A Class of Predefined-Time Stable Dynamical Systems

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Abstract

This paper introduces predefined-time stable dynamical systems which are a class of fixed-time stable dynamical systems with settling time as an explicit parameter that can be defined in advance. This concept allows for the design of observers and controllers for problems that require to fulfill hard time constraints. An example is encountered in the fault detection and isolation problem, where mode detection in a timely manner needs to be guaranteed in order to apply a recovery action. Furthermore, through the notion of strong predefined-time stability, the approach hereinafter presented permits to overcome the problem of overestimation of the convergence time bound encountered in previous methods for the analysis of finite-time stable systems, where the stabilization time is often an unbounded function of the initial conditions of the system. A Lyapunov analysis is provided together with a detailed discussion of the applications to consensus and first order sliding mode controller design. Finite-time stability, Sliding-mode control, Lyapunov stability, Robust control, Consensus.

1 Introduction

Nowadays, there exist several applications of dynamical systems that are characterized by requiring to meet hard response-time constraints while being robust to uncertainties such as external disturbances or parameter variation. A basic case is encountered in the fault detection and isolation problem, where it is of paramount importance to guarantee the mode detection in a timely manner in order to apply a recovery action (Lee and Park, 2012), because in some situations a late response may lead to a no recovery scenario. When, additionally, robustness to external disturbances and/or parameter uncertainty is required, sliding mode algorithms have been one of the most promising methods (Utkin, 1992, Drakunov and Utkin, 1992).

Nonetheless, even if various developments in *finite-time stability*, *fixed-time stability* and *deadbeat control* have been carried out to deal with these time requirements (see e.g. Roxin (1966), Weiss and Infante (1967), Michel and Porter (1972), Ryan (1991), Bhat and Bernstein (2000), Hong (2002), Orlov (2005), Moulay and Perruquetti (2006) for finite-time stability, Andrieu et al. (2008), Cruz-Zavala et al. (2010), Polyakov (2012) for fixed-time stability, and Smith (1957), Tallman and Smith (1958) for deadbeat control), the design of robust controllers and observers guaranteeing that time constrains are met is still challenging. One of the main reasons is because, in current methods, there is no explicit relationship between the system parameters and the convergence time bound. As a consequence, current settling-time estimation methods are usually conservative and inaccurate.

Such is the case in several control approaches with the finite-time feature, like Bhat and Bernstein (2000), Orlov (2005), Moulay and Perruquetti (2006), where the stabilization time is often an unbounded function of the initial conditions of the system. To make the settling time bounded for any initial condition a stronger form of stability, called *fixed-time stability*, was introduced by Andrieu et al. (2008) for homogeneous systems and by Cruz-Zavala et al. (2010), Polyakov (2012), Polyakov and Fridman (2014) for systems with sliding modes. The settling time of fixed-time stable systems presents a class of uniformity with respect to their initial conditions.

Unfortunately, when current fixed-time algorithms are applied to control or observation problems, there are still hard issues related to the convergence-time estimation. The main drawback is that the relationship between the parameters of the system and the bound of the convergence time is not explicit; thus, finding the system parameters to achieve a desired maximum stabilization time is challenging, leading to very conservative estimations of the settling-time bound and to a transient response of lower quality than necessary as consequences. Consider for example the work by Cruz-Zavala et al. (2011), where the settling-time bound estimate is approximately 100 times larger than the actual true fixed stabilization time. To overcome this parameter selection problem, a simulation-based approach has been proposed under the concept of *prescribed-time stability* (Fraguela et al., 2012); nonetheless, since the method is simulation-based, it lacks rigorous analysis and explicit formulas for the settling-time computation are not provided. A rigorous approximation of the settling time in planar systems controlled with uniform finite-time controllers is given by Oza et al. (2015); however, this approach implies cumbersome calculations and the resulting estimate is not directly related to the system parameters.

In order to cope with the problems presented above, a class of systems where an upper bound of the fixed stabilization time is a tunable parameter is proposed. Such systems are defined as *predefined-time stable systems*. Moreover, two categories are identified within this definition: *weakly predefined-time stable systems* only possesses the aforementioned property, while *strongly predefined-time stable systems* present the additional advantage that this tunable parameter is not only an upper bound for the settling time but precisely the least upper bound, thus avoiding any unnecessary overestimation of the convergence time.

Predefined-time stability is strongly related to the continuous deadbeat control; for example, a classic case of predefined-time stable controllers are those based on the posicast method (Smith, 1957, Tallman and Smith, 1958), where part of the input command is delayed to achieve deadbeat control. However, predefined-time algorithms based on deadbeat control are not robust to external disturbances or parameter uncertainty and time requirements are not guaranteed to be met when they are present.

Having defined the concept of predefined-time stability, this paper presents the analysis of a class of first-order predefined-time stable dynamical systems (Sánchez-Torres et al., 2014, 2015). In contrast to most of the fixed-time stable systems, the bound on the convergence time associated with this class of systems is not a conservative estimate but truly the minimum value that is greater than all the possible exact settling times. Moreover, this bound is not based on simulations due to the fact that all the mentioned properties are characterized by a suitable Lyapunov theorem. Furthermore, the system structure contains no delay terms, making its analysis and design easier when compared to the mentioned deadbeat methods. Contrary to previously proposed methods, under mild assumptions, the approach hereinafter presented guarantees that time constraints are met even in the presence of uncertainty.

In addition to the predefined settling time, the devised systems depend on other parameters whose values determine whether the right-hand sides of the differential equations are continuous or discontinuous and, from both cases, predefined-time controllers are derived. Besides, taking advantage of the discussed features, more general first order sliding mode controllers with predefined-time reaching phase are introduced. Finally, a predefined-time consensus algorithm is designed for complete networks.

The rest of this paper is structured in the following manner: Section 2 exposes the main results of this paper, including necessary the mathematical preliminaries, a class of predefined-time stable systems and the characterization of the Lyapunov conditions they satisfy. In Section 3, the influence of the tuning parameters on the proposed class of systems is analyzed and a suggestion for their selection is provided. In addition, a brief numerical study on the relationship between the parameter selection and the effect of noisy measurements is presented. Section 4 shows the design of first order sliding mode controllers where the reaching phase stage ends after a predefined time. Taking advantage of the strong stability features of the proposed family of systems, a consensus algorithm for complete networks is presented in Section 5. Finally, Section 6 presents the conclusions of this paper.

2 Predefined-Time Stability: Definitions and Lyapunov Characterization

2.1 Basic definitions for unperturbed systems

Consider the system

$$\dot{x} = f(t, x; \rho), \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\rho \in \mathbb{R}^b$ with $\dot{\rho} = 0$ represents the parameters of the system, and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. The time variable t is defined on the interval $[t_0, \infty)$, where $t_0 \in \mathbb{R}_+ \cup \{0\}$. For this system, the initial conditions are $x_0 = x(t_0)$.

Definition 2.1 (Globally fixed-time attraction (Polyakov, 2012))**.** *A non-empty set $M \subset \mathbb{R}^n$ is said to be globally fixed-time attractive for system (1) if any solution $x(t, x_0)$ of (1) reaches M in some finite time $t = t_0 + T(x_0)$, where the settling-time function $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is bounded by some positive number T_m, i.e., $T(x_0) \le T_m$ for all $x_0 \in \mathbb{R}^n$.*

Note that there are several possible choices for T_m; for example, if $T(x_0) \le T_\mathrm{m}$ for a positive number T_m, also $T(x_0) \le \lambda T_\mathrm{m}$ with $\lambda \ge 1$. This motivates the definition of a set which contains all the bounds of the settling-time function.

Definition 2.2 (Settling-time set)**.** *Let the set of all the bounds of the settling-time function for system (1) be defined as follows:*

$$\mathcal{T} = \{T_{max} \in \mathbb{R}_+ : T(x_0) \le T_m \, \forall \, x_0 \in \mathbb{R}^n\}. \tag{2}$$

In addition, the minimum bound for the settling-time function of (1) is defined in the following manner:

Definition 2.3 (Least upper bound for the settling time)**.** *Consider the set $\mathcal{T}$ defined in (2). The least upper bound of the settling-time function, denoted by T_f, is defined as*

$$T_f = \min \mathcal{T} = \sup_{x_0 \in \mathbb{R}^n} T(x_0). \tag{3}$$

Remark 2.1. *For several applications it could be desirable for system (1) to stabilize within a time $T_c \in \mathcal{T}$ which can be defined in advance as function of the system parameters, that is $T_c = T_c(\rho)$. The cases where this property is present motivate the definition of predefined-time stability. A strong notion of this class of stability is given when $T_c = T_f$, i.e., T_c is the true fixed-time in which the system stabilizes. A weak notion of predefined-time stability is presented when $T_c \ge T_f$, that is, if well it is possible to define an upper bound of the settling-time function in terms of the system parameters, this overestimates the true fixed-time in which the system stabilizes.*

Definition 2.4. *For the system parameters ρ and a constant $T_c(\rho) > 0$, a non-empty set $M \subset \mathbb{R}^n$ is said to be*

(i) Globally weakly predefined-time attractive for system (1) if any solution $x(t, x_0)$ of (1) reaches M in some finite time $t = t_0 + T(x_0)$, where the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$T(x_0) \le T_c \quad \forall \; x_0 \in \mathbb{R}^n.$$

In this case, T_c is called the weak predefined time.

