Summer 1960

Difference Equations: Theory and Applications

Ira T. Whipple
Fort Hays Kansas State College

DIFFERENCE EQUATIONS:

THEORY AND APPLICATIONS

being

A Thesis Presented to the Graduate Faculty
of the Fort Hays Kansas State College in
Partial Fulfillment of the Requirements for
the Degree of Master of Arts

by

Ira T. Whipple, B. S.
Kansas State University

Date July 22, 1960 Approved ______________
Major Professor

Approved ______________
Chm., Graduate Council

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Mathematics is often called the oldest science, but it might also be called the most modern of sciences. In many aspects it is so modern that practical uses have not been found. While "pure" mathematicians develop new ideas and concepts with small attention to possibilities of practical applications, the applied scientist regularly discovers uses which may never have been considered by the originator.

One very old idea in mathematics which is today finding new importance is the calculus of finite differences. Closely related to the infinitesimal calculus, the calculus of finite differences is said in Van Nostrand's Scientific Encyclopedia to have been understood by Newton and Leibniz. The first book on the subject was written by Brook Taylor in 1715.

Developmental work was done by Daniel and Jakob Bernoulli, Euler, Sterling, and many others. George Boole published A Treatise on the Calculus of Finite Differences in 1880. Milne, Thompson, Fort, and Jordon are a few more recent writers. Finite differences have traditionally found uses in approximate integration and differentiation, interpolation, and summation of finite series. Difference equations have the same relation to the calculus of finite differences as differential equations have to differential calculus. With the development of computers and numerical methods of solutions, the difference equation is taking on renewed interest. Modern application of mathematics to the field of social science has likewise found difference equations particularly suited to its needs.

A variety of sources is available for those interested in either applications or theoretical background of difference equations. Many universities are offering graduate courses, and some are offering undergraduate courses, in difference equations. Still, to the majority of undergraduate students, difference equations are probably thought to be another name for differential equations. The purpose of this paper is thus to help those who are unfamiliar with difference equations to become better acquainted with them by giving a brief summary of the types of difference equations, a discussion of the methods of solution, and a few applications to practical problems. This will include some explanation of notation and a summary of important relationships found in the calculus of finite differences. The reader should have some knowledge of differential equations if he is to attain complete understanding of this material, but he would not need any background in the calculus of finite differences.

CHAPTER II

NOTATIONS AND DEFINITIONS USED IN THE CALCULUS OF FINITE DIFFERENCES

General background. The familiar calculus is concerned with the manipulation of continuous variables, or variables which may take on changes or increments of any desired size; and with a limiting process whereby the change in increment size becomes ever smaller or approaches as closely as desired to zero. In contrast, the calculus of finite differences is associated with variables which are defined or known only at discrete, evenly spaced intervals. For instance, a psychology experiment may obtain information on repeated trials of a rat being conditioned to a stimulus; or an agronomist may obtain yields of a plant variety in successive years, in which cases the information is defined only in relation to these particular trials. Similarly, an economist may obtain census information every five years such that his data is discrete and evenly spaced although he may logically assume that the variable being studied had made a continuous change throughout the period. Many electrical and mechanical problems can be broken down into repeated sets of identical components which are advantageously studied by finite difference methods.

Notation. In the study of finite differences the independent variable is represented as x, and f(x) represents a function of x or the values which the dependent variable assumes as the independent variable, x, changes by definite intervals. Since the variable x is only defined at discrete points, f(x) is only defined at values of f(x + h), f(x +2h), to f(x + nh) where h is some constant spacing and n is zero or some whole number. That is, the dependent variable changes as x changes by various multiples of h. The value of h can be chosen as any constant amount since its size serves only to determine the scale, so it is convenient to choose its value to be 1. Many notations have been used by writers, but a popular one is to use U_x to represent f(x), the dependent variable. [(11)p. 6] This has the disadvantage of implying a continuous function which suggests to some writers the use of y_k. The k intimates discrete values but can give the impression of being a constant which it is not. Using the former notation, if the function is x^2, $U_x = x^2$

$$U_{x+1} = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1$$

and

$$U_{x+2} = (x + 2)^2 = x^2 + 4x + 4$$

with the method of finding succeeding values obvious.

The change in a function corresponding to an increase in the independent variable is represented by ΔU_x and is the difference between the new and the original value. Thus

$$\Delta U_x = U_{x+1} - U_x$$

is called the first forward difference. Forward difference is used since the difference could be

$$\nabla U_x \equiv U_x - U_{x-1}$$

or

$$\delta U_x \equiv U_{\frac{x}{2}} - U_{-\frac{x}{2}}$$

which are called backward and central differences respectively.

The second difference is defined as the change in the first difference or

$$\Delta^2 U_x = \Delta(\Delta U_x) = (U_{x+1+1} - U_{x+1}) - (U_{x+1} - U_x)$$
$$= U_{x+2} - 2U_{x+1} + U_x$$

Likewise, the third, fourth..., and nth differences are represented as $\Delta^3 U_x$, $\Delta^4 U_x$,... $\Delta^n U_x$ with the nth difference equal to the first difference of the (n-1)st difference:

$$\Delta^n U_x = \Delta(\Delta^{n-1} U_x)$$

The expansion of this in terms of successive values of the original function is related to the binomial expansion or

$$\Delta^n U_x = U_{x+n} - n\, U_{x+n-1} + \frac{n(n-1)}{1 \cdot 2}\, U_{x+n-2} - \cdots$$
$$\pm \frac{n(n-1)\cdots(2)}{1 \cdot 2 \cdots (n-1)}\, U_{x+1} \pm \cdots \pm U_x$$

with numbers added to the subscript x corresponding to the power, and signs alternating as when the binomial has opposite signs. For example,

$$\Delta^3 U_x = U_{x+3} - 3U_{x+2} + 3U_{x+1} - U_x$$

<u>Use of operators</u>. Considerable use is made of operator notation and the fact that the operators satisfy most of the laws of algebra. Thus, the delta just introduced is treated as an operator. This and other operators in common use are indicated below:

(1) $\Delta U_x = U_{x+1} - U_x$

(2) $EU_x = U_{x+1}$

(3) $DU_x = \frac{d}{dx} U_x$

(4) $kU_x = kU_x$

It is seen that the operator, E, serves to translate the function 1 unit to the right, or is the value as x is increased one increment. The operator, D, is the usual derivative of x taken at one of the spaced intervals, and k is **any constant** multiplied times the variable. As with the delta operator, successive application of all the operators is indicated by the power notation. Thus E^2 serves to translate the function 2 units to the right or

$$E^2 = U_{x+2}$$

As long as the symbols are interpreted as operators their multiplication is distributive with respect to addition. Further, operators are commutative with respect to multiplication and addition when operating on constants, and associative with respect to addition and multiplication. [(11) p. 16]

The operators are related by:

(5) $\Delta U_x = EU_x - U_x$ by substituting (2) into (1)

or

(6) $\Delta = E - 1$

With the symbolic form of Taylor's series

(7) $E = e^{hD} = e^{1 \cdot D} = e^D$

and

(8) $\ln E = D = \ln (1 + \Delta)$

where Ln indicates the principle value of the natural logarithm. Then

$$(9) \quad \Delta = e^D - 1$$

Many useful interpretations are gained from these operators, such as interpolation formulas, and differention and integration of tabulated functions.

<u>The factorial function</u>. A useful function called the factorial function is introduced in the calculus of finite differences. It is found to play the same role in finding differences as a function raised to a power does in differential calculus. The factorial function is defined as [(6) p. 262]

$$(10) \quad x^{(m)} = x(x-1)\ (x-2) \cdots (x-m+1) \equiv \frac{\Gamma(x+1)}{\Gamma(x-m+1)}$$

where x is real and continuous and m is rational.

and

$$x^{(-m)} = \frac{1}{(x+1)\ (x+2) \cdots (x+m)} \equiv \frac{\Gamma(x+1)}{\Gamma(x+m-1)}$$

The Gamma function, Γ, as defined by Euler [(9) p. 339] is the definite integral

$$(11) \quad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} d_x \qquad n > o$$

It permits some summations which would not be possible using the factorial definition alone.

