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A Study of Certain Trigonometric and Hyperbolic Transformations

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A STUDY OF CERTAIN TRIGONOMETRIC

AND HYPERBOLIC TRANSFORMATIONS

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Conformal mapping occupies a prominent position in the field of the theory of functions of a complex variable. The importance of this theory in the application of mathematics to other sciences is also noteworthy.

The general theory has been developed through the efforts of such noted mathematicians as Gauss, Argand, Dirichlet, Cauchy, Riemann, and Weierstrass. Several well-known modern mathematicians have also contributed to its present state of development.

Most works concerning complex variables include sections illustrating mapping under various functional relationships. For this purpose most authors represent the variables of the functions in either rectangular or polar coordinates. Their use is determined by the facility of calculation.

Rectangular coordinates are frequently employed for mapping under trigonometric and hyperbolic functions. Mapping of vertical and horizontal lines is frequently studied under these transformations. The nature of their images suggested to this writer the investigation of mappings of other geometrical figures.

The main problem of this paper is the study of certain trigonometric and hyperbolic transformations. The functions sin z, cos z, sinh z, and cosh z will be considered. Particular emphasis will be placed upon the images of circles and non-vertical and non-horizontal lines produced by these transformations.

CHAPTER II

FUNDAMENTAL CONCEPTS AND DEFINITIONS

Knowledge of certain facts is required as support for study of any problem that is encountered. For this purpose some fundamental concepts and definitions from the general theory of complex variables and analytic functions will be stated.

It is assumed that the reader has a working knowledge of complex numbers and of the theory of functions of a complex variable. The form of complex variables used herein will be $z = x + iy$ and $w = u + iv$, where x, y, u, and v are real variables. The real parts and the imaginary parts of the complex variable, w, will be represented by

$$u = R(w) \quad \text{and} \quad v = Im(w) \text{ , respectively.}$$

The first partial derivatives of the dependent variables, u and v, with respect to the independent variables, x and y, will be represented by u_x, v_x, u_y, and v_y. The second partial derivatives of u and v with respect to x and y will be represented by u_{xx}, v_{xx}, u_{yy}, and v_{yy}.

The equation of a circle with center at the point, z_o, and a radius of length, d, is $|z - z_o| = d$. The set of all points interior to this circle is represented by the relation $|z - z_o| < d$.

Franklin [4, pg 16] defines a neighborhood, or an ϵ neighborhood, of a point, z_o, as the set of points, z, such that

$$|z - z_o| < \epsilon \text{ ,}$$

where ϵ is a given positive number. The set of points consisting of all points in a neighborhood of z_o except z_o itself is a deleted neighborhood.

A set of points, S, is said to be connected if any two of its points can be joined by a continuous curve all of whose points belong to S. A connected set of points in a plane is usually called a region. The relationship of each of these points to the region is embodied in the definitions of the following three distinct types of points. An inner point of a region is any point, z_o, which has at least one neighborhood such that all points in this neighborhood are points of the region. A neighborhood of a point, z_o, that contains no points of the region signifies that z_o is an exterior point. Points which are neither interior nor exterior points are called boundary points. These have neighborhoods that contain both points of the region and points exterior to it, however small ϵ of the neighborhood may be.

A point, z_o, is said to be a limit point of a set of points, S, if and only if every neighborhood of z_o contains at least one point of S other than z_o. From the above definitions we can see that a limit point must necessarily be an interior point or a boundary point. Thus it may or may not be a point of S.

Regions are classified as open or closed, or neither. If every point of a region is an interior point, the region is said to be open. An open region is also referred to as a domain. A closed region [2, pg. 17] is a set of points consisting of an open region with all its limit points included. However, a set consisting of a domain with some of its boundary points included and some excluded is neither open nor closed.

In 1837 Dirichlet formulated the definition of a function of a real variable that is widely accepted today. "If for each value of a variable, x, there is determined a definite value or set of values of another variable, y, then y is called a function of x for those values of x [11, pg. 21]." If there is determined one and only one value of y the function, f(x), is single-valued. A multiple-valued function is one for which a set of values of y is determined for each value of x. Although the accepted definition of a function is swiftly being shifted from that of Dirichlet to one involving sets, the former will be used in this paper.

Later, Cauchy applied complex values to variables and from this work the theory of functions of complex variables developed. The complex variable, w, is a function of the complex variable, z, in a given region, S, if for each value of z in this region w has a definite value or set of values. The following notation is used to denote this:

$$w = f(z) = u(x,y) + iv(x,y),$$

where u and v are real functions of the two real variables, x and y.

An elementary function of the complex variable z is defined by Franklin [4, pg. 70] to be a function which can be explicitly represented in terms of complex constants and the independent variable, z, by means of the four fundamental operations and the two basic functions, using at most a finite number of operations and at most a finite number of basic functions. The two basic elementary functions of a complex variable are the exponential and logarithmic functions, e^z and log z.

The inverse of an elementary function is not necessarily elementary. For example, if

$$w = f(z) = z + e^z$$

is elementary, the inverse

$$z = g(w) \qquad \text{is not elementary.}$$

We will be concerned with two types of rational functions. A rational integral function is of the form

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_o,$$

where a_o, a_1, a_2, ..., a_n are constants and n is a positive integer. The quotient of two such rational integral functions produces a rational fractional function. All functions which are not classified under one of these two rational types are irrational functions.

Functions are also divided into algebraic and transcendental classes. Any variable, w, is an algebraic function of z if w and z are related in an irreducible equation of the form

$$f_o(z)\, w^n + f_1(z)\, w^{n-1} + \ldots + f_n(z) = 0,$$

where $f_o(z)$, $f_1(z)$, ... , $f_n(z)$ are rational integral functions of z. Some examples of algebraic functions are:

$$w = z^n, \quad w = \frac{1}{z}, \quad w = \frac{az + b}{cz + d}.$$

It is a matter of interest to note here that all rational functions are algebraic. Trigonometric, exponential, and logarithmic functions are examples of transcendental functions. Transcendental functions encompass all functions that are not algebraic.

It should also be noted here that a certain set of values may be used in defining a function. A multiple-valued function, w, may be considered single-valued in a given region if for each value of z there is one and only one value of w in that region.

Sokolnikoff [9, pg. 23] considers the functional dependence of one variable upon another and gives this definition of the limit of f(z): "The function f(z) approaches the limit L as z tends to z_0 when, corresponding to any given positive number ϵ , one can find a number δ such that $|f(z) - L| < \epsilon$ for all values of z for which $0 < |z - z_0| < \delta$." This is equivalent to the following expression:

$$\lim_{z \to z_0} f(z) = L.$$

Only those values which the function takes in the deleted neighborhood of the point, z_0, determine this limit and not the value of the function at z_0. The function must therefore be defined throughout some neighborhood of z_0, but not necessarily at z_0, for the limit to exist.

Whether or not a function is continuous is an important criterion in analyzing the function. For a function to be continuous at a point the following must hold:

1. the function must be defined at the point in question,
2. the function must have a unique limit as the variable approaches the point,
3. the value of the limit must be equal to the value of the function at the point.

This may also be represented symbolically by:

1. $f(z_o)$ is defined ,

2. $\lim_{z \to z_o} f(z)$ exists ,

3. $\lim_{z \to z_o} f(z) = f(z_o)$.

A function is continuous throughout a region, whether open or closed, if it is continuous at each point of the region.

A further property which a function may have is that of analyticity. A single-valued function, $w = f(z)$, is analytic [2, pg. 32] at a point, z_o, if and only if its derivative exists at every point in some neighborhood of z_o. This definition implies that the derivative must exist at z_o. If a function has a derivative at each point of a region it is said to be analytic throughout that region. "Regular" and "holomorphic" are often used in place of the term analytic.

Any point where f(z) is an analytic function is known as an ordinary point. Any point that is not an ordinary point is a singular point. Thus, the derivative of a function does not exist at a singular point. A function that is analytic throughout a region will not have a singular point in that region. For example, a singularity of the function,

$$f(z) = \frac{1}{z} ,$$

exists at the point where the derivative of the function,

$$f'(z) = -\frac{1}{z^2} ,$$

does not exist. This is at the point, $z = 0$. The image of the point, z_o, is undefined in the w-plane.

A zero of a function, $f(z)$, is a number, z_o, for which $f(z_o) = 0$. Churchill [2, pg. 194] has shown that the zeros of an analytic function are isolated. That is, there is some neighborhood of z_o throughout which $f(z)$ is analytic except at z_o itself.

Any point, z_o, where $f'(z_o) = 0$ is called a critical point of the function. For example, a critical point of the function

$$f(z) = z + z^{-1}$$

exists at the point, $z_o = 1$, where the derivative of the function,

$$f'(z_o) = 1 - z_o^{-2} = 0 .$$

The discontinuities of multiple-valued functions are called branch points. Branch points always occur in pairs. The line joining a pair of branch points is known as a branch cut or a branch line. The regions for which branch cuts serve as boundaries are denoted as branches, or sheets of a Riemann surface. It is also single-valued in that it assumes only one value of the multiple-valued function, $f(z)$, for each value of z. A suitable number of sheets, or branches are considered collectively as a Riemann surface. This presents the transformation under a multiple-valued function in such a manner that it can be considered as a one-to-one relation.

