left(\frac{1}{mp}+1\right)\right\}$, i.e., the boundary of $\mathcal{D}_{m,p}$.

Remark 3.4: Since predefined-time stability is a stronger form of finite-time stability, it can only be induced using non-smooth functions at the origin, due to the lack of uniqueness of the solutions in backward time once the equilibrium has been reached. From part *(iii)* of Remark 2.5, it can be noticed that the function $\boldsymbol{\phi}_{m,p}(\boldsymbol{x})$ is, in fact, non-smooth.

Example 3.1: Consider the system $\dot{\boldsymbol{x}} = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x} - \boldsymbol{x}_{ss})$, with $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{x}_{ss} = [3 \quad 0 \quad -1]^T$, $m = 1$, $p = \frac{1}{3}$ and $T_c = 0.1$ time units. Fig. 3 shows the trajectories of the system for several initial conditions. All these trajectories converge to the equilibrium point $\boldsymbol{x}_{ss}$ at least in the strong predefined time T_c.

IV. Robust First-Order Semi-Global Predefined-Time Vector Controllers

To apply the results in section III to robust first-order controller design, consider the dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{u} + \boldsymbol{\Delta}(t, \boldsymbol{x}) \tag{19}$$

with $\boldsymbol{x} \in \mathcal{D} \subseteq \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{\Delta} : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$. The objective is to stabilize system (19) at the origin in a (weak) predefined time T_c, starting from an arbitrary state $\boldsymbol{x}_0 = \boldsymbol{x}(0) \in \mathcal{D}$ and in spite of the unknown disturbance $\boldsymbol{\Delta}(t, \boldsymbol{x})$.

Theorem 4.1: Let the function $\boldsymbol{\Delta}(t, \boldsymbol{x})$ be considered as a vanishing perturbation term such that $||\boldsymbol{\Delta}(t, \boldsymbol{x})|| \le \delta\, ||\boldsymbol{x}||$, with $0 < \delta < \infty$ a known constant. Selecting the control input as

$$\boldsymbol{u} = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x}) - k\boldsymbol{x} \tag{20}$$

with $T_c > 0$, $m \ge 1$, $0 < p < 1$, and $k \ge \delta$. If $\boldsymbol{x}_0 \in \mathcal{D}_{m,p}$, then the origin is weakly predefined-time-stable for system (19) closed by (20), with T_c as the weak predefined time.

Proof: Consider the positive definite Lyapunov function candidate $V(\boldsymbol{x}) = ||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||^m$. Following similar steps to that in the proof of Theorem 3.1, its derivative along the trajectories of system (19) closed by (20) is

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= m\, ||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||^{m-1} \frac{\partial\, ||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||}{\partial \boldsymbol{x}} \Big[-\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x}) - \\ &\qquad k\boldsymbol{x} + \boldsymbol{\Delta}(t, \boldsymbol{x}) \Big] \\ &= -\frac{1}{T_c p}\exp(V(\boldsymbol{x})^p)V(\boldsymbol{x})^{1-p} - \\ &\qquad \frac{1}{p}\psi(\boldsymbol{x})\frac{\boldsymbol{x}^T}{||\boldsymbol{x}||}\left[k\boldsymbol{x} - \boldsymbol{\Delta}(t, \boldsymbol{x})\right],\end{aligned}$$

with $\psi(\boldsymbol{x}) = ||\boldsymbol{\phi}_{m,p}(\boldsymbol{x})||^{m-1}\exp(V(\boldsymbol{x})^p)V(\boldsymbol{x})^{-p} > 0$ for all $\boldsymbol{x} \in \mathcal{D}_{m,p}$.

Furthermore, using the Cauchy-Schwarz inequality, the above becomes

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &\le -\frac{1}{T_c p}\exp(V(\boldsymbol{x})^p)V(\boldsymbol{x})^{1-p} - \\ &\qquad \frac{1}{p}\psi(\boldsymbol{x})\left[k\, ||\boldsymbol{x}|| - \frac{\left|\boldsymbol{x}^T\boldsymbol{\Delta}(t, \boldsymbol{x})\right|}{||\boldsymbol{x}||}\right] \\ &\le -\frac{1}{T_c p}\exp(V(\boldsymbol{x})^p)V(\boldsymbol{x})^{1-p} - \\ &\qquad \frac{1}{p}\psi(\boldsymbol{x})\left[k\, ||\boldsymbol{x}|| - \frac{||\boldsymbol{x}||\, ||\boldsymbol{\Delta}(t, \boldsymbol{x})||}{||\boldsymbol{x}||}\right] \\ &\le -\frac{1}{T_c p}\exp(V(\boldsymbol{x})^p)V(\boldsymbol{x})^{1-p} - \\ &\qquad \frac{1}{p}\psi(\boldsymbol{x})\, ||\boldsymbol{x}||\, [k - \delta] \\ &\le -\frac{1}{T_c p}\exp(V(\boldsymbol{x})^p)V(\boldsymbol{x})^{1-p}.\end{aligned}$$

From the above and Theorem 2.1, the origin is weakly predefined-time stable for system (19) closed by (20), with T_c as a weak predefined time. ■

Example 4.1: Consider the system $\dot{\boldsymbol{x}} = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x} - \boldsymbol{x}_{ss}) - k(\boldsymbol{x} - \boldsymbol{x}_{ss}) + \boldsymbol{\Delta}(t, \boldsymbol{x} - \boldsymbol{x}_{ss})$, with $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{x}_{ss} = [3 \quad 0 \quad -1]^T$, $m = 1$, $p = \frac{1}{3}$ and $T_c = 0.1$ time units. The perturbation term is chosen as $\boldsymbol{\Delta}(t, \boldsymbol{x} - \boldsymbol{x}_{ss}) = \frac{\sin(||\boldsymbol{x}-\boldsymbol{x}_{ss}||)}{\sqrt{3}}[1 \quad 1 \quad 1]^T$ (recall that $|\sin(||x||)| \le ||x||$). Fig. 4 shows the trajectories of the system for several initial conditions. All these trajectories converge to the equilibrium point $\boldsymbol{x}_{ss}$ at least in the strong predefined time T_c.

