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Transformations of Lines and Conics in The Z-Plane

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TRANSFORMATIONS OF LINES AND CONICS IN THE z -PLANE

being

A thesis presented to the Graduate Faculty
of the Fort Hays Kansas State College in
partial fulfillment of the requirements for
the Degree of Master of Science

by

Charles R. Deeter, B.S.

Fort Hays Kansas State College

Date

Jan. 12, 1956

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TRANSFORMATIONS OF LINES AND CONICS IN THE z-PLANE

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TRANSFORMATIONS OF LINES AND CONICS IN THE z -PLANE

The purpose of the problem was to investigate the behavior of curves under some simple complex transformations. The transformations used were limited to $w = z^2$, $w = z^{\frac{1}{2}}$, and $w = \frac{1}{z}$. The curves considered were limited to straight lines and conic sections. However, the general cases of the conics were usually too complicated to be dealt with in the thesis. Therefore, most of the conics considered were special cases which were simpler and from which some indication of the behavior of more general cases might be found.

Some interesting special cases of the more complicated transformations were treated briefly, as were practical applications of complex transformations. Sketches were included showing the results of the transformations in graphic form.

It was noted that, in general, subjection of a curve to a transformation complicated that curve. Cases in which the curve was simplified were less numerous, but usually had greater chance of application.

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CHAPTER I

INTRODUCTION

Early in his mathematical studies, the student encounters the problem of expressing the square-root of a negative number. Such a value is defined as an imaginary number. Then, for the sake of simplification, the term $\sqrt{-1}$ is called i .

Later, numbers involving the sum of a real number and an imaginary number are met. These numbers may be written generally in the form $a + ib$, where a and b are real numbers and i is $\sqrt{-1}$. Expressions of this form are called complex numbers, and as such include all of the real numbers and all of the imaginary numbers. The rules for performing the fundamental operations of addition, subtraction, multiplication, division, and extraction of roots as they apply to complex numbers are established in algebra and are used extensively in applications of the quadratic formula.

Complex variables. If z is defined as a complex number such that $z = x + iy$, and if x and y are real variables, then z is called a complex variable. The real numbers x and y are known as the real part and the coefficient of the imaginary part of z , respectively. In some cases it is convenient to indicate these real and imaginary components by the notation

$$R(z) = x, I(z) = y.$$

The common rules for operations with complex numbers apply to the complex variable z .

All complex numbers can be represented geometrically by means of the Argand diagram. This is a set of rectangular coordinate axes in a plane. Each complex number $x + iy$ is represented by a point whose rectangular Cartesian coordinates are (x, y) . This coordinate system in a plane is also referred to simply as the complex plane or the z -plane.

The complex number $x - iy$ is commonly noted as \bar{z} and is called the conjugate of z .

At times it becomes necessary to think of the complex number z as a vector from the origin of the coordinate system to the point (x, y) .

The absolute value or modulus of z is defined as

$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

From a trigonometric standpoint, this value is the length of the vector which represents z . Consequently, $|z_1 - z_2|$ is the distance between the points z_1 and z_2 , since

$$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The polar form of z is obtained by substituting

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Thus,

$$z = r(\cos \theta + i \sin \theta).$$

All points in the plane may be represented in polar coordinates without using negative values of r ; so r is taken to be greater than, or equal to zero. Since

$$r = \sqrt{x^2 + y^2},$$

then,

$$r = |z|.$$

The angle θ is called the argument or amplitude of z and is commonly expressed as

$$\theta = \arg z.$$

The value of θ may be obtained from the relationship

$$\tan \theta = y/x.$$

If e^z is defined as

$$e^z = e^x(\cos y + i \sin y),^1$$

the polar form can be written in the more compact exponential form,

$$z = r e^{i\theta}.$$

If another complex variable, $w = u + iv$, is related to z so that in some part of the z -plane a definite value or set of values of w corresponds to each value of z , then w is a function of the complex variable z . Thus,

$$w = f(z).$$

Two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, are equal if, and only if, $x_1 = x_2$ and $y_1 = y_2$. Therefore, a function

$$w = f(z)$$

may be represented by

$$u = u(x,y), v = v(x,y),$$

where $u(x,y)$ and $v(x,y)$ indicate functional relationships.

If $u(x,y)$ and $v(x,y)$, together with their partial derivatives of the first order, are continuous and single valued and satisfy the

¹Ruel V. Churchill, Introduction to Complex Variables and Applications (New York: McGraw-Hill Book Company, Inc., 1948), pp. 37-38.

Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

at some point, then the function $f(z) = u + iv$ is said to be analytic at that point. The study of analytic functions in complex variables is of considerable importance.

Real functions of real variables, $y = f(x)$, may be exhibited graphically as curves in the xy -plane. When the variables are complex, such graphical representation is more complicated, since each of the complex variables w and z is represented by a point in the z -plane. It is generally simpler to use separate planes for the two variables. Thus, corresponding to each point (x,y) in the z -plane for which $f(x+iy)$ is defined, there will be a point (u,v) in the w -plane where $w = u + iv$. This correspondence between points in the two planes is called a mapping or transformation of points in the z -plane into points in the w -plane. The point in the w -plane which corresponds to a point (x,y) in the z -plane is called the image of the point (x,y) . This correspondence of points may be extended to curves, and the terms mapping and transformation are then applied in the sense that a curve in the z -plane is mapped or transformed into another curve in the w -plane. The curve in the w -plane is then the image of the curve in the z -plane. It is convenient at times to think of the mapping as occurring in one plane, even though two separate planes are used to represent w and z . This permits the use of such terms as translation and rotation. In the consideration of functions of complex variables as transformations it is possible to use the relationships

$$u = u(x,y), v = v(x,y)$$

as a real transformation with real variables, if the w -plane and the z -plane are looked upon as ordinary rectangular Cartesian coordinates. This method is commonly used in dealing with curves of higher degree because the transformed equations encountered may be recognized more readily when they are expressed in the way that is used in Cartesian geometry.

Notation. Several types of notation are found in texts on complex analysis. The following is a summary of the notation used in this thesis.

The complex variables:

$$z = x + iy = r(\cos\theta + i \sin\theta) = r e^{i\theta},$$

$$w = u + iv = \rho(\cos\phi + i \sin\phi) = \rho e^{i\phi},$$

where $x, y, r, \theta, u, v, \rho,$ and ϕ are real variables.

The complex constants:

$$w_0, w_1, w_2, z_0, z_1, \text{ and } z_2.$$

The real constants:

$$a, b, c, d, \text{ and the components of the complex constants, } u_0, v_0, x_0, y_0, \text{ etc.}$$

Any deviation from this notation or the introduction of supplementary notation is explained at the time of its occurrence.

The conics in terms of complex variables. A circle is the locus of all points equidistant from a fixed point. This suggests the representation

$$|z - z_1| = a$$

for the circle; that is, the point z moves so that the distance between z and z_1 is a constant, a . In other words, z describes a circle with center at z_1 and radius a .

Similarly, the equation of the ellipse is

$$|z - z_1| + |z - z_2| = 2a, \quad (2a > |z_1 - z_2|).$$

Its foci are z_1 and z_2 and the length of the major axis is $2a$.

The equation

$$|z - z_1| - |z - z_2| = \pm 2a, \quad (0 < 2a < |z_1 - z_2|)$$

represents a hyperbola, with the positive or negative sign applying for the branch nearer the focus z_2 or for the branch nearer the focus z_1 , respectively.

In Cartesian coordinates the equation of the ovals of Cassini is

$$(x^2 + y^2 + a^2)^2 - 4a^2x^2 = c^4.$$

This curve is the locus of a point which moves so that the product of its undirected distances from two fixed points is a constant. This relationship suggests the equation

$$|z - z_1| |z - z_2| = a, \quad (a > 0)$$

in complex variables.

These equations usually lead to difficulties under transformations, but in a few cases they greatly simplify the work involved.

The problem and its limitations. The purpose of the problem was to investigate the behavior of curves under some simple complex transformations.

The transformations used were limited to

$$w = z^2,$$

$$w = z^{\frac{1}{2}},$$

and

$$w = 1/z.$$

These are basic transformations, but in many cases the job of analyzing a complicated transformation may be made easier by expressing it as a sequence of successive transformations of these basic forms.

The curves considered were limited to straight lines and conic sections. However, the general cases of the conics were usually too complicated to be dealt with in this thesis. Therefore, most of the conics considered were special cases which were simpler and from which some indication of the behavior of more general cases might be found.

Some interesting special cases of the more complicated transformations are treated briefly in Chapter V. In Chapter VI the practical applications of complex transformations are discussed.

Within the discussions of the various curves and their transforms, sketches appear showing the appearances of the curve in the z -plane and of its image in the w -plane. The equations of the curves in each plane are given below the sketches. Pertinent points, such as the intercepts on the axes, are labeled only when that information is necessary to clarify some aspect of the transformation.

Appendix A contains the proofs of certain properties of each transformation, called here rotational properties or rotational characteristics. Appendix B consists of an outline of the procedure used in analyzing cubic and quartic equations in order to sketch the curves.

CHAPTER II

THE TRANSFORMATION $w = z^2$

The transformation $w = z^2$ is easily described in terms of polar coordinates. When $z = r e^{i\theta}$ and $w = \rho e^{i\phi}$, it becomes

$$\rho e^{i\phi} = r^2 e^{2i\theta}$$

and when the real and imaginary components are equated,

$$\rho = r^2, \phi = 2\theta.$$

That is, the point (r_1, θ_1) in the z -plane is transformed into the point in the w -plane whose polar coordinates are $\rho = r_1^2$ and $\phi = 2\theta_1$.

Geometrically then, the length of the radius vector of the point in the w -plane is equal to the square of the length of the radius vector of the point in the z -plane, and its argument is twice the argument of the point in the z -plane.

In rectangular coordinates, the transformation is

$$u + iv = x^2 - y^2 + 2ixy$$

or

$$u = x^2 - y^2, v = 2xy.$$

A rotational property of this transformation can be stated as follows:¹

If C_2 is a curve in the z -plane which is obtained by rotating another curve C_1 through an angle ψ about $z=0$, then K_2 , the image of C_2 , can be obtained by rotating K_1 , the image of C_1 , through an angle 2ψ about $w=0$.

Use of this property allows the determination of the transforms of many curves, if, and when, the transform of one such curve is known.

¹The proof of this property is given in Appendix A, p. 62.

The straight line. The transformation of the line

$$ax + by = c,$$

can be made by noting that

$$y = \frac{c-ax}{b}.$$

When this value is substituted into the expressions for $u(x,y)$ and $v(x,y)$

$$u = x^2 - y^2 = \frac{(b^2-a^2)x^2 + c(2ax-c)}{b^2},$$

$$v = 2xy = \frac{2x(c-ax)}{b}.$$

These equations essentially are the parametric equations of the transformed curve, the parameter being x . If these equations are rewritten as

$$(a^2-b^2)x^2 - 2acx + c^2 + b^2u = 0,$$

and

$$2ax^2 - 2cx + bv = 0,$$

Sylvester's Method² may be used to eliminate the parameter.

