Non-Singular Predefined-Time Stable Manifolds

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Abstract: In this paper it is introduced a class of non-singular manifolds with predefined-time stability. That is, for a given dynamical system with its trajectories constrained to this manifold it can be shown predefined-time stability to the origin. In addition, the function that defines the manifold and its derivative along the system trajectories are continuous, therefore no singularities are presented for the system evolution once the constrained motion starts. The problem of reaching the proposed manifold is solved by means of a continuous predefined-time stable controller. The proposal is applied to the predefined-time exact tracking of fully actuated and unperturbed mechanical systems. It is assumed the availability of the state and the desired trajectory as well as its two first derivatives. As an example, the proposed solution is applied over a two-link planar manipulator and numerical simulations are conducted to show its performance.

Keywords: Predefined-Time Stability, Sliding Mode Algorithms, Second Order Systems, Mechanical Systems.

1. INTRODUCTION

Several applications are characterized for requiring hard time response constraints. In order to deal with those requirements, various developments concerning to concept of *finite-time stability* have been carried out (see for example: Roxin (1966); Weiss and Infante (1967); Michel and Porter (1972); Haimo (1986); Utkin (1992); Bhat and Bernstein (2000); Moulay and Perruquetti (2005, 2006)). Nevertheless, usually this finite time is an unbounded function of the initial conditions of the system.

With the aim to eliminate this boundlessness, the notion of fixed-time stability have been studied in Andrieu et al. (2008); Cruz-Zavala et al. (2010); Polyakov (2012); Fraguela et al. (2012); Polyakov and Fridman (2014). Fixed-time stability represents a significant advantage over finite-time stability due to its desired feature of the convergence time, as a function of the initial conditions, is bounded. That makes the fixed-time stability a valuable feature in estimation and optimization problems.

For the most of the proposed fixed-time stable system, there are problems related with the convergence time. First, the bounds of the fixed stabilization time found by Lyapunov analysis constitute usually conservative estimations, i.e. they are much larger than the true fixed stabilization time (see for example Cruz-Zavala et al. (2011), where the upper bound estimation is approximately 100 times larger than the actual true fixed stabilization time). Second, and as consequence, it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time, making this time hard to tune.

To overcome the above, a class of first-order dynamical systems with the minimum upper bound of the fixed stabilization time equal to their only tuning gain has been studied (Sánchez-Torres et al., 2014; Sánchez-Torres et al., 2015). It is said that these systems exhibit the property of *predefined-time stability*.

In this sense, this paper introduces the concept of non-singular predefined-time stable manifolds. Similarly to Jiménez-Rodríguez et al. (2016), the proposed scheme allows to define second-order predefined-time stable systems as a nested application of first-order predefined-time stabilizing functions, with the difference that such function which defines the manifold and its derivative along the system trajectories are continuous, therefore no singularities are presented for the system evolution.

Finally, this idea is used to solve the problem of predefined-time exact tracking in fully actuated mechanical systems, assuming the availability of the state and the desired trajectory and its two first derivatives measurements.

In the following, Section 2 presents the mathematical preliminaries needed to introduce the proposed results. Section 3 exposes the main result of this paper, which is the non-singular predefined-time stable manifold design. Section 4 presents a non-singular second-order predefined-time tracking controller for fully actuated mechanical systems. Section 5 describes the model of a planar two-link manipulator, where the proposed controller is applied. The

simulation results of the example are shown in Section 6. Finally, Section 7 presents the conclusions of this paper.

2. PRELIMINARIES

2.1 Mathematical Preliminaries

Consider the system

$$\dot{x} = f(x; \rho) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\rho \in \mathbb{R}^b$ represents the parameters of the system and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. The initial conditions of this system are $x(0) = x_0$.

Definition 2.1. (Polyakov, 2012) The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(x_0) : x(t, x_0) = 0$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$.

Definition 2.2. (Polyakov, 2012) The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}$.

