Semi-Global Predefined-Time Stable Systems

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***Abstract*—A Lyapunov-based construction of a predefined-time stabilizing function (a function that stabilizes a system in fixed-time with settling time as function of the controller parameters) for scalar systems is considered in this paper. The constructed function involves the inverse incomplete gamma function, causing this function to be semi-global, i.e., the domain of definition of the function can be made as large as wanted with an appropriate parameter selection. Finally, the constructed function is used to design predefined-time stabilizing controllers which are robust against vanishing and non-vanishing perturbations.**

I. Introduction

The various developments concerning the concept of *finite-time stability* permit to solve different applications which are characterized for requiring hard time response constraints. Some important works on this topic and its application to control systems have been carried out in [1]–[5].

However, generally this finite time is an unbounded function of the initial conditions of the system. A desired feature is to eliminate this boundlessness, for example, in estimation or optimization problems. This gives rise to a stronger form of stability called *fixed-time stability*, where the settling-time function, is bounded. The notion of fixed-time stability have been investigated in [6]–[10].

Although the fixed-time stability concept represents a significant advantage over finite-time stability, it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time. To overcome the above, another class of dynamical systems which exhibit the property of *predefined-time stability*, have been studied [11]–[15]. For this systems, an upper bound (sometimes the least upper bound) of the fixed stabilization time appears explicitly in their tuning gains.

Due to its importance, a Lyapunov-based construction of a predefined-time stabilizing function for scalar systems is considered in this paper. The constructed function involves the inverse incomplete gamma function, causing this function to be semi-global. Finally, the constructed function is used to design predefined-time stabilizing controllers which are robust against vanishing and non-vanishing perturbations. Simulation examples are included.

II. Preliminaries

A. On finite-time, fixed-time and predefined-time stability

Consider the system

$$\dot{x} = f(x;\rho), \qquad x_0 = x(0), \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\rho \in \mathbb{R}^b$ represents the parameters of the system, $f : \mathcal{D} \to \mathbb{R}^n$ is continuous on a neighborhood $\mathcal{D} \subseteq \mathbb{R}^n$ of the origin, and $f(0;\rho) = 0$. The initial conditions of this system are $x_0 = x(0) \in \mathcal{D}$.

Definition 1 (*Finite-time stability* [4]). The origin is said to be a *finite-time-stable equilibrium* of (1) if it is asymptotically stable and any solution $x(t, x_0) \in \mathcal{D}$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(x_0) : x(t, x_0) = 0$, where $T : \mathcal{N} \to \mathbb{R}_+ \cup \{0\}$, with $\mathcal{N} \subseteq \mathcal{D}$ a neighborhood of the origin, is called the *settling-time function*.

The origin is said to be a *globally finite-time-stable equilibrium* if it is a finite-time-stable equilibrium with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Definition 2 (*Fixed-time stability* [10]). The origin is said to be a *fixed-time-stable equilibrium* of (1) if it is finite-time-stable and the settling-time function $T(x_0)$ is bounded on $\mathcal{N}$, i.e. $\exists T_{\text{max}} > 0 : \forall x_0 \in \mathcal{N} : T(x_0) \leq T_{\text{max}}$.

The origin is said to be a *globally fixed-time stable equilibrium* if it is a fixed-time-stable equilibrium with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Remark 1. Note that there are several possible choices for T_{max}; for example, if $T(x_0) \leq T_{\text{m}}$ for a positive number T_m, also $T(x_0) \leq \lambda T_{\text{m}}$ with $\lambda \geq 1$. This motivates the definition of a set which contains all the bounds of the settling-time function.

Definition 3 (*Settling-time set and its minimum bound* [11], [12]). Let the origin be fixed-time-stable for the system (1). The set of all the bounds of the settling-time function is defined as:

$$\mathcal{T} = \{T_{\text{max}} \in \mathbb{R}_+ : T(x_0) \leq T_{\text{m}},\ \forall\, x_0 \in \mathcal{N}\}\,. \tag{2}$$

In addition, the least upper bound of the settling-time function, denoted by T_f, is defined as

$$T_f = \min \mathcal{T} = \sup_{x_0 \in \mathcal{N}} T(x_0). \tag{3}$$

Remark 2. For several applications it could be desirable for system (1) to stabilize within a time $T_c \in \mathcal{T}$ which can be defined in advance as function of the system parameters, that is $T_c = T_c(\boldsymbol{\rho})$. The cases where this property is present motivate the definition of predefined-time stability. A strong notion of this class of stability is given when $T_c = T_f$, i.e., T_c is the true fixed-time in which the system stabilizes. A weak notion of predefined-time stability is presented when $T_c \geq T_f$, that is, if well it is possible to define an upper bound of the settling-time function in terms of the system parameters, this overestimates the true fixed-time in which the system stabilizes.