Thus, the equation of the image of $ax + by = c$ becomes

$$4a^2b^2u^2 - 4ab(a^2-b^2)uv + (a^2-b^2)^2v^2 + 4c^2(a^2-b^2)u + 8abc^2v - 4c^4 = 0.$$

When this conic is compared with the general conic

$$Au^2 + 2Huv + Bv^2 + 2Gu + 2Fv + C = 0,$$

the value³

$$H^2 - AB = 0.$$

This is the condition which indicates that the conic is a parabola.

²Nelson Bush Conkwright, Introduction to the Theory of Equations (Boston: Ginn and Company, 1941), pp. 162-164.

³Alan D. Campbell, Advanced Analytic Geometry (New York: John Wiley & Sons, Inc., 1938), p. 8.

Further, when the discriminant⁴ of this parabola is evaluated, it is seen that

$$\Gamma = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = -4c^4(a^2+b^2)^4,$$

and $\Gamma = 0$ only when $c = 0$. This shows that the image of $ax + by = c$ will be degenerate only when $c = 0$, or when the line passes through the origin.

In summary, under the transformation $w = z^2$, all straight lines are transformed into parabolas as is shown in Figure 1 and in Figure 2.

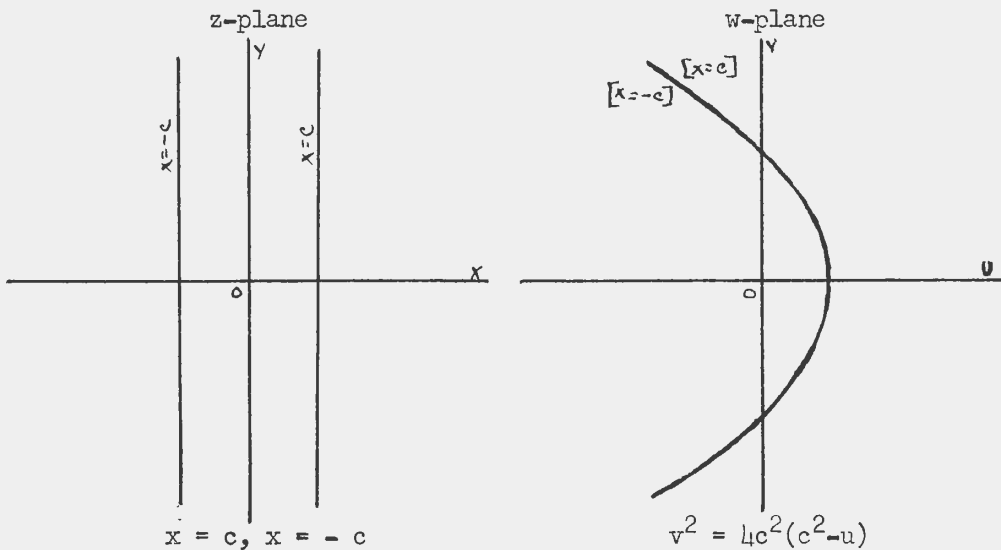


Figure 1. Transformation of $x = c$ and $x = -c$ under $w = z^2$.

⁴Ibid., p. 28.

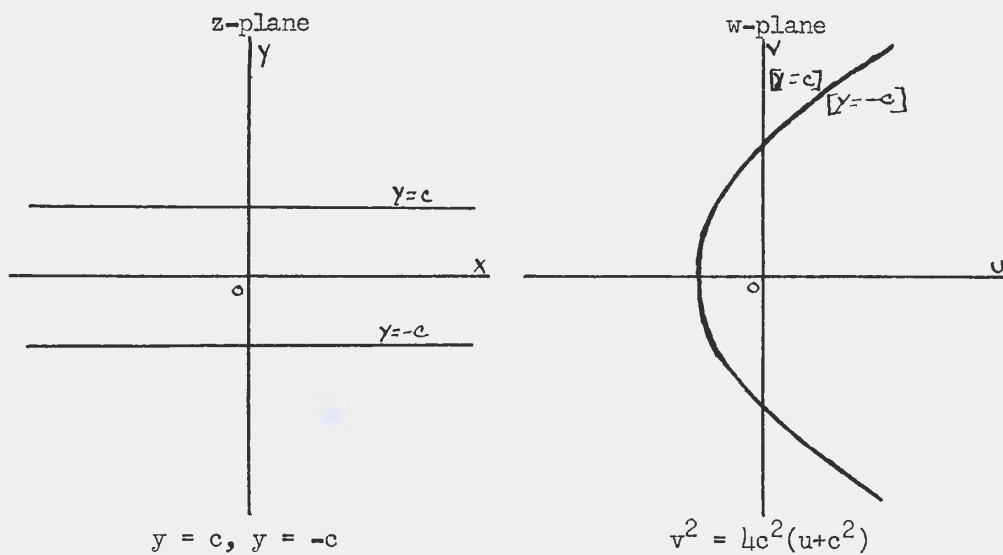


Figure 2. Transformation of $y = c$ and $y = -c$ under $w = z^2$.

Figure 3 shows the case in which $c = 0$ and the parabola degenerates into a pair of coincident straight lines.

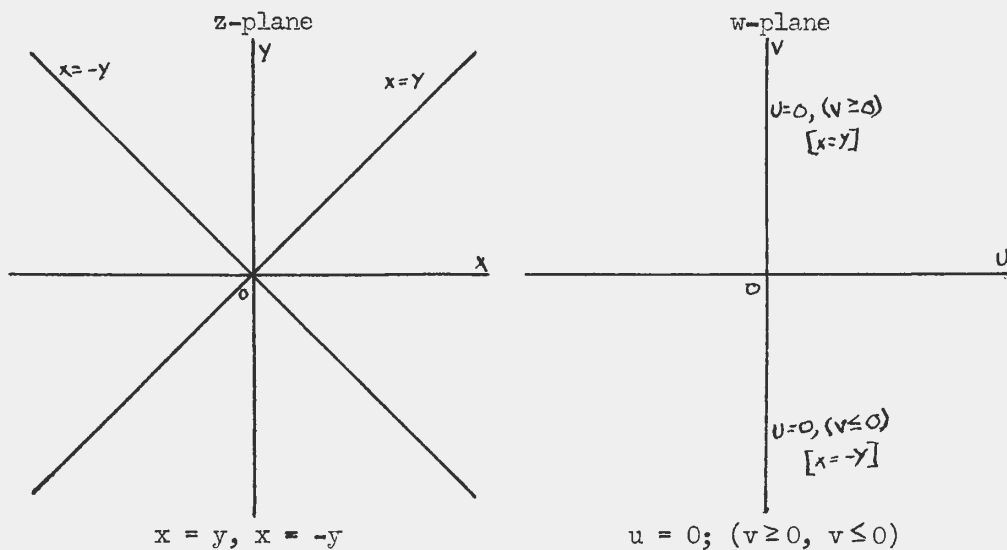


Figure 3. Transformation of $x = y$ and $x = -y$ under $w = z^2$.

The circle. From the equation

$$x^2 + y^2 = a^2$$

it is seen that

$$x^2 = a^2 - y^2,$$

or

$$x = \pm\sqrt{a^2 - y^2}.$$

Substitution into the rectangular coordinate form of the transformation gives

$$u = x^2 - y^2 = a^2 - 2y^2,$$

$$v = 2xy = \pm 2y\sqrt{a^2 - y^2}.$$

Elimination of the parameter y results in the equation

$$u^2 + v^2 = a^4.$$

Seemingly, this equation simply represents a circle about $w = 0$, whose radius is the square of the radius of the circle in the z -plane.

However, upon examination of the parametric equations of the circle,

$$x = a \cos \psi, \quad y = a \sin \psi,$$

it is seen that the transformation results in

$$u = a^2 \cos^2 \psi - a^2 \sin^2 \psi, \quad v = 2a^2 \cos \psi \sin \psi,$$

or

$$u = a^2 \cos 2\psi, \quad v = a^2 \sin 2\psi.$$

To be sure, the curve is a circle with radius a^2 , but the parameter 2ψ indicates that, while the circle in the z -plane is being generated once, the circle in the w -plane is generated twice. This conclusion follows from the condition that, to generate the circle $x^2 + y^2 = a^2$, ψ must take all values in the interval

$$0 \leq \psi \leq 2\pi,$$

and hence, 2ψ must take all the values in the interval

$$0 \leq 2\psi \leq 4\pi.$$

Figure 4 shows the physical appearance of this transformation.

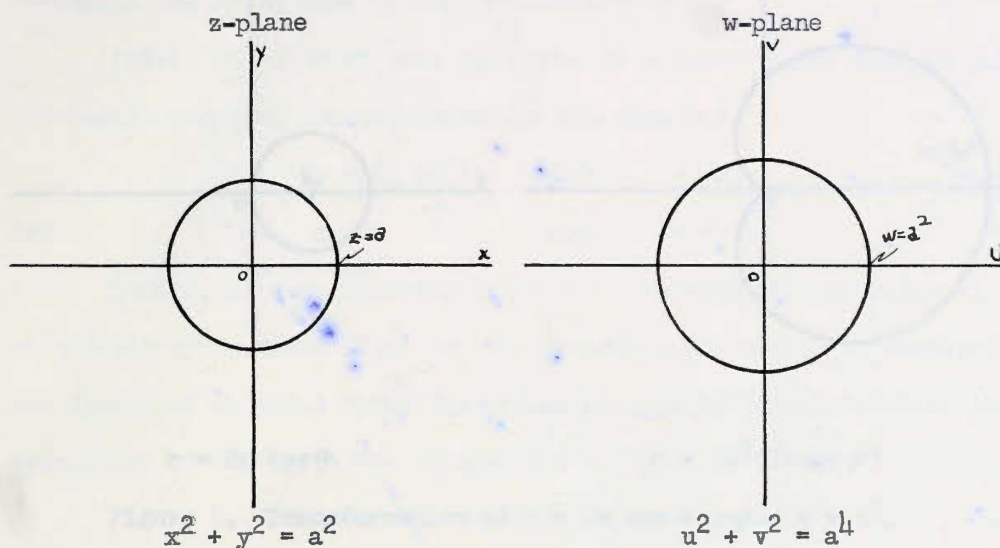


Figure 4. Transformation of $x^2 + y^2 = a^2$ under $w = z^2$.

An interesting circle which illustrates the use of the polar form of the transformation is

$$x^2 + y^2 - 2ax = 0.$$

This is a circle whose center is on the x-axis at $(a, 0)$ and which passes through the origin. In polar coordinates the equation is

$$r = 2a \cos \theta.$$

When this equation is transformed

$$\rho = r^2 = 4a^2 \cos^2 \theta = 2a^2(1 + \cos 2\theta),$$

but $\phi = 2\theta$; so the transformed equation becomes

$$\rho = 2a^2(1 + \cos \phi).$$

This curve is the cardioid shown in Figure 5.