CHAPTER III

DIFFERENCE EQUATIONS

Definition, types, and properties. A difference equation is defined by Richardson [(11) p. 98] as, "An equation which expresses a relation between an independent variable x and successive differences or successive values of a dependent variable U_x ..." An example is:

$$(12) \quad a_o \Delta^2 U_n - a_1 \Delta U_x + a_2 U_x = a_3$$

where the a_1 are constants or functions of the independent variable x. Due to equalities of operators this may be written as

$$a_o (U_{x+2} - 2U_{x+1} + U_x) - a_1 (U_{x+1} - U_x) + a_2 U_x = a_3$$

or

$$a_o U_{x+2} - (2a_o + a_1) U_{x+1} + (a_o + a_1 + a_2) U_x = a_3$$

By substituting new coefficients this becomes

$$(13) \quad b_o U_{x+2} - b_1 U_{x+1} + b_2 U_x - b_3 = 0$$

or equivalently

$$(14) \quad b_o E^2 U_x - b_1 EU_x + b_2 U_x - b_3 = 0$$

By similar methods any combination of differences and the function may be expressed in terms of sums of the successive values of the function or the value of the function at various intervals. The form of equation (13) is the most convenient and commonly used form of the difference equation.

Difference equations are classified and studied according to standard classifications of order, degree, and homogeneity. If an equation written in the form of successive values as (13) contains U_x and successive values out to U_{x+m}, the equation is said to be of order m. Equation (13) is of the second order. An example of a third order equation is

$$(14) \quad U_{x+3} + 2U_{x+2} - U_{x+1} - 2U_x = 6$$

It is instructive to note here that an identical equation would be

$$(15) \quad U_{x+1} + 2U_x - U_{x-1} - 2U_{x-2} = 6$$

since if the equation is to hold for all x, we can think of the x in (14) as being diminished by 2. Otherwise stated, the equation specifies a relation among successively spaced values of the independent variable regardless of the starting point within the specified interval. A more complete definition of order, then, would be to say it is the algebraic difference of the final and initial increment. The use of "successive" above does not mean to imply that all intermediate values must be included. An example is

$$(16) \quad U_{x+3} - U_{x-2} = x$$

which is a fifth order equation.

The degree of an equation relates to the power of the dependent variable and its successive values. If all such variables are to the first power and no products between the variables are present, the equation is of first degree, normally referred to as linear. Linear equations are the most frequently encountered, are most easily solved, and have received the most study.

Equations are further classified as homogeneous if the only terms involved are terms of the dependent variable or its differences or successive values. That is,

$$(17)\quad a_0 U_{x+n} + a_1 U_{x+n-1} + \cdots + a_{n-1} U_{x+1} + a_n U_x = R_x$$

is a homogeneous linear equation of order n if R_x is zero. If R_x is a function of x the equation is non-homogeneous.

Solutions and solution methods. U_x is called a solution of a difference equation if substitution of the value into the equation makes it an equality for all values of x. Solutions of two general types are found, explicit solutions and step by step solutions. [(6) p. 230] Explicit solutions may be in a finite series form or in a closed form. It is proven in the literature that an explicit solution may always be found when the equation is linear with constant coefficients and the independent variable (Rx in equation 17) is zero or one of the standard forms a^x or e^{bx}, sin cx, cos cx, or x^r ($r = 0,1,2,\ldots$) or combinations of their products or sums. Further, if the function is defined at consecutive values, the most general solution will contain exactly "n" arbitrary constants, where "n" is the order of the equation. Second order equations of certain special forms with variable coefficients have explicit solutions by methods analogous to those of differential equation such as integrable combinations, substitution, and separable equations.

Step by step solutions may be found to a large number of equations not otherwise solvable. This type of solution has found considerable application in the solution of differential equation problems and, particularly, partial differential equation boundary value problems. By substituting an analogous difference equation for the differential or partial differential equation, an iterative formula may be derived whereby values may usually be found to any desired accuracy.

Explicit solutions. Methods for obtaining explicit solutions are analogous to those used in differential equations. The methods will be discussed according to types of equations to which they are applicable. One classification of types is:

1. First order linear equations,
2. Linear equations of nth order with constant coefficients, and
3. Special forms and simple difference equations.

First order linear equations. Methods are available for solving completely linear equations of the first order either with constant or variable coefficients. Simple first order equations such as $\Delta U_x = A_x$ which are expressed as a difference (or which may be put in this form by substitution or rearrangement) are solved by summation or the reverse process of differencing. These formulas for summation or antidifferences are developed in the calculus of finite differences and may be found tabulated in those references or in handbooks such as Cogan and Norman's Handbook of Calculus, Difference and Differential Equations. [(1) Tables 14.2] Included in this handbook is a table of Stirling numbers which is useful in transforming powers of x into factorial powers, previously mentioned. Factorial powers obey analogous rules in differences and antidifferences as simple powers do in differentiating and integrating. Compare

(18) $\Delta x^{(n)} = nx^{(n-1)}$ to $D_x x^n = nx^{n-1}$

(19) $\Delta(ax + b)^{(n)} = an(ax + b)^{n-1}$ to

$$D_x (ax + b)^n = an(ax + b)^{n-1}$$

It is shown [(2) p. II 61] that powers of x may always be expressed as factorial powers with the coefficients found by using Stirling numbers.

To solve the equation

$$\Delta U_x = x^3$$

express x^3 as $x^{(1)} + 3\, x^{(2)} + x^{(3)}$ [(1) Table 5.8], the table lists the value of each coefficient, so that

$$\Delta U_x = x^{(1)} + 3\, x^{(2)} + x^{(3)}$$

Taking the antidifference of each term by using (18) gives

$$= \frac{x^{(2)}}{2} + \frac{3x^{(3)}}{3} + \frac{x^{(4)}}{4} + C$$

which by use of the equalities listed in Table 5.7 of the handbook becomes

$$U_x = \tfrac{1}{2}(-x + x^2) + (2x - 3x^2 + x^3) + \tfrac{1}{4}(-6x + 11x^2 - 6x^3 + x^4)$$

and collecting terms gives

$$U_x = \frac{x^2}{4} - \frac{x^3}{2} + \frac{x^4}{4} + C$$

A check shows it is indeed the solution:

$$\Delta U_x = U_{x+1} - U_x = \tfrac{1}{4}(x + 1)^2 - \tfrac{1}{4}x - \tfrac{1}{2}(x + 1)^3 + \tfrac{1}{2}(x)^3$$
$$+ \tfrac{1}{4}(x + 1)^4 - \tfrac{1}{4}x^4 + C - C = x^3$$

Solutions may be obtained by a variety of methods. If the first order equation has constant coefficients, it obviously is included under the theory of those of nth order equations to be discussed later and may be solved by those methods. A method somewhat easier to apply in this case, however, is outlined by Goldberg [(5) p. 63]. Taking the general equation with constant coefficients.

$$U_{x+1} - A\, U_x = B \qquad x = 0,1,2,\cdots$$

where A and B are constants, we can rewrite it as

$$(20) \quad U_{x+1} = A\, U_x + B$$

We assume $A \neq 0$ since the equation with $A = 0$ would be simply solved as $U_x = B$.