Remark 2.1. Note that there are several choices for $T_{\max}$. For instance, if the settling-time function is bounded by T_{m}, it is also bounded by λT_{m} for all $\lambda \geq 1$. This motivates the following definition.

Definition 2.3. (Sánchez-Torres et al., 2014; Sánchez-Torres et al., 2015) Let $\mathcal{T}$ be the set of all the bounds of the settling time function for the system (1), i.e.,

$$\mathcal{T} = \{T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}\}. \tag{2}$$

The minimum bound of the settling-time function T_f, is defined as:

$$T_f = \inf \mathcal{T} = \sup_{x_0 \in \mathbb{R}^n} T(x_0). \tag{3}$$

Remark 2.2. In a strict sense, the time T_f can be considered as the true fixed-time in which the system (1) stabilizes.

Definition 2.4. (Sánchez-Torres et al., 2014; Sánchez-Torres et al., 2015) For the case of fixed time stability when the time T_f defined in (3) can be tuned by a particular selection of the parameters ρ of the system (1), it is said that the origin of the system (1) is *predefined-time stable*.

Definition 2.5. Let $h \geq 0$. For $x \in \mathbb{R}$, define the function

$$\lfloor x \rceil^h = |x|^h \operatorname{sign}(x),$$

with $\operatorname{sign}(x) = 1$ for $x > 0$, $\operatorname{sign}(x) = -1$ for $x < 0$ and $\operatorname{sign}(0) \in [-1, 1]$.

Remark 2.3. For $x \in \mathbb{R}$, some properties of the function $\lfloor \cdot \rceil^h$ are:

(i) $\lfloor x \rceil^h$ is continuous for $h > 0$.

(ii) $\lfloor x \rceil^0 = \operatorname{sign}(x)$.

(iii) $\lfloor x \rceil^1 = \lfloor x \rceil = x$,

(iv) $\lfloor 0 \rceil^h = 0$ for $h > 0$.

(v) $\frac{d|x|^h}{dx} = h \lfloor x \rceil^{h-1}$ and $\frac{d \lfloor x \rceil^h}{dx} = h |x|^{h-1}$.

(vi) For $h_1, h_2 \in \mathbb{R}$, it follows:

    · $|x|^{h_1} |x|^{h_2} = |x|^{h_1 + h_2}$

    · $\lfloor x \rceil^{h_1} |x|^{h_2} = |x|^{h_1} \lfloor x \rceil^{h_2} = \lfloor x \rceil^{h_1 + h_2}$

    · $\lfloor x \rceil^{h_1} \lfloor x \rceil^{h_2} = |x|^{h_1 + h_2}$

(vii) For $h_1, h_2 > 0$, then $\left\lfloor \lfloor x \rceil^{h_1} \right\rceil^{h_2} = \lfloor x \rceil^{h_1 h_2}$.

Definition 2.6. (Sánchez-Torres et al., 2014; Sánchez-Torres et al., 2015) For $x \in \mathbb{R}$, the *predefined-time stabilizing function* is defined as:

$$\Phi_p(x; T_c) = \frac{1}{T_c p} \exp\left(|x|^p\right) \lfloor x \rceil^{1-p} \tag{4}$$

where $T_c > 0$ and $0 < p \leq 1$.

Remark 2.4. It can be checked, using Remark 2.3, that the derivative of the predefined-time stabilizing function (4) is given by

$$\frac{d\Phi_p(x; T_c)}{dx} = \frac{\exp\left(|x|^p\right)}{T_c p} \left[p + (1-p)\frac{1}{|x|^p}\right], \quad \forall x \neq 0 \tag{5}$$

To handle vector systems, the above definitions are extended.