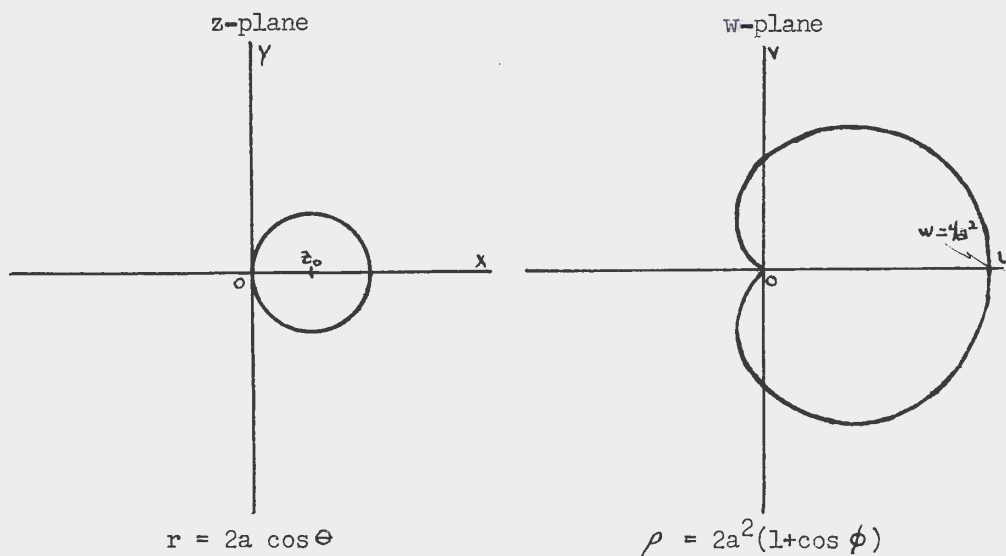


Figure 5. Transformation of $r = 2a \cos \theta$ under $w = z^2$.

Algebraically the general circle presents a complicated problem when the transformation is applied. However, in the Dictionary of Conformal Representations⁵, it is shown that, in general, a circle is transformed into a limaçon with the cardioid and circle as limiting cases.

The parabola. When the parabola

$$y^2 = 2px$$

is transformed under $w = z^2$,

$$u = x^2 - 2px, \quad v = \pm 2x\sqrt{2px}$$

since

$$y = \pm\sqrt{2px}.$$

Elimination of the parameter x reduces these equations to

$$v^4 - 64p^4v^2 - 48p^2uv^2 - 64p^2u^3 = 0.$$

⁵H. Kober, Dictionary of Conformal Representations (New York: Dover Publications, Inc., 1952), p. 38.

A procedure established in Cartesian geometry⁶ is followed to determine the appearance of the transformed curve.

First, it is noted that the curve is symmetric with respect to the u -axis and that its intercepts on the axes are

$$v = 0, \pm 8p^2 \quad \text{when} \quad u = 0$$

and $u = 0 \quad \text{when} \quad v = 0.$

Second, if p is considered greater than zero, the equation is solved for v^2 in terms of u , and the resulting expression is examined for limits of extent. Thus, the following conditions which affect the appearance of the curve are determined:

$$v \text{ has four real values, when } -p^2 \leq u \leq 0;$$

$$v \text{ has two real values, when } 0 < p < \infty;$$

and $v \text{ has no real value, when } -\infty < u < -p^2.$

Then, if the equation of the transformed curve is solved simultaneously with the equation of a line $v = ku$, and k is allowed to approach zero, it is seen that the curve forms a cusp at the origin.

This combined information is sufficient to allow a sketch of the curve to be made as in Figure 6. The images of parabolas, where p is less than zero, along with the rotated forms of the parabola $y^2 = 2px$, may be determined by use of the rotational property of this transformation as stated on page 8.

⁶An outline of that procedure is presented in Appendix B, p. 68.

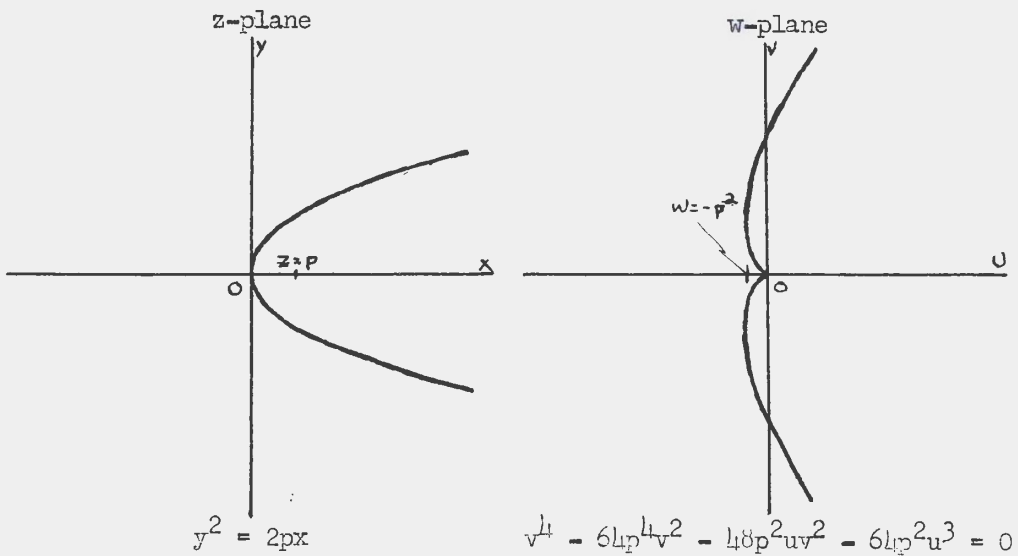


Figure 6. Transformation of $y^2 = 2px$ under $w = z^2$.

The ellipse. If the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is transformed under $w = z^2$, then

$$u = \frac{a^2b^2 - (a^2+b^2)y^2}{b^2},$$

$$v = \frac{\pm 2ay\sqrt{b^2 - y^2}}{b^2}.$$

After elimination of the parameter, these equations become

$$\frac{v^2}{a^2b^2} + \frac{\left[u - \frac{(a^2-b^2)}{2} \right]^2}{\frac{(a^2+b^2)^2}{4}} = 1.$$

This equation is obviously the equation of the ellipse which is illustrated in Figure 7.

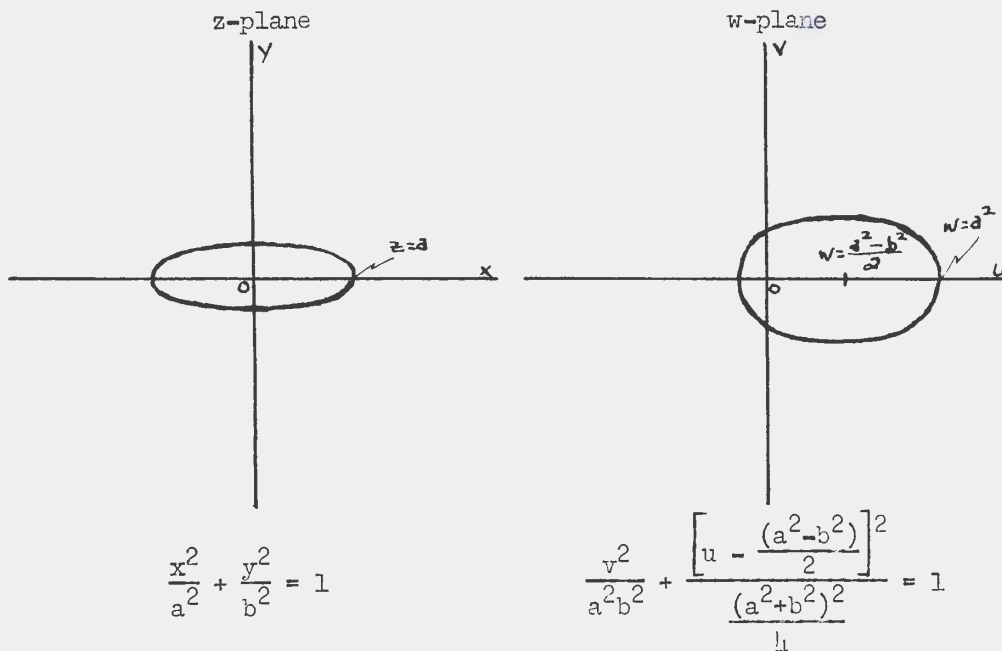


Figure 7. Transformation of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ under $w = z^2$.

The hyperbola. In a manner similar to that used in transforming the ellipse, it can be shown that the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is transformed into the curve

$$\left[u - \frac{(a^2+b^2)}{2} \right]^2 - \frac{(a^2-b^2)^2v^2}{4a^2b^2} = \frac{(a^2-b^2)^2}{4}.$$

Now, when $a \neq b$, it is permissible to simplify this further to

$$\frac{\left[u - \frac{(a^2+b^2)}{2} \right]^2}{\frac{(a^2-b^2)^2}{4}} - \frac{v^2}{a^2b^2} = 1.$$

This equation is obviously the equation of the hyperbola shown in Figure 8.

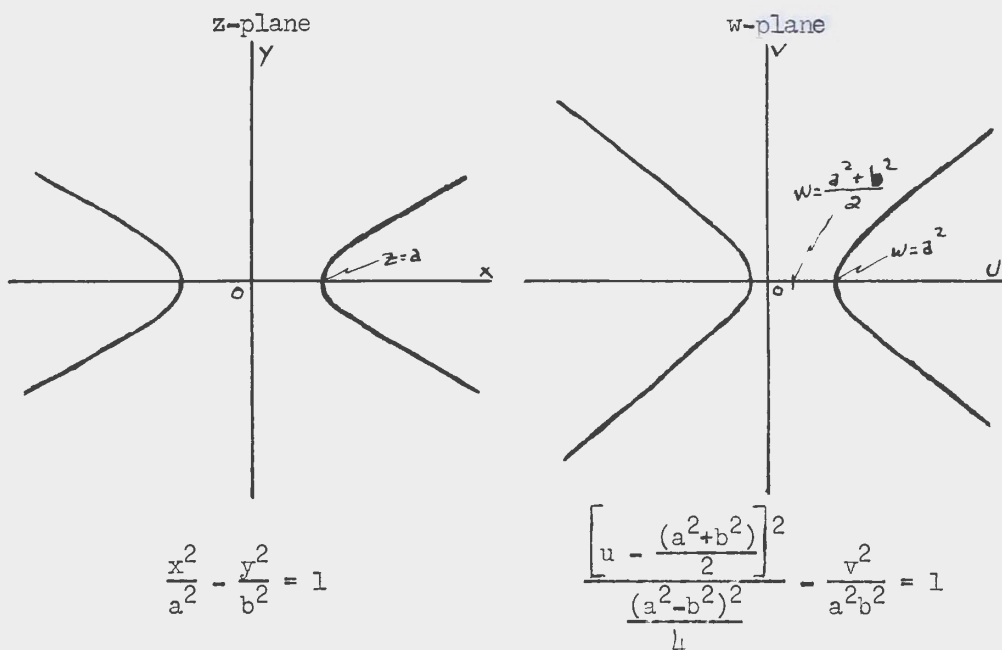


Figure 8. Transformation of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ under $w = z^2$.

