

Predefined-Time Tracking of a Class of Mechanical Systems

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Abstract—In this paper the problem of predefined-time exact tracking of fully actuated and unperturbed mechanical systems is solved by means of a continuous controller. It is assumed the availability of the state and the desired trajectory as well as its two first derivatives. This is accomplished introducing the idea of second-order predefined-time stable systems, which is based on the nested application of the first-order predefined-time stabilizing function. As an example, the proposed solution is applied over a two-link planar manipulator and numerical simulations are conducted to show its performance.

I. INTRODUCTION

The various developments concerning the concept of *finite-time stability* permit to solve different applications which are characterized for requiring hard time response constraints. Some important works of this topic and its application to control systems have been carried out in [1]–[6].

However, generally this finite time is an unbounded function of the initial conditions of the system. A desired feature is to eliminate this boundlessness, for example, in estimation or optimization problems. This gives rise to a stronger form of stability called *fixed-time stability*, where the convergence time, as a function of the initial conditions, is bounded. The notion of fixed-time stability have been investigated in [7]–[11].

Although fixed-time stability represents a significant advantage over finite-time stability, it has two major drawbacks. The first of them is that it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time; the other one is that the bounds of the fixed stabilization time found by Lyapunov analysis constitute usually conservative estimations, i.e. they are much larger than the true fixed stabilization time (see for example [12], where the upper bound estimation is approximately 100 times larger than the actual true fixed stabilization time). To overcome the above, another class of dynamical systems which exhibit the property of *predefined-time stability*, have been studied [13], [14]. For this systems, the minimum upper bound of the fixed stabilization time appears explicitly in their tuning gains.

Nevertheless, until now, this predefined-time property have been studied only for first-order systems (systems of relative degree one). In this sense, this paper introduces the concept of *second-order predefined-time* as a nested application of first-order predefined-time stabilizing functions [14]. Furthermore,

this idea is used to solve the problem of predefined-time exact tracking in fully actuated mechanical systems, assuming the availability of the state and the desired trajectory (as well as its two first derivatives) measurements.

In the following, Section II presents the mathematical preliminaries needed to introduce the proposed results. Section III states the problem which will be solved in this paper. Section IV exposes the main result of this paper, which is the second-order predefined-time tracking controller for fully actuated mechanical systems. Section V describes the model of a planar two-link manipulator, where the proposed controller is applied. The simulation results of the example are shown in Section VI. Finally, Section VII presents the conclusions of this paper.

II. MATHEMATICAL PRELIMINARIES

Consider the system

$$\dot{x} = f(x; \rho) \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state, $\rho \in \mathbb{R}^b$ represents the parameters of the system and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The initial conditions of this system are $x(0) = x_0$.

Definition 2.1 (Global finite-time stability [9]): The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(x_0) : x(t, x_0) = 0$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$.

Definition 2.2 (Fixed-time stability [9]): The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}$.

Remark 2.1: Note that there are several choices for T_{\max} . For instance, if the settling-time function is bounded by T_m , it is also bounded by λT_m for all $\lambda \geq 1$. This motivates the following definition.

Definition 2.3 (Settling-time set and its minimum bound [13], [14]): Let \mathcal{T} be the set of all the bounds of the settling time function for the system (1), i.e.,

$$\mathcal{T} = \{T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}\}. \quad (2)$$

The minimum bound of the settling-time function T_f , is defined as:

$$T_f = \inf \mathcal{T} = \sup_{x_0 \in \mathbb{R}^n} T(x_0). \quad (3)$$

Remark 2.2: In a strict sense, the time T_f can be considered as the true fixed-time in which the system (1) stabilizes.

Definition 2.4 (Predefined-time stability [13]): For the case of fixed time stability when the time T_f defined in (3) can be tuned by a particular selection of the parameters ρ of the system (1), it is said that the origin of the system (1) is *predefined-time stable*.

Definition 2.5 (Predefined-time stabilizing function [14]): For $x \in \mathbb{R}^n$, the *predefined-time stabilizing function* is defined as

$$\Phi_p(x; T_c) = \frac{1}{T_c p} \exp(\|x\|^p) \frac{x}{\|x\|^p} \quad (4)$$

where $0 < p \leq 1$ and $T_c > 0$.

Remark 2.3: Since $\lim_{x \rightarrow 0} \Phi_p(x; T_c) = 0$ for $0 < p < 1$, it is considered that $\Phi_p(0; T_c) = 0$. Therefore, the function defined in (4) is continuous for $0 < p < 1$ and discontinuous in $x = 0$ for $p = 1$.