Definition 2.7. Let $h \geq 0$, $T_c > 0$, $0 < p \leq 1$ and $v = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}^T \in \mathbb{R}^k$. Then, the functions $\operatorname{sign}(\cdot)$, $|\cdot|^h$, $\lfloor \cdot \rceil^h$ and $\Phi_p(\cdot; T_c)$ are extended component-wise, as follows:

(i) $\operatorname{sign}(v) = \begin{bmatrix} \operatorname{sign}(v_1) & \cdots & \operatorname{sign}(v_k) \end{bmatrix}^T$

(ii) $|v|^h = \begin{bmatrix} |v_1|^h & \cdots & |v_k|^h \end{bmatrix}^T$;

(iii) $\lfloor v \rceil^h = \begin{bmatrix} \lfloor v_1 \rceil^h & \cdots & \lfloor v_k \rceil^h \end{bmatrix}^T$;

(iv) $\Phi_p(v; T_c) = \begin{bmatrix} \Phi_p(v_1; T_c) & \cdots & \Phi_p(v_k; T_c) \end{bmatrix}^T$.

Definition 2.8. Let $v = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}^T \in \mathbb{R}^k$. Then $\operatorname{diag}(v)$ will denote the $k \times k$ matrix defined as

$$\operatorname{diag}(v) = \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_k \end{bmatrix}.$$

Remark 2.5. The properties (i), (ii), (iii), (iv) and (vi) of Remark 2.3 remain the same. For $v \in \mathbb{R}^k$, the derivatives of the functions $|\cdot|^h$, $\lfloor \cdot \rceil^h$ and $\Phi_p(\cdot; T_c)$ are:

$$\frac{\partial |v|^h}{\partial v} = \operatorname{diag}\left[h \lfloor v_1 \rceil^{h-1} \quad \cdots \quad h \lfloor v_k \rceil^{h-1}\right] = h \operatorname{diag}\lfloor v \rceil^{h-1},$$

$$\frac{\partial \lfloor v \rceil^h}{\partial v} = \operatorname{diag}\left[h |v_1|^{h-1} \quad \cdots \quad h |v_k|^{h-1}\right] = h \operatorname{diag}|v|^{h-1}$$

and

$$\frac{\partial \Phi_p(v; T_c)}{\partial v} = \operatorname{diag}\left[\frac{d\Phi_p(v_1; T_c)}{dv_1} \quad \cdots \quad \frac{d\Phi_p(v_k; T_c))}{dv_k}\right],$$

respectively.

Remark 2.6. It is important to note that if $k = 1$, all the extensions reduce to the scalar case considered by Definition 2.5, Remark 2.3 and Definition 2.6.

From the Definition 2.6 of the stabilizing function, the following Lemma presents a dynamical system with the predefined-time stability property.

Lemma 2.1. (Sánchez-Torres et al., 2014; Sánchez-Torres et al., 2015) The origin of the system

$$\dot{x} = -\Phi_r(x; T_c) \tag{6}$$

with $T_c > 0$, and $0 < r < 1$ is predefined-time stable with $T_f = T_c$. That is, $x(t) = 0$ for $t > T_c$ in spite of the $x(0)$ value.

2.2 Motivation

Consider the following second order system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\alpha \text{sign}\left(x_2 + \beta \lfloor x \rfloor^{\frac{1}{2}}\right) \tag{7}$$

where $x_1, x_2, u \in \mathbb{R}$ and $\alpha, \beta > 0$. The initial conditions of this system are $x_1(0) = x_{1,0}$ and $x_2(0) = x_{2,0}$.

Let the variable $\sigma = x_2 + \lfloor x \rfloor^{\frac{1}{2}}$ and its time derivative given by

$$\dot{\sigma} = -\alpha \text{sign}(\sigma) + \frac{1}{2}\beta x_2 |x_1|^{-\frac{1}{2}}. \tag{8}$$

For $\alpha > \beta/2$ a sliding motion on the manifold $\sigma = 0$ is obtained in finite time. This can verified by evaluating the dynamics of (8) when the sliding motion starts. Once the manifold $\sigma = 0$ is reached, the dynamics of (7) reduces to

$$\dot{x}_1 = -\beta \lfloor x \rfloor^{\frac{1}{2}} \tag{9}$$

that is finite-time stable. Therefore, there is a time $T = T(x_{1,0}, x_{2,0})$ such that $\sigma = 0$ and $x_1 = 0$ for every time $t \geq T$, which implies that $x_2 = 0$ for $t \geq T$.