Definition 4 (*Predefined-time stability* [15])**.** For the system parameters ρ and a constant $T_c(\rho) > 0$, the origin is said to be

(i) *A weakly predefined-time-stable equilibrium* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathcal{N} \to \mathbb{R}$ is such that

$$T(x_0) \leq T_c, \quad \forall x_0 \in \mathcal{N}.$$

In this case, T_c is called a *weak predefined time*.

(ii) *A globally weakly predefined-time-stable equilibrium* for system (1) if it is a weakly predefined-time-stable equilibrium with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

(iii) *A strongly predefined-time-stable equilibrium* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathcal{N} \to \mathbb{R}$ is such that

$$\sup_{x_0 \in \mathcal{N}} T(x_0) = T_c.$$

In this case, T_c is called the *strong predefined time*.

(iv) *A globally strongly predefined-time-stable equilibrium* for system (1) if it is a strongly predefined-time-stable equilibrium with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Theorem 1 (*Lyapunov characterization of weak predefined–time stability* [15])**.** *Assume there exist a continuous function $V : \mathcal{D} \to \mathbb{R}$, real numbers $T_c = T_c(\rho) > 0$ and $0 < p < 1$, and a neighborhood $\mathcal{V} \subseteq \mathcal{D}$ of the origin such that:*

$$V(0) = 0$$
$$V(x) > 0, \qquad \forall x \in \mathcal{V} \setminus \{0\},$$

and the derivative of V along the trajectories of the system (1) *satisfies*

$$\dot{V}(x) \leq -\frac{1}{pT_c} \exp(V(x)^p) V(x)^{1-p}, \qquad \forall x \in \mathcal{V} \setminus \{0\}. \tag{4}$$

Then, the origin is weakly predefined-time-stable for system (1)*, and a weak predefined time is T_c. If, in addition, $\mathcal{D} = \mathbb{R}^n$, V is radially unbounded, and* (4) *holds on $\mathbb{R}^n \setminus \{0\}$, then the origin is a globally weakly predefined-time-stable equilibrium of* (1).

Theorem 1 characterizes weak predefined-time stability in a very practical way since the Lyapunov condition (4) directly involves a bound on the convergence time. Nevertheless, this condition is not enough to imply strong predefined-time stability. The following theorem provides a Lyapunov characterization for strong predefined-time stability:

Theorem 2 (*Lyapunov characterization of strong predefined–time stability* [15])**.** *Assume there exist a continuous function $V : \mathcal{D} \to \mathbb{R}$, real numbers $T_c = T_c(\rho) > 0$ and $0 < p < 1$, and a neighborhood $\mathcal{V} \subseteq \mathcal{D}$ of the origin such that:*

$$V(0) = 0$$
$$V(x) > 0, \qquad \forall x \in \mathcal{V} \setminus \{0\},$$

and the derivative of V along the trajectories of the system (1) *satisfies*

$$\dot{V}(x) = -\frac{1}{pT_c} \exp(V(x)^p) V(x)^{1-p}, \qquad \forall x \in \mathcal{V} \setminus \{0\}. \tag{5}$$

Then, the origin is strongly predefined-time-stable for system (1)*, and the strong predefined time is T_c. If, in addition, $\mathcal{D} = \mathbb{R}^n$, V is radially unbounded, and* (5) *holds on $\mathbb{R}^n \setminus \{0\}$, then the origin is a globally strongly predefined-time-stable equilibrium of* (1).

B. On the incomplete gamma function inverse

Recall the definition of the Gamma function:

Definition 5 (*Gamma function* [16])**.** Let $a > 0$. The *Gamma* function is defined as

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) \mathrm{d}t. \tag{6}$$

Remark 3. The Gamma function satisfies $\Gamma(a+1) = a\Gamma(a)$, which is called the *Functional equation*. Furthermore, note that

$$\Gamma(1) = \int_0^\infty \exp(-t) \mathrm{d}t = 1.$$

Then, for $n \in \mathbb{N}$

$$\Gamma(n+1) = 1 \cdot 2 \cdot \cdots \cdot n = n!.$$

Hence, the Gamma function can be viewed as an extension of the factorial function to positive real numbers.