Theorem 4.2: Let the function $\boldsymbol{\Delta}(t, \boldsymbol{x})$ be considered as a non-vanishing bounded perturbation such that $||\boldsymbol{\Delta}(t, \boldsymbol{x})|| \le \delta$, with $0 < \delta < \infty$ a known constant. Selecting the control input as

$$u = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x}) - k\frac{\boldsymbol{x}}{||\boldsymbol{x}||} \tag{21}$$

with $T_c > 0$, $m \ge 1$, $0 < p < 1$, and $k \ge \delta$. If $\boldsymbol{x}_0 \in \mathcal{D}_{m,p}$, then the origin is weakly predefined-time-stable for system (19) closed by (21), with T_c as the weak predefined time.

Proof: Similar to the proof of Theorem 4.1. ■

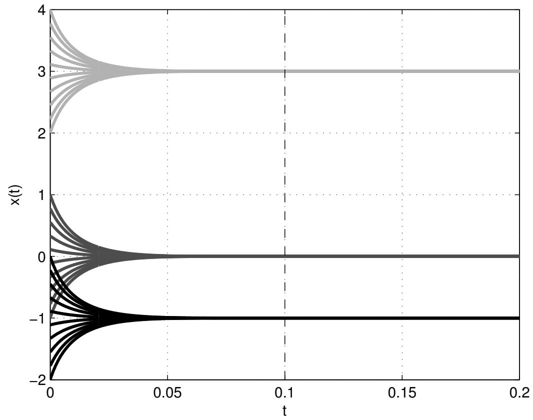

Fig. 4: Response of the system $\dot{\boldsymbol{x}} = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x} - \boldsymbol{x}_{ss}) - k(\boldsymbol{x} - \boldsymbol{x}_{ss}) + \boldsymbol{\Delta}(t, \boldsymbol{x} - \boldsymbol{x}_{ss})$ to several initial conditions.

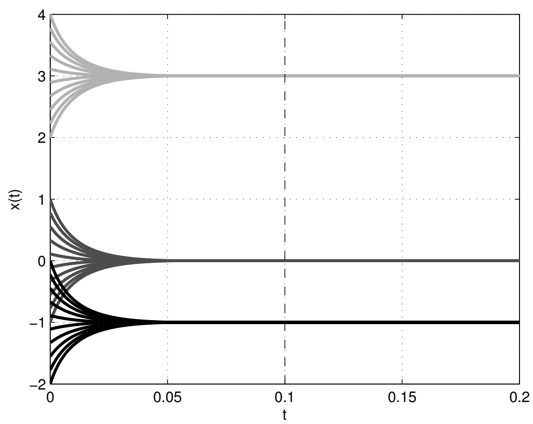

Fig. 5: Response of the system $\dot{\boldsymbol{x}} = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x} - \boldsymbol{x}_{ss}) - k\frac{(\boldsymbol{x}-\boldsymbol{x}_{ss})}{||\boldsymbol{x}-\boldsymbol{x}_{ss}||} + \boldsymbol{\Delta}(t, \boldsymbol{x} - \boldsymbol{x}_{ss})$ to several initial conditions.

Remark 4.1: The controller (21) in Theorem 4.2 contains a discontinuous term to cancel the effect of the non-vanishing perturbation term.

Example 4.2: Consider the system $\dot{\boldsymbol{x}} = -\frac{1}{T_c}\boldsymbol{\phi}_{m,p}(\boldsymbol{x} - \boldsymbol{x}_{ss}) - k\frac{(\boldsymbol{x}-\boldsymbol{x}_{ss})}{||\boldsymbol{x}-\boldsymbol{x}_{ss}||} + \boldsymbol{\Delta}(t, \boldsymbol{x} - \boldsymbol{x}_{ss})$, with $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{x}_{ss} = [3 \quad 0 \quad -1]^T$, $m = 1$, $p = \frac{1}{3}$ and $T_c = 0.1$ time units. The perturbation term is chosen as $\boldsymbol{\Delta}(t, \boldsymbol{x} - \boldsymbol{x}_{ss}) = \frac{\sin(t)}{\sqrt{3}}[1 \quad 1 \quad 1]^T$ (recall that $|\sin(t)| \leq 1$). Fig. 5 shows the trajectories of the system for several initial conditions. All these trajectories converge to the equilibrium point $\boldsymbol{x}_{ss}$ at least in the strong predefined time T_c.

V. Conclusion

A vector semi-global predefined-time stabilizing function is proposed in this paper as an extension of the scalar semi-global predefined-time stabilizing function and based on the unit control algorithm. As an important remark, the predefined-time parameter could be defined independently of the other controller parameters. Finally, the constructed function was used to design predefined-time stabilizing controllers, robust against vanishing and non-vanishing perturbations. Simulation examples were included.

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