There are two methods of showing the analyticity of a function:

1. exhibiting that the derivative exists at every point throughout the region R, or

2. determining, for the function $f(z) = u + iv$, that $u(x,y)$, $v(x,y)$, u_x, u_y, v_x, and v_y are continuous and single-valued and satisfy the Cauchy-Riemann conditions throughout R.

The Cauchy-Riemann conditions are:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \, .$$

If a function, $u(x,y)$, is assumed to have continuous partial derivatives of the first and second order in some given region and if it satisfies Laplace's equation in two variables,

$$u_{xx} + u_{yy} = 0, \tag{2.1}$$

then it is a harmonic function. Two harmonic functions, $u(x,y)$ and $v(x,y)$, are said to be conjugate harmonic functions if they satisfy the Cauchy-Riemann conditions; i.e., if and only if $u + iv$ is an analytic function.

A concept that parallels and complements that of multiple-valued functions is periodicity of functions. A function is known as simply periodic [11, pg. 126] if it remains invariant under a translation of the plane by means of the relation,

$$z' = z + nt,$$

where n is an integer and t is the fundamental period of the function. If $f(z)$ is our given function, then

$$f(z') = f(z + nt) = f(z) \qquad n = (1, 2, 3, \ldots)$$

defines it as a periodic function. A period-strip is then a counterpart of a branch.

The expression of the relation of points of one plane to points of another plane is defined as mapping. In this paper mappings in the w-plane and the z-plane are considered under the previously listed functional relation

$$w = f(z) = u(x,y) + iv(x,y) .$$

The point, $w = f(z)$, is the image or transform of the point z. The enlarged z-plane will be known as the plane consisting of all finite points of the z-plane with the addition of the ideal point, $z = \infty$. In like manner the enlarged w-plane consists of all finite points of the w-plane with the addition of the ideal point, $w = \infty$. With the inclusion of these ideal points a unique one-to-one correspondence between the points of the two planes may be established.

Churchill [2, pg. 137] proves the theorem that at each point where a function f(z) is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal. If the argument of the function, or the magnitude of the angle, is preserved in the transformation, it is said to be conformal. The terms "equiangular" and "isogonal" are also used for this condition. If the transformation preserves the magnitude of the angle but causes the angle to be reflected on the axis of reals it is said to be isogonal with reversal of angles.

Under any conformal transformation a harmonic function is transformed into a harmonic function in the other plane. Thus, the following relations hold true and satisfy Laplace's equation (2.1):

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 , \quad v_{xx}(x,y) + v_{yy}(x,y) = 0 ,$$

and

$$x_{uu}(u,v) + x_{vv}(u,v) = 0 , \quad y_{uu}(u,v) + y_{vv}(u,v) = 0 .$$

The image of each small figure near a point conforms to the original figure in the sense that it has approximately the same shape. However, large figures may transform into images that bear no resemblance to the original.

Even though the sizes and shapes are distorted by expansion, contraction, rotation, or translation, or a combination of these it must be remembered that under conformal transformations the angles are preserved. From this we conclude that sets of orthogonal curves are mapped into orthogonal curves under every conformal transformation. A system of orthogonal curves on a surface is defined by James and James [5, pg. 276] to be "a system of two one-parameter families of curves on a surface, S, such that through any point of S there passes exactly one curve of each family, and such that at each point, P, of S the tangents to the two curves of the system through P are mutually perpendicular or right-angled."

CHAPTER III

THE TRANSFORMATIONS

Properties. The function which will be considered here

$$w = \sin z \tag{3.1}$$

is an elementary transcendental integral function. It is easily recognized as an elementary function when represented in the exponential form

$$\frac{e^{iz} - e^{-iz}}{2i} . \tag{3.2}$$

If the series

$$\sin z \doteq z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots \tag{3.3}$$

is an infinite series it represents sin z but requires an infinite number of terms. Hence, it is not an algebraic expression and is known as a transcendental function. The above Maclaurin Series expansion, however, denotes that the function is integral [5, pg. 140] .

For each value of z there is one and only one value of w. The function is therefore single-valued.

Equation (3.1) may be represented in the following manner:

$$u + iv = \sin (x + iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$u + iv = \sin x \cosh y + i\cos x \sinh y. \tag{3.4}$$

Equation (3.4) implies that

$$R(w) = u(x,y) = \sin x \cosh y \tag{3.5}$$

$$Im(w) = v(x,y) = \cos x \sinh y . \tag{3.6}$$

The derivative of the function is

$$f'(z) = \cos z$$

$$= \cos (x + iy)$$

$$= \cos x \cos iy - \sin x \sin iy$$

(3.7) $$f'(z) = \cos x \cosh y - i \sin x \sinh y \;.$$

The derivative exists at every point in the z-plane. The first and second order partial derivatives of the real and imaginary parts of sin z are as follows:

(3.8) $$u_x = \cos x \cosh y \;, \quad u_y = \sin x \sinh y$$

(3.9) $$v_x = -\sin x \sinh y \;, \quad v_y = \cos x \cosh y$$

(3.10) $$u_{xx} = -\sin x \cosh y \;, \quad u_{yy} = \sin x \cosh y$$

(3.11) $$v_{xx} = -\cos x \sinh y \;, \quad v_{yy} = \cos x \sinh y \;.$$

The zeros of the function may be found by considering equations (3.5) and (3.6) and noting that the zeros must satisfy the equation

(3.12) $$w = \sin z = 0 \;.$$

Both the real and imaginary parts of w must be equal to zero. Thus,

(3.13) $$\sin x \cosh y = 0 \quad \text{and} \quad \cos x \sinh y = 0 \;.$$

Since x and y are real, cosh y never vanishes, while sin x = 0 only for the values $x = 0, \pm 1, \pm 2, \pm 3, \ldots$. The function $\cos x \neq 0$ for these given values of x. This implies that sinh y must vanish. The only value of y for which sinh y = 0 is y = 0. Thus, zeros exist for the sin z function only when z takes on the real values

$$z = 0, \pm n \qquad (n = 1, 2, 3, \ldots).$$

If the derivative of the function, (3.7), is inspected for its zeros, we find that such zeros exist only for

$$\cos x \cosh y = 0 \quad \text{and} \quad \sin x \sinh y = 0 .$$

As pointed out before, each y is never equal to zero which implies that $\cos x = 0$ must hold. This is true for values of $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots$ This makes it necessary that $\sinh y = 0$ since $\sin x$ never vanishes for these values. Hence, $y = 0$, and only zeros of $\cos z$ are real values

$$z = \pm \frac{2n + 1}{2} \pi \qquad (n = 0, 1, 2, \ldots).$$

These are the critical points of the function $w = \sin z$. It should be noted here that a critical point, z_o, implies $f'(z_o) = 0$. A conformal mapping requires that $f'(z) \neq 0$. Thus a mapping is never conformal at a critical point.

It has already been shown that the derivative of sin z exists at all points of the z-plane. The analyticity may be shown in another manner. The equations (3.5), (3.6), (3.8), and (3.9) are continuous and single-valued, and the following relationships apply:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x .$$

Thus, the Cauchy-Riemann conditions are satisfied, and the function $w = \sin z$ is analytic at all points in the region.

The conditions of Laplace's equation (2.1) are met as evidenced by inspection of the second order partial derivatives, (3.10) and (3.11), of the function. That is

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0 .$$

The function is therefore a harmonic function.

That the function sin z is periodic with a period of 2 is evidenced by the fact that

(3.14) $$\sin(z + 2\pi) = \sin z \ .$$

The z-plane may be divided into strips such that

(3.15) $$(2n - 1)\pi \leq x < (2n + 1)\pi \quad \text{and} \quad y \geq 0 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Each strip is called a period-strip. Each distinct period-strip will then map into the complete w-plane with a branch cut extending from the origin along the negative half of the imaginary axis to infinity. The z-plane may also be divided into period-strips such that

(3.16) $$n\pi \leq x < (n + 2)\pi \quad \text{and} \quad y \geq 0 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

In this case the period-strip maps into the whole w-plane with the branch cut extending from the origin along the positive half of the imaginary axis to infinity.

Thus, there are an unlimited number of period-strips in the z-plane and an unlimited number of branches in the w-plane. These superimposed branches constitute the Riemann surface for w = sin z.