(ii) Globally strongly predefined-time attractive for system (1) if any solution $x(t, x_0)$ of (1) reaches M in some finite time $t = t_0 + T(x_0)$, where the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\sup_{x_0 \in \mathbb{R}^n} T(x_0) = T_c.$$

In this case, T_c is called the strong predefined time.

2.2 Generalization to perturbed systems

Consider the system

$$\dot{x} = f(t, x, d; \rho), \tag{4}$$

where $x \in \mathbb{R}^n$ is the system state, $d \in \Omega \subset \mathbb{R}^m$, with Ω bounded, is a perturbation term, $\rho \in \mathbb{R}^b$ with $\dot{\rho} = 0$ represents the parameters of the system, and $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function. The time variable t is defined on the interval $[t_0, \infty)$, where $t_0 \in \mathbb{R}_+ \cup \{0\}$. For this system, the initial conditions are $x_0 = x(t_0)$.

Note that the settling time of system (4) to a given set depends not only on x_0 but also on the the trajectory $\omega \in \Omega^{[t_0,\infty)} = \{d : [t_0, \infty) \to \Omega \mid d \text{ is a function}\}$ followed by the perturbation $d(t)$.

Definition 2.5. *For the system parameters ρ and a constant $T_c(\rho) > 0$, a non-empty set $M \subset \mathbb{R}^n$ is said to be*

(i) Globally weakly predefined-time attractive for system (4), if any solution $x(t, x_0, \omega)$ of (4) reaches M in some finite time $t = t_0 + T(x_0, \omega)$, where the settling-time function $T : \mathbb{R}^n \times \Omega^{[t_0,\infty)} \to \mathbb{R}$ is such that

$$T(x_0, \omega) \leq T_c \quad \forall\ x_0 \in \mathbb{R}^n, \omega \in \Omega^{[t_0,\infty)}.$$

In this case, T_c is called the weak predefined time.

(ii) Globally strongly predefined-time attractive for system (4), if any solution $x(t, x_0, \omega)$ of (4) reaches M in some finite time $t = t_0 + T(x_0, \omega)$, where the settling-time function $T : \mathbb{R}^n \times \Omega^{[t_0,\infty)} \to \mathbb{R}$ is such that

$$\sup_{x_0 \in \mathbb{R}^n, \omega \in \Omega^{[t_0,\infty)}} T(x_0) = T_c.$$

In this case, T_c is called the strong predefined time.

2.3 Lyapunov stability

First, the following theorem provides a useful Lyapunov condition for weakly predefined-time attractive sets:

Theorem 2.1 (Lyapunov characterization of weak predefined-time stability (Sánchez-Torres et al., 2014, 2015)). *If there exists a continuous radially unbounded function*

$$V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$$

such that $x \in M$ if and only if $V(x) = 0$ and any solution $x(t)$ of (1) or (4) satisfies

$$\dot{V} \leq -\frac{1}{pT_c} \exp(V^p) V^{1-p} \tag{5}$$

for constants $T_c = T_c(\rho) > 0$ and $0 < p \leq 1$, then the set M is globally weakly predefined-time attractive for system (1) or (4), respectively, and the weak predefined time is T_c.

Proof. The solution to (5) is

$$V(t) \leq \left[\ln \left(\frac{1}{\frac{t-t_0}{T_c} + \exp(-V_0^p)} \right) \right]^{\frac{1}{p}}$$

where $V_0 = V(x_0)$.

Note that $V(t) = 0$ if $\frac{t-t_0}{T_c} + \exp(-V_0^p) = 1$, hence the settling-time function for the system (1) is such that

$$T(x_0) \leq T_c \left[1 - \exp(-V_0^p)\right] \quad \forall\ x_0 \in \mathbb{R}^n,$$

and for the system (4) is such that

$$T(x_0, \omega) \leq T_c \left[1 - \exp(-V_0^p)\right] \quad \forall\ x_0 \in \mathbb{R}^n, \omega \in \Omega^{[t_0,\infty)}.$$

Then, since $0 < \exp(-V_0^p) \leq 1$, T_c is an upper bound for the settling-time function and, therefore, the weak predefined time. □

Theorem 2.1 characterizes weak predefined-time stability in a very practical way since the Lyapunov condition (5) directly involves a bound on the convergence time. Nevertheless, this condition is not enough to imply strong predefined-time stability. The following theorems provide Lyapunov characterizations for strongly predefined-time attractive sets for both perturbed and unperturbed systems:

Theorem 2.2 (Lyapunov characterization of strong predefined-time stability)**.** *If there exists a continuous radially unbounded function*

$$V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$$

such that $x \in M$ *if and only if* $V(x) = 0$ *and any solution* $x(t)$ *of* (1) *satisfies*

$$\dot{V} = -\frac{1}{pT_c}\exp(V^p)V^{1-p} \tag{6}$$

for constants $T_c = T_c(\rho) > 0$ *and* $0 < p \leq 1$*, then the set* M *is globally strongly predefined-time attractive for system* (1) *and the strong predefined time is* T_c*.*

Proof. Since the equality version of (5) holds, the settling-time function is known to be exactly

$$T(x_0) = T_c\left[1 - \exp(-V_0^p)\right],$$

where $V_0 = V(x_0)$.

Besides, given that V is radially unbounded, $\sup_{x_0 \in \mathbb{R}^n}[1 - \exp(-V_0^p)] = 1$ and it follows that

$$\sup_{x_0 \in \mathbb{R}^n} T(x_0) = T_c.$$

□

The following theorem extends the Lyapunov result given in Theorem 2.2 to the strong predefined-time stability of perturbed systems:

Theorem 2.3. *If there exists a continuous radially unbounded function*

$$V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$$

such that $x \in M$ *if and only if* $V(x) = 0$ *and any solution* $x(t)$ *of* (4) *satisfies*

$$\sup_{d \in \Omega} \dot{V} = -\frac{1}{pT_c}\exp(V^p)V^{1-p} \tag{7}$$

for constants $T_c = T_c(\rho) > 0$ *and* $0 < p \leq 1$*, then the set* M *is globally strongly predefined-time attractive for system* (4) *with* T_c *as the strong predefined time.*

Proof. If (7) is satisfied, it holds that

$$\sup_{\omega \in \Omega^{[t_0,\infty)}} T(x_0, \omega) = T_c\left[1 - \exp(-V_0^p)\right] \quad \forall\ x_0 \in \mathbb{R}^n$$

with $V_0 = V(x_0)$. Then, since V is radially unbounded, it follows that

$$\sup_{x_0 \in \mathbb{R}^n, \omega \in \Omega^{[t_0,\infty)}} T(x_0, \omega) = T_c.$$

□

2.4 A Class of predefined stable systems

Finally, a class of strongly predefined-time stable systems is presented. These systems depend on the least upper bound for the settling time, i.e., the strong predefined time, as an explicit parameter.

Definition 2.6 (Predefined-time stabilizing function)**.** *For* $x \in \mathbb{R}^n$*, the predefined-time stabilizing function is defined as*

$$\Phi_{m,q}(x) = \frac{1}{mq}\exp\left(\|x\|^{mq}\right)\frac{x}{\|x\|^{mq}}, \tag{8}$$

where $m \geq 1$ *and* $0 < q \leq \frac{1}{m}$*.*

Some important properties of the function (8) are:

1. The Maclaurin representation of (8) is given by

$$\Phi_{m,q}(x) = \frac{1}{mq}\left[\frac{x}{\|x\|^{mq}} + x + x\sum_{i=2}^{\infty}\frac{\|x\|^{i}}{i!}\right], \tag{9}$$

where it is observed that $\Phi_{m,q}(x)$ is continuous and non-Lipschitz for $0 < q < \frac{1}{m}$ and, discontinuous for $q = \frac{1}{m}$.

2. For $x \in \mathbb{R}$, the predefined-time stabilizing function can be written as $\Phi_{m,q}(x) = \frac{1}{mq}\exp\left(|x|^{mq}\right)|x|^{1-mq}\mathrm{sign}(x)$ for $0 < q < \frac{1}{m}$ and, $\Phi_{m,q}(x) = \exp\left(|x|\right)\mathrm{sign}(x)$ for $q = \frac{1}{m}$.

With the definition of this stabilizing function, the following lemma presents a dynamical system with the strong predefined-time stability property.

Lemma 2.1 (A predefined-time stable dynamical system)**.** *For every initial condition x_0, the system*

$$\dot{x} = -\frac{1}{T_c}\Phi_{m,q}(x) \tag{10}$$

with $T_c > 0$, $m \geq 1$ and $0 < q \leq \frac{1}{m}$ is globally strongly predefined-time stable with strong predefined time T_c. That is, $x(t) = 0$ for all $t \geq t_0 + T_c$ in spite of the value of x_0.

Proof. Consider the Lyapunov function $V = \|x\|^{m}$, defined for $x \in \mathbb{R}^n$. The derivative of V along the trajectories of (10) is

$$\begin{aligned}
\dot{V} &= m\|x\|^{m-1}\frac{x^T}{\|x\|}\dot{x} = m\|x\|^{m-2}x^T\dot{x} \\
&= -\frac{1}{qT_c}\|x\|^{m-2}\exp\left(\|x\|^{mq}\right)x^T\frac{x}{\|x\|^{mq}} \\
&= -\frac{1}{qT_c}\exp\left(\|x\|^{mq}\right)\|x\|^{m(1-q)} \\
&= -\frac{1}{qT_c}\exp\left(V^{q}\right)V^{1-q},
\end{aligned}$$

which is negative definite. Therefore, system (10) is asymptotically stable. In addition, considering that V is a continuous radially unbounded function, from *Theorem 2.2*, the desired result follows. □

Example 2.1 (A multivariable case)**.** *Consider the system*

$$\dot{x} = -\frac{1}{T_c}\Phi_{m,q}(x - x_{ss})$$

with $x \in \mathbb{R}^3$, $x_{ss} = [3, 0, -1]^T$, $T_c = 0.1$ time units, $m = 1$ and $q = 1/2$.

