Summer 1962

A Modern Approach to Certain Algebraic Topics

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A MODERN APPROACH
TO CERTAIN ALGEBRAIC TOPICS

being

A Thesis Presented to the Graduate Faculty
of the Fort Hays Kansas State College in
Partial Fulfillment of the Requirements for
the Degree of Master of Arts

by

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(An Abstract)

The purpose on this paper is to incorporate in textbook form, notes and methods that the author has used for the past three years on an experimental basis.

Set terminology is introduced and applied to the development of the number system through the set of real numbers. Rigorous proofs are used in developing the arithmetic of the various sets of numbers.

Methods for solving equations and inequalities are discussed in set terminology. Graphs of solution sets of equations and inequalities are displayed, and the methods of plotting solution sets are discussed.

Conclusions concerning the success of applying the material submitted in this thesis are presented in the form of observations that the writer has made during his experimentation with the material in the classroom.

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INTRODUCTION

The purpose of this paper is to organize notes and elaborate on methods that the writer has used on an experimental basis for the past three years in introductory algebra. The material included in the body of this thesis is intended as a supplement to a traditional algebra text. The approach to the use of signed numbers presented in this paper is somewhat unorthodox in high school mathematics, but it is the opinion of this writer that an abstract approach to the rules of operation with signed numbers is as easily understood by most students as an approach using a mathematical model.

One purpose of this work is to demonstrate the use of mathematical proofs in algebra. The second chapter of the thesis is devoted largely to proofs, and some of these proofs are left to the student as exercises. It is the opinion of this writer that the sooner students learn to support mathematical statements with accepted reasons, the faster will be their progress in the field of mathematics.

No attempt is made to define all the technical terms used in this paper, but the student should already be familiar with most of them. An effort has been made to define any new term that is not usually used in elementary arithmetic.

Rigorous proofs of the laws of operation and many other theorems have been omitted purposely at this time. It seems to this writer that it is better to postulate many of the things that may be proved in higher mathematics.

Many of the terms of set theory are used throughout the thesis, but most of the algebra of sets and much of the material used in studying sets has been omitted. The writer has tried to use set notation wherever possible, but set terminology has been introduced and used only where that particular terminology fitted the needs of the writer and students.

Exercises are included throughout the text, but these are only specimen exercises. In actual practice many more exercises would be needed.

The material in this work does not parallel any text book, but by the use of this material in the beginning of an algebra course, the chapters in the traditional text on signed numbers, and the solution of simple equations can be omitted.

The writer does not consider the material in this thesis to be easily understood by a student without considerable explanation by the teacher. It is submitted for the purpose of complementing the work of the teacher.

CHAPTER I

DEFINITIONS AND PROPERTIES

Sets. Usually "set" is not defined mathematically, however, the idea of "set" is familiar to the student. The student is familiar with sets of dishes, sets of golf clubs, sets of checkers and other sets found in and around the home. It can be said that a set is a collection of objects or elements whose membership in the set may be determined either by naming each element in the set or by prescribing a condition that will determine whether the element is in the set or not. Listing the names of each member in the set is called tabulation. For example, a set can be designated by listing the name of each person in an algebra class and enclosing the names in brackets, or the same set could be formed by enclosing a defining property in brackets and agreeing that the persons having that property and only those persons are members of the algebra class.

A set consisting of three whole numbers is indicated in the following manner: $A = \{1,2,3\}$. It is read "A is the set containing the elements 1,2,3." The order of the elements in the set is not important. The set $B = \{3,1,2\}$ is equal to set A since the elements in A are the same as the elements in B. When this condition exists, that is, when for each element in one set there exists an identical element in a second set, and for each element in the second set there exists an identical element in the first, then the two sets are equal.

<u>The Set Builder</u>. For the sake of convenience it is often desirable to express the conditions determining membership in the set as a statement enclosed in braces. To do this mathematicians often employ what is called the set builder. The set builder is written in this manner: $A = \{x \mid F(x)\}$. It is read, "A is the set of all x such that x meets a certain condition". The set of all possible values from which x may be selected is called the <u>universe</u> of the set. The following are examples:

(a) $B = \{x \mid x\}$ is a whole number between 2 and $7\}$ read, "B is the set of all x such that x is a whole number between 2 and 7". In this case the universe is the set of whole numbers $\{1,2,3,4,5,6,7,8\}$. Another way to write the same set would be: $B = \{3,4,5,6\}$.

(b) $C = \{x \mid x>2$, read, "C is the set of all x such that x is a whole number greater than 2". (Note: The symbol $>$ is read "greater than". The symbol $<$ is read "less than".) The universe is $\{1,2,3,4, \ldots\}$. This set could also be written: $C = \{3,4,5, \ldots\}$. (Note that the three dots indicate that the sequence continues indefinitely).

EXERCISES

1. Write the set M, whose elements are the members of your family.
2. Write the set G, whose elements are the girls in your algebra class.
3. Write the following set in set builder notation: $N = \{1,3,5,7, . . .\}$.
4. Write the following set in set builder notation: $R = \{8,9,10\}$.
5. Write the following set by tabulation if the universe is $\{1,2,3,4,5,6,7,8,9,10\}$ $S = \{x / 2<x<9\}$.

Subsets. A set A is a subset of set B if each element of A is also an element of B. B would be called the universal set. Subsets with one element are called unit sets and the set which contains no elements is called the empty set. A subset of the universal set that does not contain all of the elements of the universal set is a proper subset. The subset that is equal to the universal set is called the improper subset. In listing all of the subsets of a given set the empty set and the improper subset must be included. The empty set is indicated by $\{\}$ or by $\emptyset$. The subsets of $A = \{a,b,c\}$ are $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$ and $\{\}$.

EXERCISES

1. List all of the subsets of $A = \{a,b,c,d\}$.
2. Suppose that three students, s_1, s_2, s_3, and their

teacher, t, plan a trip to a convention. Show the ways committees of students can be formed to be responsible for making hotel reservations.

The set of natural numbers. The student will recall that his first introduction to mathematics was learning to count. All numbers used to count are members of the set of natural or counting numbers. This set can be indicated by $N \equiv \{1,2,3,4, \ldots\}$. This set is called an infinite set, since there is no last element in the set. The symbol $\equiv$ is read, "is identical to".

Properties of natural numbers. The properties of natural numbers are most important in understanding the material that follows. The first property to examine is closure. A set is closed under a particular operation if the result of performing the operation on members of the set is also a member of the set. Thus, if two elements are added together, the sum is a member of the set if the set is closed under addition. If any two elements are subtracted, the difference is an element of the set if the set is closed under subtraction. As the student will discover, the set of natural numbers is closed under the operation of addition and multiplication, but not under subtraction or division.

EXERCISES

1. Given the set of natural numbers, show that the set is not closed for subtraction and division.

2. Is the set of all natural odd numbers closed for addition? Explain.

3. Is the set of all natural even numbers closed for addition? Explain.

4. Is the set of all odd numbers closed for multiplication? Explain.

5. Is the set of all even numbers closed for multiplication? Explain.

6. Is the set of natural numbers less than 10 closed for any operation? Explain.

7. Is the set of all odd numbers that are not multiples of 5 closed under addition? Under multiplication?

Fundamental Properties of Operation

If a is an element of the set of natural numbers, N, and b is also an element of N, then a + b = b + a. This is the <u>commutative property of addition</u>. This simply states that order of adding two natural numbers is of no importance, the result is the same regardless of order. Symbolically, we could write the above property as follows: If $a \in N$ and $b \in N$ then a + b = b + a. <u>$\in$ is read "is an element of"</u>.

If $a \in N$, $b \in N$ and $c \in N$ then (a + b) + c = a + (b + c). This is the associative property of addition.

Thus,

$$(2 + 3) + 4 = 2 + (3 + 4).$$

The parentheses indicate that the quantities contained within them are to be treated as a single quantity. Thus, combining the 2 and 3 gives 5 and adding 5 to 4 gives 9. On the right, 2 plus the sum of 3 and 4 is the same as 2 plus 7 which also yields 9. The student should note the necessity of using parentheses, because as the student will recall, he learned to add numbers two at a time. In fact, addition is defined for two numbers at a time. Thus, if more than two numbers are to be added, they must be paired before the addition can be performed.

If $a \in N$ and $b \in N$ then $a \cdot b = b \cdot a$. (The dot between the letters is a symbol for multiplication and is read "times".) This is the <u>commutative property of multiplication</u>. Thus $3 \cdot 4 = 4 \cdot 3$. This property simply states that the order of multiplying two elements does not affect the result.

If $a \in N$, $b \in N$ and $c \in N$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This is called the <u>associative property of multiplication</u>. As in addition, multiplication is defined for only two numbers at a time, thus the parentheses are necessary to pair the numbers so that they can be multiplied. It should be noted at this time that subtraction and division are also defined for only 2 numbers at a time. When an operation is defined for only two numbers at a time, the operation is called a <u>binary operation</u>.

If $a \in N$, $b \in N$ and $c \in N$, then $a \cdot (b + c) = a \cdot b + a \cdot c$. This is called the <u>distributive property of multiplication over addition</u>. This property states that multiplying the sum of two numbers by a number yields the same result as multiplying the number times each of the numbers to be added, then adding the results.

Thus,

$$2 \cdot (3 + 5) = 2 \cdot 8 = 16$$

and

$$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5 = 6 + 10 = 16$$

Cancellation properties. If $a, b, c \in N$ and $a + c = b + c$, then $a = b$. This is called the cancellation property of addition. Similarly, if $ac = bc$, then $a = b$. This is called the cancellation property of multiplication.

The commutative properties of addition and multiplication, the associative properties of addition and multiplication and the distributive properties of addition over multiplication are collectively called the fundamental properties of operation.

Symbols of inclusion. In the associative properties above and in the distributive property, parentheses were used for grouping numbers. Symbols used for grouping are called symbols of inclusion and include parentheses, ( ), brackets, [ ], braces, { } and the bar, ——. It is sometimes necessary to use more than one symbol of inclusion since one symbol of inclusion may be contained within another.

Examples:

(a) $(2 + 3) + [(3 + 5) + 8]$

(b) $[(1 + 2) + (3 + 2)] + \{[(4 + 1) + (5 + 6)] + (6 + 2)\}$

(c) $3 \cdot [(5 + 6) + (8 + 3)]$

In studying the above examples the student should see that the numbers are paired so that the indicated operations can be performed.

In example (a),

$$(2 + 3) + [(3 + 5) + 8] = 5 + [8 + 8] = 5 + 16 = 21.$$

In example (b),

$$[(1 + 2) + (3 + 2)] + \{[(4 + 1) + (5 + 6)] + (6 + 2)\}$$
$$= [3 + 5] + \{[5 + 11] + 8\}$$
$$= 8 + \{16 + 8\} = 8 + 24 = 32.$$

In example (c),

$$3 \cdot [(5 + 6) + (8 + 3)] = 3 \cdot [11 + 11] = 3 \cdot 22 = 66.$$

The bar is not used as often as other symbols of inclusion except when division is indicated. In algebra, division is customarily shown by writing a fraction. Thus, $a \div b$ is written a/b. $\frac{2 + 4}{3 + 1}$ means the sum of 2 and 4 is divided by the sum of 3 and 1. Here the bar serves two purposes; one to show division, and one to show that 2 + 4 and also 3 + 1 are to be treated as single quantities.

