

On Optimal Predefined-Time Stabilization

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SUMMARY

This paper addresses the problem of optimal predefined-time stability. Predefined-time stable systems are a class of fixed-time stable dynamical systems for which the minimum bound of the settling-time function can be defined a priori as an explicit parameter of the system. Sufficient conditions for a controller to solve the optimal predefined-time stabilization problem for a given nonlinear system are provided. These conditions involve a Lyapunov function that satisfies a certain differential inequality for guaranteeing predefined-time stability. It also satisfies the steady-state Hamilton-Jacobi-Bellman equation for ensuring optimality. Furthermore, for nonlinear affine systems and a certain class of performance index, a family of optimal predefined-time stabilizing controllers is derived. This class of controllers is applied to optimize the sliding manifold reaching phase in predefined time, considering both the unperturbed and perturbed case. For the perturbed case, the idea of integral sliding mode control is jointly used to ensure robustness. Finally, as a study case, the predefined-time optimization of the sliding manifold reaching phase in a pendulum system is performed using the developed methods, and numerical simulations are carried out to show their behavior. Copyright © 2016 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Several applications of dynamical systems are characterized by requiring hard-time response constraints. For instance, the design of control, observation and optimization algorithms at some industrial applications are required to achieve high performance of their time response to match certain quality or safety standards. In order to deal with these requirements, various developments concerning to the concepts of *finite-time stability* and *deadbeat control* have been carried out (see for example: [1–17]).

These time response constraints arise also in more sophisticated applications of finite-time stability in some common problems of control systems design. One fundamental case is the design of sliding mode algorithms since they are based on driving the trajectories of a dynamical system to a certain desired manifold in a limited time period [9, 18]. Another basic case is the observer design for nonlinear systems; here, finite-time convergence of the observer is a desired property since it guarantees that the separation principle holds when used in combination with a global controller. This principle allows the separate design of observers and controllers, making the closed-loop system stability analysis easier. This advantage of finite-time convergent estimation is a remarkable

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feature, since the study of the separation principle in nonlinear systems is often a challenging task. Furthermore, in the fault detection and isolation problem, the finite-time convergence property is of paramount importance to guarantee the mode detection in a timely manner in order to apply a recovery action [19], since there are situations where a late response may lead to a no recovery scenario.

Nevertheless, the settling time of finite-time stable systems is often an unbounded function of the initial conditions. This drawback can make difficult, even impossible, the appropriate design of observers and controllers satisfying some desired time constraints. Consequently, it is important and useful to make this function bounded to ensure the settling time is less than a certain quantity for any initial condition. With this purpose a stronger form of stability, called *fixed-time stability*, was introduced by [20] for homogeneous systems and it was proposed in [21–23] for systems with sliding modes. The settling time of fixed-time stable systems presents certain uniformity with respect to the initial conditions.

When fixed-time stable dynamical systems are applied to control or observation tasks, it may be difficult to find a direct relationship between the gains of the system and the upper bound of the convergence time; thus, tuning the system in order to achieve a desired maximum stabilization time is not a trivial task. A simulation-based approximation to select the values of the tuning parameters is proposed in [24] under the concept of *prescribed-time stability*; this method permits the design of robust sliding differentiators for noisy signals, by expressing the gains as functions of the desired settling time. Therefore, prescribed-time stable systems present a way to surmount the tuning problem. However, this prescribed time usually constitutes a conservative estimation of the upper bound of the convergence time; that is, the prescribed time is commonly larger, maybe quite larger, than the *true* amount of time the system takes to converge.

To overcome the above problems, a class of dynamical systems where the least upper bound of the fixed stabilization time equals a tunable parameter were proposed and defined as *predefined-time stable systems* in [25, 26]. Predefined-time stability is strongly related to the continuous deadbeat control; for example, a classical case of predefined-time stable controllers are those based on the posicast method [1, 2], where part of the input command is delayed to achieve deadbeat control. Several cases of the application of inputs with delayed parts in order to achieve deadbeat control are presented by [27, 28] for the stabilization of continuous systems and by [29–33] for observer design. In most of this approaches, the predefined stabilization time is equal to the control input delay.

In contrast to most of the fixed-time stable systems, in those presented in [25, 26], for the unperturbed case, the bound on the convergence time is not a conservative estimation but truly the minimum value that is greater than all the possible exact settling times. In addition, this bound is not based on simulations due to the fact that all the mentioned properties are characterized by a suitable Lyapunov theorem. Moreover, the system structure contains no delay terms, making its analysis and design easier when compared to the mentioned deadbeat methods.

As previously desired, the upper bound for the convergence time of the proposed class of systems appears explicitly in their dynamical equations; in particular, it equals a system parameter. This maximum stabilization time also appears in the main condition required by the proposed Lyapunov theorem that characterizes the discussed family of systems.

On the other hand, the infinite-horizon nonlinear optimal asymptotic stabilization problem was addressed in [34–36]. The main idea of the results is based on the condition that a Lyapunov function for the nonlinear system is at the same time the solution of the steady-state Hamilton-Jacobi-Bellman equation, guaranteeing both asymptotic stability and optimality. Nevertheless, returning to the opening paragraph idea, the finite-time stability is a desired property in some applications, but optimal finite-time controllers obtained using the maximum principle do not generally yield feedback controllers. In this sense, the optimal finite-time stabilization is studied in [37], as an extension of [36]. Since the results are based on the framework developed in [36], the controllers obtained are in fact feedback controllers.

Consequently, as an extension of the ideas presented in [36–38], this paper addresses the problem of *optimal predefined-time stabilization* (see [38]), namely the problem of finding a state-feedback

control that minimizes certain performance measure, guaranteeing at the same time predefined-time stability of the closed-loop system. In particular, sufficient conditions for a controller to solve the optimal predefined-time stabilization problem for a given system are provided. These conditions involve a Lyapunov function that satisfies both a certain differential inequality for guaranteeing predefined-time stability and the steady-state Hamilton-Jacobi-Bellman equation for ensuring optimality.

Finally, this result is applied to the predefined-time optimization of the sliding manifold reaching phase considering both the unperturbed and the perturbed case. For the unperturbed case, the developed result is used directly, while for the perturbed case it is used jointly with the idea of integral sliding mode control [39–43] to provide robustness.

In the following, Section 2 presents the mathematical preliminaries needed to introduce the proposed results. Section 3 exposes the main results of this paper, which are the sufficient conditions for a controller to solve the optimal predefined-time stabilization problem and the particularization to affine systems. Section 4 shows the application of the obtained results to the predefined-time inverse optimization of the sliding manifold reaching phase. This is illustrated in a pendulum system model in Section 5. Finally, Section 6 presents the conclusions of this paper.

2. MATHEMATICAL PRELIMINARIES

2.1. Predefined-Time Stability

Consider the system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function such that $f(0) = 0$, i.e. the origin $x = 0$ is an equilibrium point of (1).

First, the concepts of finite-time, fixed-time and predefined-time stability are reviewed.

Definition 2.1 (Global finite-time stability [22])

The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(x_0) : x(t, x_0) = 0$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$.

Remark 2.1

The settling-time function $T(x_0)$ for systems with a finite-time stable equilibrium point, is usually an unbounded function of the system initial condition. This is illustrated in Example 2.1.