When $a = b$, the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes

$$x^2 - y^2 = a^2,$$

the rectangular hyperbola, and its image, as seen in Figure 9, is the straight line $u = a^2$.

If the rectangular hyperbola $x^2 - y^2 = a^2$ is rotated through an angle $\Psi = \pi/4$, it becomes the rectangular hyperbola

$$2xy = a^2.$$

Then by the rotational characteristics of the transformation $w = z^2$, the image of this hyperbola will be the line $u = a^2$, after it has been rotated through an angle $\lambda = 2\Psi = \pi/2$. That curve (Figure 10) is the straight line $v = a^2$.

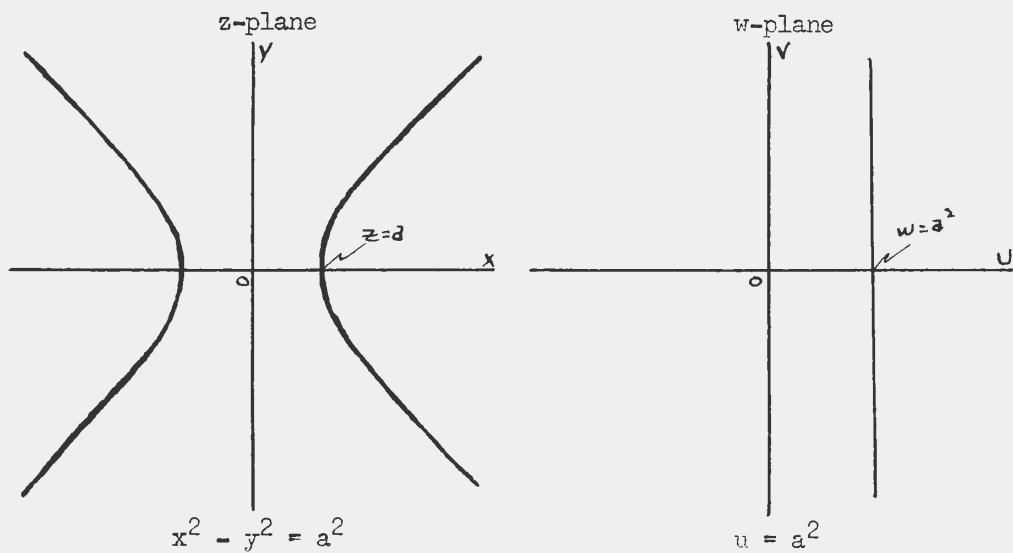


Figure 9. Transformation of $x^2 - y^2 = a^2$ under $w = z^2$.

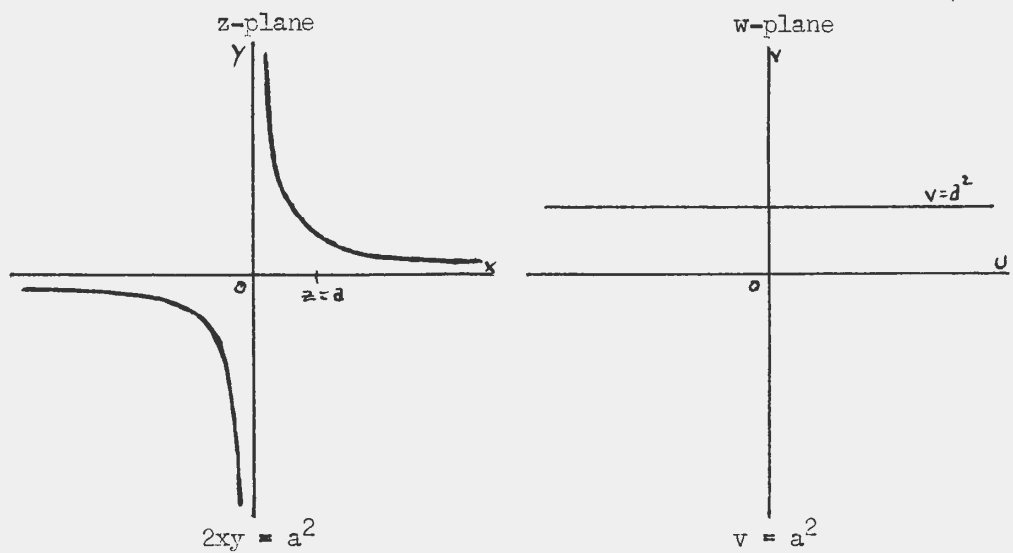


Figure 10. Transformation of $2xy = a^2$ under $w = z^2$.

Some other interesting curves. Some special curves have interesting properties under the transformation $w = z^2$.

For instance, the littuus,

$$r^2\theta = a^2,$$

is transformed into

$$\rho = r^2 = \frac{a^2}{\theta}.$$

But $\phi = 2\theta$, so

$$\rho\phi = 2a^2,$$

which is a hyperbolic spiral as seen in Figure 11.

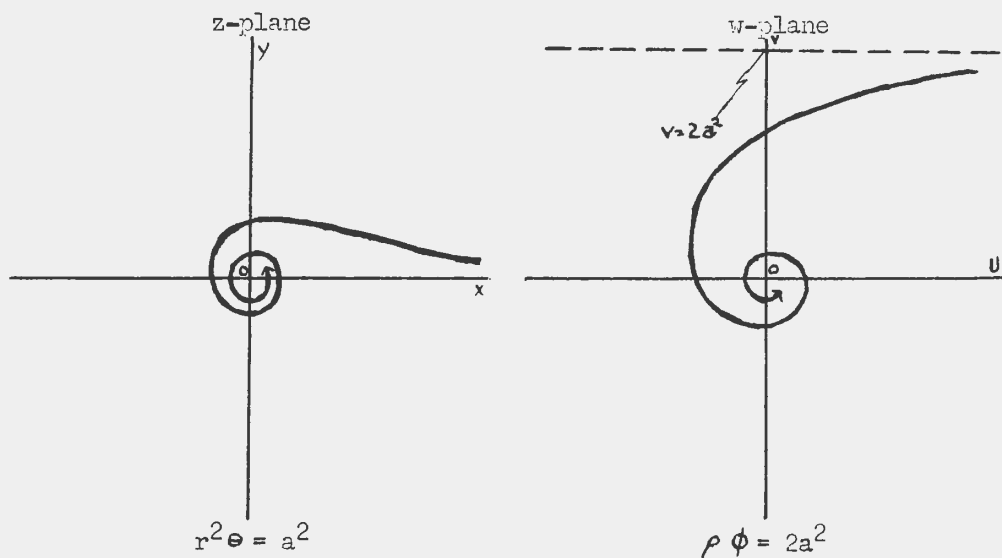


Figure 11. Transformation of $r^2\theta = a^2$ under $w = z^2$.

Figure 12 illustrates the transformation of the logarithmic spiral

$$r = e^{a\theta}.$$

There

$$\rho = r^2 = e^{2a\theta},$$

and

$$\phi = 2\theta.$$

Hence,

$$\rho = e^{a\phi}$$

which is another logarithmic spiral exactly the same as the original,

but transferred from the z -plane to the w -plane.

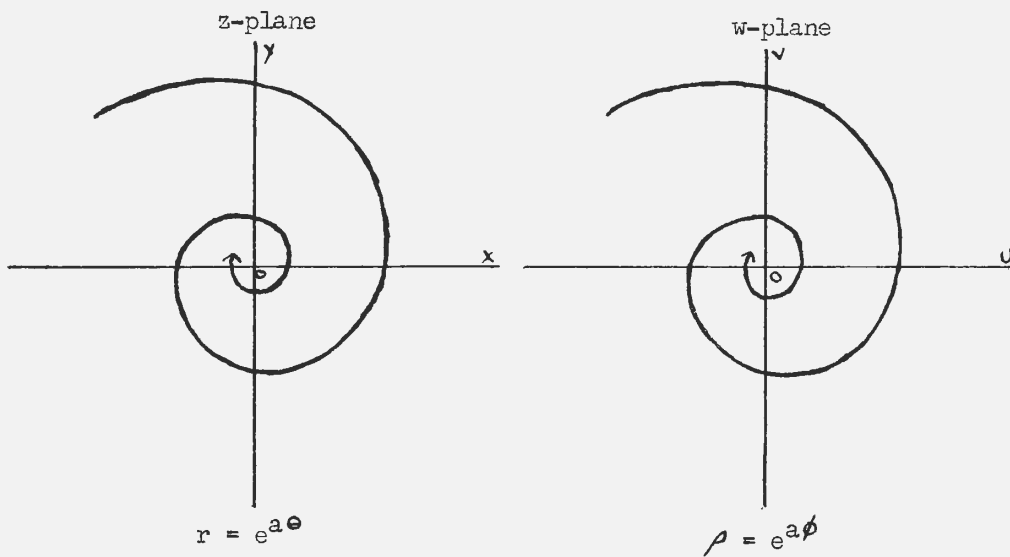


Figure 12. Transformation of $r = e^{a\theta}$ under $w = z^2$.

THE TRANSFORMATION $w = z^{\frac{1}{2}}$

The function $w = z^{\frac{1}{2}}$ is a double-valued function; that is, w assumes two values for each assigned value of z .

The transformation in polar coordinates is

$$\rho \exp(i\phi) = \sqrt{r} \exp\left[\frac{i(\theta + 2\pi k)}{2}\right]$$

where

$$k = 0, 1$$

and \sqrt{r} is the principle square-root of r .¹

Geometrically, this transformation changes the length of the radius vector of z by the factor $1/\sqrt{r}$ and rotates it through an angle $-\theta/2$ to obtain the corresponding point in the w -plane.

The transformation $w = z^{\frac{1}{2}}$ can be thought of as the inverse of the transformation $w = z^2$. Comparison of these two transformations in this way indicates that $w = z^{\frac{1}{2}}$ might be expressed as

$$x = u^2 - v^2, \quad y = 2uv.$$

This form can be shown to be correct, and it simplifies the algebraic computations involved in determining the equation of the image of a curve in the z -plane.

In some cases it is necessary to return to the polar form in order to avoid confusion because of the double-valued property of the transformation.

¹The notation $\exp \mu$ is equivalent to e^μ .

A useful rotational characteristic is connected with this transformation, also. It may be stated:²

If C_2 is a curve in the z -plane which is obtained by rotating another curve C_1 through an angle Ψ about $z=0$, then K_2 , the image of C_2 , can be obtained by rotating K_1 , the image of C_1 , through an angle $\Psi/2$ about $w=0$.