Transformations. The portion of the z-plane that will be considered in this investigation of the function w = sin z is the period-strip where x and y are limited to the following values

(3.17) $$0 \leq x < 2\pi \quad \text{and} \quad y \geq 0 \ .$$

This period-strip maps into the whole w-plane with the branch cut extending from the origin along the positive half of the imaginary axis to infinity. The corresponding letters indicate the regions of the z-plane that are mapped into quadrants of the w-plane. The correspondence of the branch cut to the boundary lines of the period-strip of the z-plane are indicated by two distinct markings. The hash marks along the boundary line indicate that the boundary points are included, while the wavy line paralleling the boundary line indicates that the boundary points are excluded. No generality is lost in the selection of this particular period-strip.

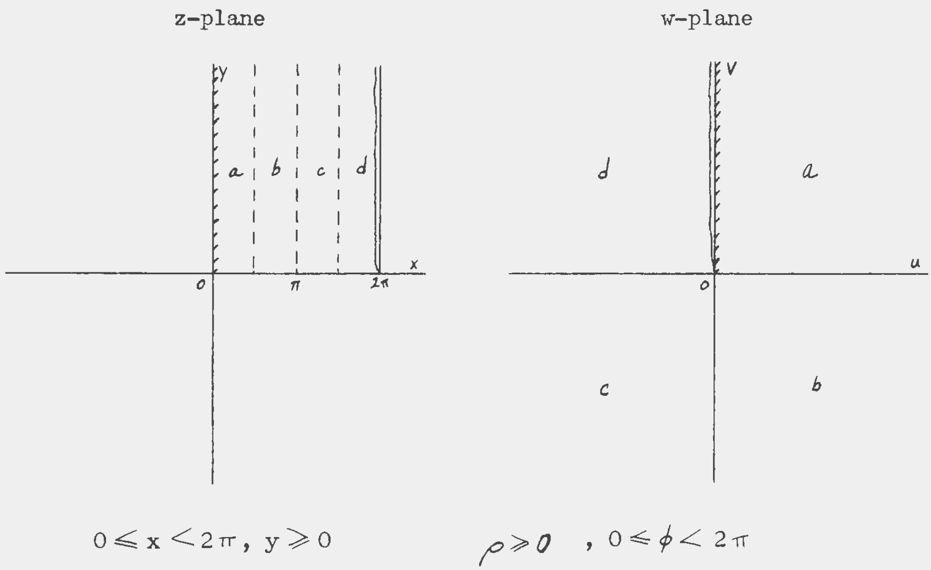


Figure 1. Transformation of period strip under w = sin z.

Some elementary figures to be considered are the general cases of parallel and horizontal line segments. The line segment

(3.18) $$y = 0 \ , \quad 0 \leqslant x < 2\pi$$

transforms into the line segment $-1 \leqslant u \leqslant 1$. It should be noted that as x increases in value from 0 to 2π, u increases in value from 0 to 1, from 1 to -1, and then from -1 to 0.

The line segment

(3.19) $$y = k \ , \quad 0 \leqslant x < 2\pi , \quad \text{where } k > 0,$$

maps into the ellipse whose parametric equations are

(3.20) $$u = \sin x \cosh k \ , \quad v = \cos x \sinh k \ .$$

An ellipse of this type is represented by the equation

(3.21) $$\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$$

and is illustrated in Figure 2. An examination of equation (3.21) shows that all such ellipses are confocal with foci at $w = \pm 1$.

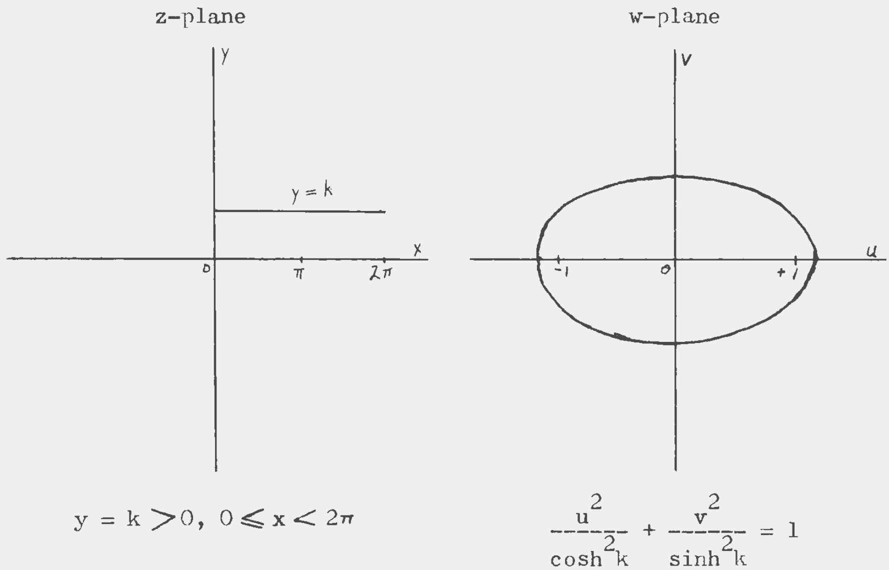


Figure 2. Transformation of $y = k$ under $w = \sin z$.

The branch cut in this case, or the line,

(3.22) $$u = 0 \ , \ v \geqslant 0$$

is the image of the line

(3.23) $$x = 0 \ , \ y \geqslant 0 \ .$$

As illustrated in Figure 3, the line

(3.24) $$x = c \ , \ y \geqslant 0$$

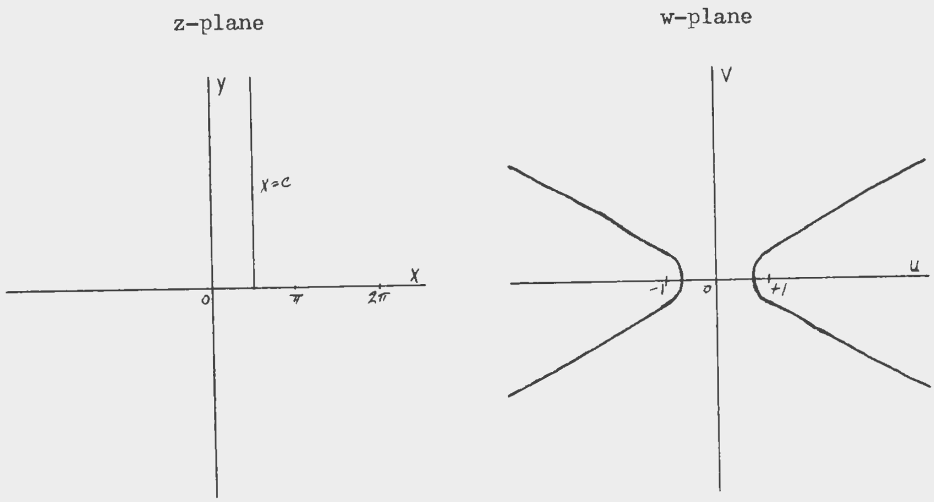
where $0 \leqslant c < 2\pi$ maps into the curve

(3.25) $$u = \sin c \cosh y \ , \ v = \cos c \sinh y$$

which is the hyperbola

(3.26) $$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1 \ .$$

An examination of (3.26) shows that all such hyperbolas are confocal with foci at $w = \pm 1$.



$0 \leqslant x = c < 2\pi, \ y \geqslant 0$ $$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1 \ .$$

Figure 3. Transformation of x = c under w = sin z.

It has been shown that the function w = sin z has a derivative that exists at all points of the region. It has also been shown to be analytic at all points in the region. Therefore, the function is conformal, which implies that orthogonal sets of curves map into orthogonal sets of curves.

The orthogonality of the curves in the w-plane may also be shown in another manner. The slope of a curve at a point is the derivative evaluated at that point. A horizontal line segment, $y = k$, is a function of the independent variable x only. Its image expressed by equation (3.20) is also a function of x only. The derivative of these parametric equations is the slope of the image of $y = k$ in the w-plane. Likewise, a vertical line segment, $x = c$, and its image are functions of y only. The derivative of the parametric equations (3.24) as a function of y only is the slope of the image of $x = c$ in the w-plane. The slopes of these two families of curves in the w-plane are then expressed as the derivatives of equations (3.5) and (3.6) taken first as a function of x and then as a function of y.

These derivatives may be denoted by

$$(3.27)\qquad \left(\frac{dv}{du}\right)_{y = k} = -\frac{u_x}{v_x} \quad , \quad \left(\frac{dv}{du}\right)_{x = c} = -\frac{u_y}{v_y}$$

$$(3.28)\qquad \left(\frac{dv}{du}\right)_{y = k} = \frac{\cos x \cosh y}{\sin x \sinh y} \, , \quad \left(\frac{dv}{du}\right)_{x = c} = -\frac{\sin x \sinh y}{\cos x \cosh y} \, .$$

The product of these slopes

$$(3.29)\qquad \left(\frac{dv}{du}\right)_{y = k} \cdot \left(\frac{dv}{du}\right)_{x = c} = -1 \, .$$

This is true for all points except the points on the lines, $x = 0$ and $y = 0$. When $x = 0$,

$$u = \sin 0 \cosh y \, , \quad v = \cos 0 \sinh y \, ,$$

and

$$u = 0 \, , \qquad v = \sinh y \, .$$

When $y = 0$,

$$u = \sin x \cosh 0 \ , \quad v = \cos x \sinh 0 \ ,$$

and

$$u = \sin x \ , \qquad v = 0 \ .$$

These equations produce derivatives of indeterminate form. The product of the slopes cannot be found in this case. Therefore, the two families of curves are orthogonal everywhere except for the values, $x = 0$ or $y = 0$.