After the student becomes familiar with handling grouping of numbers for computation, the symbols of inclusion are frequently omitted. Thus $(a + b) + c$ is frequently written $a + b + c$. But the student must be aware that before the numbers can be combined there must be at least a "mental grouping", so that $a + b + c = (a + b) + c$ or $a + b + c = a + (b + c)$. Similarly $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ may be written $a\cdot b\cdot c$.

EXERCISES

1. Place symbols of inclusion in the following so that the operations can be performed.

   a. $1 + 2 + 3 + 4 + 5 + 6$

   b. $3\cdot 2\cdot 5\cdot 6$

   c. $3\cdot 1 + 4\cdot 2 + 5\cdot 6$

   d. $4 + 3 + 7 + 8 + 6 + 9 + 7$

2. Find the value of each of the parts of exercise 1.

3. Show that the associative property of addition holds for the following: $[(2 + 3) + (4 + 6)] + (8 + 1) = (2 + 3) + [(4 + 6) + (8 + 1)]$.

4. Apply the distributive property to each of the following:

   a. $2\cdot(a + 2\ b) =$

   b. $3\cdot x + 4\cdot x =$

5. Show by using the fundamental properties that:

   a. $a\cdot(b + c) = c\cdot a + a\cdot b$

   b. $a\cdot(b\ c) = c\cdot(a\ b)$

6. State all properties involved in changing the left side of the following statements to the right side.

   a. $2\cdot(3\cdot y) = 6y$

   b. $2\cdot(a + 1) = (1 + a)\cdot 2$

$4x + 5 = 17$, $\{3\}$ is the solution set, since $4 \cdot 3 + 5 = 17$ is a true sentence.

A sentence like $5 + x > 7$ is called an <u>inequality</u>. A study of the open sentence shows that any value of x greater than 2 makes this open sentence a true statement. Then the solution set from the set of natural numbers is $x = \{3, 4, 5, \ldots\}$.

EXERCISES

Find the solution sets of each of the following. The solution set must be a subset of the set of natural numbers. Hint: The solution set may be the empty set.

1. $3x + 4 = 10$
2. $x + x + 2 = 4$
3. $x + 2 < 10$
4. $x - 6 = 4$
5. $6 - 3x = 12$
6. $x + 1 < 1$

<u>Zero defined</u>. Consider the open sentence $x + 5 = 5$. If $N = \{1, 2, 3, \ldots\}$ there can be no solution in N. When the mathematician is faced with a problem for which there is no solution, he can accept the fact that no solution exists, or he can make a definition so that a solution will exist. Rather than accept a case having no solution, it is preferable to define a new set which will contain the element or elements

needed to give the desired non-empty solution set. Therefore, the following definition is given:

> There exists an element, zero, such that when zero is added to any element in N the sum is the same element. Symbolically, $b + 0 = b$.

A new set is now defined; $Z = \{0,1,2,3, \ldots\}$. It should be noted immediately that the set, N is a subset of Z. This fact may be written $N \subset Z$, and read, "N is a subset of Z".

Z is defined so that all of the fundamental properties of operation of N will still hold for Z with the exception of the cancellation property of multiplication. In the set, Z, the cancellation property of multiplication can be stated, if $ac = bc$ than $a = b$ providing c is different from zero. (written $c \neq 0$). The necessity of this provision follows from the fact that zero times any number is zero. The proof of this statement follows later in this chapter. The student can see that $2 \cdot 0 = 3 \cdot 0$ is true, but this does not imply that $2 = 3$. So, for the set Z and all subsequent sets, the provision that $c \neq 0$ must be made in connection with the cancellation property. From the definition of 0, $a + 0 = a$, but the commutative property of addition is required to hold, therefore, $a + 0 = 0 + a = a$. The new element, 0, is called the additive identity since adding 0 to any element in the set does not change the value of the element. It will be agreed that $0 + 0 = 0$.

It is necessary to check the new set for closure under addition and multiplication. It has been shown that 0 is the additive identity, hence, the set is closed under addition.

In checking multiplication, the only possible change would come about by multiplying by zero. Subtraction of natural numbers makes it evident that $b + 0 = b$ is just another way of writing $b - b = 0$. So to investigate the product of zero and any element, a, the product can be written, $a \cdot 0$, but replacing 0 with $b - b$, $a \cdot 0 = a \cdot (b - b)$. By the distributive property, $a \cdot (b - b) = a \cdot b - a \cdot b$ and by the closure property, $a \cdot b \in N$. But a natural number subtracted from itself is zero by our definition, hence $a \cdot b - a \cdot b = 0$, and this completes the proof that $a \cdot 0 = 0$. By the commutative property of multiplication it follows that $0 \cdot a = 0$.

Subtraction. Subtraction is defined as follows: $a - b = c$ if $b + c = a$ where a, b, c are elements of N. Thus, $6 - 2 = 4$ if $4 + 2 = 6$. Consider the exercise in the previous section, $6 - 3x = 12$. If the solution set is required to be a subset of the set Z, then the solution set must be the empty set, since, applying the definition of subtraction, $6 = 12 + 3 \cdot x$, and there is no element in Z that can be multiplied by 3 and added to 12 to give 6. Clearly, the set Z is not closed for subtraction. In order to have a non-empty solution set for the above example, it is necessary to define a new set that will be closed for subtraction.

The integers. A new set will now be defined as follows: There exists a set $\bar{N}$, such that for every element a in N there corresponds an element $\bar{a}$ in $\bar{N}$ such that $a + \bar{a} = 0$. The new set is indicated by $\bar{N} = \{. . . \bar{4}, \bar{3}, \bar{2}, \bar{1}\}$. This set is an infinite set since it has no first element. The numbers in $\bar{N}$ will be called "barred numbers".

When two sets of elements are combined to form a new set which contains as element all the elements of the original sets, the sets are said to be united. If A is one set and B is the other then $A \cup B$ (read A union B) contains all of the elements which belong to A or B or to both A and B.

If we unite sets $\bar{N}$ and Z, $(\bar{N} \cup Z)$, then the new set, $(\bar{N} \cup Z)$, will be denoted by $I = \{. . . \bar{3}, \bar{2}, \bar{1}, 0, 1, 2, 3 . . .\}$. This set is called the set of integers.

Operations with the set of integers. It will be required, by definition, that all the fundamental properties of operation with the natural numbers and zero hold for the new set. Before a check can be made for closure, an investigation of the properties of the arithmetic of I must be made. To make the arithmetic of I meaningful it must be developed in terms of the elements of Z with which the student is familiar.

natural number, and by definition $(b - a) + \overline{(b - a)} = 0$.

Then

$$a - b = a + \overline{b}$$

$$= (a + \overline{b}) + 0 = (a + \overline{b}) + (b - a) + \overline{(b - a)}$$

$$= [(a + \overline{b}) + (b - a)] + \overline{(b - a)}$$ associative property of addition

$$= [a + (\overline{b} + b) - a] + \overline{(b - a)}$$ associative property of addition

$$= [a + 0 - a] + \overline{(b - a)}$$ definition of zero

$$= (a - a) + \overline{b - a}$$ definition of zero

$$= 0 + \overline{b - a}$$ definition of zero

$$= \overline{b - a}$$ definition of zero

Therefore, in the case of $a < b$, $a - b = a + \overline{b} = \overline{b - a}$ and by commutative property of addition $a - b = a + \overline{b} = \overline{b} + a = \overline{b - a}$ if $a < b$.

Examples:

(a) $3 + \overline{2} = (3 - 2) = 1$

(b) $4 + \overline{5} = \overline{(5 - 4)} = \overline{1}$

(c) $\overline{6} + 2 = \overline{(6 - 2)} = \overline{4}$

(d) $\overline{3} + 3 = 0$

(e) $5 + \overline{5} = 0$

The proofs given above establish the rules for adding the elements of set I and enable the student to discover that I is closed for addition.

CHAPTER II

THE SET OF INTEGERS

The purpose of this chapter is to derive a new set of numbers, starting with the familiar set of natural numbers, which will be closed under the operations of addition, subtraction and multiplication.

Equations. An equation is a sentence which states that two expressions represent the same number and is generally expressed in mathematical symbols. $a + b = c$ could be expressed in words by: "The sum of two numbers, a and b, is equal to another number, c".

Open sentences. If an equation is written so that some symbol such as a, b, c, x or y takes the place of a number or set of numbers, then the equation may be either true or false. Thus $5 + x = 9$ is an equation. x holds the place for some number or numbers that will make this statement true or false. For example, if x is replaced by 6, the statement is false, but if x is replaced by 4, the statement is true. Such an equation can properly be called an open sentence. When the set of numbers that will make the equation true is formed, the set is called the solution set for the open sentence. The solution set is sometimes called the truth set. In the equation, $5 + x = 9$, it can readily be seen that $\{4\}$ is the solution set, since only 4 will make the open sentence true. In the open sentence

First the rules for addition will be considered. What is the result of adding $\bar{a} + \bar{b}$? If $a + b$ is added to $\bar{a} + \bar{b}$, then, by the associative property of addition, $(\bar{a} + \bar{b}) + (a + b) = \bar{a} + (\bar{b} + a) + b$. This in turn is equal to $\bar{a} + (a + \bar{b}) + b$ using the commutative property of addition, which equals $(\bar{a} + a) + (\bar{b} + b)$ using the associative property of addition again. But by the definition of zero $\bar{a} + a = 0$ and $b + \bar{b} = 0$, hence, $(\bar{a} + \bar{b}) + (a + b) = 0 + 0 = 0$ therefore, $\bar{a} + \bar{b} = \overline{(a + b)}$. This states simply that the sum of two barred numbers is equal to the bar of the sum of two natural numbers.

Examples:

(a) $\bar{2} + \bar{3} = \overline{2 + 3} = \bar{5}$

(b) $\bar{7} + \bar{4} = \overline{7 + 4} = \overline{11}$

What is the result of adding a natural number to a barred number? Start with an open sentence $a + \bar{b} = x$, then add b to $a + \bar{b}$ and to x to obtain $(a + \bar{b}) + b = x + b$. Then $a + (\bar{b} + b) = x + b$ by the associative property of addition. But $\bar{b} + b = 0$ by the definition of zero. Hence $a + 0 = x + b$ or $a = x + b$. But if $a = x + b$ then $x = a - b$ by the definition of subtraction. Hence $a + \bar{b} = a - b$. If $a > b$ then $a - b$ is a natural number and there is no problem. If $a = b$, then $a - b = 0$, but if $a < b$ then the value of $a - b$ must be established. If $a < b$ then $b - a$ must be a

EXERCISES

1. Add the following:

   a. $(\bar{2} + \bar{3}) + (\bar{5} + \bar{6})$

   b. $(\bar{2} + 3) + (5 + \bar{2})$

   c. $(4 + \bar{2}) + \bar{7}$

   d. $6 + \bar{3} + \bar{8} + 2$

   e. $\bar{8} + \bar{6} + \bar{7} + \bar{8}$

2. Rewrite the following using barred numbers:

   a. $x - y$

   b. $7 - 3$

   c. $3 - 7$

   d. $4 - 5$

<u>Closure for subtraction</u>. To discover the rules for subtraction in I, the results of subtracting b from a, b from $\bar{a}$ and $\bar{b}$ from $\bar{a}$ and $\bar{b}$ from a in terms of natural numbers a and b must be investigated.