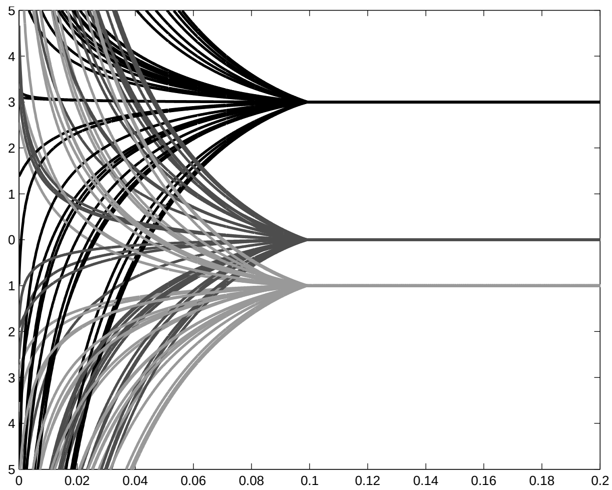

Figure 1: Time response of the system state variables.

Figure 1 shows the trajectories of the system for several initial conditions. It can be observed that all these trajectories converge to the equilibrium point x_{ss} at least in the strong predefined time $T_c = 0.1$.

3 Influence of the Tuning Parameters

In this section, the influence of the product mq, on the dynamics of the predefined-time stable system (10), as well as the effect of both mq and T_c in the performance of this system in the case of noisy measurements, are analyzed.

3.1 Influence of mq on the dynamics of the system

With the purpose of evaluating how the parameters m and q, or more precisely the product mq, affect the behavior of the strongly predefined-time stable system given by (10), the following function is introduced:

$$W(t) = \|x(t)\|.$$

The first aspect to be considered is the smoothness of the convergence of system (10) to the manifold $x = 0$. In order to do this, the time derivative of $W(t)$ is required; since $\|x\|$ is a valid Lyapunov function that satisfies the equality version of (5) with $p = mq$, this derivative is given by

$$\dot{W}(t) = -\frac{1}{mqT_c}\exp\left(W(t)^{mq}\right)W(t)^{1-mq}. \tag{11}$$

Let $T(x_0)$ be the exact settling time of system (10); that is, $T(x_0) = T_c\left[1 - \exp\left(-W_0^{mq}\right)\right]$. Then, $\dot{W}(T(x_0) + t_0)$ is the first time derivative of $\|x\|$ at the exact moment of convergence and, since $W(T(x_0) + t_0) = 0$, it follows that

$$\dot{W}(T(x_0) + t_0) = -\frac{0^{1-mq}}{mqT_c}.$$

Thus, $\|x\|$ converges to zero with a time derivative that is equal to zero for $0 < mq < 1$ and undefined for $mq = 1$. However, if $mq = 1$, it can be proved that

$$\lim_{t \to T(x_0)+t_0^-} \dot{W}(t) = -\frac{1}{T_c}.$$

In spite of this, values of mq higher than $\frac{1}{2}$ produce a visually abrupt convergence. In order to understand this phenomenon, an analysis of the second derivative of $W(t)$ at the moment of converge is performed as well.

Differentiating (11) produces

$$\begin{aligned}\ddot{W}(t) &= \frac{(mq - 1 - mqW(t)^{mq})W(t)^{-mq}}{mqT_c\exp\left(-W(t)^{mq}\right)}\dot{W}(t) \\ &= \frac{(1 - mq + mqW(t)^{mq})W(t)^{1-2(mq)}}{(mqT_c)^2\exp\left(-2W(t)^{mq}\right)}.\end{aligned}$$

Then, the second time derivative of $\|x\|$ at the exact moment of convergence is given by

$$\ddot{W}(T(x_0) + t_0) = \frac{(1 - mq)0^{1-2(mq)}}{(mqT_c)^2}.$$

Thus, $\|x\|$ converges to zero with a second time derivative that is equal to zero for $0 < mq < \frac{1}{2}$, undefined for $mq = \frac{1}{2}$, and infinite for $mq > \frac{1}{2}$. However, if $mq = \frac{1}{2}$, it can be proved that

$$\lim_{t \to T(x_0)+t_0^-} \ddot{W}(t) = \frac{2}{T_c^{\,2}}.$$

The fact that $\ddot{W}(T(x_0) + t_0)$ is infinite for $mq > \frac{1}{2}$ explains why the convergence of $\|x\|$ to zero is appreciably less smooth in this case than when $mq \le \frac{1}{2}$.

The second aspect that should be studied is the steepness of the initial response of $\|x\|$, which is characterized by $\dot{W}(t_0)$. Letting $x_0 = x(t_0)$, it follows from (11) that

$$\dot{W}(t_0) = -\frac{1}{mqT_c} \exp\left(\|x_0\|^{mq}\right) \|x_0\|^{1-mq}.$$

Figure 2 portrays $T_c\dot{W}(t_0)$ as a function of mq for several values of $\|x_0\|$ of both low and high orders of magnitude. It can be observed that, for every $\|x_0\|$, there exists a value of mq that maximizes $T_c\dot{W}(t_0)$; that is, that produces the least steep initial response of $\|x\|$. If mq is significantly far from this optimum value, the initial response might be so drastic to the extent of causing major errors in numerical simulations and digital implementations.

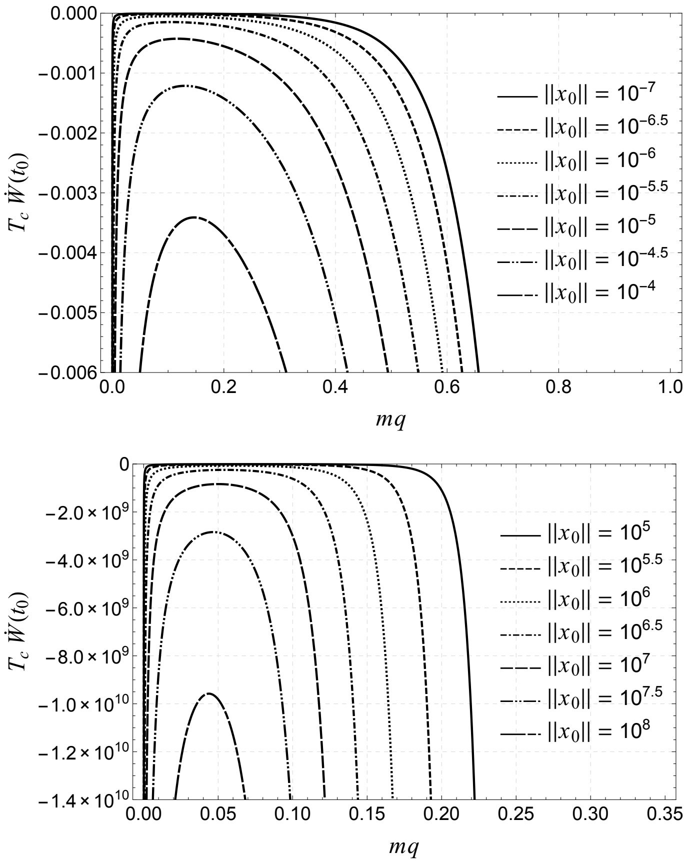

Figure 2: Steepness of $\|x\|$ at $t = t_0$ for several values of $\|x_0\|$.

A suggestion for the value of mq as a function of the norm of x_0 is provided in Figure 3. For a given $\|x_0\|$, this suggestion corresponds to the value of mq in $\left(0, \frac{1}{2}\right]$ that maximizes $T_c\dot{W}(t_0)$. The resulting suggested value is valid for any constant $T_c > 0$ since, naturally, it would also maximize $\dot{W}(t_0)$. The restriction for mq to be less than or equal to $\frac{1}{2}$ is placed in order to maintain an acceptable smoothness at the time of convergence given that, as has been discussed, a larger value would cause $\|x\|$ to reach zero with an infinite second time derivative.

Remark 3.1. *Adjusting the parameters of a predefined-time stable system in accordance with detailed knowledge of its initial conditions would certainly defeat the purpose of predefined-time stability itself. However, it should be noted that using the suggestion given by Figure 3 based on the expected order of magnitude of $\|x_0\|$ should suffice to avoid extremely steep initial responses and their related complications.*

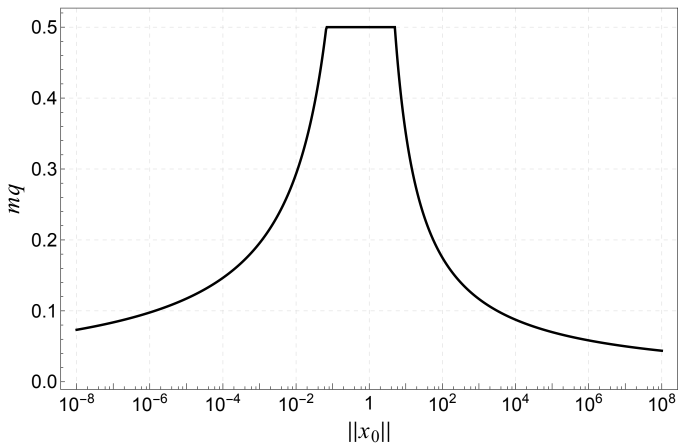

Figure 3: Suggested value for mq as a function of the initial condition.