Splitting the integral (6) at a point $x \geq 0$, two incomplete gamma functions are obtained. These incomplete gamma functions are of great interest in applied mathematics, which motivates the following definition.

Definition 6 (*Incomplete gamma functions* [16])**.** Let $a > 0$ and $x \geq 0$. The *incomplete gamma function* is defined as

$$\gamma(x; a) = \int_0^x t^{a-1} \exp(-t) \mathrm{d}t, \tag{7}$$

and the *complementary incomplete gamma function* is defined as

$$\Gamma(x; a) = \int_x^\infty t^{a-1} \exp(-t) \mathrm{d}t.$$

Remark 4. Some properties concerning the incomplete gamma function (7) are:

(i) Clearly, the following decomposition of the Gamma function (6) is satisfied

$$\Gamma(a) = \gamma(x;a) + \Gamma(x;a).$$

(ii) Since the integrand $t^{a-1}\exp(-t)$ is nonnegative ($t \geq 0$), the incomplete gamma functions are also nonnegative, i.e.,

$$\gamma(x;a) \geq 0 \text{ and } \Gamma(x;a) \geq 0.$$

(iii) From *(i)* and *(ii)*, the incomplete gamma function is bounded above by the Gamma function, i.e.,

$$\gamma(x;a) \leq \Gamma(a).$$

Moreover, $\lim_{x\to\infty}\gamma(x;a) = \Gamma(a)$ (i.e., $y = \Gamma(a)$ is an horizontal asymptote of the function $\gamma(x;a)$).

(iv) Note that $\gamma(x;a) = 0$ if and only if $x = 0$. Furthermore,

$$\frac{\mathrm{d}\gamma(x;a)}{\mathrm{d}x} = x^{a-1}\exp(-x) > 0 \text{ for } x > 0.$$

Then, the function $\gamma(\cdot;a)$ is strictly increasing in $[0,\infty)$, and thus it is injective.

(v) From *(iii)* and *(iv)*, the incomplete gamma function $\gamma(\cdot;a) : [0,\infty) \to [0,\Gamma(a))$ is bijective. Then, there exists the *inverse incomplete gamma function*.

Definition 7 (*Incomplete gamma function inverse*)**.** Let $a > 0$ and $x \geq 0$. The *incomplete gamma function inverse* $\gamma^{-1}(\cdot;a) : [0,\Gamma(a)) \to [0,\infty)$, is defined as the unique function satisfying $\gamma^{-1}(\gamma(x;a);a) = x$.

Remark 5. Some properties of the incomplete gamma function inverse in Definition 7 are:

(i) $\lim_{x\to\Gamma(a)^-}\gamma^{-1}(x;a) = \infty$ (i.e., $x = \Gamma(a)$ is a vertical asymptote of the function $\gamma^{-1}(x;a)$).

(ii) By the inverse function theorem,

$$\frac{\mathrm{d}\gamma^{-1}(x;a)}{\mathrm{d}x} = \left[\frac{\mathrm{d}\gamma(x;a)}{\mathrm{d}x}\right]^{-1} = \frac{\exp(x)}{x^{a-1}} > 0,$$

for $x \in (0,\Gamma(a))$. Thus, the function $\gamma^{-1}(\cdot;a)$ is strictly increasing in $(0,\Gamma(a))$.

(iii) From *(ii)*, for $a > 1$,

$$\lim_{x\to 0^+}\frac{\mathrm{d}\gamma^{-1}(x;a)}{\mathrm{d}x} = \infty.$$

Example 1. For $a = 5$ the plots of the incomplete gamma function and its inverse are shown in Fig. 1 and Fig. 2, respectively.

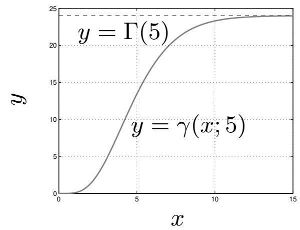

Fig. 1: Incomplete gamma function $\gamma(x;5)$ (gray solid) and horizontal asymptote $\Gamma(5)$ (black dashed).