Remark 2.7. The exposed procedure is the main idea behind of the terminal sliding mode controllers (Venkataraman and Gulati, 1992) since the motion on $\sigma = 0$ is also finite time stable and, the nested high-order sliding mode controllers (Levant, 2003) since x_1 and its derivative x_2 are driven to zero in finite time and the system has a nested structure.

3. PREDEFINED-TIME STABLE NON-SINGULAR MANIFOLDS

Similarly to the nested approach presented given in (7)-(8), in order to obtain a similar second order system but with predefined-time stability, consider the double integrator system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u \tag{10}$$

where $x_1, x_2, u \in \mathbb{R}$.

As first attempt, from (4), the variable

$$\sigma = x_2 + \frac{1}{T_{c_1} p} \exp(|x_1|^p) \lfloor x \rfloor^{1-p}. \tag{11}$$

with $T_{c_1} > 0$ can be used. However, note that the dynamics of (11) is given by

$$\dot{\sigma} = u + \frac{\exp\left(|x_1|^p\right)}{T_{c_1} p}\left[p + (1-p)\frac{1}{|x_1|^p}\right]x_2 \tag{12}$$

which has a singularity at $x_1 = 0$. Therefore, the variable σ in (11) is called a singular sliding variable.

Considering that drawback, it is desirable a variable which provides the same dynamics in $\sigma = 0$ than these presented in (11), but avoiding the singularity thus (12) exposes. With this aim, let the following variables:

$$\sigma_1 = x_1$$
$$\sigma_2 = (1-p)\lfloor x_2 \rfloor^{\frac{1}{1-p}} + (1-p)\lfloor \Phi_p(x_1; T_{c_1}) \rfloor^{\frac{1}{1-p}}, \tag{13}$$

where $\lfloor \Phi_p(x_1; T_{c_1}) \rfloor^{\frac{1}{1-p}} = \left[\frac{1}{T_c p}\right]^{\frac{1}{1-p}} \exp\left(\frac{1}{1-p}|x_1|^p\right) \lfloor x_1 \rfloor$ with $0 < p < \frac{1}{2}$.

Hence, the system (10) can be written as

$$\dot{\sigma}_1 = -\left\lfloor \lfloor \Phi_p(\sigma_1; T_{c_1}) \rfloor^{\frac{1}{1-p}} - \frac{1}{1-p}\sigma_2 \right\rfloor^{1-p}$$
$$\dot{\sigma}_2 = |x_2|^{\frac{p}{1-p}} u + \psi(\sigma_1)\lfloor x_2 \rfloor,$$

where $\psi(\sigma_1) = \left[\frac{1}{T_{c_1} p}\right]^{\frac{1}{1-p}} \exp\left(\frac{1}{1-p}|\sigma_1|^p\right)\left[p|\sigma_1|^p + (1-p)\right]$.

Remark 3.1. The variable σ_2 in (13) is based on the approach proposed in Feng et al. (2002). However, here it is not necessary to define fractional powers in terms of odd integers.

With $u = -|x_2|^{\frac{p}{p-1}}\left[\Phi_r(\sigma_2; T_{c_2}) + \psi(x_1)\lfloor x_2 \rfloor\right]$, $T_{c_2} > 0$ and $0 < r < 1$, it yields

$$\dot{\sigma}_1 = -\left\lfloor \lfloor \Phi_p(\sigma_1; T_{c_1}) \rfloor^{\frac{1}{1-p}} - \frac{1}{1-p}\sigma_2 \right\rfloor^{1-p}$$
$$\dot{\sigma}_2 = -\Phi_r(\sigma_2; T_{c_2}). \tag{14}$$

The stability analysis of the system (14) is an direct application of Lemma 2.1. For $t > T_{c_2}$, $\sigma_2 = 0$ and the system reduces to $\dot{\sigma}_1 = -\Phi_p(\sigma_1; T_{c_1})$. Then, for $t > T_{c_1} + T_{c_2}$, $(\sigma_1, \sigma_2) = (0,0)$. Consequently, from (13), $(x_1, x_2) = (0,0)$ for $t > T_{c_1} + T_{c_2}$.