What is the result of subtracting b from a where a and b are any natural numbers? It will be recalled that in the set of natural numbers, subtraction was only possible if $a > b$. Here, no restriction is made on a and b, but three cases exist. One of the following must be true: $a < b$, $a = b$ or $a > b$. If $a > b$, then the rules for subtracting natural numbers will hold, and if $a = b$, $a - b = a - a = 0$. The case that must be investigated is the one in which $a < b$. If $a < b$, then $b > a$, therefore $(b - a)$ is a natural number. By the definition of zero, if $(b - a)$ is a natural number then $(b - a) + \overline{(b - a)} = 0$.

Let

$$a - b = (a - b) + 0$$

then replacing 0 with $(b - a) + (\overline{b - a})$

$$a - b = (a - b) + \left[(b - a) + (\overline{b - a})\right]$$

$$= \left[(a - b) + (b - a)\right] + (\overline{b - a})$$ by the associative property of addition

$$= \left[(b - a) + (a - b)\right] + \overline{b - a}$$ by the commutative property of addition

$$= \left\{\left[(b - a) + a\right] - b\right\} + \overline{b - a}$$ by the associative property of addition

But it will be agreed that $a - a = - a + a = 0$

therefore, $\left[b - a + a\right] = b + 0 = b$

Hence,

$$a - b = (b - b) + (\overline{b - a})$$

but $b - b = 0$

therefore, when $a < b$

$$a - b = \overline{b - a}$$

To investigate the results of $a - \bar{b}$,

let

$$a - \bar{b} = a - \bar{b} + 0 = 0 + a - \bar{b}$$

$$= (b + \bar{b}) + (a - \bar{b})$$ by the definition of zero,

$$= b + (\bar{b} + a) - \bar{b}$$ by the associative property of addition

$$= b + (a + \bar{b}) - \bar{b}$$ by the commutative property of addition

$$= b + a + (\bar{b} - \bar{b})$$ by the associative property of addition

$= b + a + 0$ from the definition of zero,

therefore,

$a - \bar{b} = a + b$ by the commutative property of addition.

The preceding proof establishes the result that subtracting a barred number from a natural number gives the *sum of the natural number and the natural number corresponding to the barred number*.

If b is to be subtracted from $\bar{a}$,

then,

$$\bar{a} - b = (\bar{a} - b) + 0 = 0 + (\bar{a} - b),$$
$$= (b + \bar{b}) + (\bar{a} - b)$$
$$= (\bar{b} + b) + (\bar{a} - b)$$
$$= \bar{b} + (b + \bar{a}) - b$$
$$= \bar{b} + (\bar{a} + b) - b$$
$$= (\bar{b} + \bar{a}) + (b - b) = \bar{b} + \bar{a} + 0 = \bar{b} + \bar{a}$$

Therefore

$$\bar{a} - b = \bar{a} + \bar{b}$$

and finally $\bar{a} - b = \overline{a + b}$

The reasons for the various steps of the above proof will be left to the student as an exercise.

The result of subtracting a natural number from a barred number is the bar of the sum of the natural number, b, and the natural number corresponding to a.

The result of subtracting a barred number from another barred number, remains to be established.

Let

$$\bar{a} - \bar{b} = (\bar{a} - \bar{b}) + 0 = 0 + (\bar{a} - \bar{b})$$
$$= (b + \bar{b}) + (\bar{a} - \bar{b})$$
$$= b + (\bar{b} + \bar{a}) - \bar{b}$$
$$= b + (\bar{a} + \bar{b}) - \bar{b}$$
$$= (b + \bar{a}) + (\bar{b} - \bar{b})$$
$$= b + \bar{a} + 0$$
$$= \bar{a} + b$$

The reasons for the above proof will be left to the student as an exercise.

The result of subtracting a barred number from a barred number is equal to the sum of the bar of the first and the natural number corresponding to the second barred number. The method of adding barred numbers and natural numbers has already been established.

The student should see that the results of subtracting in the set of integers is always defined in terms of addition, therefore, it will be convenient to always think of combining numbers in terms of addition and the term "subtraction" will not be used in general.

To summarize the rules of subtraction, let $\bar{a}$ be called the additive inverse of a, and a be called the additive inverse of $\bar{a}$. Then subtraction can always be defined to be the sum of the minuend and the additive inverse of the subtrahend.

Thus, if the student wishes to subtract $\overline{4}$ from 8, he should think, "I will add the additive inverse of $\overline{4}$ to 8". Hence, the result would be $8 + 4 = 12$.

The above investigation shows that the set of integers is closed under subtraction.

EXERCISES

1. Complete the discussion of subtracting a natural number from a barred number, giving the reason for each step.

2. Complete the discussion of subtracting two barred numbers, giving the reason for each step.

3. Using the rules developed for subtraction in the above section, subtract the following:

   (a) $4 - 8$

   (b) $5 - \overline{4}$

   (c) $\overline{4} - 6$

   (d) $5 - \overline{6}$

   (e) $8 - \overline{3}$

   (f) $\overline{6} - \overline{4}$

   (g) $\overline{14} - \overline{3}$

   (h) $\overline{24} - 30$

   (i) $44 - \overline{62}$

   (j) $\overline{56} - 43$

   (k) $\overline{104} - \overline{356}$

   (l) $\overline{87} - 15$

4. Applying the rules for addition and subtraction, give the results of the following:

   (a) $(\overline{4} + 6) - (3 + \overline{7})$

   (b) $(\overline{4} - \overline{7}) + (6 - \overline{7})$

   (c) $(\overline{4} + \overline{8}) - (6 + \overline{13})$

   (d) $(6 - 7) + (8 - 4)$

   (e) $(8 - 9) - (7 - \overline{4})$

   (f) $(7 - 3) - (3 - 7)$

   (g) $(6 + \overline{8}) - (6 - \overline{8})$

   (h) $(\overline{7} + \overline{6}) - (5 - 9)$

<u>Closure for multiplication.</u> To investigate closure under multiplication, it is necessary to find the results of each of the following: $a \cdot b$, $a \cdot \bar{b}$, $\bar{a} \cdot b$, $\bar{a} \cdot \bar{b}$. The first case is simply the product of two natural numbers, and it is known that the product of two natural numbers yields a natural number.

When there is no danger of confusion, the dot indicating multiplication may be omitted. Thus, $a \cdot b$ may be written ab. $a \cdot (b + c)$ may be written $a(b + c)$ and $(a + b) \cdot (c + d)$ may be written $(a + b)(b + c)$. Obviously, the dot cannot be omitted when two numerals are multiplied, since $2 \cdot 2$ is not equal to 22. The dot may be omitted when a numeral is multiplied by a letter which represents another number. For example, $2 \cdot a$ can be written $2a$.

Before developing the properties of multiplication for the remaining cases, it is necessary to prove that $b \cdot 0 = 0$. To do this, let $ab = (a + 0) \cdot b$. Then applying the distributive property, $ab = ab + 0 \cdot b$, and applying the cancellation property of addition, $0 = 0b$, but since the commutative property of multiplication is to hold, $0 = 0 \cdot b = b \cdot 0$. It will be agreed that $0 \cdot 0 = 0$.

Having shown that 0 times any natural number is 0, it can be noted that if b were replaced with $\bar{b}$, the same procedure could be used to show that $0 \cdot \bar{b} = 0$, hence $0x = 0$ where x is <u>any</u> integer.

It is now easy to develop the results in the remaining cases. Since $0 = 0a$, it follows that $0 = (b + \bar{b}) \cdot a$. Applying the distributive property, it can be seen that $0 = ba + \bar{b}a$, but from the definition of 0, $0 = ba + \overline{ba}$. Replacing 0 with $ba + \overline{ba}$, the equation becomes $ba + \overline{ba} = ba + \bar{b}a$, hence, applying the cancellation property of addition, $\overline{ba} = \bar{b}a$. Similarly, it can be shown that $\bar{a}b = \overline{ab}$, therefore, it can be concluded that the product of a natural number by a barred number is equal to the bar of the product of the natural number and the natural number corresponding to the barred number. Thus, $4 \cdot \bar{2} = \overline{4 \cdot 2}$ and $\bar{5} \cdot 3 = \overline{5 \cdot 3}$.

It remains to investigate the case of the product of two barred numbers.

Let

$$ab = ab + 0, \text{ then } ab = ab + \bar{a}0,$$

since it has been shown that 0 times any integer is 0. Then replacing 0 with $b + \bar{b}$, the equation becomes $ab = ab + \bar{a}(b + \bar{b})$. Applying the distributive property, which has been assumed for all integers,

$$ab = ab + (\bar{a}b + \bar{a} \cdot \bar{b}).$$

Using the associative property of addition,

$$ab = (ab + \bar{a}b) + \bar{a} \cdot \bar{b}$$

In the previous section, it was shown that $ab + \bar{a}b = ab + \overline{ab} = 0$.

Therefore,

$$ab = 0 + \bar{a}\cdot\bar{b} \text{ or } ab = \bar{a}\cdot\bar{b}.$$

In words, this says that the product of any two barred numbers gives a product equal to the product of the natural numbers corresponding to the barred numbers.

Since it has been demonstrated that the product of two elements in set I gives an element belonging to set I, it can be said that the set of integers is closed for multiplication

EXERCISES

Using the rules developed for the equations with integers, find the following:

1. $3\cdot 4$
2. $\bar{3}\cdot 6$
3. $\bar{7}\cdot\bar{4}$
4. $6\cdot\bar{5}$
5. $\bar{6}\cdot\bar{7}$
6. $\bar{5}(7 + \bar{6})$
7. $(8 + \bar{7})\ (9 + \overline{13})$
8. $(6 - \bar{9})(\overline{10} + \overline{16})$
9. $(\overline{43} + \overline{21})(\bar{4} + 13)$
10. $\dfrac{\{[6(8 + 15] + 3\,[4(6 + 12)]\}}{[6(5 - 7)]}$

<u>Division</u> <u>of</u> <u>integers</u>. It has been shown that the set of integers is closed for addition, subtraction and multiplication. Can it be shown that the set is closed for division? The question means, "Can one divide any integer by any other non-zero integer and be always assured that the quotient is an integer?" If the student can think of a single example

where one integer is divided by another and the result is something other than an integer, then the set is not closed for division. Try, for example, dividing 3 by 4. Assume that the result is x. Then $4 \cdot x$ should equal 3, but no integer x exists for which this is true. Hence, the set of integers is not closed for division.

Negative integers. In the section on subtraction, it was pointed out to the student that subtraction is defined in terms of addition. Since there is no need to use the word "subtraction", it follows that there is no need to use the minus sign to mean subtraction. Since the minus sign is not to be used to indicate subtraction, it may be put to use to indicate something else.

Let the student imagine that he could push the bar from the top of one of the barred integers so that it falls in front of the integer. Then $\bar{a}$ would become $-a$. The "$-a$" can then be called a *negative integer*. Since this is the same integer as $\bar{a}$, the rules developed for barred numbers apply for negative numbers. Also, the natural numbers then are called *positive integers*.