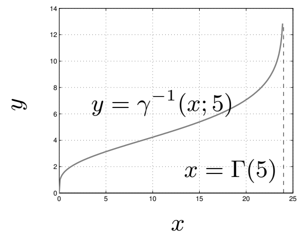

Fig. 2: Incomplete gamma function inverse $\gamma^{-1}(x;5)$ (gray solid) and vertical asymptote $\Gamma(5)$ (black dashed).

Properties stated on Remarks 4 and 5 can be observed in Fig. 1 and Fig. 2.

For instance, the incomplete gamma function image is the interval $[0,\Gamma(5)) = [0,24)$, it strictly is increasing in $[0,\infty)$, it is bijective and $\lim_{x\to\infty}\gamma(x;5) = \Gamma(5) = 4! = 24$ (see Fig. 1).

On the other hand, the incomplete gamma function inverse domain is the interval $[0,\Gamma(5)) = [0,24)$, it is strictly increasing in $(0,\Gamma(5))$, $\lim_{x\to\Gamma(5)^-}\gamma^{-1}(x;5) = \infty$ and $\lim_{x\to 0^+}\frac{\mathrm{d}\gamma^{-1}(x;5)}{\mathrm{d}x} = \infty$ (see Fig. 2).

C. Odd extension of the power function

Definition 8. Let $h \geq 0$. For $x \in \mathbb{R}$, define the function

$$\lfloor x \rceil^h = |x|^h \operatorname{sign}(x),$$

with $\operatorname{sign}(x) = 1$ for $x > 0$, $\operatorname{sign}(x) = -1$ for $x < 0$ and $\operatorname{sign}(0) \in [-1,1]$. Furthermore, it is defined $\lfloor 0 \rceil^h = 0$ for $h > 0$, so that $\lfloor x \rceil^h$ is continuous in $\mathbb{R}$ for $h > 0$.

Remark 6. For $x \in \mathbb{R}$, some properties of the function $\lfloor \cdot \rceil^h$ are:

(i) $\lfloor x \rceil^0 = \operatorname{sign}(x)$.

(ii) $\lfloor x \rceil^1 = \lfloor x \rceil = x$,

(iii) $\frac{d|x|^h}{dx} = h\lfloor x \rceil^{h-1}$ and $\frac{d\lfloor x \rceil^h}{dx} = h|x|^{h-1}$.

(iv) For $h_1, h_2 \in \mathbb{R}$, it follows:
- $|x|^{h_1}|x|^{h_2} = |x|^{h_1+h_2}$
- $\lfloor x \rceil^{h_1}|x|^{h_2} = |x|^{h_1}\lfloor x \rceil^{h_2} = \lfloor x \rceil^{h_1+h_2}$
- $\lfloor x \rceil^{h_1}\lfloor x \rceil^{h_2} = |x|^{h_1+h_2}$

(v) For $h_1, h_2 > 0$, then $\left\lfloor \lfloor x \rceil^{h_1} \right\rceil^{h_2} = \lfloor x \rceil^{h_1 h_2}$.

III. Lyapunov-Based Construction of a Semi-Global Predefined-Time Stabilizing Function

Consider the scalar system

$$\dot{x} = -k\phi(x), \tag{8}$$

where $x \in \mathbb{R}$ is the system state, $\phi : \mathcal{D} \to \mathbb{R}$ is an odd continuous function to be constructed such that the origin is a predefined-time-stable equilibrium of (8), and $k > 0$ is a parameter.

Remark 7. Two immediate properties of the function ϕ are:

(i) $\phi(x) = 0$ if and only if $x = 0$, since the origin is the only desired equilibrium of (8).

(ii) $\text{sign}(\phi(x)) = \text{sign}(x)$. If this were not the case, the origin would not be a stable equilibrium of (8).

Let $m \geq 1$ and consider the Lyapunov function candidate $V : \mathcal{D} \to \mathbb{R}$, defined by

$$V(x) = |\phi(x)|^m . \tag{9}$$

Note that it is, in fact, a positive definite function. This is, $V(0) = 0$ and $V(x) = |\phi(x)|^m > 0$ for $x \in \mathcal{D} \setminus \{0\}$.