Remark 3.2. From (14), it can be noted x_2 cannot be zero before $\sigma_2 = 0$. Besides, once $\sigma_2 = 0$, the control signal becomes

$$u_{\sigma_2 = 0} = -|x_2|^{\frac{p}{p-1}} \psi(x_1)\lfloor x_2 \rfloor = -\psi(x_1)\lfloor x_2 \rfloor^{\frac{2p-1}{p-1}},$$

which is continuous since $0 < p < \frac{1}{2}$. In addition, it can be observed that the term $|x_2|^{\frac{p}{p-1}}$ in the controller vanishes in predefined time T_{c_2}, avoiding a singularity at $x_2 = 0$.

4. PREDEFINED-TIME TRACKING CONTROLLER OF A CLASS OF MECHANICAL SYSTEMS

4.1 Problem Statement

A generic model of second-order, fully actuated mechanical systems of n degrees of freedom has the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + P(\dot{q}) + \gamma(q) = \tau, \tag{15}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the position, velocity and acceleration vectors in joint space; $M(q) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n \times n}$ is the Coriolis and centrifugal effects matrix, $P(\dot{q}) \in \mathbb{R}^n$ is the damping effects vector, usually from viscous and/or Coulomb friction and $\gamma(q) \in \mathbb{R}^n$ is the gravity effects vector.

Defining the variables $x_1 = q$, $x_2 = \dot{q}$ and $u = \tau$, the mechanical model (15) can be rewritten in the following state-space form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x_1, x_2) + B(x_1, x_2)u, \tag{16}$$

where $f(x_1, x_2) = -M^{-1}(x_1)\left[C(x_1, x_2)x_2 + P(x_2) + \gamma(x_1)\right]$, $B(x_1, x_2) = M^{-1}(x_1)$ are continuous maps and the initial conditions are $x_1(0) = x_{1,0}$, $x_2(0) = x_{2,0}$.

Remark 4.1. The matrix function $M(x_1)$ is, in fact, invertible since $M(x_1) = M^T(x_1)$ is positive definite.

A common problem in mechanical systems control is to track a desired time-dependent trajectory described by the triplet $(q_d(t), \dot{q}_d(t), \ddot{q}_d(t))$ of desired position

$q_d(t) = [q_{d_1}(t) \cdots q_{d_n}(t)]^T \in \mathbb{R}^n$, velocity $\dot{q}_d(t) = [\dot{q}_{d_1}(t) \cdots \dot{q}_{d_n}(t)]^T \in \mathbb{R}^n$ and acceleration $\ddot{q}_d(t) = [\ddot{q}_{d_1}(t) \cdots \ddot{q}_{d_n}(t)]^T \in \mathbb{R}^n$, which are all assumed to be known.

To be consequent with the state space notation, the desired position and velocity vectors are redefined as $x_{1,d} = q_d$ and $x_{2,d} = \dot{q}_d = \dot{x}_{1,d}$, respectively. Then, defining the error variables as $e_1 = x_1 - x_{1,d}$ (position error) and $e_2 = x_2 - x_{2,d}$ (velocity error), the error dynamics are:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= f(x_1, x_2) + B(x_1, x_2)u - \ddot{x}_{1,d}, \end{aligned} \tag{17}$$

with initial conditions $e_1(0) = e_{1,0} = x_{1,0} - x_{1,d}(0)$, $e_2(0) = e_{2,0} = x_{2,0} - x_{2,d}(0)$.