Example 2.1 (Globally finite-time stable scalar system)

Consider the scalar system $\dot{x} = -|x|^{\frac{1}{3}} \text{sign}(x)$ with initial condition $x(0) = x_0 \in \mathbb{R}$. Its solution is given by

$$x(t) = \begin{cases} \left(|x_0|^{\frac{2}{3}} - \frac{2}{3}t \right)^{\frac{3}{2}} \text{sign}(x_0) & \text{if } 0 \leq t \leq T(x_0) \\ 0 & \text{if } t > T(x_0), \end{cases}$$

where $T(x_0) = \frac{3}{2} |x_0|^{\frac{2}{3}}$. Thus, the origin is a globally finite-time stable equilibrium for this system.

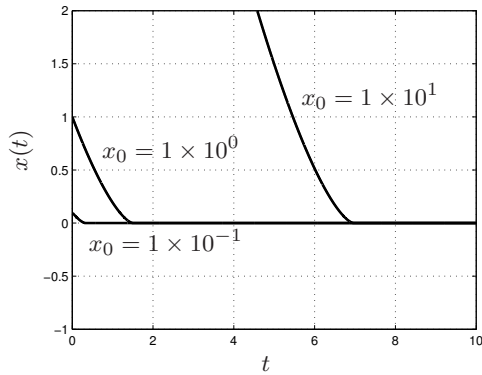


Figure 1. Solutions $x(t)$ for different initial conditions.

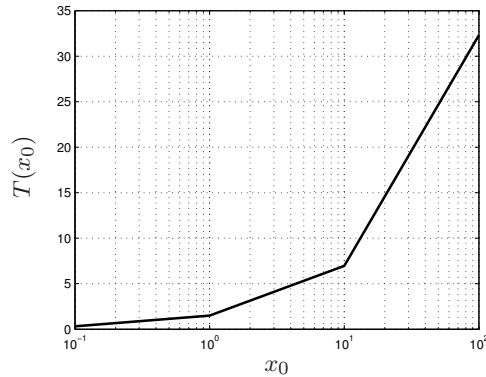


Figure 2. Settling-time function $T(x_0)$. Note that this function is unbounded.

It can be seen, from Figure 2, that the settling-time function $T(x_0)$ is finite for every x_0 , however, it is unbounded.

Definition 2.2 (Fixed-time stability [22])

The origin of the system (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}$.

Remark 2.2

The settling-time function $T(x_0)$ for systems with a fixed-time stable equilibrium point, presents certain uniformity with respect to the initial conditions. However, it is hard to find a direct correspondence between the parameters of the system and a desired upper bound of the convergence time. This is illustrated in Example 2.2.

Example 2.2 (Fixed-time stable scalar system)

Consider the scalar system $\dot{x} = -|x|^{\frac{1}{3}} \text{sign}(x) - |x|^3 \text{sign}(x)$ with initial condition $x(0) = x_0 \in \mathbb{R}$. The origin is a fixed-time equilibrium for this system, and it can be proved that $T_{\max} = 2.5$ is a bound for the settling-time function $T(x_0)$, using Lyapunov analysis [22].

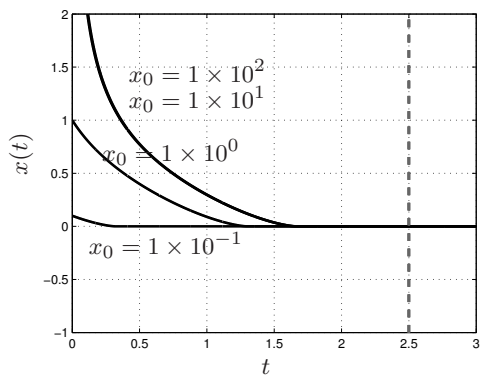


Figure 3. Solutions $x(t)$ for different initial conditions.

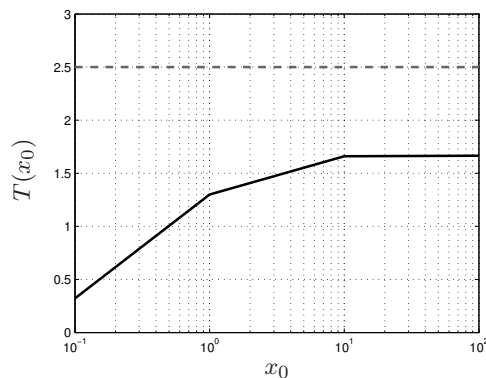


Figure 4. Settling-time function $T(x_0)$. Note that this function is bounded.

It can be seen, from Figure 4, that the settling-time function $T(x_0)$ is bounded. However, there is no a direct relation between the bound T_{\max} and the system parameters, and also, $T_{\max} = 2.5$ is a conservative bound for $T(x_0) < 1.7$.

Remark 2.3

Note that there are several choices for T_{\max} . For instance, if the settling-time function is bounded by T_m , it is also bounded by λT_m for all $\lambda \geq 1$. This motivates the following definition.

Definition 2.3 ([25, 26])

Let \mathcal{T} be the set of all the bounds of the settling-time function for the system (1), i.e.,

$$\mathcal{T} = \{T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}\}. \quad (2)$$

Then, the minimum bound of the settling-time function T_f , is defined as

$$T_f = \inf \mathcal{T} = \sup_{x_0 \in \mathbb{R}^n} T(x_0). \quad (3)$$

Remark 2.4

In a strict sense, the time T_f can be considered as the true fixed-time in which the system (1) is stabilized.

Remark 2.5

When T_{\max} or T_f can be expressed in terms of the system (1) parameters, it will be referred to the notion of *predefined-time stability*, since the fixed time in which the system (1) is stabilized can be defined in advance. The following two definitions are intended to characterize the aforementioned cases.

Definition 2.4 (Strongly predefined-time stable systems [25, 26])

For the case of fixed time stability when the time T_f defined in (3) can be expressed in terms of the system (1) parameters, it is said that the origin of the system (1) is *strongly predefined-time stable*.

The strong notion of predefined-time stability given in Definition 2.4 requires complete knowledge about the system. Therefore, the given notion can not be easily applied to uncertain systems since T_f is the minimal upper-bound of the settling-time function. Hence, it is presented also a notion of weak predefined-time stability when an upper-bound of the settling-time function, depending only on the parameters of the system known part, is available. The uncertain system part is considered as bounded, and the bound is known.

Definition 2.5 (Weakly predefined-time stable systems)

For the case of fixed time stability, when the time T_{\max} defined in (3) can be expressed in terms of the parameters of the system (1) known part, it is said that the origin of the system (1) is *weakly predefined-time stable*.

In some cases, Lyapunov functions used for stability analysis are required to satisfy some smoothness conditions. In this sense, the first differentiability class of functions C^1 is introduced.

Definition 2.6

Let $\mathcal{D} \subset \mathbb{R}^n$ be an open set and $f : \mathcal{D} \rightarrow \mathbb{R}$. It is said that the function f is (of differentiability class) C^1 in \mathcal{D} if all of the first-order partial derivatives exist and are continuous in \mathcal{D} .

With the above definition, the following lemma provides the following Lyapunov condition for weak predefined-time stability of the origin:

Lemma 2.1 (Lyapunov characterization of weak predefined-time stability [25, 26])

Assume there exist a C^1 radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, and real numbers $\alpha > 0$ and $0 < p \leq 1$, such that

$$V(0) = 0 \quad (4)$$

$$V(x) > 0, \quad \forall x \neq 0, \quad (5)$$

and the time derivative of V along the trajectories of the system (1) satisfies

$$\dot{V} \leq -\frac{\alpha}{p} \exp(V^p) V^{1-p}. \quad (6)$$

Then, the origin of the system (1) is weakly predefined-time stable and $T_{\max} = \frac{1}{\alpha}$.