The straight line. When the general line

$$ax + by = c$$

is transformed according to the function $w = z^{\frac{1}{2}}$, it becomes

$$au^2 + 2buv - av^2 - c = 0.$$

Comparison of this equation and the equation of the general conic and the evaluation of³

$$H^2 - AB = b^2 + a^2 > 0,$$

and

$$\Gamma = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = -c(a^2 + b^2),$$

shows that the curve is a hyperbola which becomes degenerate only when $c = 0$.

Cases of degeneracy stem from lines which pass through the origin. These cases indicate the need for caution arising from the use of a double-valued function as a transformation.

In transforming the line $x = 0$ by the transformation $w = z^{\frac{1}{2}}$ in the rectangular form, the degenerate hyperbola

$$u^2 - v^2 = 0$$

²The proof of this property is given in Appendix A, p. 64.

³Campbell, op. cit., pp. 8, 28.

is obtained. This degenerate curve is the same as the two lines

$$u = v \quad \text{and} \quad u = -v.$$

However, if the half-line

$$x = 0, \quad 0 \leq y < \infty$$

is considered in its polar form

$$\theta = \pi/2, \quad r \geq 0,$$

it is seen that, when it is transformed, it becomes

$$\phi = \frac{\theta + 2k\pi}{2} = \frac{\pi}{4} + k\pi, \quad k = 0, 1;$$

$$\rho = r^2 \geq 0.$$

This is the complete line $\phi = \frac{\pi}{4}$ or $u = v$.

Similarly, the half-line

$$x = 0, \quad -\infty < y \leq 0$$

is transformed into the complete line $u = -v$.

These straight lines are shown in Figure 13.

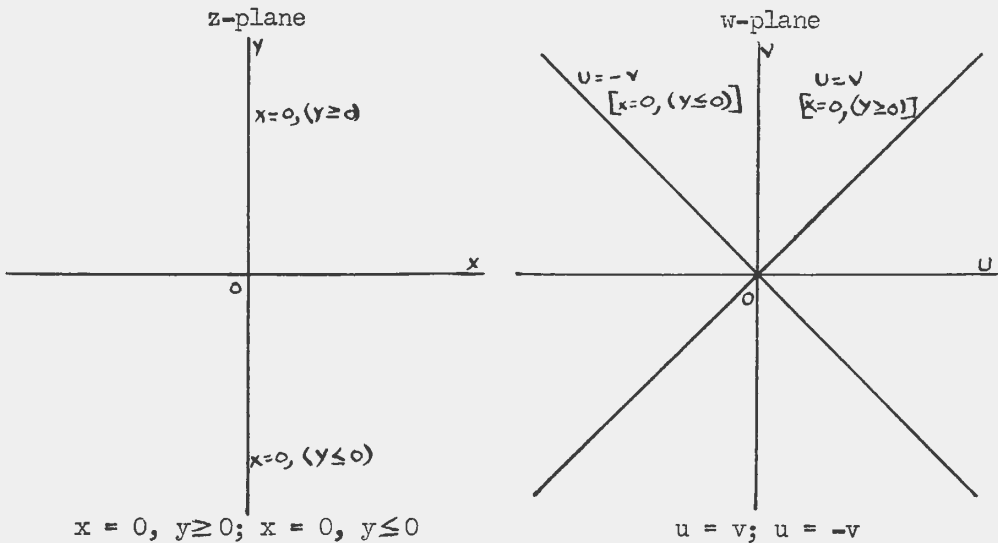


Figure 13. Transformation of $x = 0$ under $w = z^{\frac{1}{2}}$.

This same reasoning applies to other lines through the origin. Although an analysis of this sort is necessary to determine the point-wise correspondence of curves in the z -plane and in the w -plane, the rectangular representation of the transformation will suffice to establish the correspondence of entire curves in each of the planes.

Thus, the application of the transformation to the lines $x = c$ and $y = c$ obviously results in the rectangular hyperbolas

$$u^2 - v^2 = c \quad \text{and} \quad 2uv = c$$

as illustrated in Figure 14.

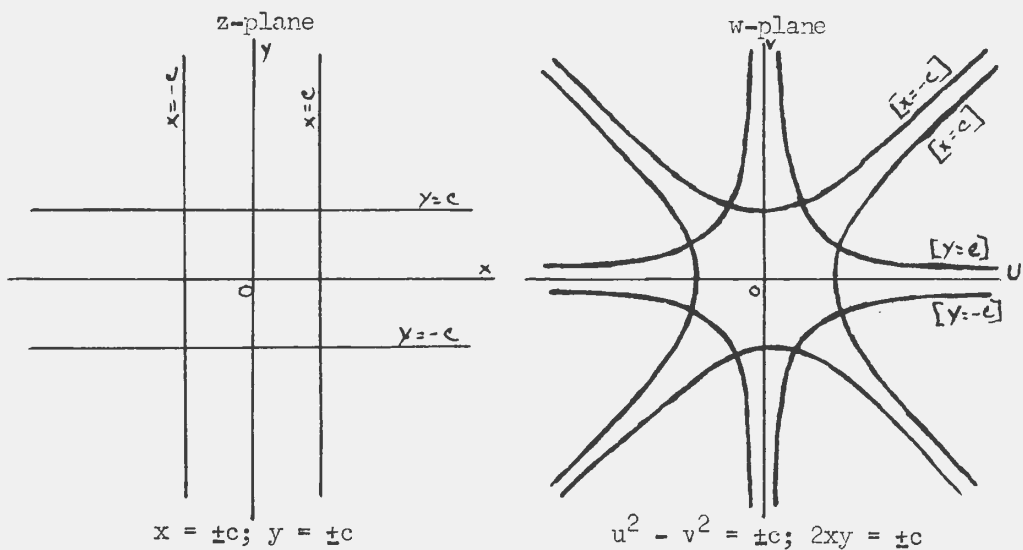


Figure 14. Transformation of $x = c$ and $y = c$ under $w = z^{\frac{1}{2}}$.

The circle. The equation of the circle

$$x^2 + y^2 = a^2$$

in polar coordinates is

$$r = a, 0 \leq \theta < 2\pi.$$

Transformed, this circle becomes

$$\rho = \sqrt{a}, \quad \pi k \leq \phi < (k+1)\pi; \quad (k=0,1).$$

The limitations on ϕ may also be stated

$$0 \leq \phi < \pi$$

and

$$\pi \leq \phi < 2\pi.$$

They are interpreted to indicate that each point on the circle $r = a$ is transformed into two points on the circle $\rho = \sqrt{a}$. Thus, while the circle $r = a$ is generated once by a moving point, the upper and lower halves of the circle $\rho = \sqrt{a}$ are generated simultaneously.

The general circle offers an opportunity for demonstrating the use of the complex variable notation. In this form the equation of the general circle

$$|z - z_0| = a$$

is transformed into

$$|w^2 - z_0| = a.$$

This equation may then be factored and written

$$|w - w_0| |w + w_0| = a, \quad \text{where } w_0 = \pm\sqrt{z_0}.$$

In Chapter I such an equation was shown to be the equation of the ovals of Cassini. The Dictionary of Conformal Representations⁴ distinguishes three separate cases and provides sketches drawn for circles whose centers lie on the x-axis.

Figure 15 shows the first case, in which $a < |z_0|$; that is, the origin is outside the circle.

⁴See pp. 38-39.

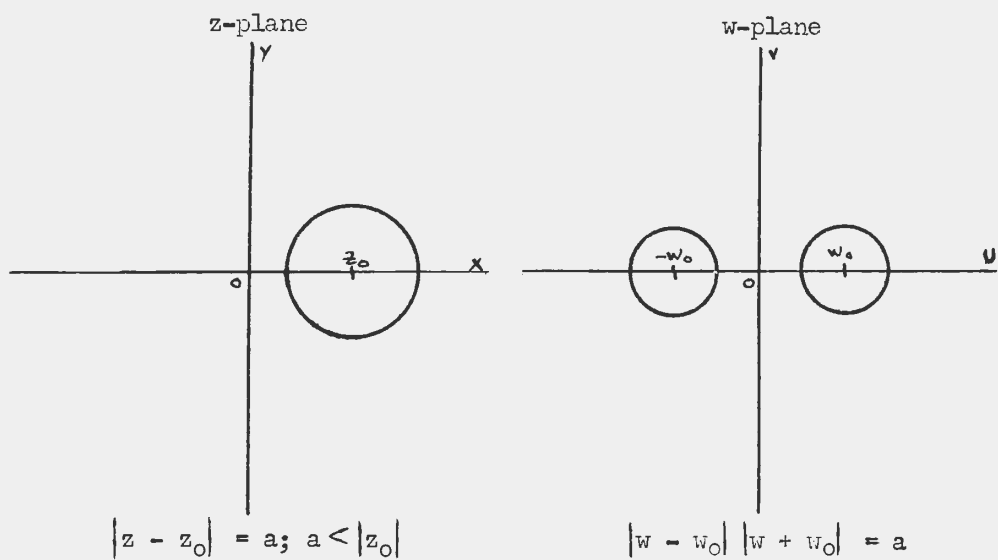


Figure 15. Transformation of $|z - z_0| = a, a < |z_0|$ under $w = z^{\frac{1}{2}}$.

The second case, $a > |z_0|$, is shown in Figure 16. There the origin is contained within the circle.

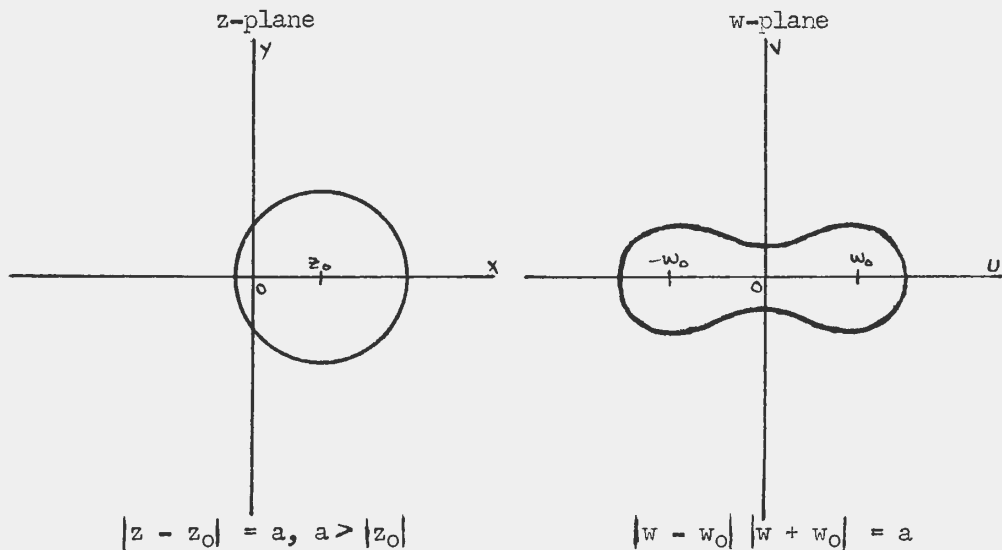


Figure 16. Transformation of $|z - z_0| = a, a > |z_0|$ under $w = z^{\frac{1}{2}}$.