The following examples illustrate how the barred numbers can be written as negative numbers.

Examples:

(a) $\bar{a} + \bar{b} = (-a) + (-b)$

(b) $\bar{a} + b = (-a) + b$

(c) $a + \bar{b} = a + (-b)$

In order to conserve as much writing as possible, the plus sign in examples a and c could be omitted. This is, (-a) + (-b) can be written -a -b, but the student should recall the meaning of the expression. It should be thought of as, "negative a plus negative b".

EXERCISES

Write the following as negative numbers and evaluate each of the following expressions:

1. $\bar{3} + \bar{4}$
2. $5 + \bar{4} + \bar{6}$
3. $7 + \bar{8} + \bar{9} + \overline{10}$
4. $\bar{4} + \bar{6} + \bar{8}$

Find the value of each of the following:

5. - 6 + 8 - 7 - 8
6. - 5 - 7 - 8 - 4
7. 8 - 9 - 3 - 7
8. (8 - 10) + (9 - 8)

CHAPTER III

THE RATIONAL NUMBERS

The discussion in the previous chapter demonstrated that the set of integers was not closed for division. Consider the open sentence, $5x = 4$. What is the solution set for this open sentence? If the solution set is required to be a subset of the set of integers, then it must be $\emptyset$ since no integer multiplied by 5 gives 4. In order to have a solution to the open sentence, it is necessary to define a new set such that a non-empty subset of the new set will be the solution set.

<u>The set of rational numbers</u>. A rational number is defined to be any number that can be expressed in the form $\frac{a}{b}$, where a and b are integers, except b can not be zero. The set of rational numbers is the set of all elements of the form $\frac{a}{b}$. This set, designated R_1, will have the following defined properties:

1. $\left(\frac{a}{b}\right)b = a$

2. $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1$ provided $a \neq 0$ and $b \neq 0$.

If $\frac{a}{b}$ is an element of R_1, then $\frac{b}{a}$ is called the <u>multiplicative inverse</u> of $\frac{a}{b}$.

As before it will be required by definition that the fundamental properties of operation hold for R_1.

It can be shown that the set of rationals is closed for addition, subtraction, multiplication and division.

Addition in the set of rationals is defined as follows:

$$a/b + c/d = \frac{ad + bc}{bd}, \text{ where } b \neq 0 \text{ and } d \neq 0$$

This definition conforms to the definition of set R_1. a,b,c and d are integers, and since the set of integers is closed for multiplication, then ad, bc, bd are integers, and from the closure property of addition in the set of integers ad + bc is an integer, hence, $\frac{ad + bc}{bd}$ is an integer over an integer which conforms to the definition of a rational number. Therefore, the set of rationals is closed for addition. The properties of subtraction have been defined in terms of addition, so R_1 is closed for subtraction as well as addition.

Multiplication of rational numbers if defined as follows:

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd} \quad \text{if } b \neq 0 \text{ and } d \neq 0$$

Since a, b, c and d are integers, ac and bd are integers by the closure property of multiplication in the set of integers. The product of $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)$ yields an integer over an integer which conforms to the definition of a rational number. Therefore, the set of rationals is closed for multiplication.

Division is defined as follows:

$a \div b = c$ (usually written $a/b = c$)

if and only if $a = bc$ where $b \neq 0$.

The student can see the necessity of the provision that $b \neq 0$, since it has been proved that zero times any number is zero. Consider the following open sentence:

$4/0 = x$

Applying the definition of division, then

$4 = 0 \cdot x$.

But it has been shown that $0 \cdot x = 0$, hence, a non-empty solution set for x which makes $0 \cdot x = 4$ does not exist. Therefore, <u>division by 0 is not defined</u>.

By the definition of division, it can be seen that if a is divided by b, the quotient is written $\frac{a}{b}$. This is an element of R_1, hence, R_1 is closed for division whenever a and b are integers and $b \neq 0$.

<u>Ratio</u>. When one number is divided by another number, the quotient is called a <u>ratio</u>. A ratio is usually written as a fraction, such as a/b, and is read, "the ratio of a to b". It may also be written a:b and is read in the same manner. In a ratio both a and b must be expressed in the same units. Thus the ratio of one foot to a yard could be written 12/36 or 1/3.

<u>Equivalence classes</u>. In the set of rationals, a particular element can be expressed in many different ways.

For instance, the number 4 can be expressed as 4/1, 8/2, 12/3, 16/4 . . ., the fraction 1/2 can be expressed as 2/4, 3/6, 4/8, 6/12 . . . . These sets of numbers, which can be used to express a particular element in R_1 are called <u>equivalence classes</u>.

EXERCISES

1. Write an equivalence class for "5".
2. What is the ratio of 1 pint to 1 gallon?
3. Write an equivalence class for 2/3.
4. Using the rule for adding rational numbers, add 3/5 and 2/7.
5. Add: - 5/8 + 6/11
6. Add: 5/8 - 3/13
7. Add: (3/5 + 6/7) + (2/3 + 2/5)

<u>Repeating decimals</u>. When a number is expressed as the ratio of two numbers, the ratio can be changed to a decimal such that a digit or a group of digits in the decimal part will repeat. For example, 1/3 = .3333 . . . . Here the digit three repeats. 1/4 = .25000 . . . . Here the zero repeats. 1/7 = .142857142857 . . . . Notice that the group of digits 142857 will continue to repeat indefinitely.

In higher mathematics it is shown that any repeating decimal can be expressed as a ratio of two numbers, and hence, it is a rational number.

EXERCISES

Change the following rational numbers to repeating decimals:

1. 1/8
2. 3/5
3. 4/7
4. 1/9
5. 3/13

What rational number do the following repeating decimals represent?

6. .666 . . .
7. .625 . . .
8. .87500 . . .
9. .16666 . . .
10. .375000 . . .

CHAPTER IV

THE REAL NUMBER SYSTEM

The set of rational numbers does not meet all of the needs of first year algebra. That is, there are numbers which cannot be expressed as a ratio of two numbers. For example, the student will recall from arithmetic that when the circumference of a circle is divided by its diameter, the result is π which is equal to 3.141592653589 . . . . It should be noted that the decimal part of π does not appear to be a repeating decimal, and hence, is apparently not a rational number. It is necessary to define a set, then, that will contain the number, π , and other numbers similar to π that cannot be placed in the set of rational numbers.

Real numbers. For the present, the set of real numbers, R, is defined to be the union of R_1 and the set of all numbers expressed by means of non-repeating decimals. Also, since the set of integers is a subset of the set of rational numbers, and the set of natural numbers is a subset of the set of integers, the set of integers and the set of natural numbers are subsets of the set of real numbers. Symbolically, this could be written: $N \subset R$, $I \subset R$, $R_1 \subset R$.

The subset of numbers in R which are not in R_1 are called irrational numbers. This set includes numbers like π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and many more. Indeed, it can be shown in advanced courses that there are more irrational numbers than there are rational numbers.

The set of real numbers will be the last set of numbers defined in this course. The student should not get the idea, however, that this is the extent of the number system. It can be shown that the set of real numbers form a subset of a larger set that is defined in higher mathematics, but the set of real numbers will be adequate for this course.

CHAPTER V

SOLUTION OF EQUATIONS AND INEQUALITIES

Equations were defined in Chapter II and the student found the truth sets for several open sentences by inspection. This chapter will introduce methods of finding solution sets for equations and inequalities which are not so easily found by inspection.

There are several definitions that are used in solving equations with which the student must become familiar.

Consider the equation

$$ax + b = c.$$

An equation of this type is called a <u>literal equation</u>. x is the placeholder for the solution set that will make the open sentence true. a, b and c represent numbers whose values are known. ax, b and c are called <u>terms</u>. ax is called an <u>algebraic term</u> and b and c re called <u>arithmetic terms</u> or <u>constant terms</u>. When reference is made to "number" in this chapter and later chapters, the set of real numbers will be the universe unless it is specified otherwise.

In the term, ax, a and x are <u>factors</u> of ax. <u>Factors</u> are any of the numbers which when multiplied together form a product. The factors of an algebraic term are called <u>coefficients</u>. Thus, in the given equation, a is the coefficient of x and x is the coefficient of a. The term is most often used when referring to the coefficient of a placeholder or placeholders.

In literal equations, it is customary to let the letters near the first of the alphabet represent known values or constants, while the letters near the end of the alphabet represent placeholders for the solution set.

EXERCISES

1. In the equation $3x + 5 = 8$, what are the constants? What are the constant terms? What are the algebraic terms? What is the coefficient of x?

2. In the equation $5x + 7x + 9x = 4 - 6x$ list the coefficients of x. What is the constant term?

3. Using the definition of <u>factors</u>, give a set of factors for the following numbers.

   (a) 24 (e) 14

   (b) 12 (f) 27

   (c) 18 (g) 32

   (d) 48 (h) 15

   Which of the above numbers have only one set of factors?

<u>Similar terms</u>. In algebraic terms such as 3x, 4y, 3xy, the letters (or letter) in the term which represent a placeholder for some numbers (or number) are called <u>literal coefficients</u>. Algebraic terms having the same literal coefficients are called similar terms or like terms. Thus, 4x and 7x are like terms. 9xy and 14xy are similar terms, but 9y and 11x are unlike terms. Unlike terms cannot be combined.

Similar terms can be combined. This follows from the distributive property of multiplication over addition. Given the expression $5x + 7x$, from the distributive property,

$$5x + 7x = (5 + 7)x = 12x.$$

From this example it can be seen that similar terms can be combined by adding the coefficients of the similar terms and multiplying the sum times the common literal coefficient.

Examples:

(a) $4x + 9x - 3x = (4 + 9 - 3)x = 10x$

(b) $7xy - 4xy - 13xy = (7 - 4 - 13)xy = -10xy$

(c) $8z + 15z - 3z = (8 + 15 - 3)z = 20z$

(d) $8x + 5y - 4x + 7y = 8x - 4x + 5y + 7y =$

$(8 - 4)x + (5 + 7)y = 4x + 12y$

Methods for finding solutions of equations and inequalities will now be developed. When one or more operations are performed on an expression containing x such that the resulting expression contains only the x term with a coefficient of 1, x is said to be "isolated".

If two or more operations are indicated in an expression, then to isolate x, an inverse of each indicated operation must be performed on the expression. Thus, in the expression, $ax + b$, to isolate x the additive inverse of b must be added to the expression which gives ax, and the multiplicative inverse of a must be multiplied times ax to give x with a coefficient of 1.

Thus,

$$\left(\frac{1}{a}\right)\cdot(ax) + b + (-\ b) = x$$

In the expression $\frac{ax}{b} + c$, to isolate x, $\frac{ax}{b}$ must be multiplied by the multiplicate inverse of $\frac{a}{b}$ and the additive inverse of c must be added to the expression.