Lemma 2.1 characterizes fixed-time stability in a very practical way since the Lyapunov condition (6) directly involves a bound on the convergence time. Nevertheless, this condition is not enough to imply strong predefined-time stability. The following corollary provides a Lyapunov characterization for strong predefined-time stability:

Lemma 2.2 (Lyapunov characterization of strong predefined-time stability [25, 26])

Assume there exist a C^1 radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, and real numbers $\alpha > 0$ and $0 < p \leq 1$, such that

$$V(0) = 0 \quad (7)$$

$$V(x) > 0, \quad \forall x \neq 0, \quad (8)$$

and the time derivative of V along the trajectories of the system (1) satisfies

$$\dot{V} = -\frac{\alpha}{p} \exp(V^p) V^{1-p}. \quad (9)$$

Then, the origin of the system (1) is strongly predefined-time stable and $T_f = \frac{1}{\alpha}$.

Definition 2.7 ([25, 26])

For $x \in \mathbb{R}^n$, the *predefined-time stabilizing function* is defined as

$$\Phi_p(x; T_c) = \frac{1}{T_c p} \exp(\|x\|^p) \frac{x}{\|x\|^p} \quad (10)$$

where $0 < p \leq 1$ and $T_c > 0$.

Remark 2.6

The function $\Phi_p(x; T_c)$ is continuous and non-Lipschitz for $0 < p < 1$, and discontinuous for $p = 1$.

The following two lemmas give meaning to the name "predefined-time stabilizing function".

Lemma 2.3 ([25, 26])

For every initial condition x_0 , the origin of the system

$$\dot{x}(t) = -\Phi_p(x(t); T_c), \quad x(0) = x_0, \quad (11)$$

with $T_c > 0$, and $0 < p \leq 1$ is **strongly** predefined-time stable with $T_f = T_c$, i.e., $x(t) = 0$ for $t > t_0 + T_c$ in spite of the x_0 value.

Example 2.3 (Strongly predefined-time stable scalar system)

Consider the scalar system $\dot{x} = -\Phi_p(x; T_c)$ with $p = 1/2$, $T_c = 1$ and initial condition $x(0) = x_0 \in \mathbb{R}$. From Lemma 2.3, it follows that the origin is a strongly predefined-time stable equilibrium for this system and $T_f = T_c = 1$.

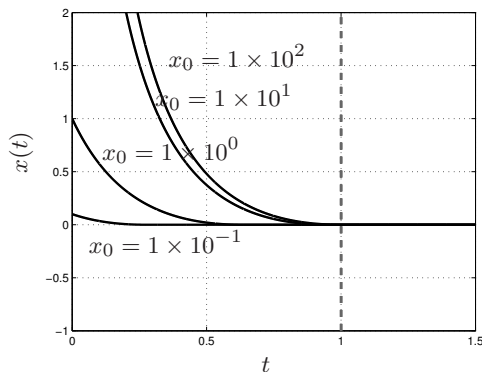


Figure 5. Solutions $x(t)$ for different initial conditions.

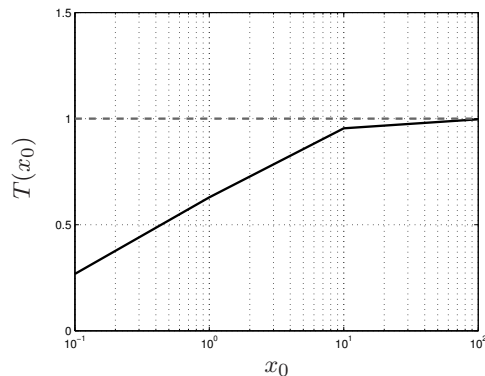


Figure 6. Settling-time function $T(x_0)$. Note that this function is bounded and $\sup T(x_0) = T_c$.

It can be seen, from Figure 6, that the settling-time function $T(x_0)$ is bounded. Moreover, the supreme of the settling-time function $T(x_0)$ corresponds to the parameter $T_c = 1$.

The previous results have been applied to design a robust predefined-time controller for the perturbed system

$$\dot{x}(t) = \Delta(t, x) + u(t), \quad x(0) = x_0, \tag{12}$$

with $x, u \in \mathbb{R}^n$ and $\Delta : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

The objective is to drive the system (12) state to the point $x = 0$ in a predefined-time in spite of the unknown perturbation $\Delta(t, x)$.

Lemma 2.4 ([25, 26])

Let the function $\Delta(t, x)$ be considered as an unknown non-vanishing perturbation bounded by

$$\|\Delta(t, x)\| \leq \delta, \quad \delta > 0.$$

Then, selecting the control input as

$$u = -k \frac{x}{\|x\|} - \Phi_p(x; T_c) \tag{13}$$

with $T_c > 0$, $0 < p < 1$ and $k \geq \delta$, ensures the closed-loop system (12)-(13) origin is weakly predefined-time stable with settling-time $T_{\max} = T_c$.

Example 2.4 (Weakly predefined-time stable scalar system)

Consider the scalar system $\dot{x} = -k \text{sign}(x) - \Phi_p(x; T_c) + \sin(t)$ with $p = 1/2$, $T_c = 1$, $k = 1$ and initial condition $x(0) = x_0 \in \mathbb{R}$. From Lemma 2.4, the origin is a weakly predefined-time stable equilibrium for this system with $T_{\max} = T_c = 1 \geq T(x_0)$.

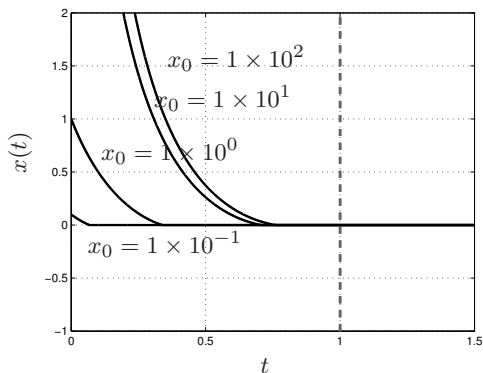


Figure 7. Solutions $x(t)$ for different initial conditions.

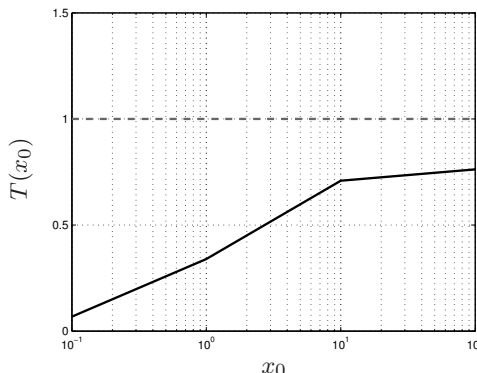


Figure 8. Settling-time function $T(x_0)$. Note that this function is bounded and $T(x_0) \leq T_c$.

It can be seen, from Figure 8, that the settling-time function $T(x_0)$ is bounded. Moreover, an upper bound of the settling-time function $T(x_0)$ corresponds to the parameter $T_c = 1$. However, T_c is not the least upper bound of the settling-time function $T(x_0)$.

Remark 2.7

As mentioned before, the exact knowledge of the system is required to apply the strong notion of predefined-time stability. However, in practical cases, this is never accomplished. Consequently, in the foregoing, the predefined-time stability notion will make reference to the weak one.

2.2. Optimal Control

Consider the controlled nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \tag{14}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the system control input, which is restricted to belong to a certain set $\mathcal{U} \subset \mathbb{R}^m$ of the admissible controls, and $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a nonlinear function with $f(0, 0) = 0$.

The control objective is to design a control law for the system (14) such that the following performance measure

$$J(x_0, u(\cdot)) = \int_0^{t_f} L(x(t), u(t)) dt, \quad (15)$$

is minimized. Here, $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a continuous function, assumed to be convex in u .