Now, the derivative of (9) along the trajectories of system (8) is

$$\begin{aligned}\dot{V}(x) &= -km \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} |\phi(x)|^m \\ &= -km \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} V(x).\end{aligned} \tag{10}$$

A sufficient condition for the origin to be a predefined-time stable equilibrium of (8), can be obtained through Theorem 2, equating the right sides of the equations (10) and (5), as follows:

$$\begin{aligned}-km \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} V(x) &= -\frac{1}{pT_c} \exp(V(x)^p) V(x)^{1-p} \\ \Leftrightarrow \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} &= \frac{1}{kmpT_c} \exp(V(x)^p) V(x)^{-p} \\ \Leftrightarrow \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} &= \frac{1}{kmpT_c} \exp(|\phi(x)|^{mp}) |\phi(x)|^{-mp} .\end{aligned} \tag{11}$$

Thus, if a function $\phi(x)$ can be found such that the condition (11) is satisfied together with the requirement of $\phi(0) = 0$ (part *(i)* of Remark 7), a predefined-time stabilizing function would have been constructed.

An integral version of (12) is

$$\int_0^{\phi(x)} \exp(-|\xi|^{mp}) |\xi|^{mp} \mathrm{d}\xi = \int_0^x \frac{1}{kmpT_c} \mathrm{d}y. \tag{12}$$

Using the substitution $u = |\xi|^{mp}$ on the left side of (12), a solution to this integral is

$$\int_0^{\phi(x)} \exp(-|\xi|^{mp}) |\xi|^{mp} \mathrm{d}\xi = \frac{1}{mp} \gamma \left(|\phi(x)|^{mp} ; \frac{1}{mp} + 1 \right) \text{sign}(\phi(x)). \tag{13}$$

Then, from (12) and (13), a function ϕ that satisfies

$$\frac{1}{mp} \gamma \left(|\phi(x)|^{mp} ; \frac{1}{mp} + 1 \right) \text{sign}(\phi(x)) = \frac{1}{kmpT_c} x$$

makes the origin a predefined-time-stable equilibrium of (8). From part *(ii)* of Remark 7 and taking $k = \frac{1}{T_c}$ for simplicity, the above is equivalent to

$$\gamma \left(|\phi(x)|^{mp} ; \frac{1}{mp} + 1 \right) = |x| . \tag{14}$$

Finally, a function ϕ satisfying (14), i.e., an odd continuous predefined-time stabilizing function for (8) is

$$\phi(x) = \phi_{m,p}(x) := \left[\gamma^{-1} \left(|x| ; \frac{1}{mp} + 1 \right) \right]^{\frac{1}{mp}} \text{sign}(x). \tag{15}$$

Remark 8. By Definition 7, the constructed function (15) is defined for x such that $|x| < \Gamma\left(\frac{1}{mp} + 1\right)$. Thus,

$$\mathcal{D} = \mathcal{D}_{m,p} := \left(-\Gamma\left(\frac{1}{mp} + 1\right), \Gamma\left(\frac{1}{mp} + 1\right) \right). \tag{16}$$

All the above construction of the function (15) is summarized in the following definition, lemma and theorem.

Definition 9 (*Semi-global predefined-time stabilizing function*)**.** Let $m \geq 1$, $0 < p < 1$ and $T_c > 0$. The *semi-global predefined-time stabilizing function* $\phi_{m,p}(\cdot; T_c) : \mathcal{D}_{m,p} \to \mathbb{R}$ is defined as (15).

Lemma 1. *Let $m \geq 1$ and $0 < p < 1$. The function $\phi_{m,q}(x)$ (15) in Definition 9 satisfies the condition*

$$\frac{d\phi_{m,q}(x)}{dx} = \frac{1}{mp} \exp(|\phi_{m,q}(x)|^{mp}) |\phi_{m,q}(x)|^{-mp} .$$

Theorem 3 (*Semi-global predefined-time stable system*)**.** *Let $m \geq 1$, $0 < p < 1$ and $T_c > 0$. If $x_0 \in \mathcal{D}_{m,p}$, the origin of the system*

$$\dot{x} = -\frac{1}{T_c} \phi_{m,p}(x)$$

is strongly predefined-time-stable with strong predefined time T_c.

Remark 9. The semi-global property refers to the fact that even though the region $\mathcal{D}_{m,p}$ (16) is a proper subset of $\mathbb{R}$, it can be made as large as wanted with an appropriate selection of the parameters m and p. For instance, for a given $m \geq 1$, select $p = \frac{1}{rm}$ with $r > 1$. Thus, with this selection

$$\mathcal{D}_{m,\frac{1}{rm}} = (-\Gamma(r+1), \Gamma(r+1)).$$

Since the Gamma function (6) grows very fast (even faster than exponential), so does this region. Furthermore, in the limit $r \to \infty$, $\mathcal{D}_{m,\frac{1}{rm}}$ becomes $\mathbb{R}$.