If we suppose that U_o is known, then taking $x = o$ and substituting in (20)

$$U_1 = A\, U_o + B$$

Then with $x = 1$

$$U_2 = A\, U_1 + B = A(A\, U_o + B) + B$$

or

$$U_2 = A^2\, U_o + B\,(1 + A)$$

and with $x = 2$

$$U_3 = A\, U_2 + B = A^3\, U_o + B\,(1 + A + A^2)$$

Continuing with $x = k - 1$

$$U_k = A^k\, U_o + B\,(1 + A + A^2 + \ldots + A^{k-1}),\ k = 0,\ 1,2\ldots$$

that is

$$U_x = A^x\, U_o + B\,(1 + A + A^2 + \ldots + A^{x-1}),\ x = 0,\ 1,\ 2\ldots$$

The quantity in parenthesis is a finite geometric series which has the sum, as shown in many standard calculus books, of

$$1 + A + A^2 + \dots + A^{x-1} = \begin{cases} \frac{1-A^x}{1-A} & \text{if } A \neq 1 \\ x & \text{if } A = 1 \end{cases}$$

The solution to (20) may now be written as

$$U_x = \begin{cases} A^x U_o + B\frac{1-A^x}{1-A} & \text{if } A \neq 1 \quad (21) \\ U_o + Bx & \text{if } A = 1 \quad (22) \end{cases} \qquad x = 0,1,2\dots$$

The proof is completed by Goldberg by directly substituting this solution back into the equation to give an identity. That the solution is unique is proven by letting U_o be an arbitrary constant, C, and thus putting it in the pattern of a previously proven theorem.

Solving an equation such as

$$U_{x+1} = 3U_x + 2 \qquad x = 0,1,2\dots$$

with the initial condition $U_o = 4$, now is routine substitution. From (20) and (21) A = 3 and B = 2, and

$$U_x = 3^x(4) + 2\frac{1-3^x}{-2}$$

which gives

$$U_x = 3^x(4) + 3^x - 1$$

or

$$U_x = 5(3)^x - 1 \qquad x = 0,1,2\dots$$

Variable coefficients. The general linear first order equation with variable coefficients in the form

$$U_{x+1} - A_x U_x = B_x \qquad x = 0,1,2\dots$$

where A_x and B_x are functions of x with $A_x \neq 0$ is solved by Richardson [(11) p. 103]. Consider first where $B_x = 0$ the homogeneous equation. If x = 0, then

$$U_1 = A_o U_o$$

and with x=1,2,3...k-1, respectively,

$$U_2 = A_1 U_1 = A_o A_1 U_o$$

$$U_3 = A_2 U_2 = A_o A_1 A_2 U_o$$

. . . .

$$U_k = A_{k-1} U_{k-1} = U_o A_o A_1 A_2 \cdots A_{k-1}$$

or

$$U_x = U_o A_o A_1 A_1 A_2 \cdots A_{x-1}$$

If we let the value of U_o be an arbitrary constant, C, and express the continued product using the symbol π

$$A_o A_1 A_2 \cdots A_{x-1} = \prod_{x=o}^{x-1} A_x$$

the solution may be written as

$$(23) \quad U_x = C \prod_{x=o}^{x-1} A_x$$

Since a product may be expressed as the sum of the logarithm of the factors, this may alternately be written as

$$(24) \quad \log U_x = \log C + \sum_{x=o}^{x-1} \log A_x$$

The equation

$$U_{x+1} - 3U_x = o$$

solved by this method has $A_x = 3$.

Then from (23)

$$U_x = C \prod_{x=o}^{x-1} 3 = C\,(3 \cdot 3 \cdot 3 \cdots 3),$$

where the number of factors is x. Hence, we have

$$U_x = C3^x$$

To check

$$U_{x+1} = C(3)^{x+1} = C3^x \cdot 3$$

and

$$U_{x+1} - 3U_x = C3^x \cdot 3 - 3C3^x = 0$$

For any value of C this is an identity and thus a solution.

The solution of the nonhomogeneous equation

(25) $U_{x+1} - A_x U_x = B_x$

where $B_x \neq 0$ makes use of the homogeneous solution by means of what is called the associated homogeneous equation or reduced equation. First assume the solution, U_x

(26) $U_x = Z_x V_x$

where the functions Z_x and V_x are to be determined. The method used here [(11) p. 105] is to show that if Z_x can be found so that it is a solution of the reduced equation, then V_x can be determined. Substituting $U_x = Z_x V_x$ into (25) it becomes

(27) $Z_{x+1} V_{x+1} - A_x Z_x V_x = B_x$

From the definition

$$\Delta V_x = V_{x+1} - V_x$$

we may substitute $\Delta V_x + V_x$ for V_{x+1} in (27) giving

$$Z_{x+1} (\Delta V_x + V_x) - A_x Z_x V_x = B_x$$

This is rearranged to give

$$V_x (Z_{x+1} - A_x Z_x) + Z_{x+1} \Delta V_x = B_x$$

Let

(28) $Z_x = \prod_{x=o}^{x-1} A_x$

then Z_x is a solution of the homogeneous or reduced part of the original equation (25), and the part in the parenthesis must identically equal zero. Then this gives

$$Z_{x+1} \Delta V_x = B_x$$

Solving this for Δv_x and taking the antidifference or indefinite sum, we have

$$(29)\quad V_x = \Delta^{-1} \frac{B_x}{Z_{x+1}} + C$$

The complete solution then is

$$(30)\quad U_x = Z_x V_x = CZ_x + Z_x \Delta^{-1} \frac{B_x}{Z_{x+1}}$$

Since division by zero is not allowed the solution Z_x, hence Z_{x+1}, must be nonzero.

To summarize: first find the solution to the reduced equation given by $Z_x = \prod_{x=0}^{x-1} A_x$, find V_x by taking the anti-difference,

$$V_x = \Delta^{-1} \frac{B_x}{Z_{x+1}} + C$$

then multiply to give

$$U_x = Z_x V_x$$

The equation

$$U_{x+1} - 3U_x = 2$$

used as an example earlier, has the reduced equation

$$U_{x+1} - 3U_x = 0$$

This has the solution

$$Z_x = \prod_{0}^{x-1} 3 = 3^x$$

From (29)

$$V_x = \Delta^{-1} \frac{2}{3^{x+1}} + C$$

and

$$V_x = 2/3\ \Delta^{-1} (1/3)^x + C$$

From table 14.10 of Cogan and Norman we get

$$V_x = 2/3 \cdot \frac{1}{1/3-1} (1/3)^x + C$$

which simplifies to

$$V_x = -(1/3)^x + C$$

Hence, the general solution is

$$U_x = Z_xV_x = -(3)^x\ (1/3)^x + 3^x + C = 3^xC - 1$$

If we take $U_o = 4$ as an initial condition as in the solution by the previous method,

$$U_o = 4 = 3^oC - 1$$

Solving for C gives

$$C = 5$$

and

$$U_x = 5(3)^x - 1$$

as before.