The task is to design a state-feedback, second-order, predefined-time controller to track the desired trajectory. In other words, the error variables e_1 and e_2 are to be stabilized in predefined time with available measurements of x_1, x_2, $x_{1,d}$, $x_{2,d} = \dot{x}_{1,d}$ and $\ddot{x}_{1,d}$.

4.2 Controller Design

With basis on Definition 2.7, consider the non-singular transformation

$$\begin{aligned} s_1 &= e_1 \\ s_2 &= (1-p)\lfloor e_2 \rceil^{\frac{1}{1-p}} + (1-p)\lfloor \Phi_p(e_1; T_{c_1}) \rceil^{\frac{1}{1-p}}, \end{aligned} \tag{18}$$

with $0 < p < \frac{1}{2}$.

From (17), the dynamics of the system in the new coordinates (s_1, s_2) can be written as

$$\dot{s}_1 = -\left\lfloor \lfloor \Phi_p(s_1; T_{c_1}) \rceil^{\frac{1}{1-p}} - \frac{1}{1-p}s_2 \right\rceil^{1-p}$$

$$\dot{s}_2 = \text{diag}\,|e_2|^{\frac{p}{1-p}}\left[f(x_1,x_2) + B(x_1,x_2)u - \ddot{x}_{1,d} \right] + \Psi(s_1)e_2, \tag{19}$$

where $\Psi(s_1) = \text{diag}\,|\Phi_p(s_1; T_{c_1})|^{\frac{p}{1-p}} \frac{\partial \Phi_p(s_1; T_{c_1})}{\partial s_1}$.

Hence, for the system (19) the following controller is proposed:

$$u = -B^{-1}(x_1, x_2)\bigg[f(x_1, x_2) - \ddot{x}_{1,d} +$$

$$\text{diag}\,|e_2|^{\frac{p}{p-1}}\left[\Psi(s_1)e_2 + \Phi_r(s_2; T_{c_2}) \right]\bigg], \tag{20}$$

with $0 < r < 1$ and $T_{c_2} > 0$.

Thus, the system (19) closed-loop with the controller (20) has the form

$$\begin{aligned} \dot{s}_1 &= -\left\lfloor \lfloor \Phi_p(s_1; T_{c_1}) \rceil^{\frac{1}{1-p}} - \frac{1}{1-p}s_2 \right\rceil^{1-p} \\ \dot{s}_2 &= -\Phi_r(\sigma_2; T_{c_2}). \end{aligned} \tag{21}$$

Taking into account the structure of the system (21), the following theorem states the tracking of the system (15).

Theorem 4.1. For the system (15), $q = q_d$ and $\dot{q} = \dot{q}_d$ for $t > T_{c_1} + T_{c_2}$.

Proof. The proof is similar to the stability analysis carried out in Section 3 and, hence, is omitted. ∎

5. EXAMPLE: TRAJECTORY TRACKING FOR A TWO-LINK MANIPULATOR

Consider a planar, two-link manipulator with revolute joints as the one exposed in Utkin et al. (2009) (see Fig. 1). The manipulator link lengths are L_1 and L_2, the link masses (concentrated in the end of each link) are M_1 and M_2. The manipulator is operated in the plane, such that the gravity acts along the z-axis.

Examining the geometry, it can be seen that the end-effector (the end of the second link, where the mass M_2 is concentrated) position (x_w, y_w) is given by

$$\begin{aligned} x_w &= L_1\cos(q_1) + L_2\cos(q_1 + q_2) \\ y_w &= L_1\sin(q_1) + L_2\sin(q_1 + q_2), \end{aligned}$$

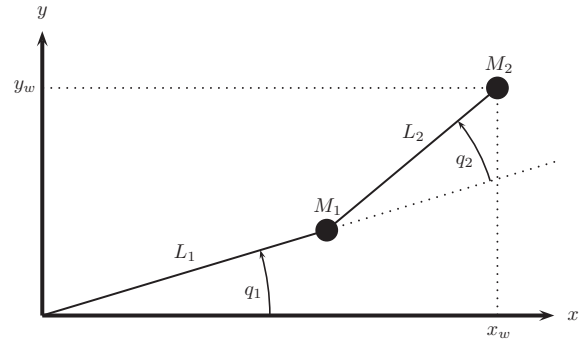where q_1 and q_2 are the joint positions (angular positions).