Hence,

$$\left(\frac{b}{a}\right)\left(\frac{a}{b}\right)\cdot x + c + (-\ c) = x$$

EXERCISES

Isolate x in each of the following by applying the inverse of each indicated operation:

1. $x + 3$
2. $2x$
3. $2x + 4$
4. $x/5 + 6$
5. $3x - 4$
6. $5x - 6$
7. $\frac{4x}{5} + 8$
8. $\frac{3x}{2} - 7$

To find the solution set of an equation, one must find the value or values of the placeholder that will make the equation a true statement. By inspection, the student can see that $\{2\}$ is the truth set for the open sentence $x + 3 = 5$. It is convenient to say that $x = 2$ is a solution or a root to the equation $x + 3 = 5$, and in general that each element of a solution set for a given equation is a solution of the equation.

Axiom. An axiom is a basic assumption that will be accepted without proof. The following axiom of equality will be used to find the solutions or roots of equations.

Solution axiom. The same operation may be performed on both sides of an equation without changing the equality. For example,

if $ax = b$, then $ax + c = b + c$ or

if $x = d$, then $ax = ad$ or

if $ax = b$, then $ax - c = b - c$, or

if $x/a = b/c$, then $\frac{dx}{a} = \frac{db}{c}$

Now reconsider the equation given at the beginning of this chapter; $ax + b = c$. To find a solution of the equation, examine the left side of the equation. The student should consider the steps necessary to isolate x. The steps that are required to isolate x are applied to both sides of the equation. To isolate x, ax must be multiplied by the multiplicative inverse of a and the additive inverse of the constant term must be added to the left side of the expression.

Applying these steps to both sides of the equation will yield a solution for x. These steps should be applied one at a time. Usually applying the additive inverse to both sides of the equation first will make the solution easier to find, but applying the multiplicative inverse to both sides of the equation first would result in a correct solution.

The method of finding a solution of $ax + b = c$ will now be demonstrated.

Given:

$$ax + b = c$$

adding the additive inverse of b to both sides

$$ax + b + (-b) = c + (-b)$$

results in

$$ax = c - b$$

multiplying by the multiplicative inverse of a

$$1/a(ax) = 1/a(c - b)$$

results in

$$x = 1/a(c - b)$$

applying the distributive property,

$$x = c/a - b/a$$

applying the rule for adding rational numbers,

$$x = c/a - b/a = \frac{ac - ab}{a\ a}$$

applying the distributive property,

$$x = \frac{a(c - b)}{a \cdot a}$$

from the definition of multiplication of rational numbers,

$$x = \frac{a}{a} \cdot \frac{c - b}{a},$$

but a/a = 1, hence,

$$x = \frac{c - b}{a}$$

Usually these steps are not all shown. When the student gains proficiency in solving equations only the most important steps are shown.

Several examples will now be presented. Supplying a reason for each step in the solution will be left to the student as an exercise.

Examples:

(a)
$$x + 4 = 8$$
$$x + 4 + (-4) = 8 + (-4)$$
$$x = 4$$

(b)
$$3x = 12$$
$$(1/3)(3x) = (1/3)12$$
$$x = 4$$

(c)
$$2x + 4 = 16$$
$$2x + 4 + (-4) = 16 + (-4)$$
$$2x = 12$$
$$(1/2)(2x) = (1/2)(12)$$
$$x = 6$$

(d)
$$x/4 = 24$$
$$(4)(x/4) = (4)(24)$$
$$x = 96$$

(e)
$$\frac{3x}{5} + 4 = 16$$
$$\frac{3x}{5} + 4 + (-4) = 16 + (-4)$$
$$\frac{3x}{5} = 12$$
$$(5/3)(3/5)x = (5/3)(12)$$

(f) $7x - 3 = 25$
$7x - 3 + (3) = 25 + 3$
$7x = 28$
$(1/7)(7x) = (1/7)(28)$
$x = 4$

In many cases, terms containing the placeholder will be found on both sides of the equation. Applying the additive inverse of one of the terms to both sides of the equation will result in getting the placeholder terms on the same side of the equation. The terms can then be combined and the equation can be solved by consecutive applications of inverse operations. For example, consider the equation:

$$3x + 5 = 2x + 10$$

Adding the additive inverse of 2x to both sides of the equation,

$$3x + 5 + (- 2x) = 2x + 10 + (-2x)$$

gives

$$x + 5 = 10.$$

Adding the additive inverse to both sides

$$x + 5 + (- 5) = 10 + (- 5)$$

gives the result

$$x = 5$$

which is a solution to the equation.

A solution of an equation may be checked by replacing the placeholder with the solution and determining if the two sides can be shown to be the same number. For example, to check the solution of the previous example, the placeholder in the original equation is replaced with 5.

Hence, $3(5) + 5 = 2(5) + 10$

The value of the left side becomes $15 + 5 = 20$ and the value of the right side becomes $10 + 10 = 20$. Since both sides reduce to 20, 5 checks as a solution to the equation, and 5 is said to satisfy the equation.

EXERCISES

1. Give the reasons for each step in the solution of examples, a through f above.

2. Check the solution of each of the examples a through f, and show that when the placeholder is replaced by the root that the value of the left side of the equation reduces to the value of the right side.

3. Solve the following equations, giving reasons for each step, and check to see if the solution satisfies the equation.

   a. $y + 17 = 15$

   b. $x + 18 = 86$

   c. $x + 9 = 15$

   d. $5x + 7 = 18$

   e. $4x + 3 = 9$

   f. $2x - 12 = -16$

   g. $y/4 = -2$

   h. $x - 9 = 7$

   i. $9x = 63$

   j. $5x - 6 = 3x + 8$

   k. $4x + 9 = 12 - 2x$

Order. The set of real numbers was developed with no mention of the order of the numbers. That is, no investigation was made as to the sequence in which the numbers were placed. Before investigating the solution of inequalities it is desirable to order the set of real numbers. To do this, consider the subset Z of R. The first element in Z is 0. This is followed by 1,2,3, ... The student is familiar with the order of this set from his past experience. This could be written $0<1<2<3$ . . . .

In set I, the order will be defined as follows:

-1 precedes 0, -2 precedes -1, -3 precedes -2 . . . .

This could be written, . . . $-3<-2<-1<0<1<2$ . . . .

Similarly, in the set R_1, an example of order is

. . . $-3<-5/2<-2<-3/2<-1<-1/2<0<1/2<1<3/2$ . . . .

The same ideas apply to the set of real numbers.

Inequalities. The methods of solving inequalities are similar to the methods of solving equations. The axiom of equality will not hold for inequalities, however, so this axiom is replaced with axioms which apply to inequalities.

These axioms are:

(1) Equal quantities may be added to each side of an inequality without changing the order of the inequality.

(2) Each side of an inequality may be multiplied by a positive real number without changing the order of the inequality.

(3) Each side of an inequality may be multiplied by negative real numbers provided the symbol of inequality is reversed.

The following examples illustrate the above axioms:

(a) Given: $4 > 2$
By axiom 1
$4 + 3 > 2 + 3$
or $7 > 5$

(b) Given: $7 < 11$
By axiom 1
$7 + (-3) < 11 + (-3)$
or $4 < 8$

(c) Given: $8 > 5$
By axiom 2
then $(6)\cdot(8) > (6)\cdot(5)$
or $40 > 25$

(d) Given: $8 > 5$
then multiplying each side by (-6)
$(-6)\cdot(8) < (-6)\cdot(5)$
or $-48 < -40$

The student should notice the necessity of reversing the inequality sign in the last example.

The procedure for finding the solution set for an inequality is similar to that for the solution of equations. The first step is to combine the algebraic terms containing the placeholder, and then using inverse operations discover what operations need to be performed to isolate the placeholder, making use of the axioms of inequalities.

Examples:

(a) Given: $3x + 6 < 5$

Adding a negative 6 to each side, (axiom 1)

$$3x + 6 + (-6) < 5 + (-6)$$

or $3x < -1.$

Multiplying by the multiplicative inverse of 3, (axiom 2)

$$(1/3)(3x) < (1/3)(-1)$$

$$x < -1/3$$

The solution set then is $\{x \mid x < -1/3\}$ Any value of x less than $-1/3$ will make the original equation a true statement. For example, $-1 < -1/3$. Replacing x with -1,

$$(3)(-1) + 6 < 5$$

or $3 < 5$ which is a true statement.

Consider this example:

(b) $5 - 6x > 4$

$5 - 6x + (-5) > 4 + (-5)$ by axiom 1

or $-6x > -1$

$(-1/6)(-6x) < (-1/6)(-1)$ by axiom 3

or $x < 1/6$

Hence, the solution set is $\{x \mid x < 1/6\}$,

$0 < 1/6$, so putting 0 in place of x,

$5 - 6 \cdot 0 > 4$

or $5 > 4$ which is a true statement.

Any value of x which is less than 1/6 may be used to check the solution of this inequality.

EXERCISES

Find the solution set to each of the following inequalities and give a reason for each step in the solution.

1. $4x > 8$
2. $5y + 9 > 4$
3. $3x - 2 < 7$
4. $2x + 5 < 27$
5. $1/x < 3$
6. $2 - 5x < 27$
7. $7 - 2/3x > 15$
8. $3(x + 4) > 9 + 2x$
9. $x/2 + 5 < x - 5$
10. Check the solutions in each of the above inequalities.

CHAPTER VI

GRAPHICAL REPRESENTATION

An alternate definition of the set of real numbers R will now be considered. Membership in the set will be determined as follows; for every point P on a given line l there is a number a belonging to the set which corresponds to P. On the other hand, for each number in the set there is a corresponding point on the line.

The set of real numbers can then be represented on a straight line which will be called the <u>graph</u> of the real numbers, or more commonly, the <u>real number line</u>. Figure 1 is a graph of all of the real numbers, x, where $-3 < x < 3$. (written $\{x \mid -3 < x < 3 \text{ and } x \in R\}$ and read, "the set of all x such that x is between -3 and 3, inclusive and x is a real number.)

Fig. 1

In general, a graph is a diagram or picture that represents mathematical relationships. In this chapter, graphs will represent the relation between sets of points.

If the set, $\{x \mid -3 \leq x \leq 3, x \in N\}$ is graphed on the real number line as shown in Figure 2, the graph shows a relationship between the set of real numbers and the set of integers.

Fig. 2

The circles about the points in Figure 2 indicate the integers. This illustrates graphically that the set of integers is a subset of the set of real numbers.

It is sometimes convenient to graph the solution set of equations or inequalities. The procedure for graphing the solution set is to locate points on the real number line corresponding to the numbers which are in the solution set. This process is called <u>plotting</u> the points.

Consider the equation $3x + 2 = 8$. The solution set is $\{2\}$. But $\{2\} \subset R$, therefore, $\{2\}$ can be graphed or plotted on the real number line. In plotting solution sets a circle about a point will indicate that the number corresponding to the point belongs to the solution set. Figure 3 is the graph of the solution set, $\{x \mid 3x + 2 = 8, x \in R\}$.

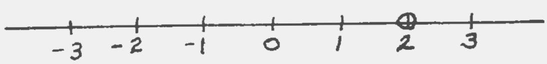

Fig. 3

For the equation, $x - 3 = 3$, $\{6\}$ is the solution set. The graph of the solution set is shown in Figure 4.

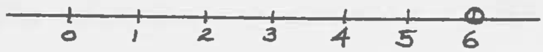

Fig. 4

The solution set of an inequality can also be shown on the graph of the real numbers. Consider the inequality, $x + 2 < 6$. If the universal set is the set of natural numbers, then the solution set would be $\{1,2,3\}$. The graph of the solution set is shown in Figure 5.