Define the minimum cost function $J^*(x(t), t)$ as

$$J^*(x(t), t) = \min_{u \in \mathcal{U}} \left\{ \int_t^{t_f} L(x(\tau), u(\tau)) d\tau \right\}. \quad (16)$$

Then, defining the *Hamiltonian*, for $p \in \mathbb{R}^n$ called usually as the costate,

$$\mathcal{H}(x, u, p) = L(x, u) + p^T f(x, u), \quad (17)$$

the *Hamilton-Jacobi-Bellman* (HJB) equation can be written as

$$0 = \min_{u \in \mathcal{U}} \left\{ \mathcal{H} \left(x, u, \frac{\partial J^*(x, t)}{\partial x} \right) \right\} + \frac{\partial J^*(x, t)}{\partial t}, \quad (18)$$

that provides a sufficient condition for optimality.

For infinite-horizon problems (limit as $t_f \rightarrow \infty$), the cost does not depend on t anymore and the partial differential equation (18) reduces to the steady-state HJB equation

$$0 = \min_{u \in \mathcal{U}} \mathcal{H} \left(x, u, \frac{\partial J^*(x)}{\partial x} \right). \quad (19)$$

which will be used in foregoing.

3. OPTIMAL PREDEFINED-TIME STABILIZATION

The main result of the paper is presented in this section. First, the notion of optimal predefined-time stabilization is defined.

Definition 3.1

Consider the optimal control problem for the system (14)

$$\min_{u \in \mathcal{U}(T_c)} J(x_0, u(\cdot)) = \int_0^\infty L(x(t), u(t)) dt \quad (20)$$

where

$$\mathcal{U}(T_c) = \{u(\cdot) : u(\cdot) \text{ stabilizes (14) in a predefined time } T_c\}.$$

This problem is called as the *optimal predefined-time stabilization problem* for the system (14).

The following theorem gives sufficient conditions for a controller to solve this problem.

Theorem 3.1

Assume there exist a C^1 radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, real numbers $T_c > 0$

and $0 < p < 1$, and a control law $\phi^* : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$V(0) = 0 \quad (21)$$

$$V(x) > 0, \quad \forall x \neq 0, \quad (22)$$

$$\phi^*(0) = 0 \quad (23)$$

$$\frac{\partial V}{\partial x} f(x, \phi^*(x)) \leq -\frac{1}{T_c p} \exp(V^p) V^{1-p} \quad (24)$$

$$\mathcal{H}\left(x, \phi^*(x), \frac{\partial V}{\partial x}\right) = 0 \quad (25)$$

$$\mathcal{H}\left(x, u, \frac{\partial V}{\partial x}\right) \geq 0, \quad \forall u \in \mathcal{U}(T_c). \quad (26)$$

Then, with the feedback control

$$u^*(\cdot) = \phi^*(x(\cdot)) = \arg \min_{u \in \mathcal{U}(T_c)} \mathcal{H}\left(x, u, \frac{\partial V}{\partial x}\right), \quad (27)$$

the origin $x = 0$ of the closed-loop system

$$\dot{x}(t) = f(x(t), \phi^*(x(t))) \quad (28)$$

is predefined-time stable with $T_{\max} = T_c$. Moreover, the feedback control law (27) minimizes $J(x_0, u(\cdot))$ (20) in the sense that

$$J(x_0, \phi^*(x(\cdot))) = \min_{u \in \mathcal{U}(T_c)} J(x_0, u(\cdot)) \quad (29)$$

$$= V(x_0). \quad (30)$$

Thus, the feedback control law (27) solves the optimal predefined-time stabilization problem for the system (14).

Proof

Applying Lemma 2.1 to the closed-loop system (28), predefined-time stability with predefined time T_c follows directly from the conditions (21)-(24).

To prove (30), let $x(t)$ be a solution of the system (28). Then,

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x} f(x(t), \phi^*(x(t))).$$

From the above and (25) it follows

$$\begin{aligned} L(x(t), \phi^*(x(t))) &= L(x(t), \phi^*(x(t))) + \frac{\partial V}{\partial x} f(x(t), \phi^*(x(t))) - \dot{V}(x(t)) \\ &= \mathcal{H}\left(x(t), \phi^*(x(t)), \frac{\partial V}{\partial x}\right) - \dot{V}(x(t)) \\ &= -\dot{V}(x(t)). \end{aligned}$$

Hence,

$$\begin{aligned} J(x_0, \phi^*(x(\cdot))) &= \int_0^\infty -\dot{V}(x(t)) dt \\ &= -\lim_{t \rightarrow \infty} V(x(t)) + V(x_0) \\ &= V(x_0). \end{aligned}$$

Now, to prove (29), let $u(\cdot) \in \mathcal{U}(T_c)$ and let $x(t)$ be the solution of (14), so that

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x} f(x(t), u(t)).$$

Then,

$$\begin{aligned} L(x(t), u(t)) &= L(x(t), u(t)) + \frac{\partial V}{\partial x} f(x(t), u(t)) - \dot{V}(x(t)) \\ &= \mathcal{H}\left(x(t), u(t), \frac{\partial V^T}{\partial x}\right) - \dot{V}(x(t)). \end{aligned}$$

Since $u(\cdot)$ stabilizes (14) in predefined time T_c , using (25) and (26) we have

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty \left[\mathcal{H}\left(x(t), u(t), \frac{\partial V^T}{\partial x}\right) - \dot{V}(x(t)) \right] dt \\ &= -\lim_{t \rightarrow \infty} V(x(t)) + V(x_0) + \int_0^\infty \mathcal{H}\left(x(t), u(t), \frac{\partial V^T}{\partial x}\right) dt \\ &= V(x_0) + \int_0^\infty \mathcal{H}\left(x(t), u(t), \frac{\partial V^T}{\partial x}\right) dt \\ &\geq V(x_0) \\ &= J(x_0, \phi^*(x(\cdot))). \end{aligned}$$

□

Remark 3.1

It is important that the optimal predefined-time stabilizing controller $u^* = \phi^*(x)$ characterized by Theorem 3.1 is a feedback controller.

Remark 3.2

Note that the conditions (21)-(26) involve a predefined-time Lyapunov function (see Lemma 2.1) that is also a solution of the steady state Hamilton-Jacobi-Bellman equation (19). As usual in optimal control theory, these existence conditions are quite restrictive. However, these conditions are very useful to obtain an inverse optimal predefined-time stabilizing controller, for instance, for a class nonlinear affine control systems with relative degree one. This is a typical case in sliding mode control design, and it will be considered in foregoing.

Remark 3.3

The main contribution of the paper is the proposed *optimal predefined-time control* concept, that can be considered as an extension of the results presented in [36] and [37].

As a matter of fact, the hypothesis in [36]

$$V'(x)f(x, \phi(x)) < 0$$

implies asymptotic stability, while the hypothesis in [37]

$$V'(x)f(x, \phi(x)) \leq -c(V(x))^\alpha$$

implies finite-time stability, and the hypothesis (24) in this paper

$$\frac{\partial V}{\partial x} f(x, \phi^*(x)) \leq -\frac{1}{T_c^p} \exp(V^p) V^{1-p}$$

implies predefined-time stability (see Lemma 2.1).

Although Theorem 3.1 provides the sufficient conditions (21)-(26) for a controller to solve the optimal predefined-time stabilization problem for a given system, it does not provide a closed form expression for the feedback controller. Instead, the feedback controller can be obtained by solving (27). To derive a closed form expression for the controller, the result of Theorem 3.1 is specialized to nonlinear affine control systems of the form

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t), \quad x(0) = x_0, \quad (31)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the system control input, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function with $f(0) = 0$ and $B: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$.