If the circle passes through the origin, then $a = |z_0|$.

This is the third and final case and is illustrated in Figure 17.

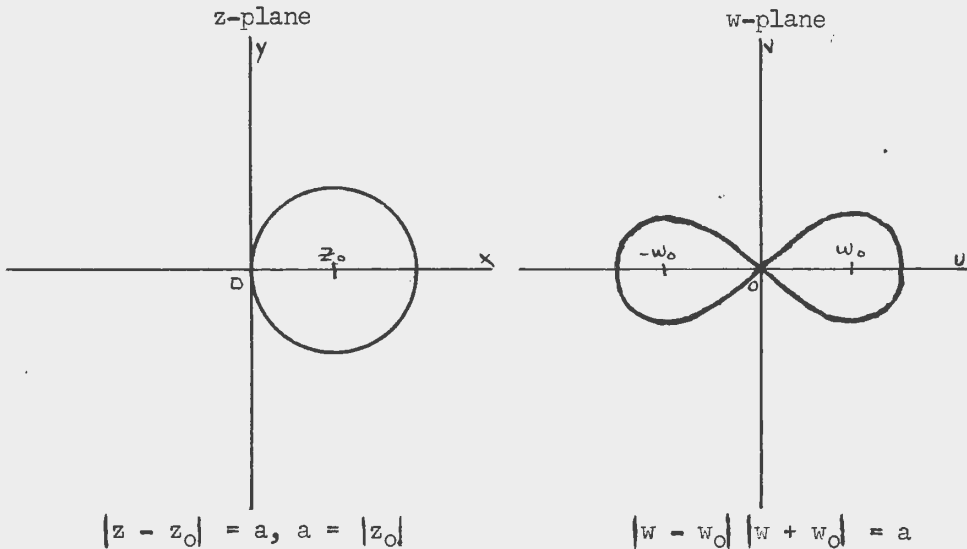


Figure 17. Transformation of $|z - z_0| = a, a = |z_0|$ under $w = z^{\frac{1}{2}}$.

The same results as shown with the use of complex variable notation may be obtained by using the rectangular representations, but the process is very cumbersome.

The parabola. When the transformation $w = z^{\frac{1}{2}}$ is applied to the parabola

$$(y-k)^2 = 2p(x-h),$$

it becomes necessary to investigate at least eight specific cases, depending upon the relationships of h , k , and p , to obtain results which may be used to draw any conclusions. This parabola is, therefore, an example that is best omitted here.

Now, consider the simpler parabola

$$y^2 = 2px.$$

Transformed, it becomes the curve

$$2u^2v^2 - pu^2 + pv^2 = 0.$$

When $p \geq 0$, this curve is symmetric with respect to the u -axis, the v -axis, and the origin. Its intercepts on the axes are

$$u = 0, v = 0$$

and

$$v = 0, u = 0.$$

Use of the procedure outlined in Appendix B reveals that the curve approaches the lines

$$v = \pm\sqrt{p/2}$$

asymptotically. Furthermore, the curve is limited to points between these asymptotes, and for all such values of v , u assumes two values. It may also be noted that v assumes two values for all finite values of u . Hence, the curve may be sketched as shown in Figure 18.

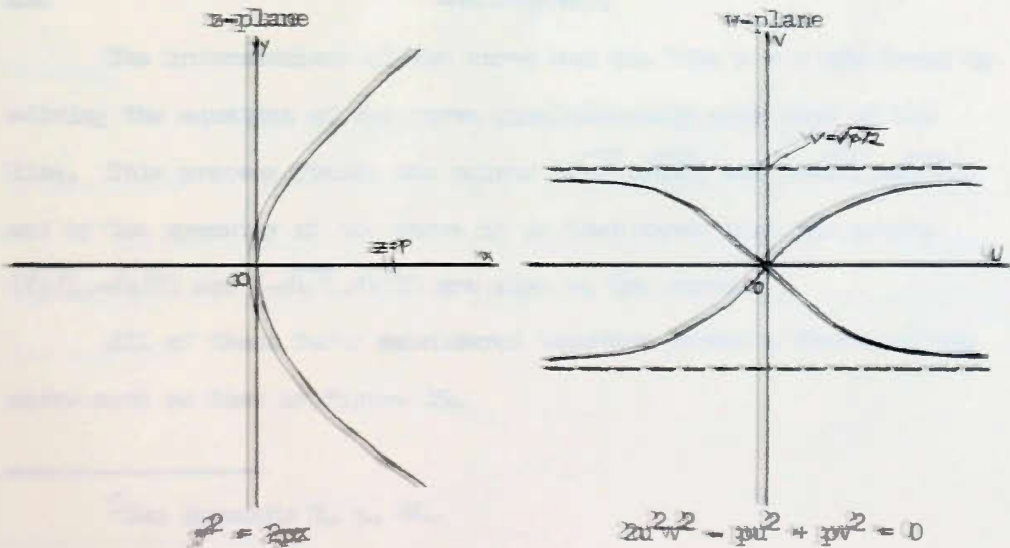


Figure 18. Transformation of $z^2 = 2px$ under $w = \frac{z}{\sqrt{2}}$.

The ellipse. If the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is transformed according to the transformation $w = \frac{1}{z^2}$, it becomes

$$b^2(u^2 - v^2)^2 + 4a^2u^2v^2 = a^2b^2,$$

or $b^2u^4 + 2(2a^2 - b^2)u^2v^2 + b^2v^4 - a^2b^2 = 0.$

Use of the established method⁵ for analyzing this curve, shows that the curve is symmetric with respect to the u -axis, the v -axis, the origin, and the lines $u = v$ and $u = -v$. The intercepts on the axes are

$$v = 0, u = \pm\sqrt{a}$$

and

$$u = 0, v = \pm\sqrt{a}.$$

The curve is limited to the intervals

$$-\sqrt{a} \leq u \leq \sqrt{a}$$

and

$$-\sqrt{a} \leq v \leq \sqrt{a}.$$

The intersections of the curve and the line $u = v$ are found by solving the equation of the curve simultaneously with that of the line. This process yields the points $(\sqrt{b/2}, \sqrt{b/2})$ and $(-\sqrt{b/2}, -\sqrt{b/2})$, and by the symmetry of the curve it is then known that the points $(\sqrt{b/2}, -\sqrt{b/2})$ and $(-\sqrt{b/2}, \sqrt{b/2})$ are also on the curve.

All of these facts considered together permit a sketch of the curve such as that in Figure 19.

⁵See Appendix B, p. 68.

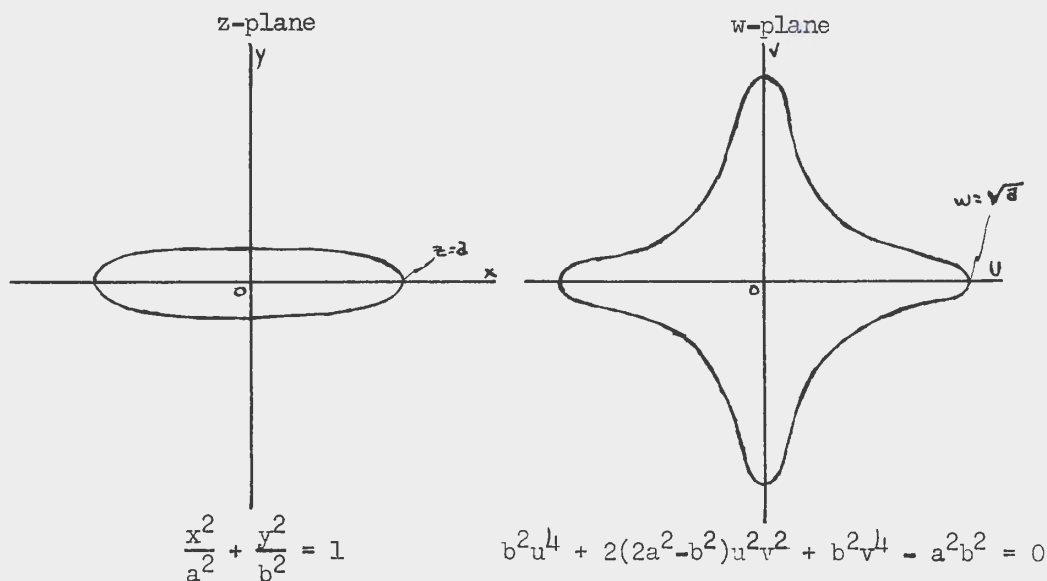


Figure 19. Transformation of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ under $w = z^{\frac{1}{2}}$.

The hyperbola. When transformed, the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

becomes

$$b^2u^4 - 2(2a^2 + b^2)u^2v^2 + b^2v^4 - a^2b^2 = 0.$$

Determination of the physical characteristics proceeds as before.⁶

The transformed curve is symmetric with respect to the u -axis, the v -axis, the origin, and the lines $u = v$ and $u = -v$. It intersects the axes at

$$v = 0, u = \pm\sqrt{a}$$

and

$$u = 0, v = \pm\sqrt{a}.$$

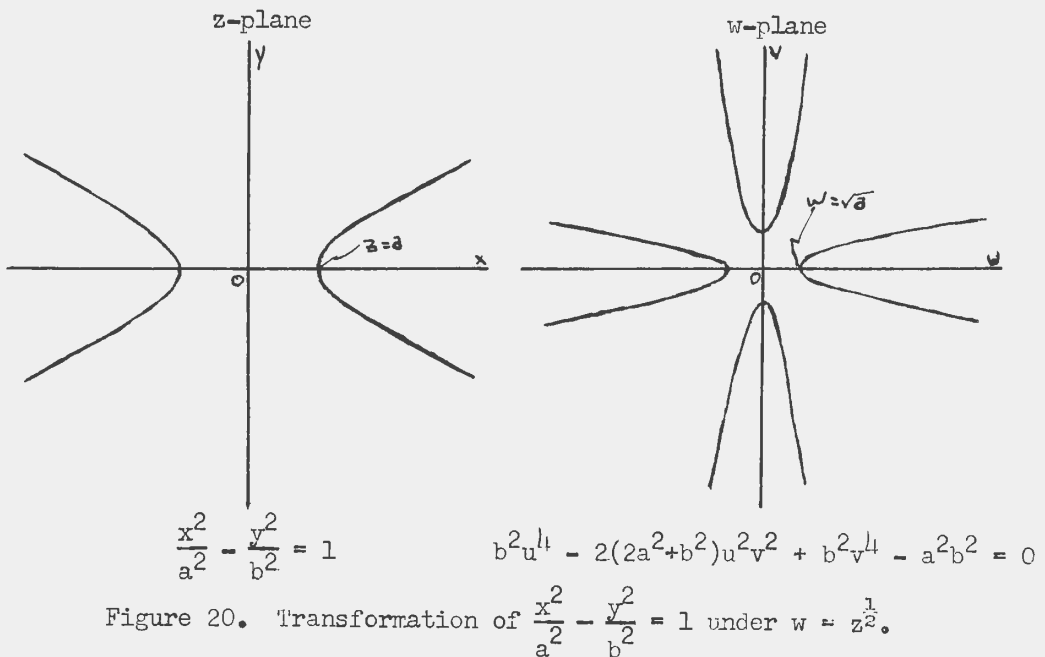
⁶See Appendix B, p. 68.