Another example illustrates the method where A_x and B_x are actually variables. Solve [(11) p. 107] the equation

$$U_{x+1} - (x+1)\ U_x = 2^x\ (x-1)$$

Here

$$A_x = x + 1$$

and

$$B_x = 2^x\ (x - 1)$$

From (28) we have

$$Z_x = \prod_{o}^{x-1} (x + 1)$$

but

$$A_o = 1,\ A_1 = 2,\ A_2 = 3,\ \ldots\ A_{x-1} = x$$

hence

$$Z_x = x\,!$$

and

$$Z_{x+1} = (x+1)!$$

From (29) comes

$$V_x = \Delta^{-1} \frac{2^x(x-1)}{(x+1)!} + C$$

Here the finite integral is not one of the tabulated forms. It may be found by a method of undetermined coefficents and undetermined functions [(11) p. 31] as illustrated. We wish to find V_x such that

$$\Delta V_x = W_x$$

where

$$W_x = \frac{2^x(x-1)}{(x+1)!}$$

Assume V_x contains some arbitrary function, f(x), so that

$$V_x = \frac{f(x)\ 2}{x!}$$

then

$$\Delta V_x = V_{x+1} - V_x = W_x$$

Substituting values into the equation gives

$$\frac{f(x+1)\cdot 2^{x+1}}{(x+1)!} - \frac{f(x)\cdot 2^x}{x!} = \frac{2^x(x-1)}{(x+1)!}$$

or

$$2f(x+1) - (x+1)\ f(x) = x-1$$

since it must be an identity. The right hand side of this equation is of the first degree in x, so it is evident that f(x) must be a constant if the left hand side of the equation is to be of the first degree in x. This gives us

$$f(x) = k$$

$$f(x+1) = k$$

and substituting in the preceding equation,

$$2k - (x+1)k = x - 1$$

from which k = -1

Then

$$V_x = \frac{-1\cdot 2^x}{x!} + C$$

Since the general solution is

$$U_x = Z_x V_x$$

then

$$U_x = x!\ \frac{-1\cdot 2^x}{x!} + Cx\ !$$

A check of this solution in the original equation gives:

$$U_{x+1} = -(2)^{x+1} + C(x+1)!$$

$$-(x+1)U_x = (x+1)2^x - C(x+1)!$$

and

$$U_{x+1} - (x+1)\ U_x = 2^x\ (-2+x+1) + (x+1)!\ (C-C) = 2^x(x-1)$$

This shows the solution is correct.

<u>Nth order linear equations with constant coefficients</u>. The general method for solving linear equations of nth order is found in many publications. The homogeneous equation is first solved by use of an auxiliary equation giving the general solution. The nonhomogeneous equation is solved by finding the solution of the associated homogeneous equation, called the complementary solution, and adding a particular solution to give the general solution.

The general linear homogeneous equation with constant coefficients is written as

$$(30)\quad a_o\ U_{x+n} + a_1\ U_{x+n-1} + \dots\ a_{n-1}\ U_{x+1} + a_n\ U_x = 0$$

where the a_i are constants. By assuming a solution of the type $U_x = \beta^x$ and substituting into the equation, it becomes

$$a_o\ \beta^{x+n} + a_1\ \beta^{x+n-1} + \dots + a_{n-1}\ \beta^{x+1} + a_n\ \beta^x = 0$$

and factoring out the common factor β^x it is

$$\beta^x\ (a_o\beta^n + a_1\ \beta^{n-1} + \dots + a_{n-1}\ \beta + a_n) = 0$$

Assuming β^x different from zero for any finite value of x, since this would give a trivial solution, the term in the parenthesis must be identically zero. This is a linear algebraic equation in β^n (called the auxiliary equation) and has n solutions (counting repeated roots). If the roots are real and distinct and designated $\beta_1, \beta_2, \ldots \beta_n$, then the equation has the general solution

$$U_x = C_1 \beta_1^{\,x} + C_2 \beta_2^{\,x} + C_3 \beta^{\,x} + \ldots + C_n \beta_n^{\,x}$$

It is evident that if each term here is a solution (that is, it makes the equation identically zero), then the sum of them is also zero. That it is indeed the general solution is not so obvious. The proof is not difficult, but is a little lengthy. It is given by Goldberg [(5) p. 128] for the case n = 2 with the general proof indicated as following the same pattern.

If there are some repeated roots, say $\beta_{r_1}, \beta_{r_2}, \beta_{r_3}, \beta_{r_4} \ldots \beta_n$ where the subscripts r_1, r_2, and r_3 indicate a thrice repeated root, then the general solution is

$$U_x = \beta_r \ (C_1 + C_2 \ x + C_3 \ x^2) + \beta_4^{\,x} + \ldots + \beta_n^{\,x}$$

The derivation of this result due to Seliwanoff is indicated in Richardson [(11) p. 117]. In the case of conjugate complex roots, $\beta_1 = a + cb$ and $\beta_2 = a - cb$, the Euler relations may be used to write the roots in the more standard form

$$U_x = \rho^x \ (A \cos \emptyset \ x + B \sin \emptyset \ x)$$

where

$$\rho = \sqrt{a^2 + b^2} \quad \text{and} \quad \emptyset = \tan^{-1} \frac{b}{a}$$

Repeated complex roots and combinations are written in an obvious manner.

The general linear nonhomogeneous equation with constant coefficients is written as

$$(31) \quad a_o \ U_{x+n} + a_1 \ U_{x+n-1} + \ldots + a_{n-1} \ U_{x+1} + a_n \ U_x = R_x$$

where R_x may be either a constant or a function of x. The general solution is the sum of the complementary and particular solution to the reduced or associated homogeneous equation (when R_x is taken as zero.) The complementary solution is found as just indicated for homogeneous equations. Various methods may be used to find the particular solution, a solution which makes the complete equation an identity. Some of these are variation of parameters, the method of operators, and the method of undetermined coefficients.

Undetermined coefficients. The most useful of these is probably the method of undetermined coefficients. Briefly, the procedure is:

1. Assume a solution containing the types or families represented by the variable term R_x. The families are β^x, $\cos \phi x + \sin \phi x$, or x^r.

2. If the family is represented by the homogeneous solution, multiply it by x^{nr}, where n is one greater than the greatest power of x^r in the homogeneous solution.

3. Substitute the assumed solution into the original equation and equate like coefficients to give an identity.

Repeating, the complete solution, then, consists of the sum of the complementary solutions and the particular solution. Constants are determined from the initial or final conditions.

An example will help clarify the procedure:

(32) $U_{x+2} - 7\, U_{x+1} + 12\, U_x = x + 3^x$

The auxiliary equation is

$$\beta^2 - 7\beta + 12 = 0$$

which has roots

$$\beta_1 = 3, \beta_2 = 4.$$

This is obtained by substituting the assumed solution β^x into the equation. More simply, we may notice equation (32) may be written

$$E^2 - 7E + 12 = x + 3^x$$

due to the equivalency of operators. The reduced equation may then be written

$$L(E) = 0$$

where L(E) is a general linear equation in E.

Then the auxiliary equation is

$$L(\beta) = 0$$

The complementary solution is

$$U_{x_c} = C_1\, 3^x + C_x\, 4^x$$

Since one of the complementary solution terms is contained in the nonhomogeneous part of the original equation (3^x), we will choose a particular solution,

$$U_{x_p} = a + bx + Cx\, 3^x$$

Substituting this into (32) gives

$$a + b\,(x+2) + C\,(x+2)\, 3^{x+2} - 7\left[a+b(x+1) + C(x+1)3^{x+1}\right] + 12\left[a+b_x + C_x\, 3^x\right] = x+3^x$$

Collecting terms and rearranging, we have

$$\left[a+2b - 7a - 7b + 12a\right] + x\left[b - 7b + 12b\right] + 3^x\left[2C\cdot 3^2 - 7C\cdot 3\right] + x\cdot 3^x\left[C\cdot 3^2 - 7C\cdot 3 + 12C\right] = x + 3^x$$

and equating like coefficients since it must be true for all x gives

$$6a - 5b = 0$$

$$6b = 1$$

$$-3C = 1$$

$$b = \frac{1}{6};\ a = \frac{5}{36};\ \text{and}\ C = -\frac{1}{3}.$$

The particular solution is

$$U_{x_p} = \frac{5}{36} + \frac{x}{6} - \frac{x}{3}\ 3^x$$

and the general solution is

$$U_x = U_{x_c} + U_{x_p}$$

$$U_x = \frac{5}{36} + \frac{x}{6} - \frac{x}{3}\ 3^x + C_1\ 3^x + C_2\ 4^x$$

If initial conditions are specified by physical or other conditions, the constants C_1 and C_2 are determined from these conditions. In this example take

$$U_o = 0$$

and

$$U_1 = 1$$

Then

$$U_o = 0 = \frac{5}{16} + \frac{0}{6} - \frac{0}{3}3^o + C_1\ 3^o + C_2\ 4^o;\ C_1 + C_2 = -\frac{5}{16}$$

$$U_1 = 1 = \frac{5}{16} + \frac{1}{6} - \frac{1}{3}\cdot\ 3^1 + C_1\ 3^1 + C_2\ 4^1;\ 3C_1 + 4C_2 = \frac{5}{8}$$

Solving these two equations simultaneously gives x

$$C_1 = -\frac{15}{8};\ C_2 = \frac{25}{16}$$

This is a specific solution to the equation (32) which satisfies the initial conditions that the function equal zero when $x = 0$ and equal one when $x = 1$.