3.2 Noisy measurements

In the non-ideal case where the measurement of x presents noise, system (10) becomes

$$\dot{x} = -\frac{1}{T_c}\Phi_{m,q}(x+v), \tag{12}$$

where v is a random variable that represents the noise. In this scenario the trajectories of the system cannot be perfectly confined to the manifold $x = 0$ and some error is unavoidable. Certainly, the magnitude of this error would depend on the values of the parameters T_c and mq and this dependence is to be assessed. In order to do so, the scalar version of system (12) was simulated for several values of T_c and mq and with a Gaussian-distributed noise v. This noise was zero-mean and with a standard deviation of 0.1. The simulations were carried out through Euler's method with a sampling period of 0.01 time units and from the initial condition $x(0) = 0$. In each case, the average absolute error from $t = 0$ to $t = 400$, given by

$$\frac{1}{400}\int_0^{400} |x(t)|\, dt,$$

was calculated. The results for this average absolute error as a function of the tuning parameters are depicted in Figure 4.

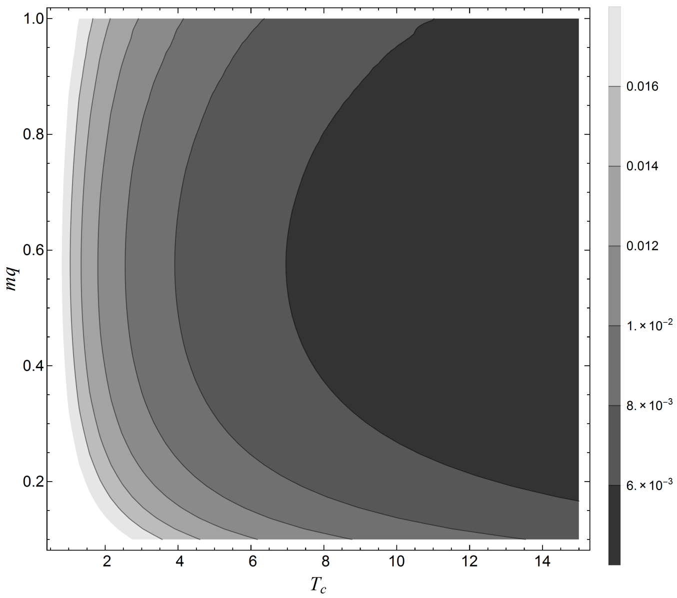

Figure 4: Average absolute error as a function of the tuning parameters for a predefined-stable system with noisy measurements.

As it was expected, the effect of the noise grows in magnitude as T_c decreases, since smaller values of T_c impose a faster and steeper convergence. The effect of mq, however, was not so clear since the several occurrences of this expression in the differential equation of the system have different influences. In particular, as mq increases, the factors $\frac{1}{mq}$ and $\frac{1}{\|x+v\|^{mq}}$ decrease but $\exp \|x+v\|^{mq}$ increases. The evidence in Figure 4 suggests that this opposing effects come to the best balance when mq is near the middle of the interval $(0, 1]$.

4 Predefined-Time Sliding Mode Controllers

A basic problem in the design of feedback control systems is the stabilization and tracking in the presence of uncertainty caused by plant parameters variation and external perturbations. In order to deal with this problem, several approaches have been proposed. Most of them are based on Lyapunov stability theory and variable structure systems with sliding modes (SM). The SM techniques are based on the idea of the sliding manifold, that is, an integral manifold with finite reaching time (Drakunov and Utkin, 1992), and have been widely used for the problems of control and observation of dynamical systems due to their characteristics of finite time convergence as well as robustness and insensitivity to uncertainties due to external bounded disturbances and parameters variation (Utkin, 1992, Utkin et al., 2009). With this idea, the aim of this section is to present a class of first order sliding mode controllers with the novel property of a predefined-time reaching phase.

4.1 Motivation

In order to apply the previous results to sliding mode controller design, consider the dynamical system

$$\dot{x} = u + \Delta(t, x) \tag{13}$$

with $x, u \in \mathbb{R}^n$, $\Delta : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $t_0 = 0$. The main objective is to stabilize system (13) at the point $x = 0$ in a predefined time T_c, starting from an arbitrary state $x_0 = x(0)$ and in spite of the unknown disturbance $\Delta(t, x)$.

Firstly, weakly and strongly predefined-time stable continuous controllers are presented. Also, a stability and robustness analysis in the presence of a class of vanishing perturbation is performed.

Lemma 4.1 (A weak predefined-time controller)**.** *Let the function $\Delta(t, x)$ be considered as a vanishing perturbation term such that $\|\Delta(t, x)\| \leq \delta \|x\|$, with $0 < \delta < \infty$ a known constant. Then, by selecting the* *control input*

$$u = -\left(\frac{1}{T_c} + kmq\right)\Phi_{m,q}(x) \tag{14}$$

with $T_c > 0$, $m \geq 1$, $0 < q < \frac{1}{m}$, and $k \geq \delta$, the system (13) closed by (14) is globally weakly predefined-time stable with T_c as the weak predefined time.

Proof. Consider the Lyapunov function $V = \|x\|^m$ defined for $x \in \mathbb{R}^n$; its derivative along the trajectories of (13)-(14) is given by $\dot{V} = m\|x\|^{m-2} x^T\dot{x}$. Therefore,

$$\begin{aligned}\dot{V} &= m\|x\|^{m-2} x^T\left[\Delta(t,x) - \left(\frac{1}{T_c} + kmq\right)\Phi_{m,q}(x)\right] \\ &= m\left[\|x\|^{m-2} x^T\Delta(t,x) - k\exp(\|x\|^{mq})\|x\|^{m(1-q)}\right] - \frac{1}{qT_c}\exp(\|x\|^{mq})\|x\|^{m(1-q)}.\end{aligned}$$

From expression (9), it follows that

$$\dot{V} \leq -\frac{1}{qT_c}\exp(\|x\|^{mq})\|x\|^{m(1-q)} + m\left[\|x\|^{m-2}|x^T\Delta(t,x)| - k\|x\|^m\right].$$

Applying the Cauchy–Schwarz inequality,

$$\begin{aligned}\dot{V} &\leq -\frac{1}{qT_c}\exp(\|x\|^{mq})\|x\|^{m(1-q)} + m\left[\|x\|^{m-2}\|x\|\|\Delta(t,x)\| - k\|x\|^m\right] \\ &\leq -\frac{1}{qT_c}\exp(\|x\|^{mq})\|x\|^{m(1-q)} - (k-\delta)m\|x\|^m.\end{aligned}$$

It is observed that the system is globally asymptotically stable. Moreover,

$$\dot{V} \leq -\frac{1}{qT_c}\exp(V^q)V^{1-q}.$$

Then, by direct application of *Theorem 2.1*, the proof is finished. □

Lemma 4.2 (A strong predefined-time controller)**.** *Let the function $\Delta(t,x)$ be considered as a vanishing perturbation term such that $\|\Delta(t,x)\| \leq \delta\|x\|$, with $0 < \delta < \infty$ a known constant. Then, by selecting the control input*

$$u = -\frac{1}{T_c}\Phi_{m,q}(x) - \delta x \tag{15}$$

with $T_c > 0$, $m \geq 1$, and $0 < q < \frac{1}{m}$, the system (13) closed by (15) is globally strongly predefined-time stable with T_c as the strong predefined time.

Proof. Consider the Lyapunov function $V = \|x\|^m$ defined for $x \in \mathbb{R}^n$; its derivative along the trajectories of (13)-(15) is given by

$$\begin{aligned}\dot{V} &= m\|x\|^{m-2} x^T\left[\Delta(t,x) - \frac{1}{T_c}\Phi_{m,q}(x) - \delta x\right] \\ &= -\frac{1}{qT_c}\exp(\|x\|^{mq})\|x\|^{m(1-q)} + m\|x\|^{m-2}\left[x^T\Delta(t,x) - \delta\|x\|^2\right].\end{aligned}$$

Since $x^T\Delta(t,x) \leq \|x\|\|\Delta(t,x)\| \leq \delta\|x\|^2$, the expression $m\|x\|^{m-2}\left[x^T\Delta(t,x) - \delta\|x\|^2\right]$ is non-positive. Furthermore, for any $x \in \mathbb{R}^n$, this expression can equal zero in the particular scenario where $x^T\Delta(t,x) = \delta\|x\|^2$; that is,

$$\sup_{\|\Delta(t,x)\|\leq\delta\|x\|} m\|x\|^{m-2}\left[x^T\Delta(t,x) - \delta\|x\|^2\right] = 0 \quad \forall\ x \in \mathbb{R}^n.$$

From this it can be concluded that

$$\sup_{\|\Delta(t,x)\|\leq\delta\|x\|} \dot{V} = -\frac{1}{qT_c}\exp(\|x\|^{mq})\|x\|^{m(1-q)} = -\frac{1}{qT_c}\exp(V^q)V^{1-q} \quad \forall\ x \in \mathbb{R}^n.$$

Then, by direct application of *Theorem 2.3*, the proof is finished. □

Secondly, a continuous controller is analyzed for the case of non-vanishing perturbations.