No real solutions are obtained when the equation of the transformed curve is solved simultaneously with the equation of the line $u = v$. This situation indicates that there are no real points of intersection of those two curves. Considerations of symmetry show that a similar situation exists concerning intersections of the transformed curve and the line $u = -v$.

Examination of the limitations of u and v shows that there are no real points of the curve within a square bounded by the four lines

$$u = \pm\sqrt{a} \quad \text{and} \quad v = \pm\sqrt{a} .$$

Now it is possible to sketch the curve that is the image of the hyperbola under the transformation $w = z^{\frac{1}{2}}$. (Figure 20)



CHAPTER IV

THE TRANSFORMATION $w = \frac{1}{z}$

In the language of complex variables, a transformation in which

$$w = f(z) \quad \text{and} \quad z = f(w)$$

is called an involutory transformation. The transformation

$$w = \frac{1}{z}$$

is such a transformation.

In rectangular coordinates this transformation may be expressed as

$$w = u + iv = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2},$$

or

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

Its inverse, $z = \frac{1}{w}$, is then

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}.$$

In polar coordinates the transformation is

$$\rho e^{i\phi} = \frac{1}{r} e^{-i\theta},$$

or

$$\rho = \frac{1}{r}; \quad \phi = -\theta.$$

Upon further consideration of this polar form, it is seen that the transformation can be expressed as the product of two successive transformations,

$$w' = \rho' e^{i\phi'} = \frac{1}{r} e^{i\theta}$$

and

$$w = \bar{w}' = \rho' e^{-i\phi'}.$$

By the first of these successive transformations, a point z is transformed into a point w' which is collinear with z and the origin. The distance of w' from the origin is

$$|w'| = \frac{1}{|z|}.$$

This equation may be written

$$|w'| |z| = 1.$$

Therefore, the product of the distances of w' and z from the origin is a constant. This property and the property of collinearity, satisfy the conditions of the definition of inversion with respect to a circle.¹ The radius of the circle of inversion is the square-root of the constant product of the distances of the points from the origin. These factors indicate that the transformation

$$w' = \frac{1}{z} e^{i\theta}$$

is an inversion with respect to the unit circle.

This inversion is then followed by the second transformation, which is easily recognized as a reflection with respect to the x -axis.

It may be said that the transformation $w = \frac{1}{z}$ maps the point $z = \infty$ into the point $w = 0$. But the behavior of a function at $z = \infty$ means, precisely, the behavior of the function at $z' = 0$ when $z = \frac{1}{z'}$. Thus it can be said that

$$w = \frac{1}{z} = \frac{1}{1/z'} = z'.$$

Consideration of this relationship when $z' = 0$ shows that $w = 0$ when

¹Levi S. Shively, An Introduction to Modern Geometry (New York: John Wiley & Sons, Inc., 1939), p. 80.

$z = \infty$. The same reasoning process is used in showing that, under this transformation $z = 0$ is transformed into $w = \infty$.

The concept of an infinite point is, of course, an abbreviation for a limiting process, and in case of doubt the direct use of limits should be applied.

In much the same manner as was used in the discussions of the transformations $w = z^2$ and $w = z^{\frac{1}{2}}$, it can be shown that:²

If C_2 is a curve in the z -plane which is obtained by rotating another curve C_1 through an angle Ψ about $z=0$, then K_2 , the image of C_2 , can be obtained by rotating K_1 , the image of C_1 , through an angle $-\Psi$ about $w=0$.

Straight lines and circles.³ Consider the equation

$$a(x^2+y^2) + bx + cy + d = 0.$$

This equation represents a circle if $a \neq 0$ or a line if $a = 0$. Under the transformation $w = \frac{1}{z}$, it becomes

$$a \left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} \right] + \frac{bu}{u^2+v^2} - \frac{cv}{u^2+v^2} + d = 0;$$

or
$$d(u^2+v^2) + bu - cv + a = 0.$$

The new equation is, in turn, a circle or a line, depending now upon whether $d \neq 0$ or $d = 0$, respectively.

The following illustrations show some particular cases of this sort.

²See Appendix A, p. 66.

³Churchill, op. cit., pp. 54-55.

In Figure 21, if $a \neq 0$ and $d \neq 0$, both the curve in the z -plane and its image are circles, and neither of them passes through the origin.

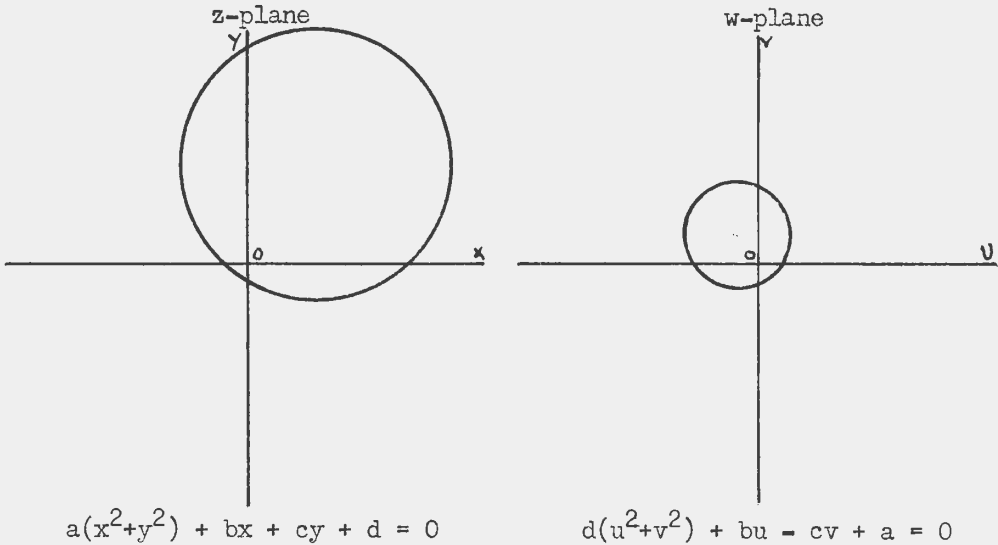


Figure 21. Transformation of $a(x^2+y^2) + bx + cy + d = 0$ under $w = 1/z$.

If $d = 0$ and $a \neq 0$, the z -plane curve is a circle through $z = 0$, and its image is a straight line which does not pass through $w = 0$. This case is shown in Figure 22.

Figure 23 shows the case where $a = 0$ and $d \neq 0$. There the z -plane curve is a straight line which does not pass through $z = 0$, and its image is a circle through $w = 0$.

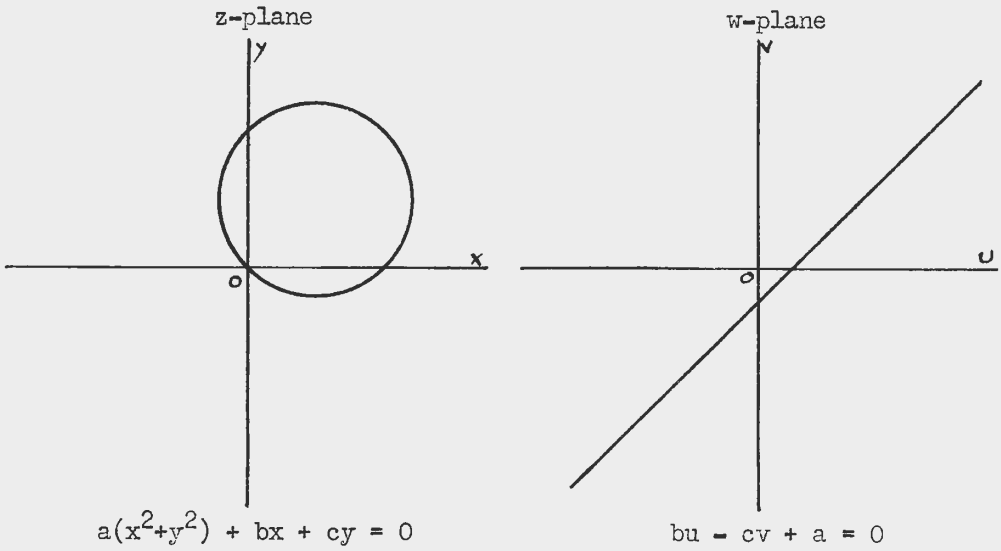


Figure 22. Transformation of $a(x^2+y^2) + bx + cy = 0$ under $w = \frac{1}{z}$.

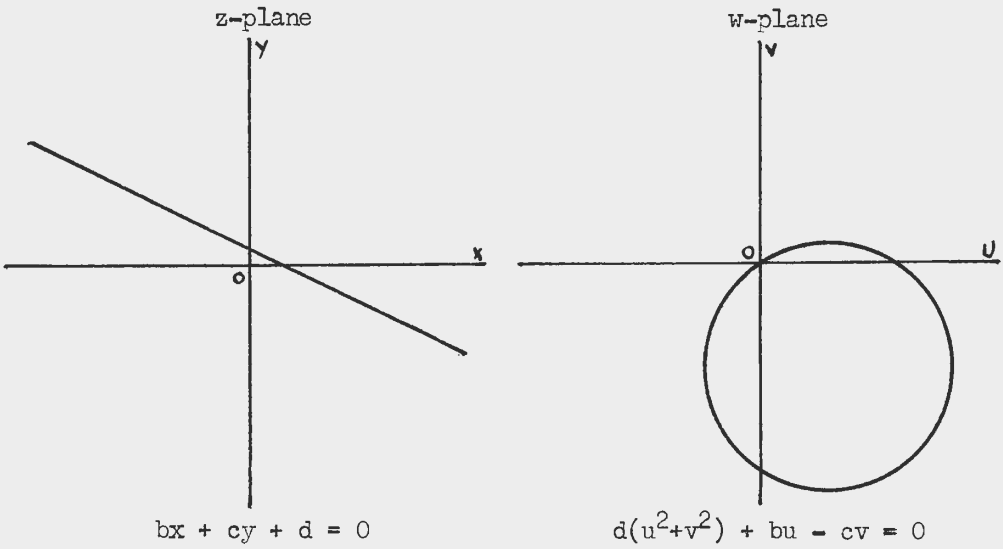


Figure 23. Transformation of $bx + cy + d = 0$ under $w = \frac{1}{z}$.