Remark 10. Note that the time parameter T_c is completely independent of the parameters m and p.

Remark 11. The function $\phi_{m,p}(x)$ takes arbitrarily large values for x sufficiently near the boundary of the region $\mathcal{D}_{m,p}$ (see Remark 5, part *(i)*). Hence, the parameters m and p should be selected carefully to ensure that x remains far enough of the points $\left\{-\Gamma\left(\frac{1}{mp} + 1\right), \Gamma\left(\frac{1}{mp} + 1\right)\right\}$, i.e., the boundary of $\mathcal{D}_{m,p}$.

Remark 12. Since predefined-time stability is a stronger form of finite-time stability, it can only be induced using non-smooth functions at the origin, due to the lack of uniqueness of the solutions in backward time once the equilibrium has been reached. From part *(iii)* of Remark 5, it can be noticed that the function $\phi_{m,q}(x)$ is, in fact, non-smooth.

Example 2. The parameters are selected as $m = 1$, $p = \frac{1}{3}$ and $T_c = 0.5$. With this selection, the region $\mathcal{D}_{m,p}$ becomes

$$\mathcal{D}_{1,\frac{1}{3}} = (-\Gamma(4), \Gamma(4)) = (-6, 6).$$

Then, the system $\dot{x} = -\frac{1}{T_c}\phi_{m,p}(x)$ is simulated for several initial conditions in the interval $[-5.9, 5.9]$ using the Euler integration method with fundamental step size of $t_s = 5 \times 10^{-4}$.

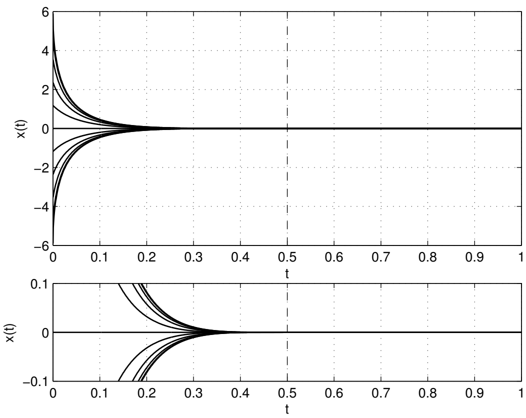

Fig. 3: Response of the system $\dot{x} = -\frac{1}{T_c}\phi_{m,p}(x)$ to several initial conditions within $[-5.9, 5.9]$.

It can be seen in Fig. 3 that every solution converges to the origin in at most $T_c = 0.5$ time units.

IV. Robust First-Order Semi-Global Predefined-Time Controllers

In order to apply the results in section III to robust first-order controller design, consider the dynamical system

$$\dot{x} = u + \Delta(t, x) \tag{17}$$

with $x \in \mathcal{D}$, $u \in \mathbb{R}$ and $\Delta : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}$. The is to stabilize system (17) at the origin in a (weak) predefined time T_c, starting from an arbitrary state $x_0 = x(0) \in \mathcal{D}$ and in spite of the unknown disturbance $\Delta(t, x)$.

Theorem 4 (A weak predefined-time controller robust against vanishing perturbations)**.** *Let the function $\Delta(t, x)$ be considered as a vanishing perturbation term such that $|\Delta(t, x)| \leq \delta |x|$, with $0 < \delta < \infty$ a known constant. Selecting the control input*

$$u = -\frac{1}{T_c}\phi_{m,p}(x) - kx \tag{18}$$

with $T_c > 0$, $m \geq 1$, $0 < p < 1$, and $k \geq \delta$. If $x_0 \in \mathcal{D}_{m,p}$, then the origin is weakly predefined-time-stable for system (17) *closed by* (18)*, with T_c as the weak predefined time.*

Proof. Consider the positive definite Lyapunov function candidate $V(x) = |\phi(x)|^m$; its derivative along the trajectories of system (17) closed by (18) is

$$\dot{V} = m\frac{\mathrm{d}\phi_{m,p}(x)}{\mathrm{d}x}\left\lfloor \phi_{m,p}(x) \right\rceil^{m-1}\left[-\frac{1}{T_c}\phi_{m,p}(x) - kx + \Delta\right].$$