Lemma 4.3 (Continuous controller in presence of of non-vanishing perturbations)**.** *Let the function* $\Delta(t,x)$ *be considered as a non-vanishing bounded disturbance such that* $\|\Delta(t,x)\| \leq \delta$*, with* $0 < \delta < \infty$ *a known constant. Then, by selecting the control input*

$$u = -\frac{1}{T_c}\Phi_{m,q}(x) \tag{16}$$

with $T_c > 0$*,* $m \geq 1$*, and* $0 < q < \frac{1}{m}$*, the system (13) closed by (16) is uniformly ultimately-bounded of the form* $\|x\| \leq \left[\frac{1-mq}{mq}\mathcal{W}\left(\frac{mq}{1-mq}(mqT_c\delta)^{\frac{mq}{1-mq}}\right)\right]^{\frac{1}{mq}}$ *with* T_c *as an upper bound for the convergence time to this region. Here,* $\mathcal{W}(\cdot)$ *stands for the Lambert function (Lambert, 1758), the inverse function of* $f(\xi) = \xi\exp(\xi)$ *for* $\xi \in \mathbb{R}$*, i.e.* $\xi = \mathcal{W}(\xi\exp(\xi))$ *(see Corless et al. (1996)).*

Proof. Consider the Lyapunov function $V = \|x\|^m$ defined for $x \in \mathbb{R}^n$; its derivative along the trajectories of (13)-(16) is given by $\dot{V} = m\|x\|^{m-2} x^T\dot{x}$. Therefore,

$$\begin{aligned}\dot{V} &= m\|x\|^{m-2} x^T\left[\Delta(t,x) - \frac{1}{T_c}\Phi_{m,q}(x)\right]\\ &= m\|x\|^{m-2}\left[x^T\Delta(t,x) - \frac{1}{mqT_c}\exp\left(\|x\|^{mq}\right)\|x\|^{2-mq}\right].\end{aligned}$$

From expression (9), it follows that

$$\dot{V} \leq -m\|x\|^{m-2}\left[\frac{1}{mqT_c}\exp\left(\|x\|^{mq}\right)\|x\|^{2-mq} - |x^T\Delta(t,x)|\right].$$

Applying the Cauchy–Schwarz inequality,

$$\begin{aligned}\dot{V} &\leq -m\|x\|^{m-2}\left[\frac{1}{mqT_c}\exp\left(\|x\|^{mq}\right)\|x\|^{2-mq} - \|x\|\,\|\Delta(t,x)\|\right]\\ &\leq -m\|x\|^{m-1}\left[\frac{1}{mqT_c}\exp\left(\|x\|^{mq}\right)\|x\|^{1-mq} - \delta\right].\end{aligned}$$

It is observed that, in order to obtain $\dot{V} < 0$, it is necessary that $\frac{1}{mqT_c}\exp\left(\|x\|^{mq}\right)\|x\|^{1-mq} > \delta$. Solving this expression for $\|x\|$ yields

$$\|x\| > \left[\frac{1-mq}{mq}\mathcal{W}\left(\frac{mq}{1-mq}(mqT_c\delta)^{\frac{mq}{1-mq}}\right)\right]^{\frac{1}{mq}}$$

and, therefore, the system is uniformly ultimately-bounded.

Moreover, for $\|x\| > \left[\frac{1-mq}{mq}\mathcal{W}\left(\frac{mq}{1-mq}(mqT_c\delta)^{\frac{mq}{1-mq}}\right)\right]^{\frac{1}{mq}}$, it holds that

$$\dot{V} \leq -\frac{1}{qT_c}\exp(V^q)V^{1-q}.$$

Then, from arguments similar to those used to prove *Theorem 2.1*, it is clear that the region $\|x\| \leq \left[\frac{1-mq}{mq}\mathcal{W}\left(\frac{mq}{1-mq}(mqT_c\delta)^{\frac{mq}{1-mq}}\right)\right]^{\frac{1}{mq}}$ is reached before $t = T_c$. □

Finally, in order to improve the robustness of the continuous controller, weak predefined-time and strong predefined-time stable discontinuous controllers, including integral sliding mode extensions, are introduced for the case of non-vanishing perturbations.

In order to obtain discontinuous controllers, the parameters m and q may be set such that $mq = 1$. In this case, the function $\Phi_{m,q}(x)$ is written as $\Phi_1(x)$.

4.2 Robust predefined-time discontinuous controllers

Lemma 4.4 (A robust weak predefined-time controller)**.** *Let the function $\Delta(t,x)$ be considered as a non-vanishing bounded disturbance such that $\|\Delta(t,x)\| \le \delta$, with $0 < \delta < \infty$ a known constant. Then, by selecting the control input*

$$u = -\left(\frac{1}{T_c} + k\right)\Phi_1(x) \tag{17}$$

with $T_c > 0$ and $k \ge \delta$, the system (13) closed by (17) is globally weakly fixed-time stable with T_c as the weak predefined time.

Proof. Consider the Lyapunov function $V = \|x\|^m$ defined for $x \in \mathbb{R}^n$; its derivative along the trajectories of (13)-(17) is given by $\dot{V} = m\,\|x\|^{m-2}\,x^T\dot{x}$. Therefore,

$$\begin{aligned}\dot{V} &= m\,\|x\|^{m-2}\,x^T\left[\Delta(t,x) - \left(\frac{1}{T_c} + k\right)\Phi_1(x)\right] \\ &= m\left[\|x\|^{m-2}\,x^T\Delta(t,x) - k\exp(\|x\|)\,\|x\|^{m-1}\right] - \frac{m}{T_c}\exp(\|x\|)\,\|x\|^{m-1}.\end{aligned}$$

From expression (9), it follows that

$$\dot{V} \le -\frac{m}{T_c}\exp(\|x\|)\,\|x\|^{m-1} + m\left[\|x\|^{m-2}\,|x^T\Delta(t,x)| - k\,\|x\|^{m-1}\right].$$

Applying the Cauchy–Schwarz inequality,

$$\begin{aligned}\dot{V} &\le -\frac{m}{T_c}\exp(\|x\|)\,\|x\|^{m-1} + m\left[\|x\|^{m-2}\,\|x\|\,\|\Delta(t,x)\| - k\,\|x\|^{m-1}\right] \\ &\le -\frac{m}{T_c}\exp(\|x\|)\,\|x\|^{m-1} - (k-\delta)m\,\|x\|^{m-1}.\end{aligned}$$

It is observed that the system is globally asymptotically stable. Moreover,

$$\dot{V} \le -\frac{1}{qT_c}\exp(V^q)V^{1-q}.$$

Then, by direct application of *Theorem 2.1*, the proof is finished. □

Lemma 4.5 (A robust strong predefined-time controller)**.** *Under the same conditions of Lemma 4.4, the selection of the control input*

$$u = -\delta\frac{x}{\|x\|} - \frac{1}{T_c}\Phi_{m,q}(x) \tag{18}$$

with $T_c > 0$, $m \ge 1$, and $0 < q \le \frac{1}{m}$, leads to the closed-loop system (13)-(18) that is globally strongly predefined-time stable with T_c as the strong predefined time.

Proof. Consider the Lyapunov function $V = \|x\|^m$ defined for $x \in \mathbb{R}^n$; its derivative along the trajectories of (13)-(18) is given by

$$\begin{aligned}\dot{V} &= m\,\|x\|^{m-2}\,x^T\left[\Delta(t,x) - \delta\frac{x}{\|x\|} - \frac{1}{T_c}\Phi_{m,q}(x)\right] \\ &= -\frac{1}{qT_c}\exp(\|x\|^{mq})\,\|x\|^{m(1-q)} + m\,\|x\|^{m-2}\left[x^T\Delta(t,x) - \delta\,\|x\|\right].\end{aligned}$$

Since $x^T\Delta(t,x) \le \|x\|\,\|\Delta(t,x)\| \le \delta\,\|x\|$, the expression $m\,\|x\|^{m-2}\left[x^T\Delta(t,x) - \delta\,\|x\|\right]$ is non-positive. Moreover, for any $x \in \mathbb{R}^n$, this expression can equal zero in the particular scenario where $x^T\Delta(t,x) = \delta\,\|x\|$; that is,

$$\sup_{\|\Delta(t,x)\| \le \delta\|x\|} m\,\|x\|^{m-2}\left[x^T\Delta(t,x) - \delta\,\|x\|\right] = 0 \quad \forall\; x \in \mathbb{R}^n.$$

From this it can be concluded that

$$\sup_{\|\Delta(t,x)\| \le \delta\|x\|} \dot{V} = -\frac{1}{qT_c}\exp(\|x\|^{mq})\,\|x\|^{m(1-q)} = -\frac{1}{qT_c}\exp(V^q)V^{1-q} \quad \forall\; x \in \mathbb{R}^n.$$

Then, by direct application of *Theorem 2.3*, the proof is finished. □

Based on the integral sliding mode approach proposed by Matthews and DeCarlo (1988), Utkin and Shi (1996), the following result presents an integral controller with predefined-time stability.