A check that this actually is the solution is as follows:

$$U_{x+2} = \frac{5}{36} + \frac{1}{6}(x+2) - \frac{1}{3}(x+2)(3)^{x+2} - \frac{15}{8}(3)^{x+2} + \frac{25}{16}(4)^{x+2}$$

$$= \frac{5}{36} + \frac{x}{6} + \frac{1}{3} - \frac{1}{3}(x+2)3^x.3^2 - \frac{15}{8}(3)^x.3^2 + \frac{25}{16}(4)^x.4^2$$

$$= \frac{17}{36} + \frac{x}{6} - 3x \cdot 3^x - \frac{183}{8}.3^x + 25.4^x$$

$$-7\,U_{x+1} = -7\left[\frac{5}{36} + \frac{1}{6}(x+1) - \frac{1}{3}(x+1)3^{x+1} - \frac{15}{8}(3)^{x+1} + \frac{25}{16}(4)^{x+1}\right.$$

$$= -\frac{77}{36} - \frac{7}{6}x + 7x.3^x + \frac{371}{8}.3^x - \frac{175}{4}.4^x$$

$$12U_x = \frac{60}{36} + \frac{12}{6}x - 4x.3^x - \frac{180}{8}.3^x + \frac{175}{4}.4^x$$

Adding these together gives

$$U_{x+2} - 7U_{x+1} + 12U_x = \left(\frac{17}{36} - \frac{77}{36} + \frac{60}{36}\right) + x\left(\frac{1}{6} - \frac{7}{6} + \frac{12}{6}\right)$$

$$+ x \cdot 3^x(-3 + 7 - 4) + 3^x\left(-\frac{183}{8} + \frac{371}{8} - \frac{180}{8}\right)$$

$$+ 4^x\left(25 - \frac{175}{4} + \frac{75}{4}\right)$$

$$U_{x+2} - 7U_{x+1} + 12U_x = 0 + x + 3^x + 0 \cdot 4^x = x + 3^x.$$

This is an identity which was to be proved.

Generating functions. A method of solving difference equations using what is called generating functions is described by Goldberg [(5) p. 189]. This is one of many transformation methods for solving such equations. The function $Y(s)$, in some real interval including zero, of the series in s,

$$(33) \quad Y(s) = y_0 + y_1 s + y_2 s^2 + \ldots + y_k s^k + \ldots$$

is defined as the generating function for the sequence y_0, y_1, y_2, ... designated as $\{y_k\}$. Then the sequence of $y_o = 1$, $y_1 = 1$, ... $y_k = 1$ has for its generating function

$$Y(s) = 1 + s + s^2 + \ldots + s^k + \ldots = \frac{1}{1-s}$$

as can be demonstrated by performing the division. Again by division, or other methods of infinite series,

$$\frac{1}{(1-s)^2} = 1 + 2s + 3s^2 + \ldots + (k + 1)\ s^k + \ldots$$

This is the generating function for the sequence with the general term (k + 1).

In a similar manner, a small table of general terms $\{y_k\}$ with their generating terms Y(s) is established. From the definition of Y(s) it is readily seen that by rearranging (33) the generating function for $\{y_{k+1}\}$, $\{y_{k+2}\}$,..., can be found if it is known for $\{y_k\}$. Thus

(34) $$\frac{Y(s)-y_o}{s} = y_1 + sy_2 + y_3\ s^2 + \ldots + y_{k-1}\ s^k + \ldots$$

is obtained as the generating function for $\{y_{k+1}\}$ by transposing y_o and dividing by s.

This transformation has the property of transforming the difference equation into a linear algebraic equation which may be solved for Y(s) by standard algebraic procedures. The reverse transformation then gives the actual solution. It has as advantage over some transformations in that its close relationship to the desired solution often allows some analysis of the solution before, or even when the reverse transformation cannot be performed.

The application of the procedure is quite simple when a table of transforms is available. The steps are:

1. Replace each variable in the equation by its transform.

2. Solve the resulting equation for Y(s). This is usually simplified by use of partial fractions.

3. Perform the inverse transform to give the solution y_k.

The following example, previously solved, will demonstrate the procedure.

$$U_{x+2} - 7U_{x+1} + 12U_x = x + 3^2$$

From the Tables of transforms 4.1 and 4.2 in Goldberg [(5) p. 190] this is transformed into

$$\frac{Y(s) - 4_0 - y_1 s}{s^2} - 7\left[\frac{Y(s) - y_0}{s}\right] + 12\, Y(s) = \frac{s}{(1-s)^2} + \frac{1}{(1-35)}$$

Solving for Y(s) gives

$$Y(s)\left[1 - 7_s + 12s^2\right] = \frac{s^3}{(1-s)^2} + \frac{s^2}{(1-3s)} + y_o + y_1 s + 7y_o s$$

$$Y(s) = \frac{s^3}{(1-s)^2\,(1-4s)\,(1-3s)} + \frac{s^2}{(1-3s)^2\,(1-4s)} + \frac{y_0\,(1-7s)}{(1-4s)\,(1-3s)} + \frac{y_1\, s}{(1-4s)\cdot(1-3s)}$$

Using the initial conditions of the problem

$$U_o = 0 = y_o$$

and

$$U_1 = 1 = y_1$$

the second term from the right is zero and the last one is

$$\frac{s}{(1-4s)(1-3s)}$$

By partial fractions the equation is further simplified to

$$Y(s) = \frac{\frac{1}{6}}{(1-s)} + \frac{\frac{7s-1}{36}}{(1-s)^2} + \frac{\frac{1}{9}}{(1-4s)} + \frac{-\frac{1}{4}}{(1-3s)} + \frac{-\frac{1}{3}}{(1-3s)} + \frac{2s-2/3}{(1-3s)^2} + \frac{1}{(1-4s)} + \frac{1}{(1-4s)} + \frac{-1}{(1-3s)}$$

By combining like terms we have

$$Y(s) = \frac{\frac{1}{6}}{(1-s)} + \frac{\frac{7s-1}{36}}{(1-s)^2} + \frac{\frac{6s-2}{3}}{(1-3s)^2} + \frac{-\frac{19}{12}}{(1-3s)} + \frac{\frac{19}{9}}{(1-4s)}$$

Since the table of transforms is incomplete, this presents some difficulty in performing the inverse transformation.

The inverses of the first, fourth, and fifth terms are:

$$\frac{1}{6}\ ; \ -\frac{19}{12}\cdot 3x\ ; \quad \text{and } \frac{19}{9}\,4x\ .$$

By expanding into series andarranging into combinations of sums and products, the second and third terms can be shown to be generating functions for

$$-\frac{1}{36}+\frac{1}{6}x \text{ and } -\frac{x}{3}\,(3^x)-\frac{2}{3}\,(3^x)$$

respectively.

Adding these to the other 3 terms, the solution becomes

$$U_x=\frac{5}{36}+\frac{x}{6}-\frac{x}{3}\cdot 3^x-\frac{9}{4}\cdot 3^x+\frac{19}{9}\cdot 4^x$$

which is the same solution arrived at previously by undetermined coefficients.