This transformation will now be considered for a more general case. If the real variable, y, can be expressed as an analytic function of the real variable, x, the resulting curve can be expressed in a form from which the parameters have been eliminated.

Beginning with the parametric equations of the function, $w = \sin z$,

(3.30) $$u = \sin x \cosh y \quad \text{and} \quad v = \cos x \sinh y \ ,$$

squaring, rearranging and combining gives the elliptical form of the equation,

(3.31) $$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1 \ .$$

Multiplying both sides of the equation by $\cosh^2 y \sinh^2 y$, and substituting the exponential form of the hyperbolic function produces

(3.32) $$u^2\left(\frac{e^y - e^{-y}}{2}\right)^2 + v^2\left(\frac{e^y + e^{-y}}{2}\right)^2 = \left(\frac{e^y + e^{-y}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \left(\frac{e^{2y} - e^{-2y}}{4}\right)^2 \ .$$

Squaring and rearranging terms gives

(3.33) $$4u^2(e^{2y} + e^{-2y} - 2) + 4v^2(e^{2y} + e^{-2y} + 2) = e^{4y} + e^{-4y} - 2 \ .$$

Substitution of $s = e^{2y}$ yields

(3.34) $4u^2(s + s^{-1} -2) + 4v^2(s + s^{-1} + 2) = s^2 + s^{-2} + 2 - 4$, and

(3.35) $4u^2(s + s^{-1}) - 8u^2 + 4v^2(s + s^{-1}) + 8v^2 = (s + s^{-1})^2 - 4$.

Letting $(s + s^{-1}) = t$, and solving for t by use of the quadratic formula gives,

(3.36) $$t^2 - 4(u^2 + v^2)t + 8(u^2 - v^2) - 4 = 0 \text{ , and}$$

(3.37) $$\tfrac{1}{2}t = (u^2 + v^2) \pm\sqrt{(u^2 + v^2)^2 - 2(u^2 - v^2) + 1} .$$

Replacing t by its equal, $(s + s^{-1})$, and letting the right hand part of equation (3.37) be represented by d,

(3.38) $$\tfrac{1}{2}(s + s^{-1}) = d .$$

The substituion, $e^{2y} = s$, now gives a hyperbolic form,

(3.39) $$\frac{e^{2y} + e^{-2y}}{2} = d \text{ , or}$$

(3.40) $$\cosh 2y = d.$$

This can be arranged in the form

(3.41) $$y = \tfrac{1}{2} \cosh^{-1} d.$$

Letting

(3.42) $$a = \cosh^{-1} d = \ln(d \pm \sqrt{d^2 - 1}) ,$$

then,

(3.43) $$y = \tfrac{1}{2}a.$$

Thus, when $y = f(x)$,

(3.44) $$y = f(x) = \tfrac{1}{2}a \quad \text{and} \quad x = g(a) ,$$

where g(a) is some function of a.

The hyperbolic functions, sinh y and cosh y, may be evaluated as follows:

(3.45)
$$\sinh y = \sinh \tfrac{1}{2}a = \pm\sqrt{\tfrac{1}{2}(\cosh a - 1)} = \sqrt{\tfrac{1}{2}(d - 1)}\ ,$$

and

(3.46)
$$\cosh y = \cosh \tfrac{1}{2}a = \sqrt{\tfrac{1}{2}(\cosh a + 1)} = \sqrt{\tfrac{1}{2}(d + 1)}\ .$$

The parametric equations (3.30) can now be expressed as,

(3.47)
$$u = \sqrt{\tfrac{1}{2}(d + 1)}\ \sin[g(a)] \quad \text{and} \quad v = \pm\sqrt{\tfrac{1}{2}(d - 1)}\ \cos[g(a)]\ .$$

The family of straight lines which pass through the origin,

(3.48)
$$y = f(x) = mx\ ,$$

where m is the slope of the line, will now be considered. Combining equations (3.43) and (3.48) yields

(3.49)
$$mx = \tfrac{1}{2}a \quad \text{and} \quad x = g(a) = \frac{a}{2m}\ .$$

The equation of the images of this family of lines may then be expressed in the form

(3.50)
$$\frac{u}{v} = \sqrt{\frac{d + 1}{d - 1}}\ \tan\frac{a}{2m}\ ,$$

or

(3.51)
$$\frac{u}{v} = \sqrt{\frac{d + 1}{d - 1}}\ \frac{e^{i\frac{a}{m}} - 1}{e^{i\frac{a}{m}} + 1}\ .$$

The equation of the curves which are the images of any general line of the form,

(3.52)
$$y = f(x) = mx + b\ ,$$

where m is the slope of the line and b is the y-intercept, is similar to equation (3.51). From equations (3.43) and (3.52),

(3.53) $$mx + b = \tfrac{1}{2}a \quad \text{and} \quad x = \frac{a - 2b}{2m} .$$

The parametric equations (3.30) are expressed in the form

$$u = \sin x \cosh (mx + b)$$

(3.54) $$= \sin x (\cosh mx \cosh b + \sinh mx \sinh b) ,$$

$$v = \cos x \sinh (mx + b)$$

(3.55) $$= \cos x (\sinh mx \cosh b + \cosh mx \sinh b) .$$

The equation without the parameters is then of the form

(3.56) $$\frac{u}{v} = \sqrt{\frac{d + 1}{d - 1}} \; \frac{e^{i \frac{a - 2b}{m}} - 1}{e^{i \frac{a - 2b}{m}} + 1} .$$

If the real variables, x and y, are expressed in polar coordinates

(3.57) $$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta ,$$

the parametric equations of the transformation are of the form

(3.58) $$u = \sin(r \cos \theta) \cosh(r \sin \theta)$$
$$v = \cos(r \cos \theta) \sinh(r \sin \theta) .$$

The straight lines passing through the origin can then be represented by letting θ be a constant and letting r vary. Hence, $\sin \theta$ and $\cos \theta$ are constants, which will be represented by A and $\sqrt{1 - A^2}$ respectively, and

(3.59) $$x = r \cos \theta = r \sqrt{1 - A^2} \quad \text{and} \quad y = r \sin \theta = rA.$$

Equations (3.58) are then expressed by the equations

(3.60) $$u = \sin rA \cosh rB$$
$$v = \cos rA \sinh rB , \text{ where } B = \sqrt{1 - A^2} .$$

If r remains constant such that

$$r = R ,$$

and θ varies, the parametric equations

$$(3.61) \qquad \begin{aligned} u &= \sin(R \cos \theta) \cosh (R \sin \theta) \\ v &= \cos(R \cos \theta) \sinh (R \sin \theta) \end{aligned}$$

express the shape of the images of a set of concentric circles with their centers at the origin of the z-plane.

The general equation,

$$(3.62) \qquad K(x^2 + y^2) + Nx + Py + Q = 0 ,$$

where x and y are real variables and K, N, P, and Q are constants, may be used to represent any circle in the z-plane. Solving for y,

$$(3.63) \qquad y = \tfrac{1}{2}a = \frac{-P \pm \sqrt{P^2 - 4K(Kx^2 + Nx + Q)}}{2K} .$$

Rearranging and solving for x in terms of a produces the following form for x,

$$(3.64) \qquad x = g(a) = \frac{-N \pm \sqrt{N^2 - K(Ka^2 + 2Pa + 4Q)}}{2K} .$$

This is an especially long and unwieldy form, but in connection with equation (3.43) and the parametric equations (3.30) will give the image of the general circle under the transformation w = sin z.

The parametric equations (3.58) have been used to compile a table of values, Table I. Values or r ranging from 0 to 9 in increments of .25 of a unit were used, while θ was allowed to assume values of 0, π/32, π/16, π/8, 3π/16, π/4, 5π/16, 3π/8, 7π/16, and π/2. The quantities r cos θ and r sin θ were then calculated and substituted into equations (3.58) to obtain the tabulated values of u and v. These values were employed to plot the images of selected lines and circular arcs under the transformation $w = \sin z$. Values of Table I are accurate to four significant digits. However, the accuracy was limited so that a hundredth of a unit represents the greatest degree of accuracy. This degree of accuracy obviously cannot be read from the mappings.

The lines that were transformed are the lines extending radially from the origin with the angles of $\pi/32$, $\pi/16$, $\pi/8$, $3\pi/16$, $\pi/4$, $5\pi/16$, $3\pi/8$, and $7\pi/16$. The length of these lines, r, are such that $0 \leq r \leq \frac{2}{\cos \theta}$, where θ is their respective angle.