Remark 2.4 (Predefined-time stabilizing function derivative): It can be checked that the derivative of the predefined-time stabilizing function is given by

$$\frac{\partial \Phi_p(x; T_c)}{\partial x} = \frac{\exp(\|x\|^p)}{T_c p} \left[p \frac{xx^T}{\|x\|^2} + \left(I_n - p \frac{xx^T}{\|x\|^2} \right) \frac{1}{\|x\|^p} \right]. \quad (5)$$

From the Definition 2.5 of the stabilizing function, the following Lemma presents a dynamical system with the predefined-time stability property.

Lemma 2.1 (Predefined-time stable dynamical system [14]): The origin of the system

$$\dot{x} = -\Phi_p(x; T_c) \quad (6)$$

with $T_c > 0$, and $0 < p \leq 1$ is predefined-time stable with $T_f = T_c$. That is, $x(t) = 0$ for $t > T_c$ in spite of the x_0 value.

Remark 2.5: From (5), the time derivative of the function $\Phi_p(x; T_c)$ defined in (4) along the trajectories of the system (6) is

$$\begin{aligned} \frac{d\Phi_p(x; T_c)}{dt} &= -\frac{\partial \Phi_p(x; T_c)}{\partial x} \Phi_p(x; T_c) \\ &= -\left[\frac{\exp(\|x\|^p)}{T_c p} \right]^2 \left[p \frac{x}{\|x\|^p} + (1-p) \frac{x}{\|x\|^{2p}} \right]. \end{aligned}$$

Since $\lim_{x \rightarrow 0} \frac{d\Phi_p(x; T_c)}{dt} = 0$ for $0 < p < \frac{1}{2}$, it is considered that $\left. \frac{d\Phi_p(x; T_c)}{dt} \right|_{x=0} = 0$. Therefore, $\frac{d\Phi_p(x; T_c)}{dt}$ is continuous for $0 < p < \frac{1}{2}$.

III. PROBLEM STATEMENT

A generic model of second-order, fully actuated mechanical systems of n degrees of freedom has the form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + P(\dot{q}) + \gamma(q) = \tau, \quad (7)$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the position, velocity and acceleration vectors in joint space; $M(q) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ is the Coriolis and centrifugal effects matrix,

$P(\dot{q}) \in \mathbb{R}^n$ is the damping effects vector, usually from viscous and/or Coulomb friction and $\gamma(q) \in \mathbb{R}^n$ is the gravity effects vector.

Defining the variables $x_1 = q$, $x_2 = \dot{q}$ and $u = \tau$, the mechanical model (7) can be rewritten in the following state-space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2) + B(x_1, x_2)u, \end{aligned} \quad (8)$$

where $f(x_1, x_2) = -M^{-1}(x_1) [C(x_1, x_2)x_2 + P(x_2) + \gamma(x_1)]$, $B(x_1, x_2) = M^{-1}(x_1)$ are continuous maps and the initial conditions are $x_1(0) = x_{1,0}$, $x_2(0) = x_{2,0}$.

Remark 3.1: The matrix function $M(x_1)$ is, in fact, invertible since $M(x_1) = M^T(x_1)$ is positive definite.

A common problem in mechanical systems control is to track a desired time-dependent trajectory described by the triplet $(q_d(t), \dot{q}_d(t), \ddot{q}_d(t))$ of desired position $q_d(t) = [q_{d_1}(t) \ \dots \ q_{d_n}(t)]^T \in \mathbb{R}^n$, velocity $\dot{q}_d(t) = [\dot{q}_{d_1}(t) \ \dots \ \dot{q}_{d_n}(t)]^T \in \mathbb{R}^n$ and acceleration $\ddot{q}_d(t) = [\ddot{q}_{d_1}(t) \ \dots \ \ddot{q}_{d_n}(t)]^T \in \mathbb{R}^n$, which are all assumed to be known.

To be consequent with the state space notation, the desired position and velocity vectors are redefined as $x_{1,d} = q_d$ and $x_{2,d} = \dot{q}_d = \dot{x}_{1,d}$, respectively. Then, defining the error variables as $e_1 = x_1 - x_{1,d}$ (position error) and $e_2 = x_2 - x_{2,d}$ (velocity error), the error dynamics are:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= f(x_1, x_2) + B(x_1, x_2)u - \ddot{x}_{1,d}, \end{aligned} \quad (9)$$

with initial conditions $e_1(0) = e_{1,0} = x_{1,0} - x_{1,d}(0)$, $e_2(0) = e_{2,0} = x_{2,0} - x_{2,d}(0)$.