Dirichlet series transform. Tomlinson Fort [(4) p. 641] has suggested a transform method of solving linear difference equations with constant coefficients using the Dirichlet series transform. The transformation, $D\{a(t)\}$, is defined, where t is an integer, as

$$f(s)=D\{a(t)\}=\sum_{t=0}^{\infty} m^{-st}\,a(t), \quad m>1, \quad s>1$$

A table of transforms has been developed by Fort. This table will be used as needed for an example, but it will not be reproduced here. The method is similar to the one used in solving differential equations by Laplace transforms. Each element in the equation is replaced by its transform giving a linear algebraic equation in f(s), the transform of the unknown. The resulting equation is solved for f(s) after which

$$U_x = -2 \cdot 3^x + \frac{3}{2} \cdot 4^x + \frac{1}{2} \cdot 2^x$$

Substituting this into the original equation to check the solution gives:

$$U_{x+2} = -2(3)^{x+2} + \frac{3}{2}(4)^{x+2} + \frac{1}{2}(2)^{x+2}$$

$$= -18 \cdot 3^x + 24 \cdot 4^x + 2 \cdot 2^x$$

$$-7U_{x+1} = -7\left[-2(3)^{x+1} + \frac{3}{2}(4)^{x+1} + \frac{1}{2}(2)^{x+1}\right]$$

$$= 42 \cdot 3^x - 42 \cdot 4^x - 7 \cdot 2^x$$

$$12U_x = -24 \cdot 3^x + 18 \cdot 4^x + 6 \cdot 2^x$$

Adding gives

$$U_{x+2} - 7U_{x+1} + 12U_x = 3^x(-18 + 42 - 24) + 4^x(24 - 42 + 18) + 2^x(2 - 7 + 6)$$

or

$$U_{x+2} - 7U_{x+1} + 12U_x = 2^x$$

This is the identical equation required to complete the check.

Special methods, the classical ruin problem. In studying the "classical ruin problem," Feller [(3) p. 313] uses the method of particular solutions to solve the difference equation,

$$(35) \quad q_z = p\, q_{z+1} + q\, q_{z-1} \qquad 1 < z < a - 1$$

This is a departure from our standard notation, which would be

$$U_x = p\, U_{x+1} + q\, U_{x-1}$$

but is justifiable due to the wide usage of p and q in probability problems. Here q_z is the probability of the gambler's losing all (ultimate ruin) and p_z is his probability of winning. His probability of winning or losing on each trial is represented by p and q respectively, z represents his present fortune and a his anticipated gain. The equation is deduced from the fact that after the first trial, the gambler's fortune is either z - 1 or z + 1. To obtain boundary conditions it is noted that if z = 1, the first trial may lead to ruin, and if z = a - 1, the first trial may result in victory. This suggests the conditions that

$q_o = 1$, and $q_a = 0$

two particular solutions, $q_z = 1$ and $q_z = \left(\frac{q}{p}\right)^z$, are verified by trial as

$$1 = p \cdot 1 + q \cdot 1$$

and

$$\left(\frac{q}{p}\right)^z = p\left(\frac{q}{p}\right)^{z+1} + q\left(\frac{q}{p}\right)^{z-1} = q\left(\frac{q}{p}\right)^z + p\left(\frac{q}{p}\right)^z$$

since it is known that

$$p_z + q_z = 1$$

Multiplying each solution by arbitrary constants, A and B, and adding them the formal solution is

$$(36) \quad q_z = A + B\left(\frac{q}{p}\right)^z$$

The constants A and B are determined from the boundary conditions which give the two equations

$$1 = A + B$$

and

$$0 = A + B\left(\frac{q}{p}\right)^a$$

That is

$$A = \frac{\left(\frac{q}{p}\right)^a}{\left(\frac{q}{p}\right)^a - 1}$$

and

$$B = -\frac{1}{\left(\frac{q}{p}\right)^a - 1}$$

The solution satisfying the boundary conditions is thus

$$(37) \quad q_z = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^z}{\left(\frac{q}{p}\right)^a - 1}$$

To prove that this is the unique solution, it must be shown that all solutions are of this form. If an arbitrary solution is assumed, the constants A and B can be chosen such that (36) is an equation for the two values $z = o$ and $z = 1$, since this represents two unknowns with two equations. But from the equation it is seen that any value is determined if the two adjacent values are known. Thus the solution is unique.

The probabilities p and q are parameters. The above argument is not valid when $p = q = \frac{1}{2}$ since then only one particular solution has been found. In this case it is seen that a second particular solution is $q_z = z$. Substituting

$$z = p(z+1) + q(z-1) = \tfrac{1}{2}(z+1) + \tfrac{1}{2}(z-1) = z$$

The formal solution is then

$$(38) \quad q_z = A + Bz.$$

The constants to satisfy the boundary conditions must satisfy the equations

$$1 = A + 0; \; A = 1$$

$$0 = A + Ba; \; B = \frac{-1}{a}$$

This gives

$$(39) \quad q_z = 1 - \frac{z}{a}$$

These results are summarized by saying that q_z is the gambler's probability of ultimate ruin, or losing all his money, when he begins with a capital, z; plays against an adversary with unlimited capital; and plays until he either loses all or increases his capital to the amount a. If q_z is his probability of losing, then $1 - q_z$ must be his probability of winning.

The tacit assumption made here in the initial difference equation is that the wager for each bet or trial is 1 unit compared to the capital. If the capital is $10, then the bet is $1. If capital is 90 cents, bet is 1 cent.

To see how this is applied, let us assume a player starts with $90 and wishes to make $10 to give him a total of $100. He is playing where the odds are against him such that he has 4 chances in 10 of winning on each trial. What are his chances of reaching his total goal?

In the solution we note that z = $90, a = $100, p = 0.4, and q = 0.6. Equation (37) gives q_z, his chance of losing, and $1 - q_z$ is his chance of winning. Substituting in (37)

$$q_z = \frac{\left(\frac{.6}{.4}\right)^{100} - \left(\frac{.6}{.4}\right)^{90}}{\left(\frac{.6}{.4}\right)^{100} - 1} = .983$$

and

$$1 - q_z = 0.017.$$

His probability of winning $10 before losing $90 is, then, 0.017.

<u>Step by step methods</u>. Step by step methods of solution of difference equations are in general solved by one of two ways, depending on the conditions of the problem. The simple equation

$$U_{x+2} + U_{x+1} + U_x = x$$

with the conditions that $U_o = 0$ and $U_1 = 1$ could be solved by steps by substituting the conditions into

$$U_{x+2} = U_{x+1} - U_x + x$$

then

$$U_2 = -1 -0 +0 = -1$$

$$U_3 = 1 - 1 + 0 = 0$$

$$U_4 = - 0 - (-1) + 0 = 1$$

$$\vdots$$

$$U_n = - U_{n-2} - U_{n-3} + n$$

from which either the specific value wanted could be obtained or the general solution deduced.

If, however, the conditions had been boundary conditions such as $U_o = 0$, and $U_n = 0$, the substitution would have given

$$U_2 = - U_1 + 0 + 0 = -U_1$$

$$U_3 = - U_2 - U_1 + 1$$

$$U_4 = - U_3 - U_2 + U_1$$

$$\vdots$$

$$U_n = - U_{n-2} - U_{n-3} + n = 0$$

The method of solution would have naturally led to a system of m simultaneous linear equations in m unknowns. The solution would then involve solving these equations, which in practice is usually accomplished by matrix methods.

Equations requiring such solutions often arise in conjunction with partial differential equations. The definition of the derivative of a function U,

$$(40) \quad \frac{du}{dx} = \lim_{h \to o} \frac{u(x+h) - u(x)}{h}$$

is compared to the difference definition,

$$(41) \quad U_x = U(x+h) - U(x)$$

returning here to the general constant difference of h instead

instead of 1. If (41) is divided by h and the limit taken as h approaches zero, it is equal to (40). This suggests that if we take h quite small in (41) the difference will approximate the derivative.