The large numerical values involved in the computations and the size of the required map made it necessary to limit the values of r still more, so that $r \leq 9.00$. The lines and their images are represented in Figures 4a, 4b, 4c, and 5. It was also necessary to make the inset, Figure 4b, for Figure 4a and the inset, Figure 4c, for Figure 4b because of the wide range of values which determine the curves. Figure 5 represents the lines $\theta = \pi/32$, $\theta = \pi/16$ and $0 \leq r \leq \frac{2}{\cos \theta} \leq 9$, in considerable detail. The purpose of this particular map was to investigate a line that was more nearly parallel to the x-axis than the others.

The circular arcs selected for transformation under $w = \sin z$ are the arcs of concentric circles with their centers at the origin. These arcs are in the first quadrant of the z-plane. Thus they are contained in the period-strip that was selected for study. They are arcs of the circles, $r = n$, $(n = 1, 2, 3, 4, 5, 6)$, and their angles are $0 \leq \theta \leq \pi/2$. It was necessary for clarity to represent their images on a map, Figure 6a, which has an inset, Figure 6b.

TABLE I

VALUES FOR THE TRANSFORMATION w = sin z.

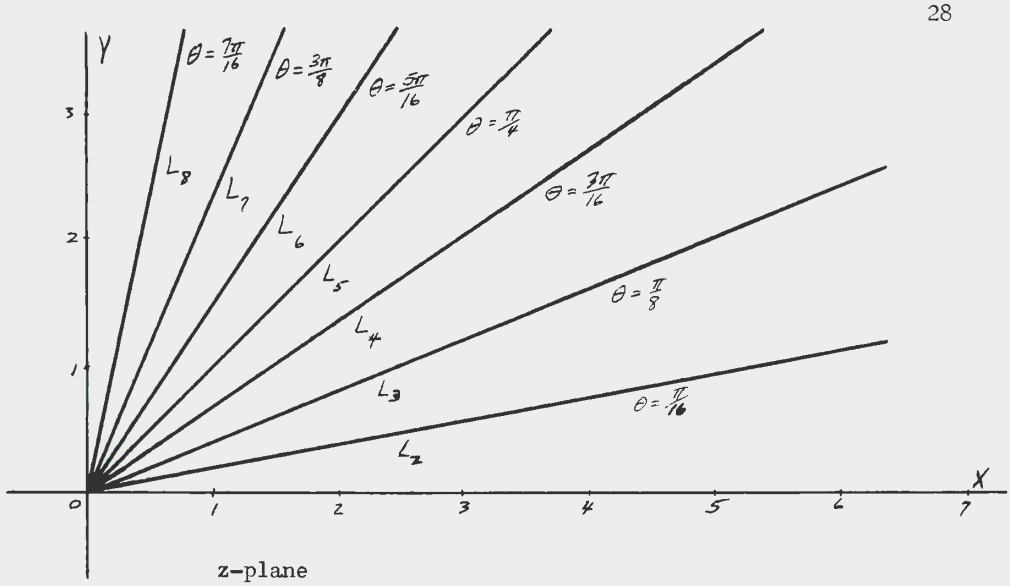
u = sin(r cos θ) cosh(r sin θ)

r \ θ	0	$\frac{\pi}{32}$	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$	$\frac{5\pi}{16}$	$\frac{3\pi}{8}$	$\frac{7\pi}{16}$	$\frac{\pi}{2}$
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
.25	.25	.25	.25	.23	.21	.18	.14	.10	.05	.00
.50	.48	.48	.47	.45	.42	.36	.30	.21	.11	.00
.75	.68	.68	.75	.66	.63	.58	.49	.36	.19	.00
1.00	.84	.84	.85	.85	.86	.82	.76	.54	.30	.00
.25	.95	.95	.97	1.02	1.11	1.09	1.01	.80	.44	.00
.50	1.00	1.01	1.04	1.15	1.30	1.41	1.39	1.15	.65	.00
.75	.98	1.00	1.05	1.23	1.50	1.77	1.87	1.63	.96	.00
2.00	.91	.93	1.00	1.26	1.67	2.16	2.44	2.27	1.38	.00
.25	.78	.80	.89	1.22	1.80	2.55	3.15	3.11	1.97	.00
.50	.60	.62	.71	1.11	1.87	2.97	4.00	4.17	2.75	.00
.75	.38	.40	.49	.91	1.82	3.31	4.98	5.57	3.84	.00
3.00	.14	.17	.23	.63	1.66	3.97	6.10	7.31	5.28	.00
.25	- .11	- .10	- .06	.26	1.33	4.14	7.26	9.55	7.34	.00
.50	- .35	- .35	- .34	- .18	.88	4.18	8.55	11.96	9.55	.00
.75	- .57	- .60	- .65	- .71	.08	3.29	9.65	16.45	13.65	.00
4.00	- .76	- .80	- .93	-1.28	- .42	2.87	10.96	20.22	17.38	.00
.25	- .90	- .97	-1.17	-1.87	- 2.13	1.40	12.14	27.00	23.88	.00
.50	- .98	-1.07	-1.35	-2.46	- 3.46	- .48	12.57	31.99	31.74	.00
.75	-1.00	-1.11	-1.46	-2.98	- 5.10	- 3.15	12.46	39.16	42.34	.00
5.00	- .96	-1.07	-1.49	-3.44	- 6.89	- 6.72	11.40	47.83	55.77	.00
.25	- .86	- .98	-1.43	-3.72	- 8.76	-11.03	8.63	58.04	73.93	.00
.50	- .70	- .83	-1.28	-3.86	-10.57	-16.68	3.97	69.33	96.67	.00
.75	- .50	- .61	-1.02	-3.67	-12.25	-23.48	- 3.61	81.70	127.3	.00
6.00	- .28	- .36	- .55	-3.36	-13.55	-30.94	- 13.88	94.82	165.4	.00
.25	-0.03	-0.07	-0.28	-2.68	-14.19	-39.81	- 29.36	109.3	216.5	.00
.50				-1.68	-14.14	-49.44	- 50.64	122.1	279.8	.00
.75				-0.33	-13.04	-59.41	- 79.03	134.9	364.2	.00
7.00					-10.86	-68.56	-114.8	143.3	472.7	.00
.25					- 6.96	-77.14	-161.5	146.9	606.0	.00
.50					- 1.30	-83.45	-220.4	136.4	774.9	.00
.75						-86.03	-291.3	115.2	998.3	.00
8.00						-84.26	-372.3	64.72	1,283.	.00
.25						-73.25	-472.6	- 10.10	1,630.	.00
.50						-52.90	-588.0	-140.8	2,065.	.00
.75						-22.03	-717.3	-308.2	2,636.	.00
9.00							-848.9	-586.9	3,356.	0.00

TABLE I (CONTINUED)

$$v = \cos(r \cos \theta) \sinh(r \sin \theta)$$

r \ θ	0	π/32	π/16	π/8	3π/16	π/4	5π/16	3π/8	7π/16	π/2
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
.25	.00	.02	.05	.10	.14	.18	.21	.24	.25	.25
.50	.00	.04	.09	.17	.26	.34	.42	.47	.51	.52
.75	.00	.05	.11	.23	.35	.48	.60	.71	.80	.82
1.00	.00	.05	.11	.24	.38	.58	.79	1.00	1.11	1.18
.25	.00	.04	.08	.20	.40	.64	.95	1.26	1.52	1.60
.50	.00	.01	.03	.11	.29	.62	1.08	1.59	1.97	2.13
.75	.00	-.03	-.05	-.04	.13	.51	1.16	1.90	2.55	2.79
2.00	.00	-.08	-.15	-.23	-.13	.31	1.13	2.23	3.22	3.63
.25	.00	-.14	-.26	-.47	-.49	-.04	1.00	2.54	4.07	4.69
.50	.00	-.20	-.39	-.75	-.94	-.54	.71	2.88	5.07	6.05
.75	.00	-.25	-.51	-1.04	-1.45	-1.23	.20	3.08	6.81	7.79
3.00	.00	-.30	-.61	-1.36	-2.04	-2.38	-.60	3.25	7.84	10.02
.25	.00	-.32	-.67	-1.57	-2.69	-3.65	-1.76	3.16	9.82	12.88
.50	.00	-.34	-.69	-1.77	-3.32	-5.09	-3.39	2.80	11.54	16.54
.75	.00	-.31	-.69	-1.89	-3.94	-6.31	-5.51	1.99	14.93	21.25
4.00	.00	-.27	-.61	-1.87	-4.46	-8.04	-8.20	.82	17.96	27.29
.25	.00	-.20	-.48	-1.71	-4.83	-9.92	-11.76	-1.25	22.50	35.04
.50	.00	-.11	-.30	-1.41	-4.99	-12.24	-16.21	-5.01	25.95	45.00
.75	.00	.10	-.05	-.94	-4.81	-14.60	-23.95	-9.71	32.86	57.79
5.00	.00	.14	.22	-.30	-4.20	-15.24	-31.20	-16.62	37.40	74.20
.25	.00	.27	.52	.51	-2.45	-17.02	-39.74	-28.04	46.66	95.28
.50	.00	.40	.82	1.46	-1.44	-18.07	-49.58	-41.51	52.18	122.3
.75	.00	.50	1.11	2.52	.99	-17.85	-60.64	-59.08	57.69	157.1
6.00	.00	.60	1.34	3.65	4.72	-15.12	-72.87	-81.66	71.21	201.7
.25	0.00	0.65	1.53	4.72	7.80	-11.66	-85.74	-120.8	76.61	259.0
.50				5.76	11.79	-5.50	-98.70	-160.6	89.16	332.6
.75				6.55	17.65	4.30	-110.8	-208.7	91.20	427.0
7.00					22.12	16.01	-120.8	-298.0	103.0	548.3
.25					29.22	33.42	-127.0	-378.7	97.02	704.1
.50					33.30	56.63	-126.8	-478.2	90.45	904.0
.75						85.24	-129.8	-658.9	60.70	1,161.
8.00						121.6	-108.4	-815.4	13.17	1,491.
.25						149.4	-64.71	-999.1	-67.18	1,914.
.50						194.9	6.54	-1,216.	-182.4	2,457.
.75						245.1	111.2	-1,461	-381.1	3,155.
9.00							259.2	-1,927	-629.1	4,052.