The performance integrand is also specialized to

$$L(x, u) = L_1(x) + L_2(x)u + u^T R_2(x)u, \quad (32)$$

where $L_1: \mathbb{R}^n \rightarrow \mathbb{R}$, $L_2: \mathbb{R}^n \rightarrow \mathbb{R}^{1 \times m}$ and $R_2: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is a positive definite matrix function.

The following theorem provides an *inverse optimal controller* which solves the optimal predefined-time stabilization problem for the system (31) with performance integrands of the form (32).

Theorem 3.2

Assume there exist a C^1 radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, and real numbers $T_c > 0$ and $0 < p < 1$ such that

$$V(0) = 0 \quad (33)$$

$$V(x) > 0, \quad \forall x \neq 0, \quad (34)$$

$$\frac{\partial V}{\partial x} \left[f(x) + B(x) \left[-\frac{1}{2} R_2^{-1}(x) \left[L_2(x) + \frac{\partial V}{\partial x} B(x) \right]^T \right] \right] \leq -\frac{1}{T_c p} \exp(V^p) V^{1-p} \quad (35)$$

$$L_2(0) = 0 \quad (36)$$

$$L_1(x) + \frac{\partial V}{\partial x} f(x) - \frac{1}{4} \left[L_2(x) + \frac{\partial V}{\partial x} B(x) \right] R_2^{-1}(x) \left[L_2(x) + \frac{\partial V}{\partial x} B(x) \right]^T = 0. \quad (37)$$

Then, with the feedback control

$$u^* = \phi^*(x) = -\frac{1}{2} R_2^{-1}(x) \left[L_2(x) + \frac{\partial V}{\partial x} B(x) \right]^T, \quad (38)$$

the origin of the closed loop system

$$\dot{x}(t) = f(x(t)) + B(x(t))\phi^*(x(t)) \quad (39)$$

is predefined-time stable with $T_{\max} = T_c$. Moreover, the performance measure $J(x_0, u(\cdot))$ is minimized in the sense of (29) and

$$J(x_0, \phi^*(x(\cdot))) = V(x_0). \quad (40)$$

Thus, the feedback control law (38) solves the optimal predefined-time stabilization problem for the system (31).

Proof

We can see that the hypotheses of Theorem 3.1 are satisfied. The control law (38) follows from $\frac{\partial}{\partial u} [\mathcal{H}(x, u, \frac{\partial V}{\partial x})] = 0$ with $L(x, u)$ specialized to (32). Then, setting $u^* = \phi^*(x)$ as in (38), the conditions (33), (34) and (35) become the hypotheses (21), (22) and (24), respectively.

On the other hand, since the function V is C^1 , and by (33) and (34) V has a local minimum at the origin, then $\frac{\partial V}{\partial x}|_{x=0} = 0$. Consequently, the hypothesis (23) follows from (36) and the fact that $\frac{\partial V}{\partial x}|_{x=0} = 0$.

Since $\phi^*(x)$ satisfies $\frac{\partial}{\partial u} [\mathcal{H}(x, u, \frac{\partial V}{\partial x})]_{u=\phi^*(x)} = 0$, and noticing that (37) can be rewritten in terms of $\phi^*(x)$ as

$$L_1(x) + \frac{\partial V}{\partial x} f(x) - \phi^{*T}(x) R_2(x) \phi^*(x) = 0 \quad (41)$$

then the hypothesis (25) is directly verified.

Finally, from (25), (38) and the positive definiteness of $R_2(x)$ it follows

$$\begin{aligned} \mathcal{H}\left(x, u, \frac{\partial V}{\partial x}\right) &= L(x, u) + \frac{\partial V}{\partial x} [f(x) + B(x)u] \\ &= L(x, u) + \frac{\partial V}{\partial x} [f(x) + B(x)u] - L(x, \phi^*(x)) - \frac{\partial V}{\partial x} [f(x) + B(x)\phi^*(x)] \\ &= \left[L_2(x) + \frac{\partial V}{\partial x} B(x) \right] (u - \phi^*(x)) + u^T R_2(x) u - \phi^{*T}(x) R_2(x) \phi^*(x) \\ &= -2 \phi^{*T}(x) R_2(x) (u - \phi^*(x)) + u^T R_2(x) u - \phi^{*T}(x) R_2(x) \phi^*(x) \\ &= [u - \phi^*(x)]^T R_2(x) [u - \phi^*(x)] \\ &\geq 0, \end{aligned}$$

which is the hypothesis (26). Applying Theorem 3.1, the result is obtained. \square

Remark 3.4

The feedback controller (38) provided by Theorem 3.2 is an inverse optimal controller in the following sense: instead of solving the steady-state HJB equation directly to minimize some given performance measure, a family of predefined-time stabilizing controllers that minimize a certain cost function is defined. In this case, one can flexibly specify $L_2(x)$ and $R_2(x)$, while from (41) $L_1(x)$ is parametrized as

$$L_1(x) = \phi^{*T}(x) R_2(x) \phi^*(x) - \frac{\partial V}{\partial x} f(x) \geq 0. \quad (42)$$

Remark 3.5

As in Theorem 3.1, it is not always easy to satisfy the hypotheses (33)-(37) of Theorem 3.2. However, for affine systems with relative degree one, the functions $L_2(x)$ and $R_2(x)$ can be chosen such that the conditions (33)-(37) are fulfilled.

Recall that there is a direct connection between the notions of relative degree and sliding order (see [44]). Then, the following section is motivated.

4. INVERSE OPTIMAL PREDEFINED-TIME STABLE REACHING LAW

In this section, first, some basic concepts corresponding to integral manifolds and sliding mode manifolds are reviewed.

Definition 4.1 ([18])

Let $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a smooth function, and define the manifold

$$\mathcal{S} = \{x \in \mathbb{R}^n : \sigma(x) = 0\}. \quad (43)$$

If for an initial condition $x_0 \in \mathcal{S}$, the solution of (1) $x(t, x_0) \in \mathcal{S}$ for all t , the manifold \mathcal{S} is called an *integral manifold*.

Definition 4.2 ([18])

If there is a nonempty set $\mathcal{N} \subset \mathbb{R}^n - \mathcal{S}$ such that for every initial condition $x_0 \in \mathcal{N}$, there is a finite time $t_s > 0$ in which the state of the system (1) reaches the manifold \mathcal{S} (43), then the manifold \mathcal{S} is called an *sliding mode manifold*.

Remark 4.1

A sliding mode on a certain sliding manifold can appear only if f is a non-smooth (possibly discontinuous) function. For this case, the solutions of (1) are understood in the Filippov sense [45].

Consider the following nonlinear affine system subject to perturbation:

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t) + \Delta(t, x), \quad x(0) = x_0, \quad (44)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ with $m \leq n$ is the system control input, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function with $f(0) = 0$, $B : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ and $\Delta : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents a system uncertainty.

With the above definitions, the main objective of the controller is to optimally drive the trajectories of the system (44) to the set \mathcal{S} (43) in a predefined time in spite of the unknown perturbation $\Delta(t, x)$. The function $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is selected so that the motion of the system (31) restricted to the sliding manifold $\sigma(x) = 0$ has a desired behavior.