Figure 1. Two-link manipulator.

Applying the Euler-Lagrange equations, a model according to (15) is obtained, with

$$m_{11} = L_1^2(M_1 + M_2) + 2(L_2^2 M_2 + L_1 L_1 M_2 \cos q_2) - L_2^2 M_2$$
$$m_{12} = m_{21} = L_2^2 M_2 + L_1 L_1 M_2 \cos q_2$$
$$m_{22} = L_2^2 M_2$$
$$h = L_1 L_2 M_2 \sin q_2$$
$$c_{11} = -h\dot{q}_2$$
$$c_{12} = -h(\dot{q}_1 + \dot{q}_2)$$
$$c_{21} = h\dot{q}_1$$
$$c_{22} = 0,$$

$$M(q) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad C(q, \dot{q}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$P(\dot{q}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \gamma(q) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The absence of gravity term is because the manipulator is operated in the plane, perpendicular to gravity. Note also that friction terms are neglected.

For this example, the end-effector of the manipulator is required to follow a circular trajectory of radius r_d and center in the origin. To solve this problem the controller exposed in Section 4 is applied.

6. SIMULATION RESULTS

The simulation results of the example in Section 5 are presented in this section. The two-link manipulator parameters used are shown in Table 1.

The simulations were conducted using the Euler integration method, with a fundamental step size of 1×10^{-4} s. The initial conditions for the two-link manipulator were selected as: $x_1(0) = \left[-\frac{3\pi}{4} \quad -\frac{\pi}{4} \right]^T$ and $x_2(0) = [0 \quad 0]^T$. In addition, the controller gains were adjusted to: $T_{c_1} = 1$, $T_{c_2} = 0.5$, $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{2}$.

The desired circular trajectory in the joint coordinates is described by the equations $q_d(t) = x_{1,d}(t) = \left[q_{d_1}(t) \quad q_{d_2}(t) \right] = \left[\frac{\pi}{2}t - \pi \quad -\frac{\pi}{2} \right]$ and it corresponds to a circumference of radius 0.2828 m.

The following figures show the behavior of the proposed controller.

Figure 4. Control signal. First component (gray and solid) and second component (black and solid).