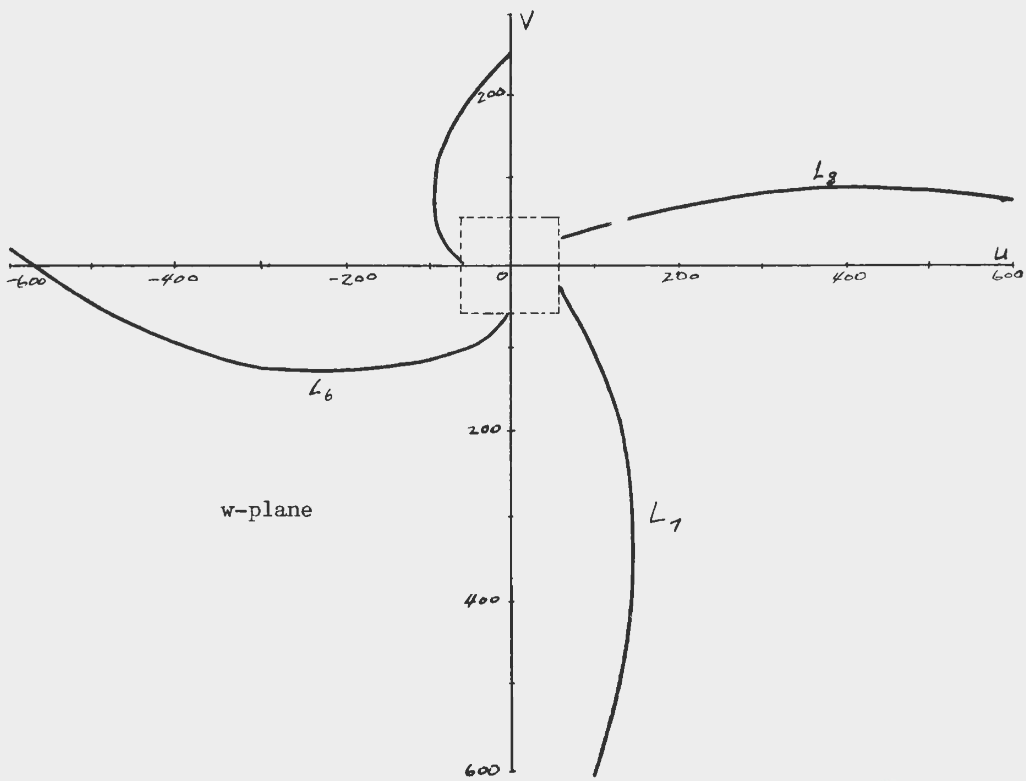


Figure 4a. Transformation of the lines $0 \leq r \leq 2\pi/\cos\theta \leq 9$ and $\theta = n\pi/16$ $(n = 1, 2, 3, 4, 5, 6, 7)$ under $w = \sin z$.

w-plane

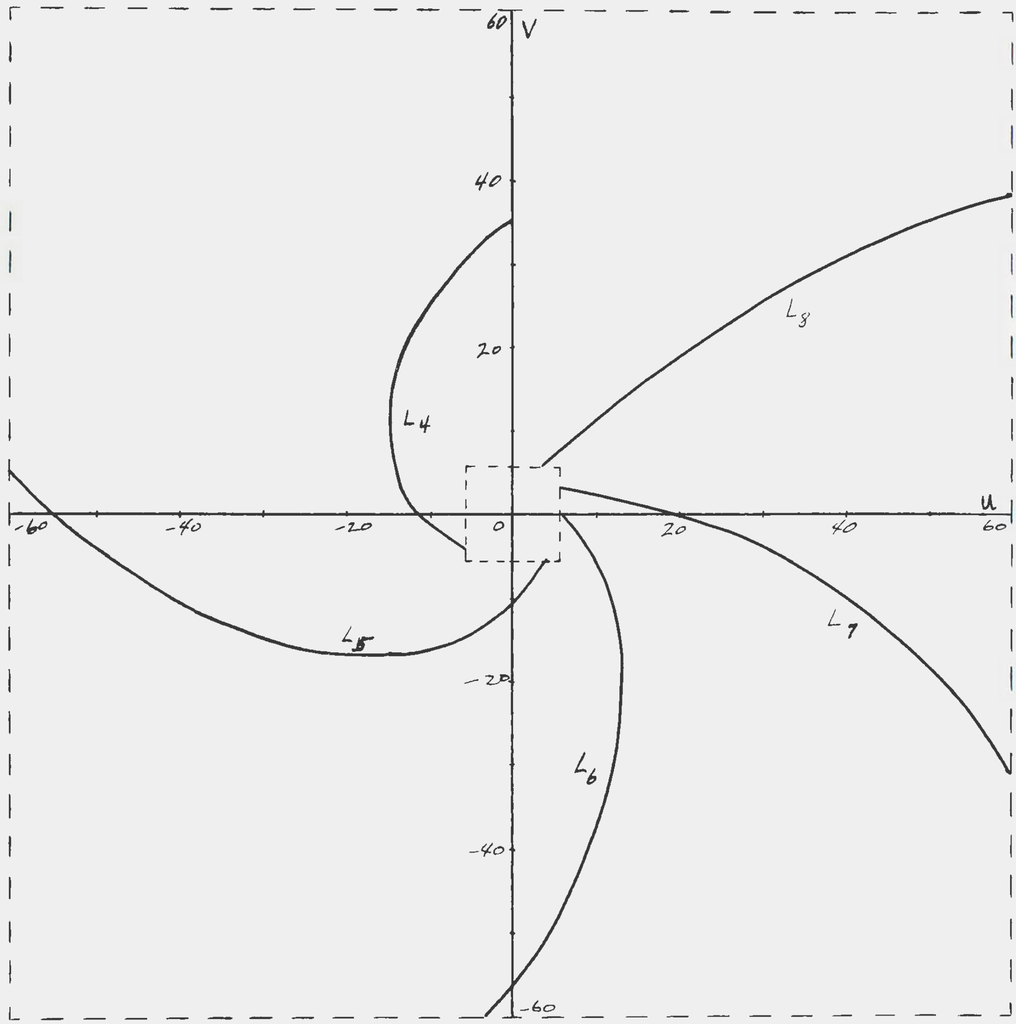


Figure 4b. Inset for Figure 4a. Transformation of the lines $0 \leq r \leq 2\pi/\cos\theta \leq 9$ and $\theta = n\pi/16$ (n = 1, 2, 3, 4, 5, 6, 7) under $w = \sin z$.