Lemma 4.6 (An integral weak predefined-time controller)**.** *Under the same conditions of Lemma 4.4, let the selection of the control input be*

$$u = -\frac{1}{T_{c_n} - T_{c_i}}\Phi_{m_n,q_n}(x) - \frac{1}{T_{c_i}}\Phi_{m_i,q_i}(\sigma) - \delta\frac{\sigma}{\|\sigma\|} \tag{19}$$

where $\sigma = x + z$ *and* $\dot{z} = \frac{1}{T_{c_n}-T_{c_i}}\Phi_{m_n,q_n}(x)$*, with* $T_{c_n} > T_{c_i} > 0$*,* $m_n, m_i \geq 1$*,* $0 < q_n \leq \frac{1}{m_n}$*, and* $0 < q_i \leq \frac{1}{m_i}$*. Then, the closed-loop system (13)-(19) is globally weakly predefined-time stable with* T_{c_n} *as the weak predefined time.*

Proof. The closed-loop system (13)-(19) is given by

$$\dot{x} = -\frac{1}{T_{c_n} - T_{c_i}}\Phi_{m_n,q_n}(x) - \frac{1}{T_{c_i}}\Phi_{m_i,q_i}(\sigma) - \delta\frac{\sigma}{\|\sigma\|} + \Delta(t,x) \tag{20}$$

$$\dot{\sigma} = -\frac{1}{T_{c_i}}\Phi_{m_i,q_i}(\sigma) - \delta\frac{\sigma}{\|\sigma\|} + \Delta(t,x) \tag{21}$$

From *Lemma 4.5*, it follows that the sub-system (21) is confined to the manifold $\sigma = 0$ within a strong predefined time T_{c_i}. Consequently, after $t = T_{c_i}$, system (20)-(21) reduces to

$$\dot{x} = -\frac{1}{T_{c_n} - T_{c_i}}\Phi_{m_n,q_n}(x). \tag{22}$$

Taking $t_0 = T_{c_i}$ as the initial time, then, from *Lemma 2.1*, the system (22) is globally strongly predefined-time stable with $T_{c_n} - T_{c_i}$ as the strong predefined time time. Therefore, the system reaches and stays at $x = 0$ by $t = T_{c_i} + T_{c_n} - T_{c_i} = T_{c_n}$; nevertheless, since the convergence of (21) to $\sigma = 0$ does not occur exactly at $t = T_{c_i}$, it cannot be assured that T_{c_n} is the least upper bound of the settling time. Thus, system (13)-(19) as a whole is weakly predefined-time stable with T_{c_n} as the weak predefined time. □

When disturbances are present, it is observed that the information available about their bounds influences the stability properties of the system. For example, the ultimate bound for the continuous controller depends on δ and the strength or weakness of the discontinuous controllers depend on the use of δ in the control law. With the aim of overcoming this dependence, the following result presents an integral controller such that strong predefined-time stability holds for any choice of $k \geq \delta$ as the gain of the discontinuous term.

Lemma 4.7 (An integral strong predefined-time controller)**.** *Under the same conditions of Lemma 4.4, let the selection of the control input be*

$$u = -\frac{h(t - T_{c_i})}{T_{c_n} - T_{c_i}}\Phi_{m_n,q_n}(x) - \frac{1}{T_{c_i}}\Phi_{m_i,q_i}(\sigma) - k\frac{\sigma}{\|\sigma\|} \tag{23}$$

where $h(\xi)$ *is the Heaviside step function,* $\sigma = x+z$ *and* $\dot{z} = \frac{h(t-T_{c_i})}{T_{c_n}-T_{c_i}}\Phi_{m_n,q_n}(x)$*, with* $k \geq \delta$*,* $T_{c_n} > T_{c_i} > 0$*,* $m_n, m_i \geq 1$*,* $0 < q_n \leq \frac{1}{m_n}$*, and* $0 < q_i \leq \frac{1}{m_i}$*. Then, the closed-loop system (13)-(23) is globally strongly predefined-time stable with* T_{c_n} *as the strong predefined time.*

Proof. The closed-loop system (13)-(23) is given by

$$\dot{x} = -\frac{h(t - T_{c_i})}{T_{c_n} - T_{c_i}}\Phi_{m_n,q_n}(x) - \frac{1}{T_{c_i}}\Phi_{m_i,q_i}(\sigma) - k\frac{\sigma}{\|\sigma\|} + \Delta(t,x) \tag{24}$$

$$\dot{\sigma} = -\frac{1}{T_{c_i}}\Phi_{m_i,q_i}(\sigma) - k\frac{\sigma}{\|\sigma\|} + \Delta(t,x). \tag{25}$$

From the hypothesis $k \geq \delta$, it follows that the sub-system (25) stabilizes in a time $T_{c_i}^-$ such that $T_{c_i}^- \leq T_{c_i}$. If $T_{c_i}^- = T_{c_i}$, the strong predefined-time stability follows from Lemma 4.6. Otherwise, system (24)-(25) reduces to

$$\begin{cases} \dot{x} = 0 \text{ for } T_{c_i}^- \leq t < T_{c_i} \\ \dot{x} = -\frac{1}{T_{c_n}-T_{c_i}}\Phi_{m_n,q_n}(x) \text{ for } t \geq T_{c_i}. \end{cases}$$

Therefore, from Lemma 2.1, system (24)-(25) is strongly predefined-time stable with $t = T_{c_i} + T_{c_n} - T_{c_i} = T_{c_n}$ as the strong predefined time. □

The following example illustrates the controller proposed in *Lemma 4.5* and how the exact settling time of the corresponding closed-loop system approaches the strong predefined time as the trajectory of the disturbance and the magnitude of the initial condition worsen.

Example 4.1. *Consider the following family of scalar dynamical systems indexed by parameter $\alpha \in \mathbb{R}_+$:*

$$\dot{x} = \cos\left(\frac{t}{\alpha}\right) - \text{sign}(x) - \frac{1}{T_c}\Phi_{m,q}(x), \qquad x(0) = \alpha, \tag{26}$$

where $x \in \mathbb{R}$ and the initial time is $t_0 = 0$. Each system of this family is of the form (13) closed by (18), with $\Delta(t,x) = \cos(t/\alpha)$ as the disturbance term and with $\delta = 1$. Given that $\|\Delta(t,x)\| \leq \delta = 1$, it follows from Lemma 4.5 that all systems of family (26) converge to $x = 0$ within a strong predefined time T_c. However, since both the disturbance trajectory and the initial condition are completely determined by α, the exact settling time will depend solely on this parameter and can be seen as a function of the form $T(\alpha)$.

Figure 5 portrays trajectories of (26) for several values of the parameter α, while Figure 6 depicts the exact settling time as a function of α. In both cases, the strong predefined time was set to $T_c = 1$ and is marked by a gray, dashed line. The other parameters were chosen so that $mq = 0.4$.

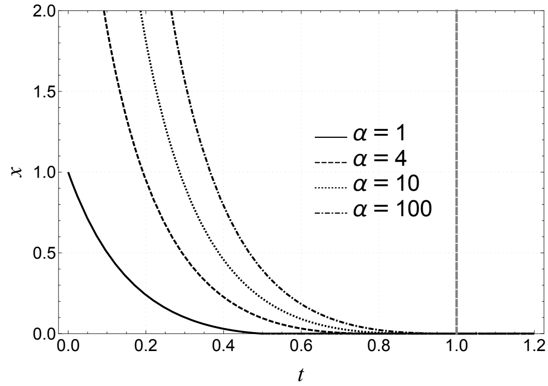

Figure 5: Trajectories of (26) for several values of α.

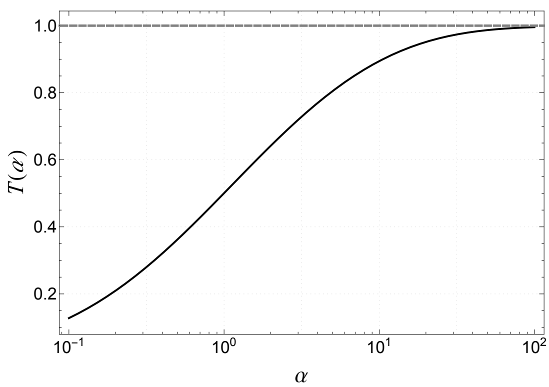

Figure 6: Exact settling time of (26) as a function of α.

As it was expected, Figure 6 shows that the supremum of the settling-time function is T_c. This strong predefined time is approached by the exact settling time for large values of α because, as α increases, the initial condition grows in magnitude and the disturbance term approaches the constant function $\Delta(t,x) = 1$, which is the worst-case scenario considered by the restriction $\|\Delta(t,x)\| \leq 1$ for a positive initial condition.

4.3 First order predefined-time sliding mode controllers

Consider the system

$$\dot{x} = f(t,x) + B(t,x)u + \Delta(t,x) \tag{27}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the system state and $\mathcal{X}$ is a non-empty set, $u \in \mathbb{R}^r$ with $r \leq n$ is the control input of the system, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n \times r}$, and $\Delta : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. The time variable t is defined on the interval $[t_0, \infty)$, where $t_0 \in \mathbb{R}_+ \cup \{0\}$. For this system, the initial conditions are $x_0 = x(t_0)$. In addition, let the function $\sigma : \mathbb{R}^n \to \mathbb{R}^r$.

The main objective of the controller is to drive the trajectories of system (27) to the manifold $\sigma(x) = 0$. The function σ is selected so that the motion on the sliding manifold $\sigma(x) = 0$ has a desired behavior.