Furthermore, from Lemma 1, the above becomes

$$\begin{aligned}
\dot{V} &= -\frac{1}{T_c p}\exp(V^p)V^{1-p} \\
&\quad - kx\left\lfloor \phi_{m,p}(x) \right\rceil^{m-1} + \Delta\left\lfloor \phi_{m,p}(x) \right\rceil^{m-1} \\
&\leq -\frac{1}{T_c p}\exp(V^p)V^{1-p} \\
&\quad - k|x|\left|\phi_{m,p}(x)\right|^{m-1} + |\Delta|\left|\phi_{m,p}(x)\right|^{m-1} \\
&\leq -\frac{1}{T_c p}\exp(V^p)V^{1-p} \\
&\quad - k|x|\left|\phi_{m,p}(x)\right|^{m-1} + \delta|x|\left|\phi_{m,p}(x)\right|^{m-1} \\
&= -\frac{1}{T_c p}\exp(V^p)V^{1-p} - (k-\delta)|x|\left|\phi_{m,p}(x)\right|^{m-1} \\
&\leq -\frac{1}{T_c p}\exp(V^p)V^{1-p}.
\end{aligned}$$

From the above and Theorem 1, the origin is weakly predefined-time stable for system (17) closed by (18), with T_c as the weak predefined time. □

Example 3. The parameters are selected as $m = 1$, $p = \frac{1}{3}$, $T_c = 0.5$ and $k = 1.1$. Then, the system (17) closed by (18), with $\Delta(t, x) = \sin(x)$ ($|\sin(x)| \leq |x|$), is simulated for several initial conditions within the interval $[-5.9, 5.9]$ using the Euler integration method with fundamental step size of $t_s = 5 \times 10^{-4}$.

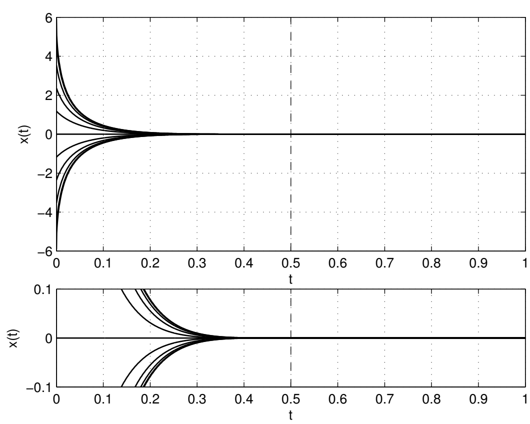

Fig. 4: Response of the system (17) closed by (18), with $\Delta(t, x) = \sin(x)$, to several initial conditions within $[-5.9, 5.9]$.

It can be seen in Fig. 4 that every solution converges to the origin in less than $T_c = 0.5$ time units.

Theorem 5 (A weak predefined-time controller robust against non-vanishing perturbations)**.** *Let the function $\Delta(t, x)$ be considered as a non-vanishing bounded perturbation such that $|\Delta(t, x)| \leq \delta$, with $0 < \delta < \infty$ a known constant. Selecting the control input*

$$u = -\frac{1}{T_c}\phi_{m,p}(x) - k\,sign(x) \tag{19}$$

with $T_c > 0$, $m \geq 1$, $0 < p < 1$, and $k \geq \delta$. If $x_0 \in \mathcal{D}_{m,p}$, then the origin is weakly predefined-time-stable for system (17) *closed by* (19)*, with T_c as the weak predefined time.*

Proof. Similar to the proof of Theorem 5. □

Remark 13. The controller (19) in Theorem 5 contains a discontinuous term to cancel the effect of the non-vanishing perturbation term.

Example 4. The parameters are selected as $m = 1$, $p = \frac{1}{3}$, $T_c = 0.5$ and $k = 1.1$. Then, the system (17) closed by (19), with $\Delta(t, x) = \sin(t)$ ($|\sin(t)| \leq 1$), is simulated for several initial conditions within the interval $[-5.9, 5.9]$ using the Euler integration method with fundamental step size of $t_s = 5 \times 10^{-4}$.