The task is to design a state-feedback, second-order, predefined-time controller to track the desired trajectory. In other words, the error variables e_1 and e_2 are to be stabilized in predefined time with available measurements of x_1 , x_2 , $x_{1,d}$, $x_{2,d} = \dot{x}_{1,d}$ and $\ddot{x}_{1,d}$.

IV. SECOND-ORDER PREDEFINED-TIME TRACKING CONTROLLER

With basis on the function $\Phi_p(x; T_c)$, consider the non-singular transformation

$$\begin{aligned} \sigma_1 &= e_1 \\ \sigma_2 &= e_2 + \Phi_{p_1}(e_1; T_{c_1}). \end{aligned} \quad (10)$$

where $0 < p_1 < \frac{1}{2}$, $0 < p_2 < 1$, $T_{c_1} > 0$, and $T_{c_2} > 0$.

From (9), the dynamics of the system in the new coordinates (σ_1, σ_2) are in a block controllable form (see [15]):

$$\begin{aligned} \dot{\sigma}_1 &= \sigma_2 - \Phi_{p_1}(\sigma_1; T_{c_1}) \\ \dot{\sigma}_2 &= f(x_1, x_2) + B(x_1, x_2)u - \ddot{x}_{1,d} + \frac{\partial \Phi_{p_1}(\sigma_1; T_{c_1})}{\partial \sigma_1} [\sigma_2 - \Phi_{p_1}(\sigma_1; T_{c_1})]. \end{aligned} \quad (11)$$

with initial conditions $\sigma_1(0) = \sigma_{1,0} = e_{1,0}$, $\sigma_2(0) = \sigma_{2,0} = e_{2,0} + \Phi_{p_1}(e_{1,0}; T_{c_1})$.

Hence, for the system (11) the following controller is proposed:

$$u = -B^{-1}(x_1, x_2) \left[f(x_1, x_2) - \ddot{x}_{1,d} + \frac{\partial \Phi_{p_1}(\sigma_1; T_{c_1})}{\partial \sigma_1} [\sigma_2 - \Phi_{p_1}(\sigma_1; T_{c_1})] + \Phi_{p_2}(\sigma_2; T_{c_2}) \right]. \quad (12)$$

Thus, the system (11) closed-loop with the controller (12) has the form

$$\begin{aligned} \dot{\sigma}_1 &= -\Phi_{p_1}(\sigma_1; T_{c_1}) + \sigma_2 \\ \dot{\sigma}_2 &= -\Phi_{p_2}(\sigma_2; T_{c_2}). \end{aligned} \quad (13)$$

Taking into account the structure of the system (13), the following theorem states the tracking of the system (7).

Theorem 4.1: For the system (7), $q = q_d$ and $\dot{q} = \dot{q}_d$ for $t > T_{c_1} + T_{c_2}$.

Proof. To prove Theorem 4.1 it is sufficient to analyze the stability of the system (13). Using Lemma 2.1, $\sigma_2(t) = 0$ for $t > T_{c_2}$, in spite of the initial conditions $\sigma_{2,0}$. The motion of the system on the manifold $\sigma_2 = 0$ is given by $\dot{\sigma}_1 = -\Phi_{p_1}(\sigma_1; T_{c_1})$. Applying again Lemma 2.1, $\sigma_1(t) = 0$ for $t > T_{c_1} + T_{c_2}$, in spite of the value of $\sigma_1(T_{c_2})$. Finally, from (10), $e_1 = e_2 = 0$ for $t > T_{c_1} + T_{c_2}$, which directly imply the result. ■

Remark 4.1: The control u can be written with respect to the parameters T_{c_1} and T_{c_2} as

$$u = \begin{cases} u_{\sigma_2=0} & \text{for } t > T_{c_2} \\ u_{\sigma_1=0, \sigma_2=0} & \text{for } t > T_{c_1} + T_{c_2} \end{cases}$$

where $u_{\sigma_2=0} = -B^{-1}(x_1, x_2) \left[f(x_1, x_2) - \ddot{x}_{1,d} - \left[\frac{\exp(\|\sigma_1\|^{p_1})}{T_{c_1} p_1} \right]^2 \left[p_1 \frac{\sigma_1}{\|\sigma_1\|^{p_1}} + (1-p_1) \frac{\sigma_1}{\|\sigma_1\|^{2p_1}} \right] \right]$, which is well defined since $0 < p_1 < \frac{1}{2}$ (see Remark 2.5), and $u_{\sigma_1=0, \sigma_2=0} = -B^{-1}(x_{1,d}, x_{2,d}) \left[f(x_{1,d}, x_{2,d}) - \ddot{x}_{1,d} \right]$. Therefore, the control input (12) is a continuous function of σ_1 and σ_2 .