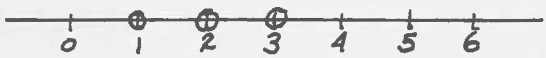

Fig. 5

The graph of $\{x \mid x - 2 > 4,\ 0 < x < 10, \text{ and } x \in N\}$ is shown in Figure 6.

Fig. 6

The graph of $\{x \mid 2x + 1 = 6 \text{ and } x \in R\}$ is shown in Figure 7.

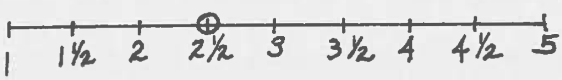

Fig. 7

The graph of $\{x | 2x + 1 = 6 \text{ and } x \in I\}$ will contain no points, since the solution set in I is empty.

The graph of $\{x | 3x > 6 \text{ and } x \in R\}$ is shown in Figure 8. The solution set contains all points covered by the double line.

Fig. 8

Notice that the point 2 is not included in the graph of the above set, as indicated by) , since the solution set is $\{x | x > 2\}$. If the inequality sign had been $\geq$, then 2 would have been included.

Consider the case where more than one solution set is graphed on the real number line. For example, the sets $A = \{x | x \leq 4 \text{ and } x \in R\}$ and $B = \{x | x > 2 \text{ and } x \in R\}$. The graphs are shown in Figure 9.

Fig. 9

The student should observe that the two graphs overlap. That is, there are elements in A that are also in B. The set of elements which is common to the two sets is called the <u>intersection</u> of A and B, written $A \cap B$ and read, "A intersection B."

The intersection of two sets can also be represented by means of Venn diagrams. It is customary in this case to let a rectangle represent the universe, and use circles to represent subsets of the universe. Consider the sets $A = \{x | 0 < x < 6, x \in R\}$ and set $B = \{x | 4 < x < 8, x \in R\}$. Using Venn diagrams the sets A and B could be represented as in Figure 10.

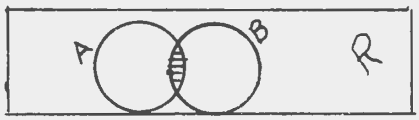

Fig. 10

R represents the set of real numbers. The circle A represents the set A and circle B represents the set B. The shaded area represents the elements in R that are in both A and B. The area in R that is outside of both circle A and circle B represents the elements in R that are neither in A nor in B.

In the Venn diagram in Figure 11, R represents the set of real numbers. A represents a subset of R. The area in R but not in A is called the <u>complement</u> of A (written $\tilde{A}$) and represents those elements in R that are not in A.

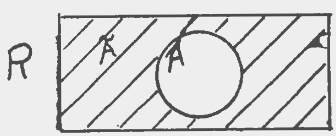

Fig. 11

Let $A = \{x \mid 2 < x < 7,\ x \in R\}$. The complement of A, ($\tilde{A}$), is graphed on the number line in Figure 12 as the heavy portions.

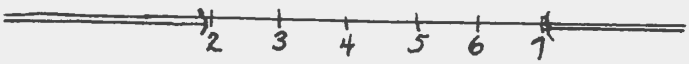

Fig. 12

Using Venn diagrams the above graph could be represented by Figure 13.

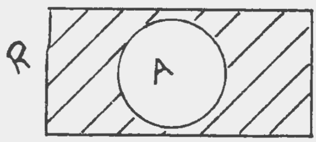

Fig. 13

The shaded area in Figure 13 represents $\tilde{A}$.

If the graphs of $A = \{x \mid 0 < X < 8,\ x \in R\}$ and $B = \{x \mid 2 < x < 5,\ x \in R\}$ are plotted on the real number line, the graph would be as shown in Figure 14.

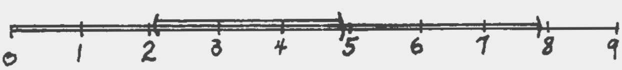

Fig. 14

Here it can be seen that B is a subset of A. That is the points included in B are also in A. Figure 15 shows a Venn diagram of the above sets.

Fig. 15

EXERCISES

1. Graph each of the following on a real number line. $x \in R$ in each set.

   (a) $\{x | x = 2\}$

   (b) $\{x | 3x = 9\}$

   (c) $\{x | x \geq 6\}$

   (d) $\{x | 2x < 4\}$

   (e) $\{x | 2x + 3 = 9\}$

   (f) $\{x | 0 < x < 5\}$

   (g) $\{x | 0 \leq x \leq 7\}$ and $\{x | 2 > x > 0\}$

   (h) $\{x | x > 3\}$ $\{x | x < 6\}$

2. Given: $A = \{x | 0 < x < 9, \quad x \in R\}$

   $B = \{x | 5 < x < 12, \quad x \in R\}$

   $C = \{x | 7 < x < 14, \quad x \in R\}$

   Draw Venn diagrams to represent the following and shade the areas represented by:

   (a) $A \cap B$

   (b) $B \cap C$

   (c) $A \cap B \cap C$

   (d) $\tilde{A}$

   (e) $\tilde{B}$

   (f) $\tilde{A} \cap \tilde{B}$

<u>Ordered</u> <u>pairs</u>. The student is familiar with giving directions. He may direct someone to go 3 blocks west and

4 blocks north. He could also direct the person to go 4 blocks north and 3 blocks west. Either set of directions would bring the person to the same point. In the study of geography, the student learned that places on the earth are designated by degrees of latitude and longitude. He may also be familiar with the fact that legal descriptions of a particular section of land is given by a township number and a range number. All of these examples illustrate that particular points are located by reference to two "perpendicular lines." In the first case, the two perpendicular lines would be streets, in the second the lines would be the equator and the prime meridian, and in the third the lines are parallels of latitude and meridians of longitude.

It is convenient to divide a plane into fourths by constructing two perpendicular lines, called axes. The horizontal line is called the x-axis and the vertical line is called the y-axis. Both axes can be considered as number lines, so that every point on each line represents a number. The point where the axes intersect is called the origin and the point is designated by two numbers (0,0). (see Figure 16) This means that the point on the x-axis is 0, and also that the point on the y-axis is 0.

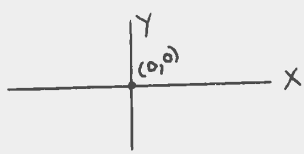

Fig. 16

Every point on the plane can be located by using ordered pairs. (The term "ordered pair" will be explained in more detail below.) The ordered pairs are written (x,y). The numbers in the parentheses are called coordinates of the point. The first coordinate always indicates the distance from the y-axis to the point, and the second coordinate indicates the distance from the x-axis to the point.

Points having positive x-coordinates are located to the right of the y-axis and points having negative x-coordinates are located to the left of the y-axis. Points having positive y-coordinates are located above the x-axis and those having negative y-coordinates are located below the x-axis. Because the point is located by first determining the value of x-coordinate and then determining the value of the y-coordinate, the number pairs are called ordered pairs, since locating the desired point requires a specific order.

The point (4,3) is located by going right from the origin 4 units and then going up from the x-axis 3 units. The point (-2, 1) is located by going to the left from the origin 2 units along the x-axis, and then going up 1 unit. The point (-3, -2) is located by going left from the origin 3 units along the x-axis and then down 2 units. A point (4, -3) is located by going to the right from the origin 4 units along the x-axis and then down 3 units. Graphs of the above points are shown in Figure 17.

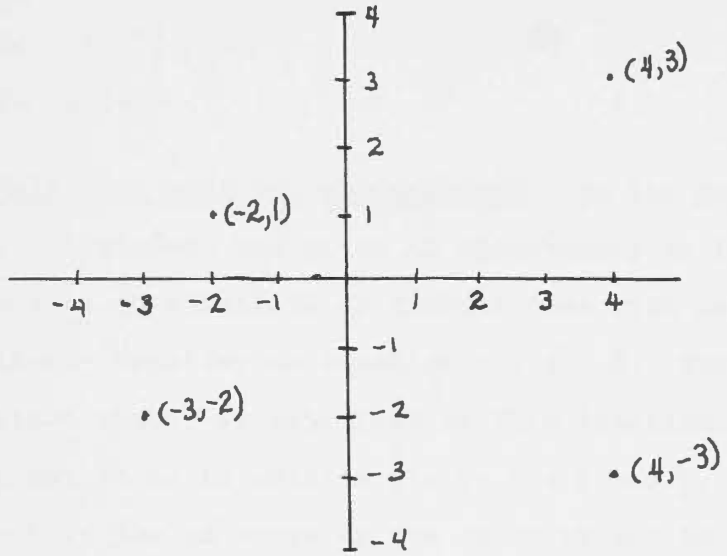

Fig. 17

Every point on the plane can be represented by an ordered pair of coordinates and every distinct ordered pair represents a point on the plane.

EXERCISES

Locate each of the following ordered pairs on a graph:

1. (1,1)
2. (0,1)
3. (3,0)
4. (4,4)
5. (2,1)

6. (-3,-3)
7. (-1,4)
8. (5,-1)
9. (4,-3)
10. (0,-5)

<u>Relations with two placeholders</u>. In the previous chapter, the student was given an opportunity to find solution sets to equations or inequalities with one placeholder. Consider an equation $x + y = 6$. There are an unlimited number of solutions to this equation. The solution set could be written $\{(x,y)\ x + y = 6\}$. First consider that the universe of the solution set is the set of natural numbers. The solutions could then be any of the following:

$x = 1$ when $y = 5$

$x = 2$ when $y = 4$

$x = 3$ when $y = 3$

$x = 4$ when $y = 2$

$x = 5$ when $y = 1$

If the solutions were written as ordered pairs, then $x = 1$ when $y = 5$ could be written (1,5), the entire solution set could be written:

$$\{(1,5),\ (2,4),\ (3,3),\ (4,2),\ (5,1)\}$$

These ordered pairs are plotted on a graph in Figure 18.

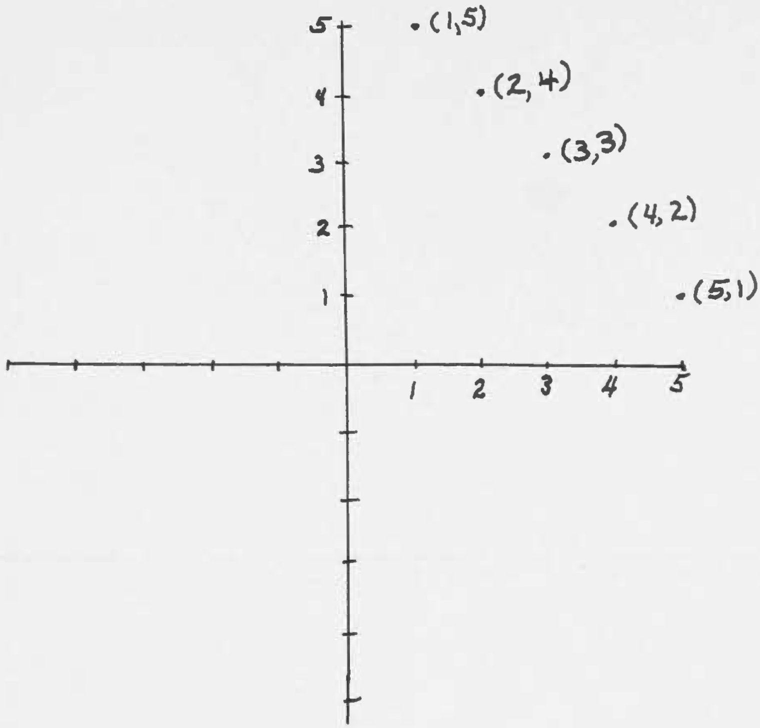

Fig. 18

If the universe is the set of integers, the solution set could not be written by tabulation. There would be an infinite number of solutions. The set below shows only a few of the many elements that belong to the solution set.