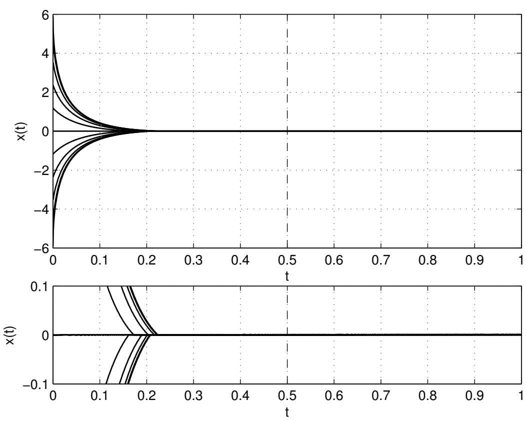

Fig. 5: Response of the system (17) closed by (19), with $\Delta(t, x) = \sin(t)$, to several initial conditions within $[-5.9, 5.9]$.

It can be seen in Fig. 5 that every solution converges to the origin in less than $T_c = 0.5$ time units.

V. Conclusion

A Lyapunov-based construction of a predefined-time stabilizing function for scalar systems was considered in this paper. The constructed function arose as a solution of an initial value problem, yielding an expression involving the inverse incomplete gamma function. As an important remark, the predefined-time parameter could be defined independently of the other controller parameters. Finally, the constructed function was used to design predefined-time stabilizing controllers, robust against vanishing and non-vanishing perturbations. Simulation examples were included.

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References

[1] E. Roxin, "On finite stability in control systems," *Rendiconti del Circolo Matematico di Palermo*, vol. 15, no. 3, pp. 273–282, 1966.

[2] V. Haimo, "Finite time controllers," *SIAM Journal on Control and Optimization*, vol. 24, no. 4, pp. 760–770, 1986.

[3] V. I. Utkin, *Sliding Modes in Control and Optimization*. Springer Verlag, 1992.

[4] S. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[5] E. Moulay and W. Perruquetti, "Finite time stability conditions for non-autonomous continuous systems," *International Journal of Control*, vol. 81, no. 5, pp. 797–803, 2008. [Online]. Available: http://www.tandfonline.com/doi/abs/10.1080/00207170701650303

[6] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous approximation, recursive observer design, and output feedback," *SIAM Journal on Control and Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

[7] E. Cruz-Zavala, J. Moreno, and L. Fridman, "Uniform second-order sliding mode observer for mechanical systems," in *Variable Structure Systems (VSS), 2010 11th International Workshop on*, june 2010, pp. 14 –19.

[8] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2106–2110, 2012.

[9] L. Fraguela, M. Angulo, J. Moreno, and L. Fridman, "Design of a prescribed convergence time uniform robust exact observer in the presence of measurement noise," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, Dec 2012, pp. 6615–6620.

[10] A. Polyakov and L. Fridman, "Stability notions and Lyapunov functions for sliding mode control systems," *Journal of the Franklin Institute*, vol. 351, no. 4, pp. 1831 – 1865, 2014, special Issue on 2010-2012 Advances in Variable Structure Systems and Sliding Mode Algorithms.

[11] J. D. Sánchez-Torres, E. N. Sánchez, and A. G. Loukianov, "A discontinuous recurrent neural network with predefined time convergence for solution of linear programming," in *IEEE Symposium on Swarm Intelligence (SIS)*, 2014, pp. 9–12.

[12] J. D. Sanchez-Torres, E. N. Sanchez, and A. G. Loukianov, "Predefined-time stability of dynamical systems with sliding modes," in *American Control Conference (ACC), 2015*, July 2015, pp. 5842–5846.

[13] E. Jimenez-Rodriguez, J. D. Sanchez-Torres, D. Gomez-Gutierrez, and A. G. Loukianov, "Predefined-Time tracking of a class of mechanical systems," in *2016 13th International Conference on Electrical Engineering,Computing Science and Automatic Control, CCE 2016*, sep 2016, pp. 1–5.

[14] E. Jiménez-Rodríguez, J. D. Sánchez-Torres, and A. G. Loukianov, "On optimal predefined-time stabilization," *International Journal of Robust and Nonlinear Control*, pp. n/a—-n/a, 2017. [Online]. Available: http://dx.doi.org/10.1002/rnc.3757

[15] J. D. Sánchez-Torres, D. Gómez-Gutiérrez, E. López, and A. G. Loukianov, "A class of predefined-time stable dynamical systems," *IMA Journal of Mathematical Control and Information*, 2017.

[16] H. Bateman and A. Erdlyi, *Higher transcendental functions*, ser. Calif. Inst. Technol. Bateman Manuscr. Project. New York, NY: McGraw-Hill, 1955.