Letting $G(x) = \frac{\partial \sigma(x)}{\partial x}$, define $D(x)$ as

$$D(x) = G(x)B(t,x)$$

It is assumed that the matrix $D(x)$ has an inverse for all $x \in \mathcal{X}$.

The following lemmas provide controllers which induce a sliding mode in $\sigma(x) = 0$ in a strong predefined time T_c. Three scenarios are presented depending on the perturbation nature. The first case is for the non-perturbed system, the second one for vanishing perturbations, and the third one for non-vanishing perturbations. For the last two cases, the perturbation is considered to be matched (Drazenovich, 1969).

Lemma 4.8 (Controller for an unperturbed system)**.** *For system* (27) *with* $\Delta(t,x) = 0$*, the selection of the control input*

$$u = -D^{-1}(x)\left[G(x)f(t,x) + \frac{1}{T_c}\Phi_{m,q}(\sigma)\right] \tag{28}$$

with $T_c > 0$*,* $m \geq 1$*, and* $0 < q \leq \frac{1}{m}$*, induces a strong predefined-time sliding mode in* $\sigma(x) = 0$ *with* T_c *as the least upper bound for the settling time.*

Proof. The dynamics of $\sigma(x)$ are given by the first order system

$$\dot{\sigma} = G(x)f(t,x) + D(x)u. \tag{29}$$

Equation (29) with the controller presented in (28) reduces to

$$\dot{\sigma} = -\frac{1}{T_c}\Phi_{m,q}(\sigma).$$

Therefore, from *Lemma 2.1*, the manifold $\sigma(x) = 0$ is reached in strong a predefined time T_c. □

Lemma 4.9 (Systems with vanishing perturbation)**.** *Let the function* $\Delta(t,x)$ *be considered as a matched and vanishing perturbation term. Hence, there exists a function* $\bar{\Delta}(t,x)$ *such that* $\Delta(t,x) = B(t,x)\bar{\Delta}(t,x)$ *and* $\left\|D(x)\bar{\Delta}(t,x)\right\| \leq \delta\left\|x\right\|$*, where* $0 < \delta < \infty$ *is a known constant. Then, the control input*

$$u = -D^{-1}(x)\left[G(x)f(t,x) + \frac{1}{T_c}\Phi_{m,q}(\sigma) - \delta\sigma\right] \tag{30}$$

with $T_c > 0$*,* $m \geq 1$*, and* $0 < q \leq \frac{1}{m}$*, induces a sliding mode in* $\sigma(x) = 0$ *with strong predefined time* T_c*.*

Proof. The dynamics of $\sigma(x)$ are given by

$$\dot{\sigma} = G(x)f(t,x) + D(x)\left(u + \bar{\Delta}(t,x)\right). \tag{31}$$

Equation (31) with the controller presented in (30) reduces to

$$\dot{\sigma} = -\frac{1}{T_c}\Phi_{m,q}(x) - \delta\sigma + D(x)\bar{\Delta}(t,x).$$

In this way, by direct application of *Lemma 4.2*, the manifold $\sigma(x) = 0$ is reached in a strong predefined time T_c. □

Lemma 4.10 (Systems with non-vanishing perturbation)**.** *For this case, let the function* $\Delta(t,x)$ *be considered as a matched and non-vanishing perturbation term. Hence, there exists a function* $\bar{\Delta}(t,x)$ *such that* $\Delta(t,x) = B(t,x)\bar{\Delta}(t,x)$ *and* $\left\|D(x)\bar{\Delta}(t,x)\right\| \leq \delta$*, with* $0 < \delta < \infty$ *a known constant. Then, the control input*

$$u = -D^{-1}(x)\left[G(x)f(t,x) + \delta\frac{x}{\|x\|} + \frac{1}{T_c}\Phi_{m,q}(x)\right] \tag{32}$$

with $T_c > 0$*,* $m \geq 1$*, and* $0 < q \leq \frac{1}{m}$*, induces a strong predefined-time sliding mode in* $\sigma(x) = 0$ *with* T_c *as the least upper bound for the settling time.*

Proof. The dynamics of $\sigma(x)$ are given by

$$\dot{\sigma} = G(x)f(t,x) + D(x)\left(u + \bar{\Delta}(t,x)\right). \tag{33}$$

Equation (33) with the controller presented in (32) reduces to

$$\dot{\sigma} = -\delta\frac{x}{\|x\|} - \frac{1}{T_c}\Phi_{m,q}(x) + D(x)\bar{\Delta}(t,x).$$

Thus, by direct application of *Lemma 4.5*, the manifold $\sigma(x) = 0$ is reached in a strong predefined time T_c. □

Example 4.2 (Control of the double integrator)**.** *Consider the system*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \Delta \end{aligned} \tag{34}$$

where Δ is a bounded disturbance such that $|\Delta| \leq \delta$ with $\delta > 0$. For this case, the main objective is to design a controller that drives the state x_1 to a constant trajectory x_{1r}.

Defining the variables $e_1 = (x_1 - x_{1r})/k$ and $e_2 = e_1 + x_2/k^2$ with $k > 0$, from (34) it follows that

$$\begin{aligned} \dot{e}_1 &= k(e_2 - e_1) \\ \dot{e}_2 &= k(e_2 - e_1) + u/k^2 + \Delta/k^2. \end{aligned} \tag{35}$$

Based on (18), the following control law is proposed:

$$u = -k^3(e_2 - e_1) - k^2\left(k_c sign(e_2) + \frac{1}{T_c}\Phi_{m,q}(e_2)\right). \tag{36}$$

Hence, system (35) closed by (36) is

$$\begin{aligned} \dot{e}_1 &= k(e_2 - e_1) \\ \dot{e}_2 &= -k_c sign(e_2) - \frac{1}{T_c}\Phi_{m,q}(e_2) + \Delta/k^2. \end{aligned} \tag{37}$$

Therefore, with $k_c = \delta/k^2$ and by Lemma 4.5, this control law guarantees that sliding motion on the manifold $e_2 = 0$ occurs in a strong predefined time T_c. The motion on this manifold is given by $\dot{e}_1 = -ke_1$ and, consequently, system (37) is exponentially stable.

For this case, let $T_c = 0.5$ time units, $k = 10$, $k_c = 0.01$, $m = 1$, $q = 1/2$, $x_{1r} = 5$, $\Delta = \sin(3t)$, and $\delta = 1$.

Figure 7 shows the trajectories of the variable e_2 for several initial conditions. It can be observed that all these trajectories converge to zero at least in the predefined time $T_c = 0.5$. Figure 8 shows the trajectories of the system variables x_1 and x_2 for several initial conditions. It can be observed that, once $e_2 = 0$, the trajectories of x_1 converge exponentially to five and the trajectories of x_2 converge exponentially to zero.

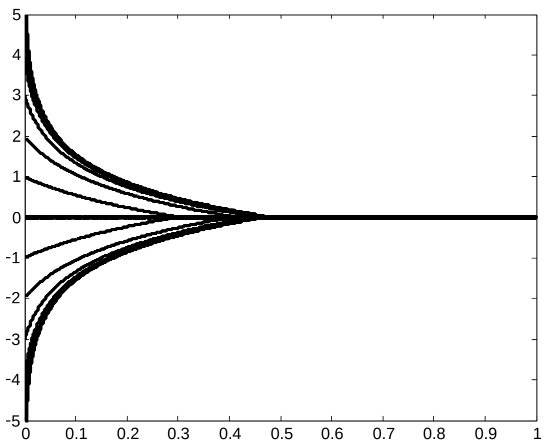

Figure 7: Time response of the sliding variable e_2.

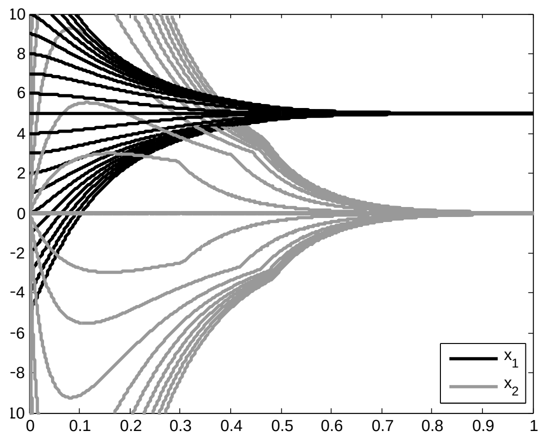

Figure 8: Time response of the system state variables.

Remark 4.1. *Similar results to those presented in Example 4.2 using Lemma 4.5 can be obtained with the application of the integral controllers given in Lemmas 4.6-4.7.*

5 A Predefined-Time Consensus Algorithm for Complete Networks

In the last decade there has been a great deal of attention placed on algorithms that achieve a goal in a self-organizing and distributed fashion. One of such algorithms is the consensus algorithm (see for example Olfati-Saber and Murray (2004), Cai (2012), Wang and Xiao (2010), Zuo and Tie (2014), and the references therein), in which a network agrees on a common value in a distributed fashion by communicating with its nearest neighbors. Such an algorithm has applications, for instance, in wireless sensor networks. In this regard, contributions have been presented in asymptotic consensus (Olfati-Saber and Murray, 2004, Cai, 2012), finite-time consensus (Cortés, 2006, Wang and Xiao, 2010), and fixed-time consensus (Zuo and Tie, 2014, Tian et al., 2016). However, a drawback of such approaches is that the convergence time, which is known to depend on the graph topology, is hard to estimate and the existing estimation methods are too conservative (Zuo and Tie, 2014). For this reason, methods for the design of consensus algorithms with predefined convergence time are of great interest.