The dynamics of σ are described by

$$\dot{\sigma}(t) = a(x(t)) + G(x(t))u(t) + \frac{\partial \sigma}{\partial x} \Delta(t, x), \quad \sigma(x(0)) = \sigma_0, \quad (45)$$

where $a(x) = \frac{\partial \sigma}{\partial x} f(x)$ and $G(x) = \frac{\partial \sigma}{\partial x} B(x)$.

It is assumed that $\sigma(x)$ is selected such that the matrix $G(x) \in \mathbb{R}^{m \times m}$ has inverse for all $x \in \mathbb{R}^n$. It means that the system (45) has relative degree one.

Remark 4.2

In systems with first-order sliding modes, in the case when the relative degree is greater than one, the finite (fixed or predefined)-time convergence can be achieved only for the designed sliding variable (reaching phase), while the vector $x(t)$ is stabilized asymptotically (see [9] and [18]).

The following results provide controllers which solve the optimal predefined-time stabilization problem (20) for the system (45). There are presented two cases. The first scenario corresponds to the non-perturbed system, and the second deals with matched non-vanishing perturbation.

4.1. Unperturbed case

Considering the case when $\Delta(t, x) = 0$, the following result, obtained as a direct consequence of Theorem 3.2, gives an explicit form of the functions V , R_2 and L_2 which characterize the optimal predefined-time stabilizing feedback controller (38).

Corollary 4.1 (Unperturbed system)

Consider the system (45) in absence of the perturbation term, i.e., $\Delta(t, x) = 0$. The feedback controller (38) with the functions V , R_2 and L_2 selected as

$$V(\sigma) = c^{\frac{1}{p+1}} (\sigma^T \sigma)^{\frac{1}{p+1}}, \quad (46)$$

$$R_2(x) = \frac{T_c p}{2} \exp(-V^p(\sigma(x))) [G^T(x)G(x)], \quad (47)$$

$$L_2(x) = 2a^T(x) [G^{-1}(x)]^T R_2(x), \quad (48)$$

with $T_c > 0$, $0 < p < 1$ and $4c = (p+1)^2$, stabilizes the system (45) in predefined time $T_f = T_c$. Moreover, this controller solves the optimal predefined-time stabilization problem (20) for the system (45) with the performance integrand specialized to $L(x, u) = L_1(x) + L_2(x)u + u^T R_2(x)u$, where L_2 and R_2 are given by (48) and (47), respectively, and L_1 is given by (42).

Proof

It is easy to see that all the conditions of Theorem 3.2 are satisfied. Indeed, note that the function V in (46) is C^1 (see Remark 4.3), and satisfies the hypotheses (33) and (34). In the same manner, the function L_2 in (48) satisfies the hypothesis (36), and defining the function L_1 as in (42), the hypothesis (37) is also satisfied.

On the other hand, the derivative of V along the trajectories of the closed loop system

$$\dot{\sigma} = a(x) + G(x)\phi^*(x),$$

is calculated as (note that $\left\|\frac{\partial V}{\partial \sigma}\right\|^2 = V^{1-p}$)

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial \sigma} [a(x) + G(x)\phi^*(x)] \\ &= \frac{\partial V}{\partial \sigma} \left[a(x) + G(x) \left(-\frac{1}{2}R_2^{-1}(x)L_2^T(x) - \frac{1}{2}R_2^{-1}(x)G^T(x)\frac{\partial V^T}{\partial \sigma} \right) \right] \\ &= -\frac{1}{T_c p} \exp(V^p) \left\| \frac{\partial V}{\partial \sigma} \right\|^2 \\ &= -\frac{1}{T_c p} \exp(V^p) V^{1-p}. \end{aligned}$$

Thus, the hypothesis (35) is satisfied. Then, the result is obtained by direct application of Theorem 3.2. \square

Remark 4.3

Note that the partial derivatives of the function V defined in (46) are $(\sigma = [\sigma_1 \ \dots \ \sigma_m])$:

$$\frac{\partial V(\sigma)}{\partial \sigma_k} = \frac{2c^{\frac{1}{p+1}} \sigma_k}{(p+1) \left(\sigma_k^2 + \sum_{j \neq k} \sigma_j^2 \right)^{\frac{p}{p+1}}}, \text{ for } \sigma \neq 0 \text{ and } k = 1, \dots, m.$$

Furthermore, note that as $\sigma(t)$ approaches to the origin the above expression vanishes

$$\lim_{\sigma \rightarrow 0} \left| \frac{\partial V(\sigma)}{\partial \sigma_k} \right| = \frac{2c^{\frac{1}{p+1}}}{p+1} \lim_{\sigma \rightarrow 0} \frac{|\sigma_k|}{\left(\sigma_k^2 + \sum_{j \neq k} \sigma_j^2 \right)^{\frac{p}{p+1}}} \leq \frac{2c^{\frac{1}{p+1}}}{p+1} \lim_{\sigma \rightarrow 0} \frac{|\sigma_k|}{\sigma_k^{\frac{2p}{p+1}}} = \frac{2c^{\frac{1}{p+1}}}{p+1} \lim_{\sigma \rightarrow 0} |\sigma_k|^{\frac{1-p}{1+p}} = 0.$$

Now, note that the partial derivatives of V at the origin do exist and are equal to zero

$$\begin{aligned} \left. \frac{\partial V(\sigma)}{\partial \sigma_k} \right|_{\sigma=0} &= \lim_{h \rightarrow 0} \frac{V([0 \ \dots \ 0+h \ \dots \ 0]^T) - V([0 \ \dots \ 0 \ \dots \ 0]^T)}{h} \\ &= c^{\frac{1}{p+1}} \lim_{h \rightarrow 0} \frac{h^{\frac{2}{p+1}} - 0}{h} = c^{\frac{1}{p+1}} \lim_{h \rightarrow 0} h^{\frac{1-p}{1+p}} = 0. \end{aligned}$$

Since the above derivatives exist and are continuous at every point $\sigma \in \mathbb{R}^m$, the function V in (46) is C^1 in \mathbb{R}^m . Thus, its gradient is expressed in the vector form as

$$\frac{\partial V(\sigma)}{\partial \sigma} = \begin{cases} \frac{2c^{\frac{1}{p+1}} \sigma^T}{(p+1)(\sigma^T \sigma)^{\frac{p}{p+1}}} & \text{for } \sigma \neq 0 \\ 0 & \text{for } \sigma = 0. \end{cases}$$

4.2. Perturbed case

Considering the case when $\Delta(t, x)$ is a matched non-vanishing perturbation, the following result provides a controller that rejects the perturbation term $\bar{\Delta}(t, x)$ in predefined-time. Moreover, once the perturbation term is rejected, this controller optimally stabilizes the system (45) in predefined-time. This will be achieved by applying the idea of integral sliding mode control [39, 40, 46].

Remark 4.4

From Remark 4.3, it is required $p < 1$ for V to be C^1 . However, in this case, it is not possible to derive a discontinuous optimal predefined-time controller which can reject the perturbation. Therefore, it is used the integral SM control idea, where the control is composed of an optimal and a discontinuous parts.