V. EXAMPLE: TRAJECTORY TRACKING FOR A TWO-LINK MANIPULATOR

Consider a planar, two-link manipulator with revolute joints as the one exposed in [16] (see Fig. 1). The manipulator link lengths are L_1 and L_2 , the link masses (concentrated in the end of each link) are M_1 and M_2 . The manipulator is operated in the plane, such that the gravity acts along the z -axis.

Examining the geometry, it can be seen that the end-effector (the end of the second link, where the mass M_2 is concentrated) position (x_w, y_w) is given by

$$\begin{aligned} x_w &= L_1 \cos(q_1) + L_2 \cos(q_1 + q_2) \\ y_w &= L_1 \sin(q_1) + L_2 \sin(q_1 + q_2), \end{aligned} \quad (14)$$

where q_1 and q_2 are the joint positions (angular positions).

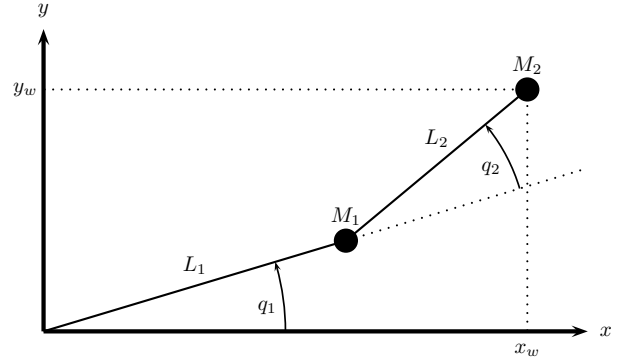


Fig. 1. Two-link manipulator.

Applying the Euler-Lagrange equations, a model according to (7) is obtained, with

$$\begin{aligned} m_{11} &= L_1^2(M_1 + M_2) + 2(L_2^2 M_2 + L_1 L_2 M_2 \cos q_2) - L_2^2 M_2 \\ m_{12} &= m_{21} = L_2^2 M_2 + L_1 L_2 M_2 \cos q_2 \\ m_{22} &= L_2^2 M_2 \\ h &= L_1 L_2 M_2 \sin q_2 \\ c_{11} &= -h \dot{q}_2 \\ c_{12} &= -h(\dot{q}_1 + \dot{q}_2) \\ c_{21} &= h \dot{q}_1 \\ c_{22} &= 0, \end{aligned}$$

$$\begin{aligned} M(q) &= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad C(q, \dot{q}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\ P(\dot{q}) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \gamma(q) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

The absence of gravity term is because the manipulator is operated in the plane, perpendicular to gravity. Note also that friction terms are neglected.

For this example, the end-effector of the manipulator is required to follow a circular trajectory of radius r_d and center in the origin. To solve this problem the controller exposed in Section IV is applied.

VI. SIMULATION RESULTS

The simulation results of the example in Section V are presented in this section. The two-link manipulator parameters used are shown in Table I.

TABLE I
PARAMETERS OF THE TWO-LINK MANIPULATOR MODEL.

Parameter	Values	Unit
M_1	0.2	kg
M_2	0.2	kg
L_1	0.2	m
L_2	0.2	m

The simulations were conducted using the Euler integration method, with a fundamental step size of 1×10^{-4} s. The

initial conditions for the two-link manipulator were selected as: $x_1(0) = [-\frac{3\pi}{4} \quad -\frac{\pi}{4}]^T$ and $x_2(0) = [0 \quad 0]^T$. In addition, the controller gains were adjusted to: $T_{c_1} = 1$, $T_{c_2} = 0.5$, $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{2}$.

The desired circular trajectory in the joint coordinates is described by the equations

$$q_d(t) = x_{1,d}(t) = \begin{bmatrix} q_{d_1}(t) \\ q_{d_2}(t) \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2}t - \pi \\ -\frac{\pi}{2} \end{bmatrix}, \quad (15)$$

and it corresponds to a circumference of radius 0.2828 m.

The following figures show the behavior of the proposed controller.

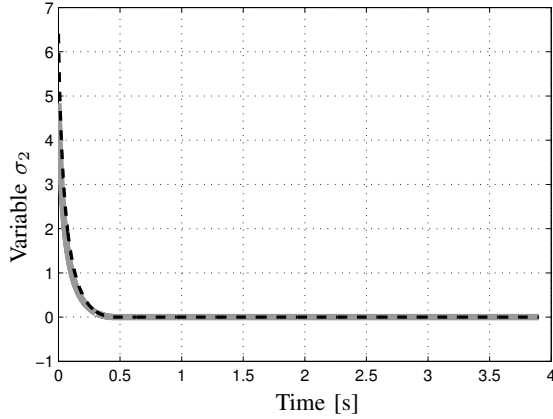


Fig. 2. Variable σ_2 . First component (gray and solid) and second component (black and dashed). Note that $\sigma_2(t) = 0$ for $t > T_{c_2} = 0.5$ s.