{. . . (-19,25) . . . (10,-4) . . . (12,-6) . . . (350,-344) . . .}

A portion of the graph of this solution set is shown in Figure 19.

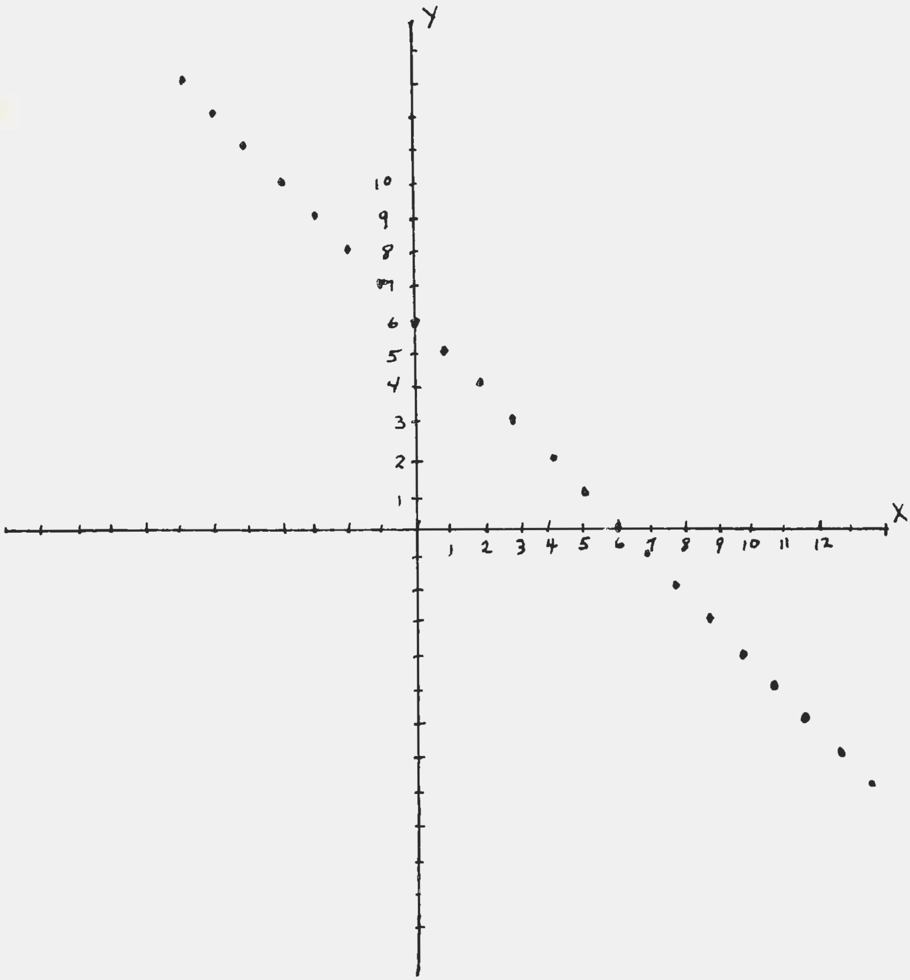

Fig. 19

If the universe of the solution set were the real numbers then the points of the solution set would fill a line. The graph is shown in Figure 20.

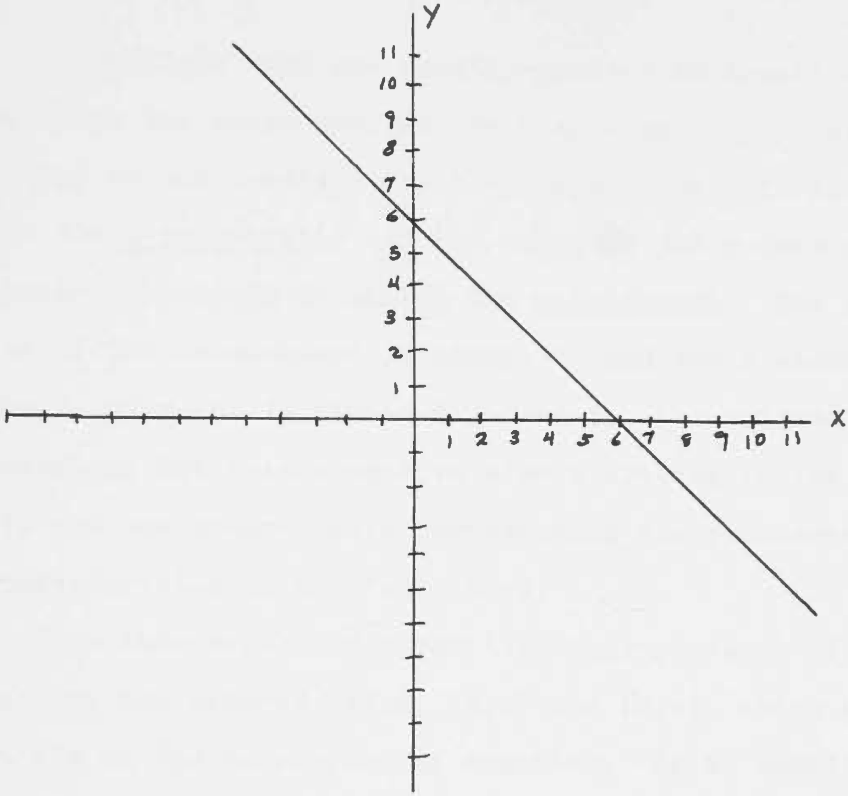

Fig. 20

When the universe of the solution set of an equation containing two placeholders is the set of real numbers, it would be impossible to locate every point corresponding to the numbers in the solution set individually, since there would be an infinite number of points to locate. It is sufficient to locate two or three points which correspond to elements of the solution set and then to draw a line through these points. Actually, two points determine the line, but it is a good idea to plot a third point to insure that a mistake was not made in determining the first two points.

The points that are usually easiest to locate are those where the graph crosses the x-axis and the y-axis. The point on the x-axis where the graph crosses it is called the x-intercept, and the point on the y-axis where the graph crosses it is called the y-intercept. The y-coordinate of the x-intercept is always 0, and the x coordinate of the y-intercept is always 0. That is, the ordered pair representing the x-intercept is always written in the form (x,0), and the ordered pair representing the y-intercept is always written in the form (0,y).

The intercepts for a particular graph can be found by finding the ordered pairs, (x,0) and (0,y), which are solutions of the corresponding equation. In an equation such as $2x + y = 4$, the ordered pair, (x,0), can be found by replacing y with zero, and solving the equation for x. Thus

$$2x + 0 = 4$$
$$2x = 4$$
$$x = 2$$

The x-intercept is then (2,0). Similarly the y-intercept may be found by replacing x with 0, and solving the resulting equation for y. The y-intercept in this case is (0,4).

If the intercepts are plotted on the graph and a line is drawn through the intercepts, the line represents the solution set of the given equation.

To check, a third point can be located by arbitrarily selecting a value for either x or y. x or y is then replaced by this value and the resulting equation is then solved to find the other coordinates.

Suppose that 1 is selected as an arbitrary value of x. Replacing x with 1, the given equation becomes

$$2 \cdot 1 + y = 4$$

$$y = 2$$

The ordered pair is then (1,2). This is an element of the solution set, and should fall on the line representing the solution set. This point is then plotted on the graph. If the point falls on the line representing the solution set, then the set has been graphed correctly. If the point fails to fall on the line representing the solution set, the student should check his work carefully to find the error in his work.

The graph of the solution set of $2x + y = 4$ is shown in Figure 21. Circles are drawn about the intercepts and the point used as a check.

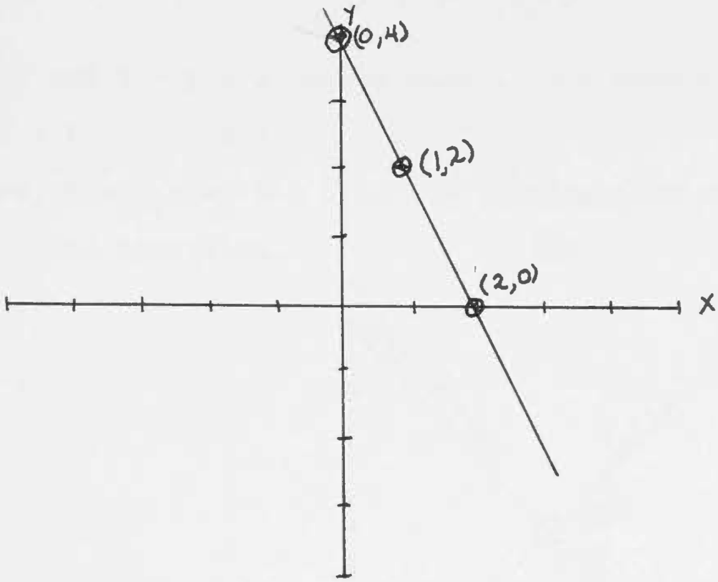

Fig. 21

When the solution sets of two equations with two placeholders are graphed on the same set of axes, and the graphs of the sets intersect, then the intersection of the two sets is called the simultaneous solution of the two equations.

The solution sets of two given equations $x + y = 4$ and $x - y = 2$ are $\{(x,y) \mid x + y = 4;\ (x,y) \in R\}$ (read, "the set of all x and y such that the sum of x and y is 4 where x and y are real numbers") and $\{(x,y) \mid x - y = 2;\ (x,y) \in R\}$. If the solution sets are graphed with reference to the same axis as shown in Figure 22, the point of intersection of the two sets is (3,1). This is the only element that is contained in both of the solution sets. Replacing x with 3 and y with 1 in each of the equations,

$x + y = 4$ and $x - y = 2$, makes each a true sentence. Thus

$$3 + 1 = 4 \quad \text{and } 3 - 1 = 2$$

Therefore, $x = 3$ when $y = 1$ is the simultaneous solution of the two given equations.