Corollary 4.2 (Non-vanishing matched perturbation)

Consider the system (45) and let the function $\Delta(t, x)$ be a matched and non-vanishing perturbation term, i.e. there exists a function $\bar{\Delta}(t, x)$ such that $\Delta(t, x) = B(x)\bar{\Delta}(t, x)$ and $\|G(x)\Delta(t, x)\| \leq \delta$, with $0 < \delta < \infty$ a known constant. Then, splitting the control function u into two parts, $u = u_0 + u_1$, and selecting

- (i) u_0 as the optimal predefined-time stabilizing feedback controller (38), with the functions V , R_2 and L_2 as in Corollary 4.1 with parameters $T_{c_2} > 0$ and $0 < p_2 < 1$, and
- (ii) $u_1 = -G^{-1}(x) \left[k \frac{s}{\|s\|} + \Phi_{p_1}(s; T_{c_1}) \right]$, with $T_{c_1} > 0$, $0 < p_1 < 1$, $k \geq \delta$, and the auxiliary sliding variable $s = \sigma + z$, where z is an integral variable, solution of $\dot{z} = -a(x) - G(x)u_0$,

the system perturbation term $\bar{\Delta}(t, x)$ is rejected in predefined time T_{c_1} , and once the perturbation term is rejected the system (45) is optimally predefined-time stabilized with predefined time $T_f = T_{c_2}$, with respect to the performance $L(x, u) = L_1(x) + L_2(x)u + u^T R_2(x)u$, where L_2 and R_2 are given by (48) and (47), respectively, and L_1 is given by (42).

Proof

By the definition of s , σ , z and u_1 , the dynamics of s are obtained as

$$\begin{aligned} \dot{s} &= \dot{\sigma} + \dot{z} \\ &= a(x) + G(x)(u_0 + u_1 + \bar{\Delta}) + \dot{z} \\ &= G(x)(u_1 + \bar{\Delta}) \\ &= - \left[k \frac{s}{\|s\|} + \Phi_{p_1}(s; T_{c_1}) \right] + G(x)\bar{\Delta}. \end{aligned}$$

By direct application of Lemma 2.4, a sliding mode appears on the manifold $s = 0$ for $t > T_{c_1}$ in spite of the initial value $s(0) = \sigma(0) + z(0)$. Once the dynamics of (45) are constrained to the manifold $s = 0$, then, from $\dot{s} = 0$, the following equivalent control value is obtained as [9]:

$$u_{1,eq} = -\bar{\Delta}.$$

Therefore, the sliding mode dynamics of σ

$$\dot{\sigma} = a(x) + G(x)u_0.$$

are invariant with respect to the perturbation. By the definition of u_0 and direct application of Corollary 4.1, the desired result is obtained. \square

5. EXAMPLE

Consider a pendulum system with viscous friction and an external perturbation [47]

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g}{L} \sin(x_1) - \frac{V_s}{J} x_2 + \frac{1}{J} u + v(t), \end{aligned} \quad (49)$$

where x_1 is the angular position, x_2 is the angular velocity, u is the input torque, J is the moment of inertia, g is the gravity acceleration, L is the length of the pendulum, V_s is the viscous friction constant and $v(t)$ is an uncertain external perturbation, bounded by $|v(t)| < \delta$. Note that the pendulum model (49) can be expressed as the affine perturbed system (44) by defining $\Delta(t) = [0 \quad 1]^T v(t)$.

The pendulum parameters are shown in Table I.

Table I. Parameters of the pendulum model (49).

Parameter	Values	Unit
M	1	kg
L	1	m
$J = ML^2$	1	$\text{kg} \cdot \text{m}^2$
V_s	0.2	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
g	9.8	$\text{m} \cdot \text{s}^{-2}$

Due to the structure of the model (49), the function σ can be selected as $\sigma(x) = x_2 + k_1 x_1$, with $k_1 > 0$. The dynamics of σ are described by the equation (45) with $a(x) = -\frac{g}{L} \sin(x_1) - \frac{V_s}{J} x_2 + k_1 x_2$ and $G(x) = \frac{1}{J} \neq 0$.

5.1. Unperturbed case

Assuming $v(t) = 0$, the functions V , R_2 and L_2 are selected, according to Corollary 4.1, as

$$\begin{aligned} V(\sigma) &= c^{\frac{1}{p+1}} \sigma^{\frac{2}{p+1}}, \\ R_2(x) &= \frac{T_c p}{2J^2} \exp(-c^{\frac{p}{p+1}} \sigma^{\frac{2p}{p+1}}), \text{ and} \\ L_2(x) &= \frac{T_c p}{J} \exp(-c^{\frac{p}{p+1}} \sigma^{\frac{2p}{p+1}}) \left[-\frac{g}{L} \sin(x_1) - \frac{V_s}{J} x_2 + k_1 x_2 \right], \end{aligned}$$

and $u^* = \phi^*(x)$ is implemented as in (38).

Finally, according to (42), the resulting function L_1 becomes

$$L_1(x) = \frac{2J^2}{4T_c p} \exp(c^{\frac{p}{p+1}} \sigma^{\frac{2p}{p+1}}) \left[L_2(x) + \frac{1}{J} \frac{\partial V}{\partial \sigma} \right]^2 - \frac{\partial V}{\partial \sigma} a(x).$$

To show the performance of the optimal predefined-time controller scheme, simulations were conducted using the Euler integration method, with a fundamental step size of 1×10^{-4} s. The initial conditions for the system (49) were selected as: $x_1(0) = \pi/2$ rad and $x_2(0) = 0$ rad/s. In addition, the controller gains were adjusted to: $T_c = 1$, $k_1 = 1$ and $p = 1/2$.

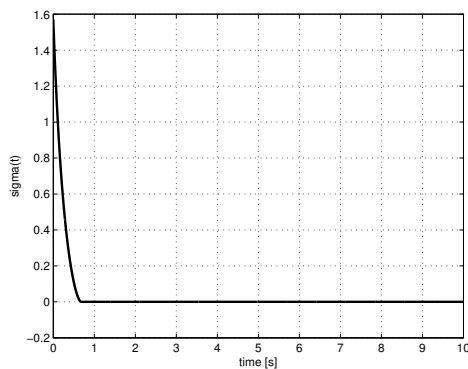
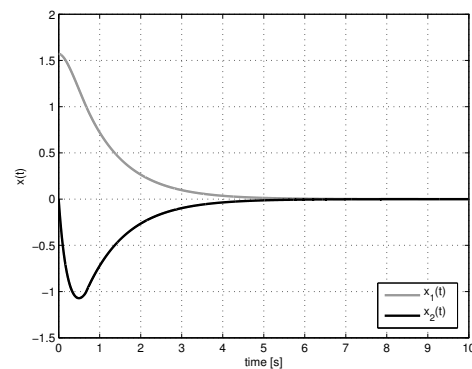
Figure 9. Function $\sigma(x(t))$.

Figure 10. Evolution of the states.

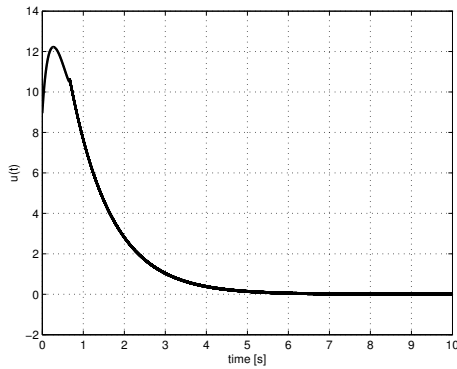


Figure 11. Control input.

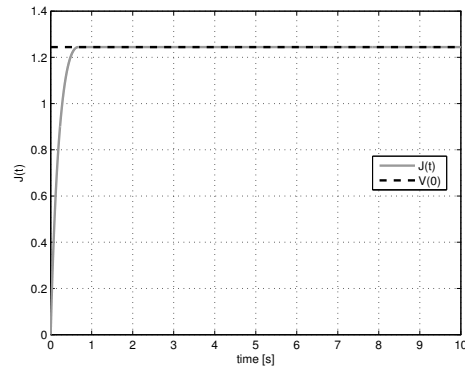


Figure 12. Function $\int_0^t [L_1 + L_2 u + u^T R_2 u] d\tau$.