This is the type of approximation used in solving boundary problems in partial differential equations by numerical methods. The partial derivatives

$$\frac{\partial U}{\partial x} = \lim_{h \to o} \frac{U(x+h, y) - U(x, y)}{h}$$

$$\frac{\partial^2 U}{\partial x^2} = \lim_{h \to o} \frac{U(x+h, y) - 2U(x, y) + U(x-h, y)}{h^2}$$

are replaced by

$$\Delta_x U = \frac{1}{h} \quad U(x+h, y) - U(x, y)$$

and

$$\Delta_{xx} U = \frac{1}{h^2} \left[U(x+h, y) - 2U(x, y) + U(x-h, y) \right]$$

The parabolic type of partial differential equation, so called because of its similarity to the equation of a parabola, [(13) p. 512]

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y}$$

is approximated by

$$\Delta_{xx} U = \Delta_y U$$

or

$$\frac{1}{h^2} \left[U(x+h, y) - 2U(x, y) + U(x-h, y) \right] = \frac{1}{k} \left[U(x, y+k) - U(x, y) \right]$$

which may be simplified to

$$U(x+h, y) - (2 - \frac{h^2}{k}) U(x, y) + U(x-h, y) = \frac{h^2}{k} U(x, y+k)$$

the ratio , $\frac{h^2}{k}$ is taken as a constant (often as 1) so that the change in x is the same as the change in y. This equation can be thought of in terms of points on a graph or rectangular net, as indicated in Figure 1.

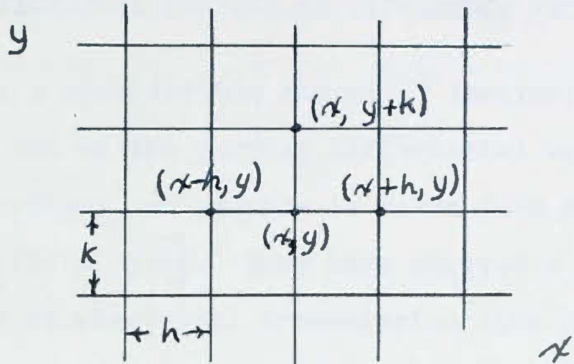

figure 1

If three of the four points are known, then the fourth can be determined. It can be seen that if this process were started, say at the bottom boundary, it could be continued until all points of the rectangular net or lattice were known. If the accuracy were not satisfactory, the spacings could be made smaller. The specific solution could be made general by fitting a general equation to the data. There are numerous methods of actually carrying out this process including substituting higher order difference equations as approximations to the differential equation, different mesh shapes, and special techniques for finding the point value approximations. Hildebrand [(6) p.295] describes one of these techniques called the relaxation method. It is essentially a process of trial and error averaging and weighing or giving more value in the average to the central point being computed. This tends to speed up the process of finding the best value.

CHAPTER 4

AN ELECTRICAL PROBLEM BY DIFFERENCE METHODS

Since a considerable amount of theoretical physics is required to set up the partial differential equations of most practical problems, an example is taken from Sokolnikoff and Redheffer [(13) p. 516]. They have derived a differential equation for an electrical transmission line problem and have then found a solution by Fourier series. It is interesting to compare this solution to one obtained by difference equation methods.

The problem is one of a submarine cable L miles in length. The voltage at the source is 12 volts, and, under steady state conditions, 6 volts at the receiving end. At time t = 0 the receiving end is grounded, but the source remains constant. The problem is to find the voltage at any time and distance from the source, (V(x, t), subject to the boundary conditions

$$V(0, t) = 12, \quad V(L, t) = 0, \text{ and } t \geq 0.$$

The derivation of the problem classifies this as one most closely approximated by the parabolic type partial differential equation

$$\frac{\partial U}{\partial t} = \alpha^2 \frac{\partial^2 U}{\partial x^2}$$

Here

$$U = \begin{cases} I, & \text{the current (amperes) or} \\ V, & \text{the voltage (volts),} \end{cases}$$

and

$$\alpha = (RC)^{-\frac{1}{2}}$$

where R is the resistance (ohms per mile), C is the capacitance (farads per mile). The solution is equivalent to that of the one-dimensional heat flow problem and is shown to have the solution in Fourier series

$$(42) \quad V(x, t) = 12 - 12\frac{x}{L} + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L \frac{6}{L} x_1 \sin\frac{n\pi x_1}{L}\,dx\right)\left(e^{-\left(\frac{1}{RC}\right)\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}\right]$$

The approximate difference equation is, as previously shown,

$$(43) \quad U(x=h, y) - (2 - \frac{h^2}{\alpha^2 k}\; U(x, y) + U(x-h, y) = \frac{h^2}{\alpha^2 k}\, U(x, y+k)$$

If we make a particularly convenient choice of spacing [(6) p. 283] such that the relationship is

$$h^2 = 2\,\alpha^2 k$$

the term $2 - \frac{h^2}{\alpha^2 k} = 0$.

Substituting the variable t for y and rearranging (43), it becomes

$$U(x, y+k) = \tfrac{1}{2}\left[U(x+h, y) + U(x-h, y)\right]$$

This relation can be taken to mean that within the prescribed boundaries, each point, x, is the average of the diagonally neighboring points below.

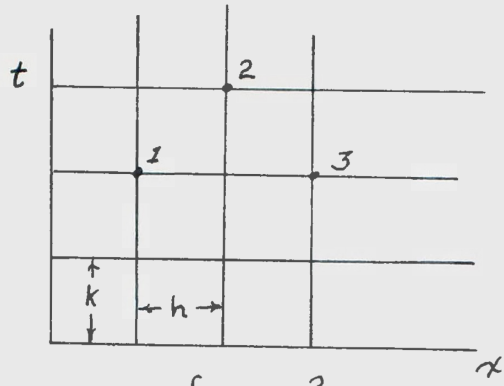

figure 2

From figure 2, point 2 is the average of points 1 and 3. This network represents a graph of the values of the function with values to be found at each intersection. In this problem each point is the voltage at x miles along the cable at time t seconds.

From the boundary conditions it is seen that when x = 0 the voltage is 12 volts at all times. The steady state conditions show that the voltage drop is linear from 12 to 6 when t = 0. Finally, when x = L, at the receiving end of the cable, and $t > 0$, the voltage is at zero. It is convenient to choose 6 division points in the x direction making the voltages at successive points along the x boundary 12, 11, 10, 9, 8, 7, and 6. The left boundary, when $x = 0$, is 12 at all points, and the right boundary is zero at all points except when t = 0. The computation is straightforward according to the formula

$$V_2 = \frac{1}{2}(V_1 + V_3)$$

from figure 2 except for one adjustment. There can be considered a discontinuity in the lower right hand corner where in thinking of moving along x the value is 6, but in moving down the right boundary, the value is zero except at the exact point. Hilderbrand [(6) p. 284] suggests a correction here of using an average of the two values for computing the rest of the values. Note the 3 in parenthesis which was used in computing the other terms.

The distance, L, is obviously divided into 6 divisions, giving

$$6h = L$$

$$h = \frac{L}{6}$$

Since the spacing was assumed such that

$$h^2 = 2\alpha^2 k$$

and

$$\alpha^2 = \frac{1}{RC}$$

then

$$k = \frac{L^2 RC}{72}$$

Some assumed values may make this more meaningful.

Let

L = 60 miles,

R = .1 ohms per mile,

and

C = 10 μflm = 1 x 10^{-5} farads per mile.