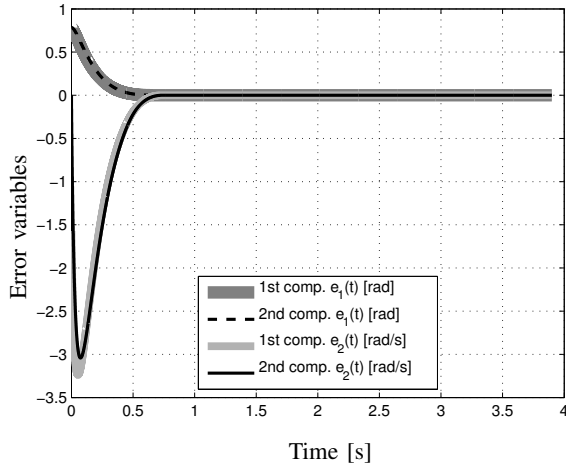


Fig. 3. Error variables. First component of e_1 (dark gray and thick), second component of e_1 (black and dashed), first component of e_2 (light gray and solid) and second component of e_2 (black and solid).

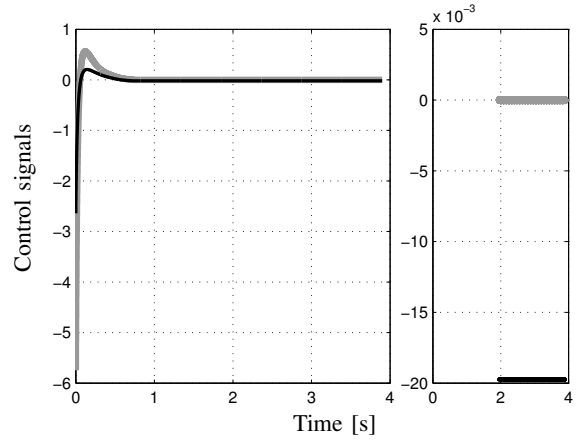


Fig. 4. Control signal. First component (gray and solid) and second component (black and solid).

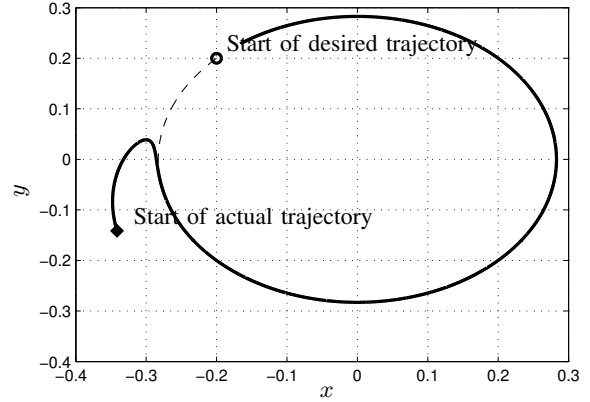


Fig. 5. Actual trajectory (x_w, y_w) (black and solid) and desired trajectory ($x_{w,d}, y_{w,d}$) (black and dashed).

Note that $\sigma_2(t) = 0$ for $t \geq 0.47$ s $< T_{c_2} = 0.5$ s (Fig. 2). Once the error variables slide over the manifold $\sigma_2 = 0$, this motion is governed by the reduced order system

$$\dot{e}_1 = e_2 = -\Phi_{p_1}(e_1; T_{c_1}).$$

This imply that the error variables are exactly zero for $t > T_{c_1} + T_{c_2} = 1.5$ s. In fact, from Fig. 3, it can be seen that $e_1(t) = e_2 = 0$ for $t \geq 0.74$ s $< T_{c_1} + T_{c_2} = 1.5$ s. Fig. 4 shows the control signal (torque) versus time. Finally, from Fig. 5, it can be seen the reference tracking in rectangular coordinates.

VII. CONCLUSION

In this paper the problem of predefined-time exact tracking in fully actuated mechanical systems was solved by means of a controller which induces second-order predefined-time stability in the tracking error. This controller was constructed as a nested application of continuous first-order predefined-time stabilizing functions.

To show the feasibility of the proposed controller, it was implemented over a two-link planar manipulator. The numerical simulations showed a good performance.

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