w-plane

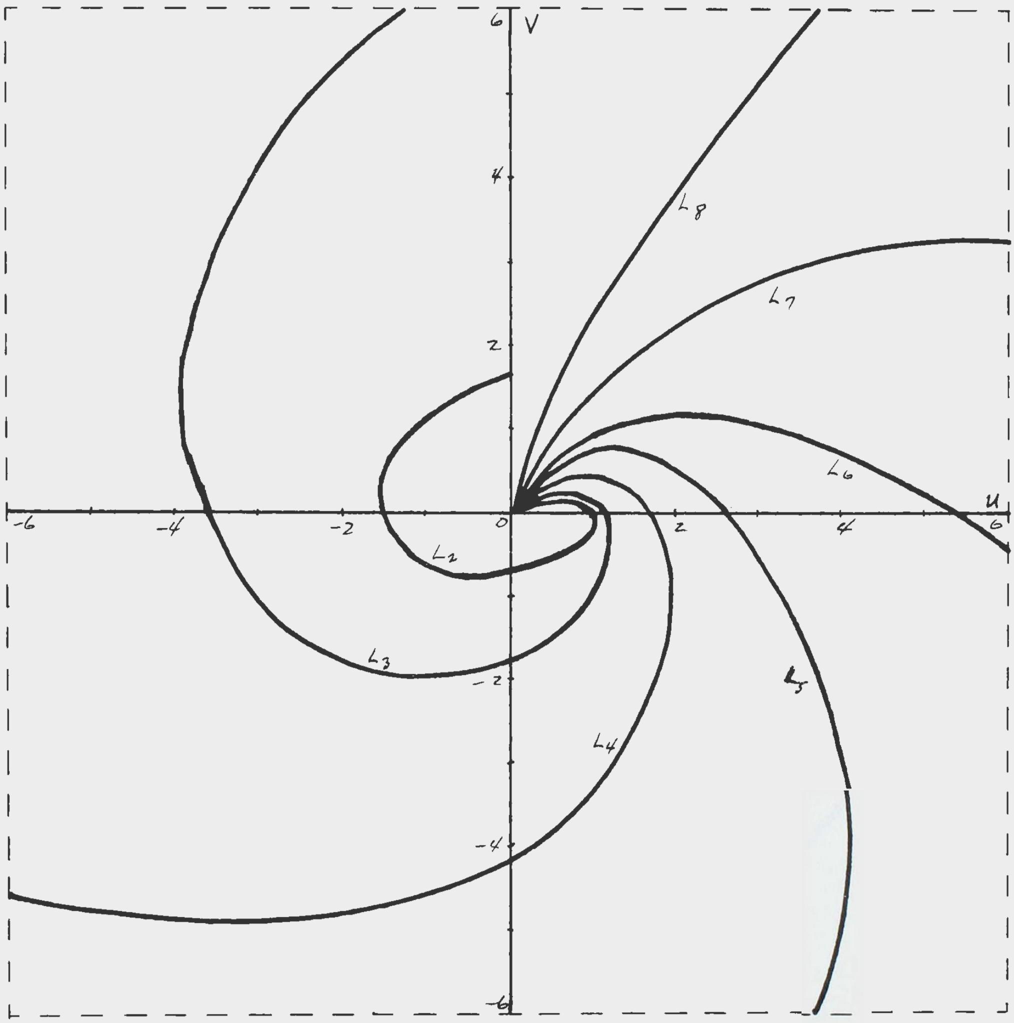


Figure 4c. Inset for Figure 4b. Transformation of the lines $0 \leq r \leq 2\pi/\cos\theta \leq 9$ and $\theta = n\pi/16$ (n = 1, 2, 3, 4, 5, 6, 7) under $w = \sin z$.

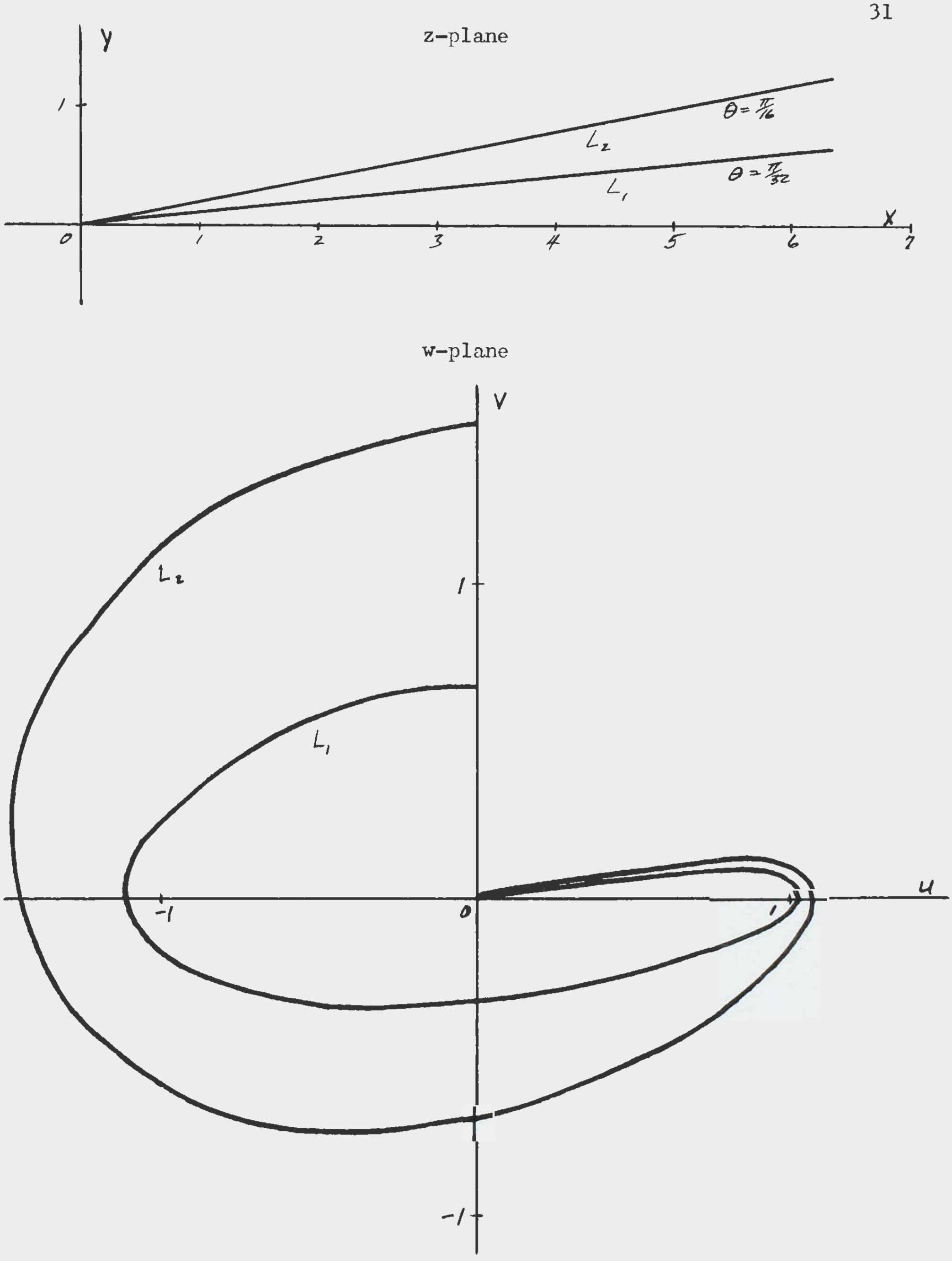


Figure 5. Transformation of the lines $\theta = \pi/32$, $\theta = \pi/16$ and $0 \leq r \leq 2\pi/\cos\theta \leq 9$ under $w = \sin z$.