Then

h = 10 miles

and

k = 50 micro-seconds

These values can be substituted into (42) to give actual values at points of the approximate solution. For instance, the voltage at 30 miles out after 250 microseconds is given from figure 3 as 7.9 volts. This is the point where

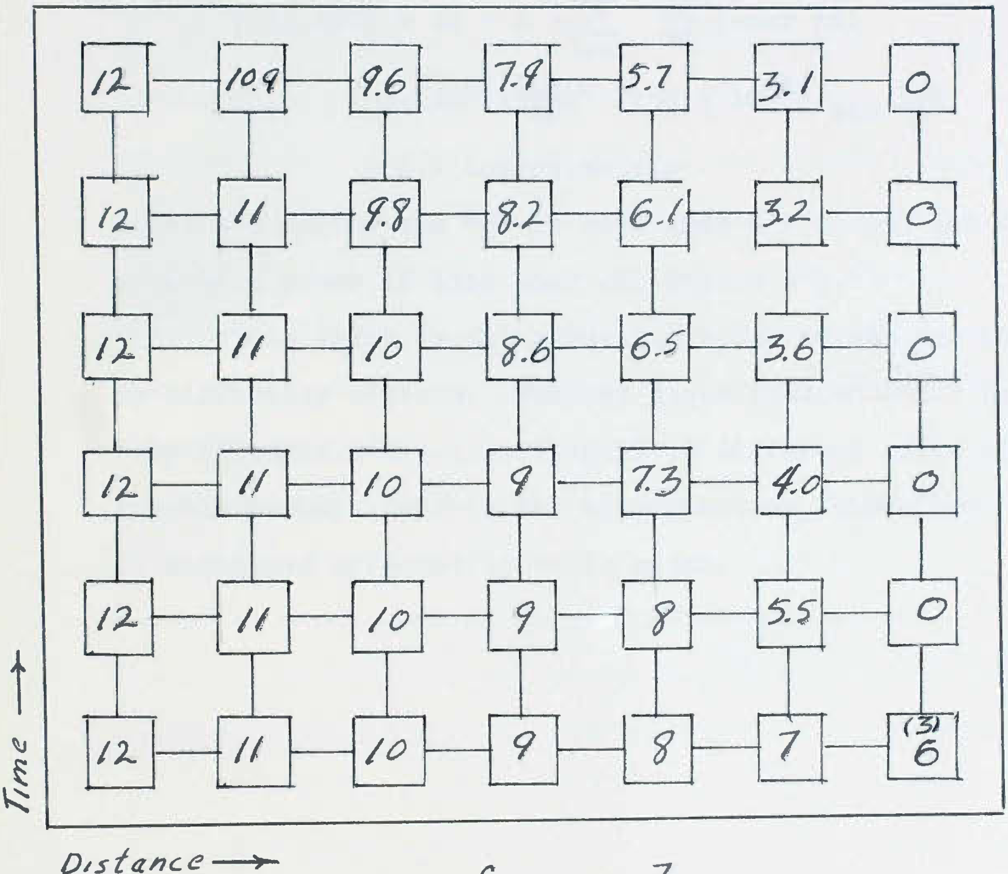

figure 3

Voltage at distances and times

$$k = \frac{250}{50} = 5 \text{ spaces}$$

and

$$h = \frac{30}{10} = 3 \text{ spaces}$$

The actual value here by calculation

$$V(30,250) = 12 - 6 + \sum_{n=1}^{\infty} \frac{12}{60} (-\cos n\pi)$$

$$e^{-(10^6)(\frac{1.\pi}{60})^2 (250 \times 10^{-6})} \sin \frac{1.\pi}{2}$$

$$= 6.1 \text{ approximately}$$

Since the factor sin $\frac{n\ x}{L}$ is zero when n = 2, and the factor of e to a power is less than .01 when n = 3.

This value is not agreeably close to the one obtained by difference methods. Further investigation would be necessary to explain the discrepancy. A different ratio of h and k would perhaps improve the approximation, since convergence is sometimes affected by their ratio.

CHAPTER V

SUMMARY

Difference equations have applications in solving problems in the social sciences, physical sciences, and in numerical solutions to differential equations. Solutions obtained are explicit or of a step by step approximation. Methods of solutions may always be found to first order linear equations. Linear equations of higher order than one with constant coefficients are solvable under most conditions. They are solved by finding a complementary solution through use of the auxiliary equation and adding it to a particular solution. Particular solutions may be found by various methods, but the method of undetermined coefficients is most useful. Various transform methods such as generating functions and Dirichlet series transforms may be used to solve linear equations with constant coefficients.

Approximate solutions to differential equations are often found by replacing the differential equations by an analogous or equivalent difference equation. The difference equation gives a method for successive calculations to obtain the approximate solution.

Chapters I and II give some of the background of difference equations and the calculus of finite differences. Although much of the theory was developed quite early, the applications are more recent. The analogies between difference equations and differential equations are extensive, and a study of one aids understanding of the other. The use of operator notation not only simplifies notation, but makes possible expanded solutions.

Chapter III illustrates the classification of equations according to order, degree, and whether homogeneous or nonhomogeneous. Solutions to first order linear equations are first found by summation or antidifferencing where the equations are simple. A formula is developed which is useful for finding the solution to first order linear equations that are either homogeneous or nonhomogeneous and which have variable or constant coefficients. Examples help clarify the procedure. Equations of order greater than one must have constant coefficients or be of special form to have a solution.

Homogeneous equations of nth order with constant coefficients are solved by finding the solution to an auxiliary algebraic equation. Methods for solving nonhomogeneous equations are then given with the particular solution being found by undetermined coefficients, generating functions, and Dirichlet series transforms.

The "classical ruin problem" illustrates one of the special methods of finding solutions, one called the method of particular solutions.

Step by step methods are shown by finding the approximate solution to a differential equation. If the equation is sufficiently well behaved, solutions may be found to any desired accuracy, but the work is often laborious. This is one of many numerical methods in use today.

A specific application of this procedure is given in finding an approximate solution to an electrical transmission line problem. The solution is compared to an exact one found by Fourier series.

The social scientist is making the greatest use of difference equations today as he develops more theory to translate social problems into mathematical terms. The successful worker in this field in the future will need a broad understanding of difference equations.

The applied scientists find uses for difference equations, but often in altered form or converted to formula work which requires little understanding of the background. The future use of difference equations appears to be in these two areas.

BIBLIOGRAPHY

(1) Cogan, E. J. and R. Z. Norman. Handbook of Calculus, Difference And Differential Equations. New York:

(2) Dartmouth College Writing Group. Multicomponent Methods. Vol. I of Modern Mathematical Methods and Models. 2 vols. Ann Arbor, Michigan: Mathematical Association of America, 1958.

(3) Feller, William. An Introduction to Probability Theory and Its Applications. Vol. I. New York: John Wiley and Sons, Inc., 1957.

(4) Fort, Tomlinson. "Linear Difference Equations and the Dirichlet Series Transformation," American Mathematical Monthly. 62:641-45, November, 1955.

(5) Goldberg, Samuel. Introduction To Difference Equations. New York: John Wiley and Sons, Inc., 1958.

(6) Hildebrand, F. B. Methods of Applied Mathematics, New York: Prentice-Hall, 1952.

(7) O'Brian, G. G., M. A. Hyman, and S. Kaplan, "A Study of the Numerical Solution of Partial Differential Equations," Journal of Mathematics and Physics, 29:223-251, May, 1951.

(8) Ott, E. R. "Difference Equations in Average Value Problems," American Mathematical Monthly, 51:570-75, 1944.

(9) Pipes, Louis A. Applied Mathematics For Engineers and Physicists. New York: McGraw-Hill Book Company, 1958.

(10) Reddick, Harry W. and Donald E. Kibbey. Differential Equations. New York: John Wiley and Sons, Inc., 1956.

(11) Richardson, C. H. An Introduction to The Calculus of Finite Differences. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1954.

(12) Scarborough, James B. Numerical Mathematical Analysis. Baltimore: The Johns Hopkins Press, 1955.

(13) Sokolnikoff, I. S. and R. M. Redheffer. Mathematics of Physics and Modern Engineering. New York: McGraw Hill Company, 1958.