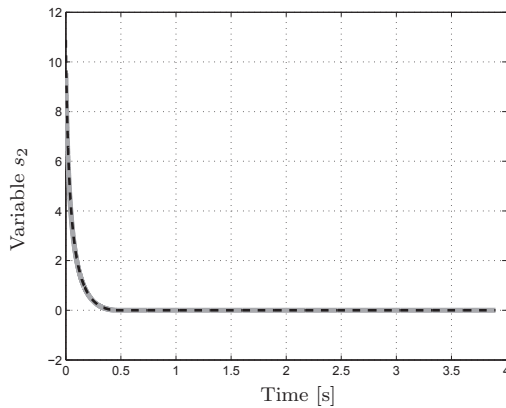

Figure 2. Variable s_2. First component (gray and solid) and second component (black and dashed). Note that $s_2(t) = 0$ for $t > T_{c_2} = 0.5$ s.
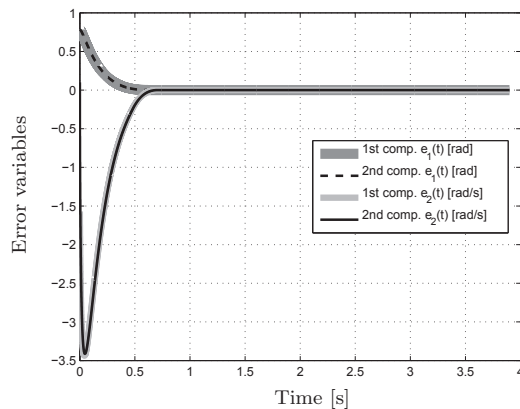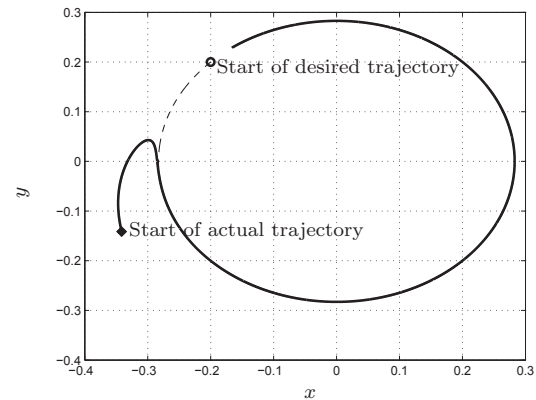

Figure 5. Actual trajectory (x_w, y_w) (black and solid) and desired trajectory $(x_{w,d}, y_{w,d})$ (black and dashed).

Note that $\sigma_2(t) = 0$ for $t \geq 0.47$ s $< T_{c_2} = 0.5$ s (Fig. 2). Once the error variables slide over the manifold $\sigma_2 = 0$, this motion is governed by the reduced order system

$$\dot{e}_1 = e_2 = -\Phi_{p_1}(e_1; T_{c_1}).$$

This imply that the error variables are exactly zero for $t > T_{c_1} + T_{c_2} = 1.5$ s. In fact, from Fig. 3, it can be seen that $e_1(t) = e_2(t) = 0$ for $t \geq 0.74$ s $< T_{c_1} + T_{c_2} = 1.5$ s. Fig. 4 shows the control signal (torque) versus time. Finally, from Fig. 5, it can be seen the reference tracking in rectangular coordinates.

Figure 3. Error variables. First component of e_1 (dark gray and thick), second component of e_1 (black and dashed), first component of e_2 (light gray and solid) and second component of e_2 (black and solid).
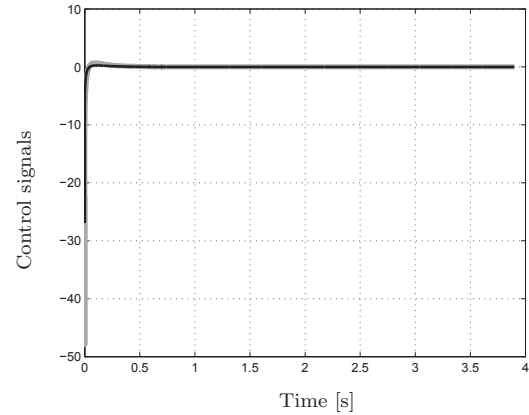
Table 1. Parameters of the two-link manipulator model.

Parameter	Values	Unit
M_1	0.2	kg
M_2	0.2	kg
L_1	0.2	m
L_2	0.2	m

7. CONCLUSION

In this paper a class of non-singular manifolds with predefined-time stability was introduced. As a result, the trajectories of a given dynamical system constrained to this class of manifolds have predefined-time stability to the origin and, in addition, the function that defines the manifold and its derivative along the system trajectories are continuous, therefore no singularities are presented for the system evolution once the constrained motion starts. Besides, the problem of reaching the proposed manifold was solved by means of a continuous predefined-time stable controller.

The proposal was applied to the predefined-time exact tracking of fully actuated and unperturbed mechanical systems as an example, assuming the availability of the state and the desired trajectory as well as its two first derivatives. Furthermore, the resulting controller is applied over a two-link planar manipulator and numerical simulations are conducted to show its performance.

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