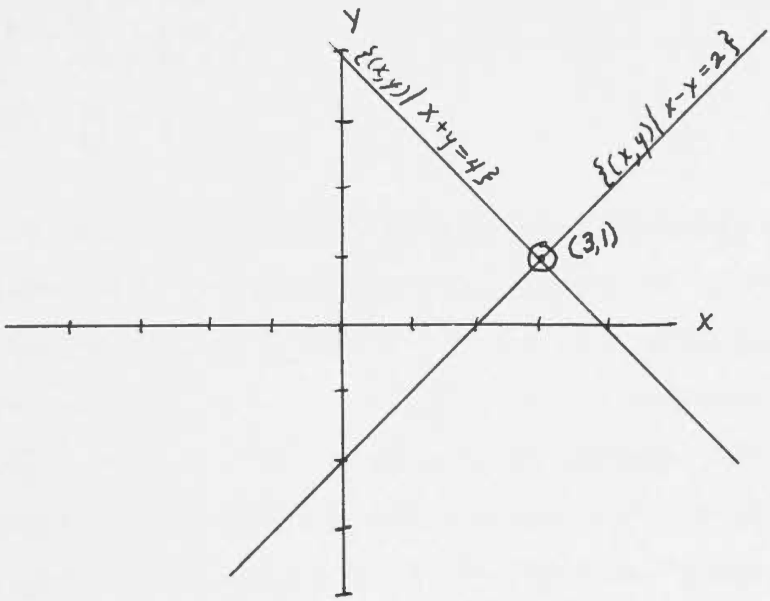

Fig. 22

EXERCISES

1. Write the coordinates of each point of the solution set shown in Figure 19.

2. Graph each of the following solution sets:

   (a) $\{(x,y) \mid x + y = 7; \quad (x,y) \in N\}$

   (b) $\{(x,y) \mid x - y = 4; \quad (x,y) \in I\}$

   (c) $\{(x,y) \mid x + 2y = 6; \quad (x,y) \in R\}$

   (d) $\{(x,y) \mid 2x - 3y = 12; \quad (x,y) \in R\}$

3. Write the solution set of each of the following equations, graph the solution sets of each pair of equations on a set of axes and find the simultaneous solution of each pair of equations.

   (a) $x + y = 6$
   $x - y = 4$

   (b) $3x + y = 8$
   $4x - y = 6$

   (c) $3x + y = 9$
   $4x + y = 12$

The solution sets of inequalities containing two placeholders can also be represented by sets of ordered pairs. In the inequality $x + y + 4 > 6$, the solution set can be written $\{(x,y) \mid x + y > 2\}$. If the universe of the solution sets is the set of natural numbers, then the solution set consists of all ordered pairs of natural numbers with the exception of (1,1). If x and y are replaced with the coordinates of this ordered pair, then $x + y = 2$, which violates the given relation. All other ordered pairs of natural numbers satisfies the given relation.

If the universe of the solution set is the set of integers or the set of real numbers, it is more difficult to describe the members of the solution set. The members of the solution set can be easily seen, however, if the solution set of the inequality is graphed. Since the set of integers is a subset of the set of real numbers, the discussion on graphing inequalities will refer to a universal set of real numbers.

In graphing the solution set of an inequality the first step is to graph its <u>boundary</u>. The boundary is the set of points obtained by changing the inequality sign to an equals sign and graphing the solution set for the new equation. The solution set for a strict inequality will lie entirely on one side of the boundary. Substituting arbitrary values for x and y will determine the side of the boundary on which the solution set will lie. The boundary is included in the soltuion set if the sign is $\geq$ or $\leq$.

Consider the solution set $\{(x,y) \mid x + y > 2, \ (x,y) \in R\}$. The boundary is determined by the solution set $\{(x,y) \mid x + y = 2, \ (x,y) \in R\}$. This set graphs on the line AB in Figure 23. If x and y are replaced with any two arbitrary values which make the inequality $x + y > 2$ a true sentence, the pair of values of x and y will determine a point. The side of the boundary on which this point falls determines the side of the boundary on which the solution set will be graphed.

For example, suppose that 1 is chosen for x and 3 is chosen for y. This determines the point (1,3). If x and y are replaced with 1 and 3, respectively, then $1 + 3 > 2$ is a true statement. If values of x and y are chosen which make the given inequality false, then more arbitrary values are chosen until an ordered pair can be found to make the inequality a true sentence.

On the graph of $\{(x,y) \mid x + y > 2, \ (x,y) \in R\}$ in Figure 23, the shaded area represents the points in the solution set, since the point (1,3) is on that side of the boundary AB.

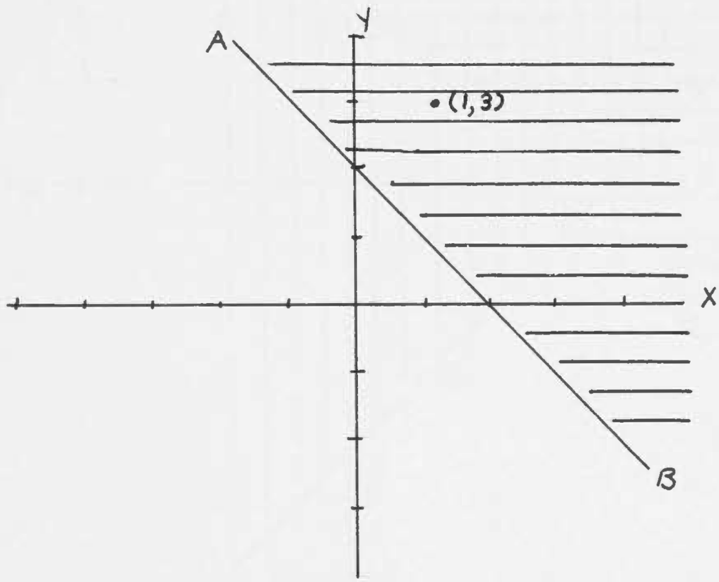

Fig. 23

When the solution sets of two inequalities are graphed on the same axis, the intersection of their solution sets is the simultaneous solution for the inequalities.

If two inequalities are $x + y > 3$ and $x - y < 4$, the solution sets would be $\{(x,y) \mid x + y > 3; \ (x,y) \in R\}$ and $\{(x,y) \mid x - y < 4; \ (x,y) \in R\}$. The graphs of these solution sets are shown in Figure 24. The area which is cross hatched is the simultaneous solution set for the two inequalities.

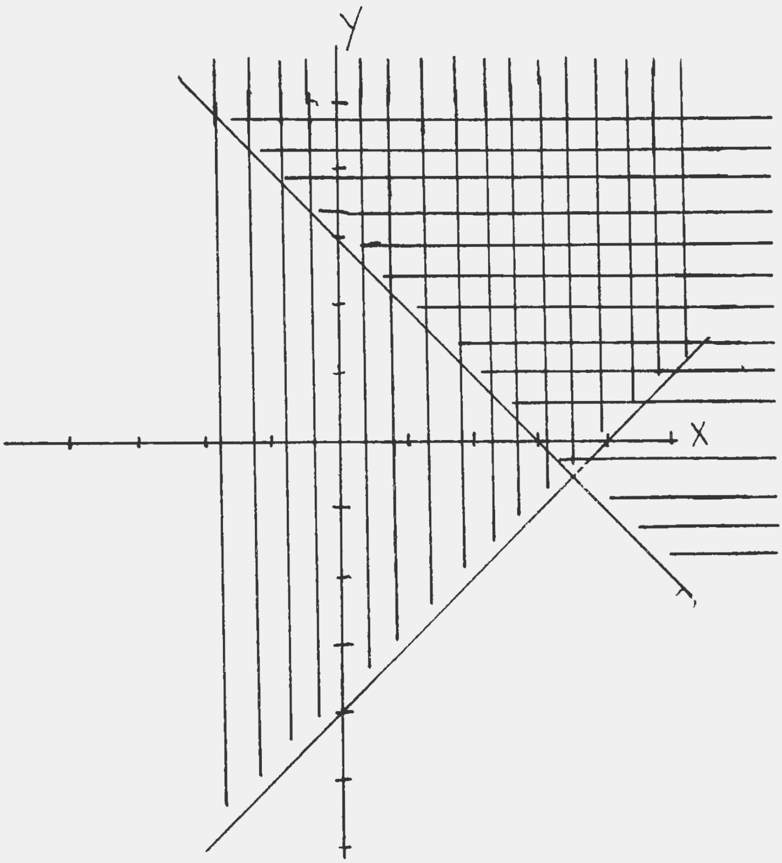

Fig. 24

EXERCISES

1. Graph the solution set for each of the following:

(a) $\{(x,y) \mid x + y > 4; \ (x,y) \in R\}$

(b) $\{(x,y) \mid x - y < 5; \ (x,y) \in R\}$

(c) $\{(x,y) \mid 3x - y > 6; \ (x,y) \in R\}$

(d) $\{(x,y) \mid 2x - y \leq 4; \ (x,y) \in R\}$

2. Write the solution set for each of the following inequalities, graph each pair of inequalities on a pair of axis, and shade the simultaneous solution set.

   (a) $x + y > 2$
   $x - y < 5$

   (b) $x + 2y < 6$
   $2x + y > 2$

3. What is the simultaneous solution set for the pair of inequalities.

   $x + y \geq 2$

   $x + y \leq 2$

CHAPTER VII

OBSERVATIONS

As the writer stated in the introduction, the material presented in the body of this thesis has been used in the classroom for the past three years on an experimental basis.

The manner of presentation was varied somewhat from year to year, but the basic material presented did not vary. Most of the material was presented through lectures from which students took notes, although certain sections were written and duplicated on a spirit duplicator and presented to the students to add to their notes. The exercises were run off on the duplicator so that each student might have a copy.

The course in Algebra was started each year with the material presented in the first five chapters of this thesis. To present this material in such a way that the students might become proficient in the use and terminology of the sets of numbers and the solutions of equations and inequalities required about four or five weeks.

A traditional text was then used for several weeks, giving the students an opportunity to progress in solving more difficult equations and to solve verbal problems.

The material in Chapter VI was then used to introduce graphing. This section was followed by a treatment of solving systems of linear equations using a traditional text.

The remainder of the course was taught from a traditional text, however, an attempt was made to use terminology throughout the course that was consistent with the terminology introduced in this thesis.

The writer chose this approach because of his dissatisfaction with the traditional presentation of signed numbers. From past experience, he had found that students seldom found difficulty in adding signed numbers, but when subtraction was introduced, students became confused with the meaning of the negative sign. The writer feels that the use of the material presented in this thesis has largely eliminated this confusion.

The traditional text usually introduces the number line as a mathematical model. The discussion of operations with signed numbers are then justified by reference to the number line. However, the writer has yet to see a convincing argument justifying the fact that the product of two negative numbers is a positive number. Through this abstract approach, this fact is proved and the students accept it without question.

The writer has observed that the modern terminology seems to appeal to students, and it is his belief that the students grasp the fundamental concepts of mathematics more readily.

The writer was a bit pessimistic about the slower students being able to grasp the concepts of this material the first year it was presented. However, it has been his observation that the slower students seem to benefit most from this approach. This does not necessarily mean that the slow students become good students, but there seems to be less confusion about certain concepts.

The writer does not recommend this approach for every teacher. He feels that there are ideas presented in this paper worthy of consideration by any teacher, but the extent that this material can be used would depend greatly upon the personality of the teacher and his mathematical background.

It is not to be implied that the writer feels that the organization of the material presented here is an end product. The organization of this material is a stepping stone toward further experimentation in search of better methods of presenting algebraic concepts.

It is difficult to draw conclusions as to the success of using this approach to algebra. There were no control groups with which to compare achievements. The conclusions of the author concerning the success of this approach are reflected in the observations made above. The author feels, through the observations he has made, that a great deal has been done in this approach to improve his students' understanding of some of the fundamental concepts of algebra. Students who were taught algebra by this approach have had average or above average success in advanced courses in mathematics.

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