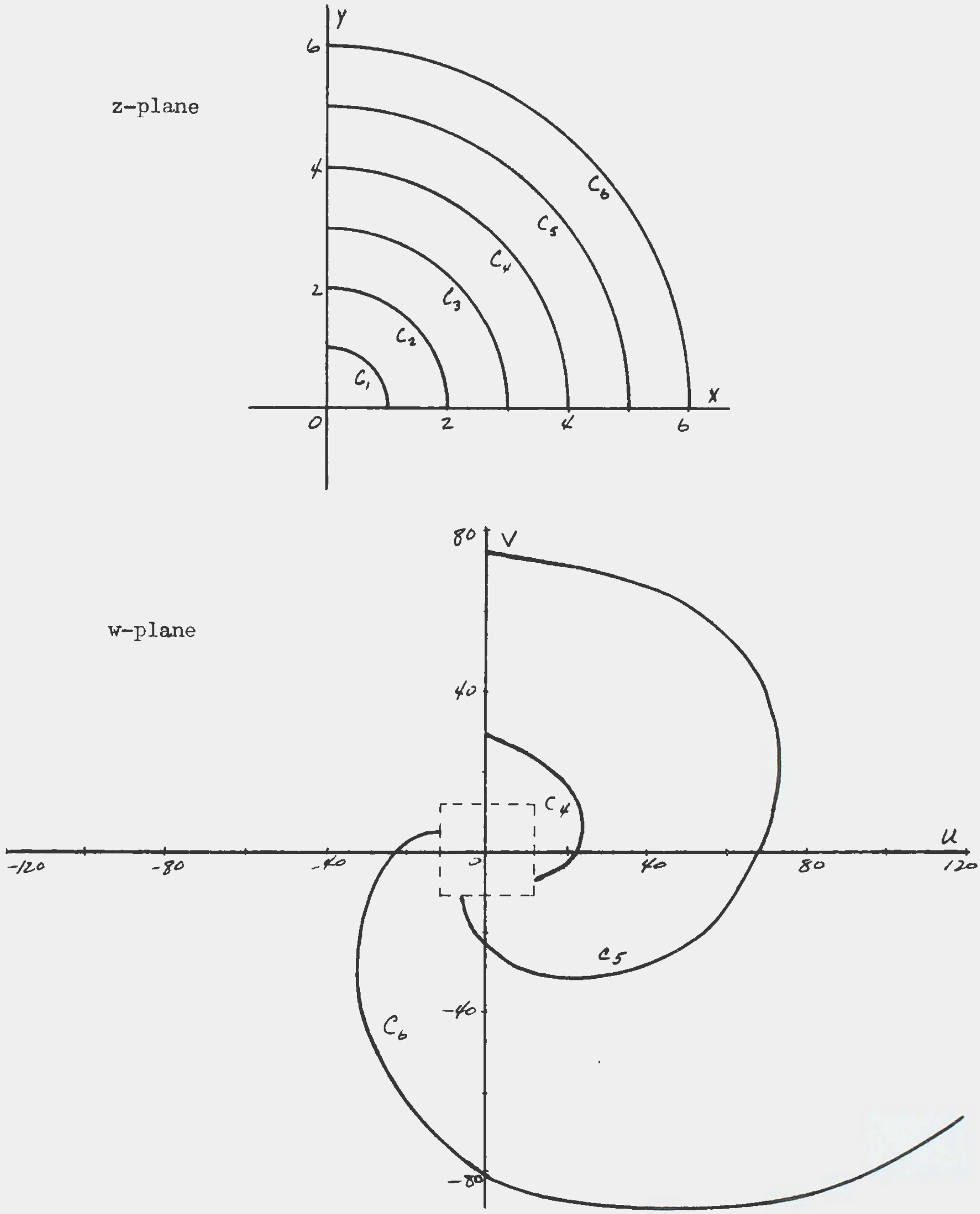


Figure 6a. Transformation of the circular arcs in the first quadrant of the z-plane, $0 \leq \theta \leq \pi/2$ and $r = n$ $(n = 1, 2, 3, 4, 5, 6)$ under $w = \sin z$.

w-plane

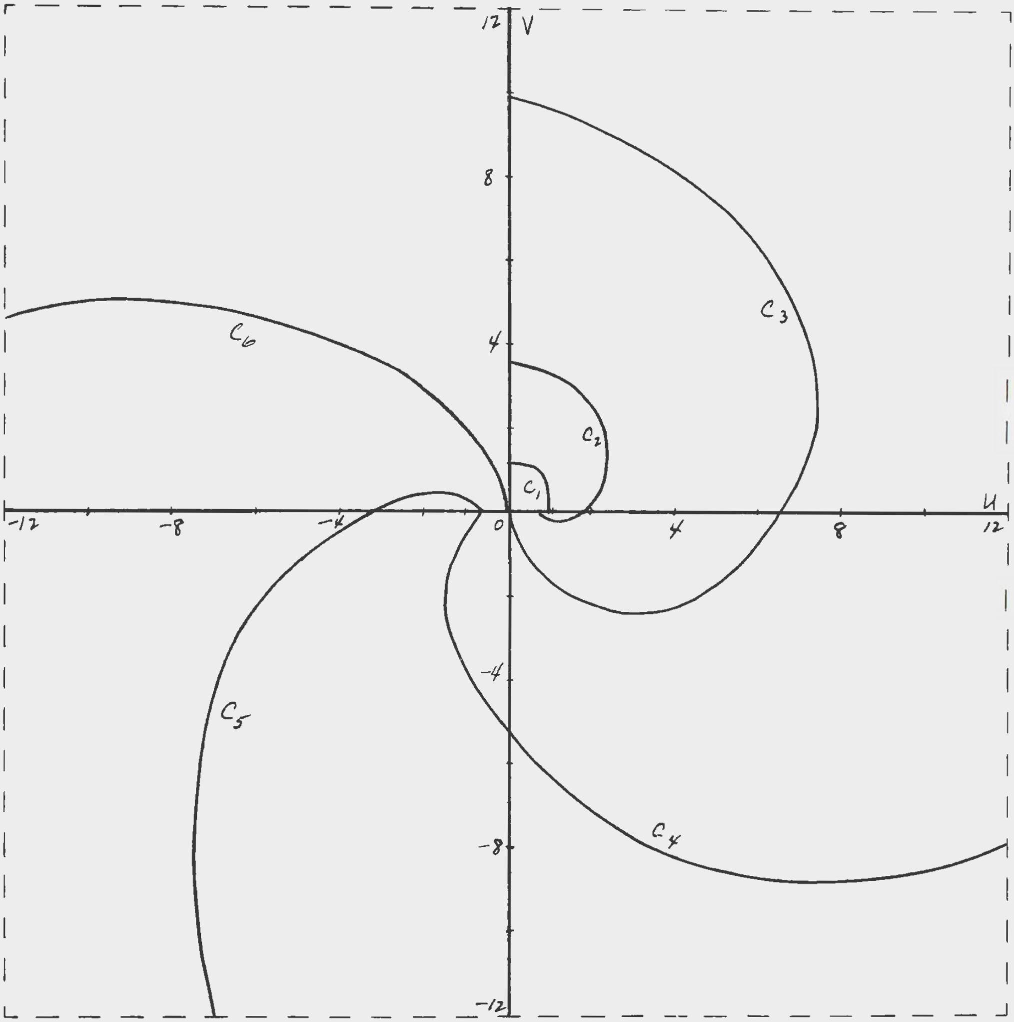


Figure 6b. Inset for Figure 6a. Transformation of the circular arcs in the first quadrant of the z-plane, $0 \leq \theta \leq \pi/2$ and $r = n$ $(n = 1, 2, 3, 4, 5, 6)$ under $w = \sin z$.

Transformations under other functions. The functions cos z, sinh z, and cosh z will now be considered. To accomplish this the tool of successive transformations will be utilized. Transforming a function of z into another variable, which is then transformed into w by the original function is known as successive transformations.

Since the function, $\sin z = \cos(z - \pi/2)$,

(3.65) $$w = \sin z = \cos z', \quad \text{where } z' = z - \pi/2.$$

Therefore, the transformation, $w = \cos z$, is the same as the transformation, $w = \sin z$, preceded by a translation to the left of each point of the z-plane through $\pi/2$ units.

The function, $w = \sinh z$, can be written

(3.66) $$iw = \sin(iz)$$

since $\sinh z = -i\,[\sin(iz)]$. Equation (3.66) can be written

(3.67) $$w' = \sin z', \quad \text{where } iw = w' \text{ and } iz = z'.$$

The axes of each plane can be rotated through the angle $\pi/2$, and then the transformation $w = \sin z$ applied to obtain the transformation $w = \sinh z$.

Following a similar procedure $w = \cosh z$ can be written in the form

$$w = \cos(iz)$$

because $\cosh z = \cos(iz)$. Then,

$$w = \cos z', \text{ where } z' = iz,$$

and $$w = \cos z' = \sin z'', \text{ where } z'' = z' + \pi/2.$$

Therefore, $w = \cosh z$ is a combination of the rotation of the axes of the z-plane through an angle $\pi/2$, the translation of all points in the z'-plane $\pi/2$ units to the right, and the transformation under the function $w = \sin z$.

The properties of the sin z function that have been discussed apply to each of these functions as well. Therefore, it will not be necessary to reiterate these properties. After the preliminary transformations have been applied to these functions, mappings identical to these of the sin z function will result.

CHAPTER IV

SUMMARY

Reference material concerning conformal transformations under trigonometric and hyperbolic functions is very limited. Apparently, very little work on these transformations has been published other than the cursory treatment presented in most complex variable and applied mathematics textbooks.

The investigation of the problem of this paper presents the probable reason for this lack of material. A discussion of the general properties of these functions presents no specific problem. The actual mapping, however, of geometrical figures by these functions is a formidable task.

It was shown that the functions being considered, sin z, cos z, sinh z, and cosh z, were very closely related. Therefore, the investigation of $w = \sin z$ was sufficient to determine the properties of each function.

It was found that the usual representation by parametric equations can be changed into an expression which is a function of u and v only. General cases of lines and circles were considered. Equations of images of lines and circular arcs were extremely complex. It was not possible by methods known to this writer to reduce the expressions into more usable forms.

Parametric equations were used in transforming selected lines passing through the origin and selected concentric circular arcs with the origin as their center. The images of both the lines and the arcs were spiral shaped. The images of the lines started at the origin of the w-plane and spiraled outwardly in a clockwise direction.

The images of the arcs spiraled from a point on the positive v-axis inwardly in a clockwise direction. These spirals ended on a line segment such that $v = 0$ and $-1 \leqslant u \leqslant 1$. The general shape of their image was determined by the length of the radius of the circular arc.

No immediate application of the results of this paper are known. The problem suggests that there is need for the development of a method of representing trigonometric and hyperbolic transformations that would greatly facilitate their evaluations.

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