Note that $\sigma(t) = 0$ for $t \geq 0.827 \text{ s} < T_c = 1 \text{ s}$ (Fig. 9). Once the system states slide over the sliding manifold $\sigma(x) = 0$, this motion is governed by the reduced order system

$$\dot{x}_1(t) = -kx_1(t).$$

This imply that the system state tends exponentially to zero at a rate of $\frac{1}{k}$ (Fig. 10). Fig. 11 shows the control signal (torque) versus time. Finally, from Fig. 12, it can be seen that the cost as a function of time grows quickly to a steady state value, corresponding to $V(\sigma(0))$.

5.2. Perturbed case

For this case, the perturbation term is taken as $v(t) = 0.5 \sin 2t + 0.5 \cos 5t$. Note that $|v(t)| < 1$. The part u_0 of the controller is selected as in the unperturbed case, according to Corollary 4.2. On the other hand, the variable z and the part u_1 of the controller are chosen according to the item (ii) of Corollary 4.2.

To show the performance of the optimal predefined-time controller scheme, simulations were conducted using the Euler integration method, with a fundamental step size of $1 \times 10^{-4} \text{ s}$. The initial conditions of the integrators were selected as: $x_1(0) = \pi/2 \text{ rad}$, $x_2(0) = 0 \text{ rad/s}$ and $z(0) = 0$. In addition, the controller gains were adjusted to: $T_{c1} = 0.5$, $T_{c2} = 1$, $k = 1$, $k_1 = 1$, $p_1 = 1/2$ and $p_2 = 1/2$.

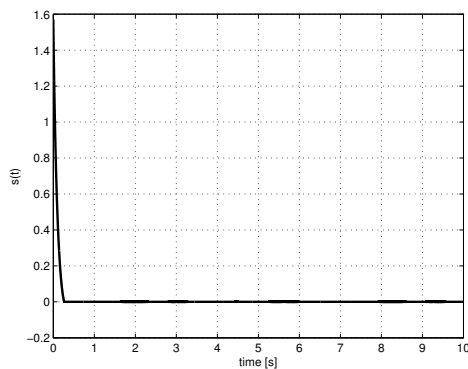


Figure 13. Function $s(x(t))$.

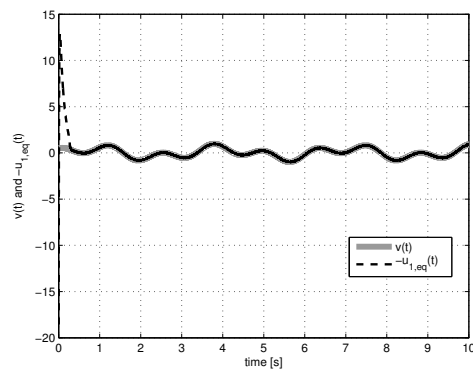


Figure 14. Perturbation cancellation.

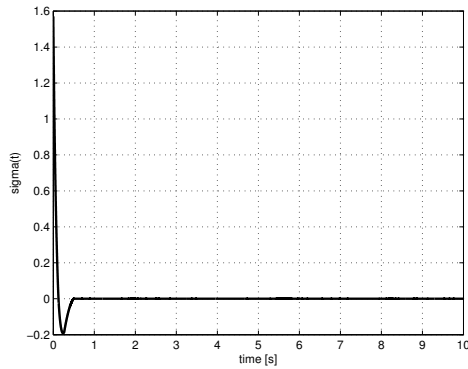
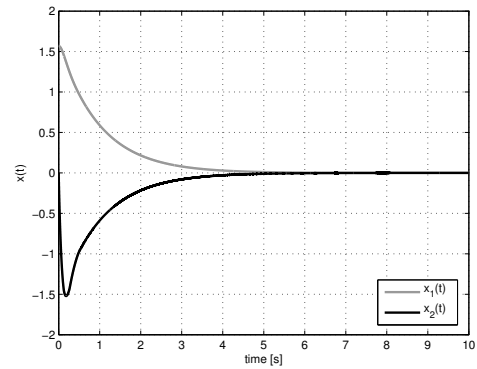
Figure 15. Function $\sigma(x(t))$.

Figure 16. Evolution of the states.

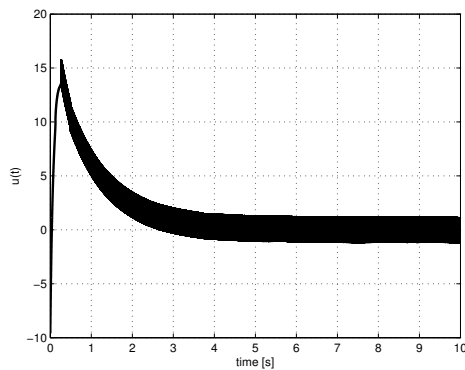
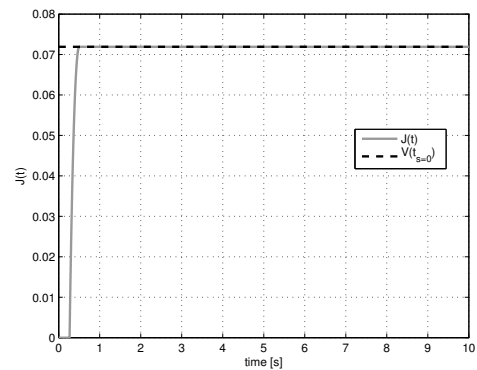


Figure 17. Control input.

Figure 18. Function $\int_{t_s=0}^t [L_1 + L_2 u + u^T R_2 u] d\tau$.

Note that $s(t) = 0$ for $t \geq 0.2658 \text{ s} := t_{s=0} < T_{c_1} = 0.5 \text{ s}$ (Fig. 13), and from that instant on, the equivalent control signal $u_{1,eq}$ (obtained using the low-pass filter $\tau \dot{u}_{1,eq} + u_{1,eq} = u_1$, with $\tau = 0.01$ [9, 46]) cancels the perturbation term $v(t)$ (Fig. 14).

Once the perturbation is canceled, the optimal predefined-time stabilization of the variable $\sigma(t)$ takes place. It can be seen that $\sigma(t) = 0$ for $t \geq 0.5 \text{ s} < T_{c_1} + T_{c_2} = 1.5 \text{ s}$ (Fig. 15). After the system states slide over the sliding manifold $\sigma(x) = 0$, this motion is governed again by the first order system

$$\dot{x}_1(t) = -kx_1(t).$$

This implies that the system state tends exponentially to zero at a rate of $\frac{1}{k}$ (Fig. 16).

Fig. 17 shows the control signal (torque) versus time. It is important to remark that this controller yields discontinuous signals in order to cancel the persistent perturbation $v(t)$. Finally, from Fig. 18, it can be seen that the cost as a function of time grows quickly to a steady state value, corresponding to $V(\sigma(t_{s=0}))$.

6. CONCLUSION

In this paper, the problem of optimal predefined-time stability was investigated. Sufficient conditions for a controller to be optimal predefined-time stabilizing for a given nonlinear system were provided. Moreover, under the idea of inverse optimal control, and considering nonlinear affine systems and a certain class of performance integrand, the explicit form of the controller was also derived. This class of controllers was applied to the predefined-time optimization of the sliding manifold reaching phase, considering both the unperturbed and the perturbed cases. For the unperturbed case, the

developed result was applied directly, while for the perturbed case it was used jointly with the idea of integral sliding mode control to provide robustness.

For illustration purposes, the developed control schemes were performed for the predefined-time optimization of the sliding manifold reaching phase in a pendulum system model. Simulation results supported the expected results.

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