In this section it is shown that, using the predefined-time stability framework presented herein, new consensus algorithms with predefined-time convergence can be proposed. For the sake of brevity, only complete graphs are considered. The results for general classes of graphs will be reported elsewhere.

5.1 Basic concepts on graph theory

Before presenting the proposed consensus algorithm and its convergence proof, some basic concepts on graph theory, which are mainly taken from Godsil and Royle (2001), are briefly introduced.

A graph $\mathcal{X}$ consists of a vertex (also called node) set $\mathcal{V}(\mathcal{X})$ and an edge set $\mathcal{E}(\mathcal{X})$ where an edge is an unordered pair of distinct vertices of $\mathcal{X}$. The notation ij is used to refer to an edge and it is said that j is a neighbor of i. The set of all neighbors of node i is denoted by $\mathcal{N}_i$ which has cardinality d_i. A graph is connected if for any two nodes i and j there is a sequence of distinct nodes starting at i and ending at j such that consecutive nodes are neighbors.

The Laplacian of $\mathcal{X}$ is $\mathcal{L} = Q - A$ where $Q = diag(d_1, \cdots, d_n)$ and $A = [a_{ij}]$ such that $a_{ij} = 1$ if the node i is a neighbor of node j and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L}$ is a positive semidefinite and symmetric matrix, thus its eigenvalues are all real and non-negative. If the graph $\mathcal{X}$ is connected, then the eigenvalue $\lambda_1(\mathcal{L}) = 0$ has algebraic multiplicity one with eigenvector $\mathbf{1} = [1 \ \cdots \ 1]^T$. For every graph $\mathcal{X}$, it holds that $x^T\mathcal{L}x = \sum_{ij\in\mathcal{E}(\mathcal{X})}(x_j - x_i)^2$ (Godsil and Royle, 2001). A graph with n nodes is complete if each node has $n-1$ neighbors. In a complete graph of n nodes, n is an eigenvalue of the Laplacian $\mathcal{L}$ with multiplicity $n-1$.

5.2 The proposed consensus algorithm

Let $\mathcal{X}$ be the underlying communication network and let

$$\mathcal{Z}_i(x) = (d_i + 1)\sum_{j\in\mathcal{N}_i}(x_j - x_i)^2.$$

Then the proposed consensus algorithm is

$$\begin{aligned}\dot{x}_i &= u_i,\\ u_i &= \frac{1}{2nT_cq}\frac{\exp\left(\mathcal{Z}_i(x)^q\right)}{\mathcal{Z}_i(x)^q}\sum_{j\in\mathcal{N}_i}\left(x_j - x_i\right),\end{aligned} \tag{38}$$

where $0 < q < \dfrac{1}{2}$ and $k \geq 0$.

In the following, it is shown that, for complete networks, (38) is a consensus algorithm with strong predefined-time convergence and T_c is the least upper bound for the settling time.

Lemma 5.1. *Let $e = -\mathcal{L}x$, then $\mathcal{L}e = ne$.*

Proof. Since $\mathcal{L}$ is symmetric, then there exist an orthonormal matrix $U = [v_1 \cdots v_n]$ formed by the eigenvectors of $\mathcal{L}$ such that $\mathcal{L} = UDU^T$, where $D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ with $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \ldots = \lambda_n = n$ because the graph is complete. Let x be expressed using the eigenvectors as a basis; then, $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ and $e = -\mathcal{L}x = -(\alpha_2\lambda_2 v_2 + \cdots + \alpha_n\lambda_n v_n)$. Following this procedure, $\mathcal{L}e = -(\alpha_2\lambda_2^2 v_2 + \cdots + \alpha_n\lambda_n^2 v_n) = \alpha_2 n^2 v_2 + \cdots + \alpha_n n^2 v_n = ne$ is obtained. □

Theorem 5.1. *Let $\mathcal{X}$ be the underlying communication network topology. Then, if $\mathcal{X}$ is a complete graph, algorithm (38) achieves consensus in the network within a strong predefined time T_c. That is, $x_1(t) = \cdots = x_n(t) \ \forall\, t \geq t_0 + T_c$.*

Proof. Let $e_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$ and $e = [e_1, \ldots, e_n]^T$ (notice that $e = -\mathcal{L}x$), and let

$$F(e) = \left[\begin{array}{ccc} u_1 & \cdots & u_n \end{array} \right]^T$$

where u_i is given by (38).

Then, the dynamics of the network under the communication topology $\mathcal{X}$ are given by

$$\dot{x} = F(e). \tag{39}$$

Let $e = -\mathcal{L}x$ be called the consensus error and notice that whenever $e = 0$, $x_1 = \cdots = x_n$, i.e., consensus is achieved. Then the consensus error dynamics are given by

$$\dot{e} = -\mathcal{L}F(e). \tag{40}$$

Notice that $F(e) \in \text{span}(\mathbf{1})$ if and only if $e \in \text{span}(\mathbf{1})$. However, since $e = -\mathcal{L}x$, then $e \perp \mathbf{1}$; thus, $e \in \text{span}(\mathbf{1})$ if and only if $e = 0$. Therefore, $e = 0$ is the only possible equilibrium point during the evolution of (40). Furthermore, since the graph is complete,

$$\mathcal{Z}_i(x) = nx^T \mathcal{L}x = x^T(ne) = x^T \mathcal{L}e = e^T e = \|e\|^2.$$

Thus,

$$u_i = \frac{1}{2nT_c q} \frac{\exp\left(\|e\|^{2q}\right)}{\|e\|^{2q}} e_i$$

and, by *Lemma 5.1*, (40) becomes

$$\dot{e} = -\frac{1}{2T_c q} \exp\left(\|e\|^{2q}\right) \frac{e}{\|e\|^{2q}} = -\frac{1}{T_c} \Phi_{m,q}(e) \tag{41}$$

with $m = 2$ and $0 < q \leq \frac{1}{m}$. Therefore, according to *Lemma 2.1*, (41) is strongly predefined-time stable with T_c as the least upper bound for the settling time. Since the graph is complete, then $e = -\mathcal{L}x = 0$ implies that $x \in \mathit{span}(\mathbf{1})$. Thus, for all $t \geq t_0 + T_c$, it holds that

$$x_1(t) = x_2(t) = \cdots = x_n(t)$$

and consensus is achieved. □

Example 5.1 (Predefined-time consensus)**.** *Consider a complete network of ten nodes under the strong predefined-time consensus algorithm* (38)*. Fig. 9 shows the dynamics of such a network with parameters* $T_c = 1$ *and* $q = 0.5$ *and initial conditions* $x(0) = x_0 = [0.2 \ \ 0.3 \ \ 0.5 \ \ 0.1 \ \ 0.4 \ \ 0.6 \ \ 0.9 \ \ 0.7 \ \ 0.8 \ \ 1]^T$*. It can be observed that the network reaches consensus before the strong predefined time* $t = T_c = 1$*.*

Remark 5.1. *Although the results presented in this paper are interesting and promising, in its present form they are not straightforwardly applicable for the design of high-order sliding mode algorithms or to the consensus for dynamic networks (Wang and Xiao, 2010). Additionally, the discretization for the application in digital systems presents some difficulties (Levant, 2013). Extensions of the presented analysis to overcome these limitations are under investigation and will be reported elsewhere.*

6 Conclusions

In this paper, a novel class of dynamical systems with predefined-time stability was introduced. The systems in this family converge within a finite time period and present the practical advantage that the least upper bound for this settling time is known through an explicit and straightforward relationship with the system gain. The Lyapunov analysis that allows for the characterization of this class of stability was also presented.

The predefined-time stability analysis was applied in two directions. First, the proposed approach was applied for the design of first order weak and strong predefined-time sliding mode controllers. Future work in this direction is concerned with design of high order sliding mode controllers with predefined-time stability. Second, a new consensus algorithm was proposed. It was shown that if the underlying topology is a complete graph, the consensus error is strongly predefined-time stable under this algorithm. Future work in this direction is concerned with the analysis and design of predefined-time consensus algorithms under general classes of communication topologies, for which the predefined-time stability feature brings advantages over existing algorithms.

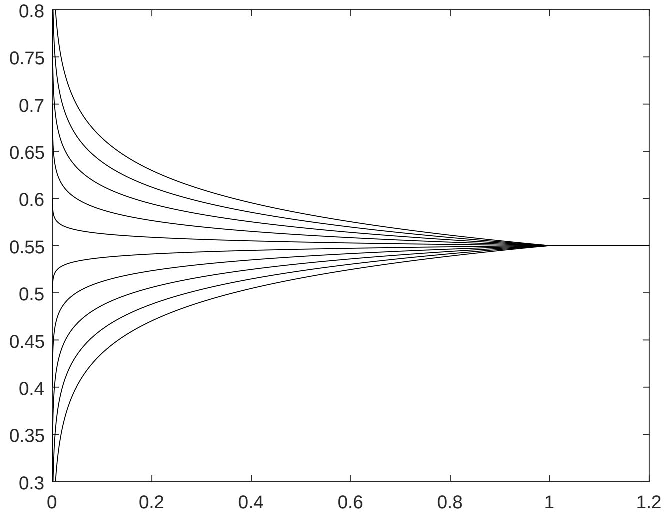

Figure 9: Predefined-time consensus for a complete network of 10 nodes with $T_c = 1$ and $q = 0.5$.

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