The case in which $a = 0$ and $d = 0$ is shown in Figure 24. Under these conditions, both the curve in the z -plane and its image are straight lines which pass through the origin in their respective planes.

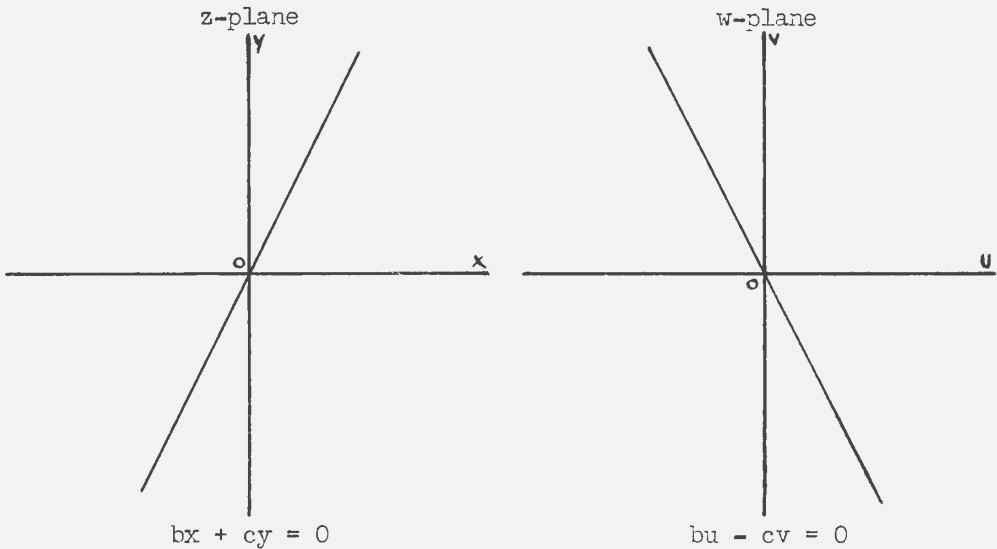


Figure 24. Transformation of $bx + cy = 0$ under $w = \frac{1}{z}$.

The line $x = a$ is transformed by $w = \frac{1}{z}$ into the circle

$$u^2 + v^2 - \frac{u}{a} = 0.$$

This circle has its center at $(\frac{1}{2a}, 0)$ and is tangent to the v -axis at the origin.

Similarly, the line $y = b$ transforms into the circle whose center is at $(0, -\frac{1}{2b})$ and which is tangent to the u -axis at the origin. The equation of such a circle is

$$u^2 + v^2 + \frac{v}{b} = 0.$$

The circle

$$x^2 + y^2 = a^2$$

is transformed into the circle

$$u^2 + v^2 = \frac{1}{a^2}.$$

In case $a = 1$, the circle in the z -plane is the unit circle, and its image is the unit circle in the w -plane.

All of these results of transforming lines and circles under the transformation $w = \frac{1}{z}$ may be summed up as follows: If lines are considered as limiting cases of circles (that is, circles of infinite radii), then the transformation $w = \frac{1}{z}$ transforms circles into circles.

The parabola. When the parabola

$$y^2 = 2px$$

is transformed by the transformation $w = \frac{1}{z}$, it becomes

$$v^2 - 2puv^2 - 2pu^3 = 0.$$

This curve is symmetric with respect to the u -axis. It intersects the axes only at the origin, and approaches the line

$$u = \frac{1}{2p}$$

asymptotically.

Then, considering only the cases where $p \geq 0$, it is found that the real values of u which satisfy the equation of the curve are limited to the interval

$$\frac{1}{2p} > u \geq 0.$$

The first derivative of v with respect to u , from the function

$$v^2 - 2puv^2 - 2pu^3 = 0,$$

vanishes when $u = 0$, which indicates that the curve forms a cusp at the origin.

Now, the curve can be sketched as shown in Figure 25.

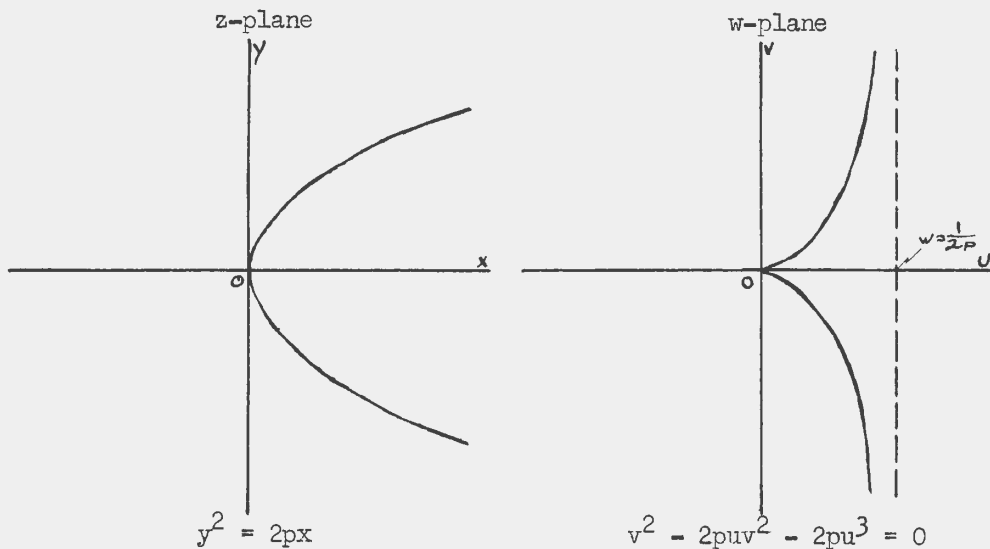


Figure 25. Transformation of $y^2 = 2px$ under $w = \frac{1}{z}$.

Application of the property of rotated curves for the transformation $w = \frac{1}{z}$ permits the examination of the images of all parabolas that have their vertices at the origin.

The ellipse. Next to be considered is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

When this ellipse is transformed, its image is found to be the curve

$$\frac{u^2}{a^2(u^2+v^2)^2} + \frac{v^2}{b^2(u^2+v^2)^2} = 1.$$

The left-hand member of this equation becomes undefined as u and v approach zero simultaneously. That shows that the origin is not included in the curve.

If the equation is expanded, it becomes

$$a^2b^2u^4 + 2a^2b^2u^2v^2 + a^2b^2v^4 - b^2u^2 - a^2v^2 = 0.$$

The point $(0,0)$ satisfies this equation, but the original equation of the transformed curve shows that this point is excluded. Therefore, in the following discussion the origin will not be considered a real point on the curve.

The quartic

$$a^2b^2u^4 + 2a^2b^2u^2v^2 + a^2b^2v^4 - b^2u^2 - a^2v^2 = 0$$

is symmetric with respect to the u -axis, the v -axis, and the origin.

It intersects the u -axis at the two points $(\frac{1}{a}, 0)$ and $(-\frac{1}{a}, 0)$. Its v -intercepts are at $v = \pm \frac{1}{b}$.

When the limits of extent of the curve are determined, it is found that when v is in the interval

$$-\frac{1}{b} \leq v \leq \frac{1}{b},$$

u assumes two real values, and when

$$v > \frac{1}{b} \quad \text{or} \quad v < -\frac{1}{b},$$

u has no real value.

When the same process is used to determine the limits on u which will give real values of v , three distinct cases appear. These cases depend upon the relationships between a and b ,

$$\frac{1}{a} > \frac{1}{\sqrt{2}b}, \quad \frac{1}{a} = \frac{1}{\sqrt{2}b}, \quad \text{and} \quad \frac{1}{a} < \frac{1}{\sqrt{2}b}.$$

These relationships may be expressed in terms of the eccentricity of the ellipse in the z -plane. Then the three cases become dependent upon

$$e < \frac{1}{\sqrt{2}}, \quad e = \frac{1}{\sqrt{2}}, \quad \text{and} \quad e > \frac{1}{\sqrt{2}},$$

respectively.

When $e < \frac{1}{\sqrt{2}}$, the following conditions are found:

v assumes two real values when $-\frac{1}{a} \leq u \leq \frac{1}{a}$;

v assumes no real value when $u > \frac{1}{a}$ or when $u < -\frac{1}{a}$.

This case is illustrated in Figure 26.

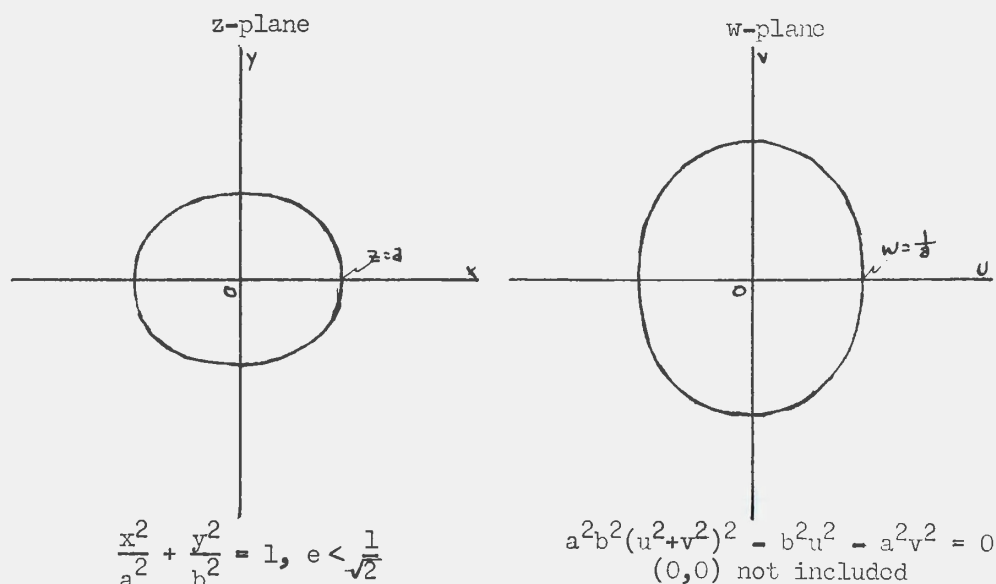


Figure 26. Transformation of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad e < \frac{1}{\sqrt{2}}$ under $w = \frac{1}{z}$.

If $e = \frac{1}{\sqrt{2}}$, a similar set of conditions is:

v assumes four real values when $u = \pm \frac{1}{a}$;

v assumes two real values when $-\frac{1}{a} < u < \frac{1}{a}$;

v assumes no real value when $u > \frac{1}{a}$ or when $u < -\frac{1}{a}$.

Physically, this curve is similar to the curve in Figure 26, except that the values of u where the curve intersects the u -axis provide four real values of v rather than two such values. These four values are all equal to zero.

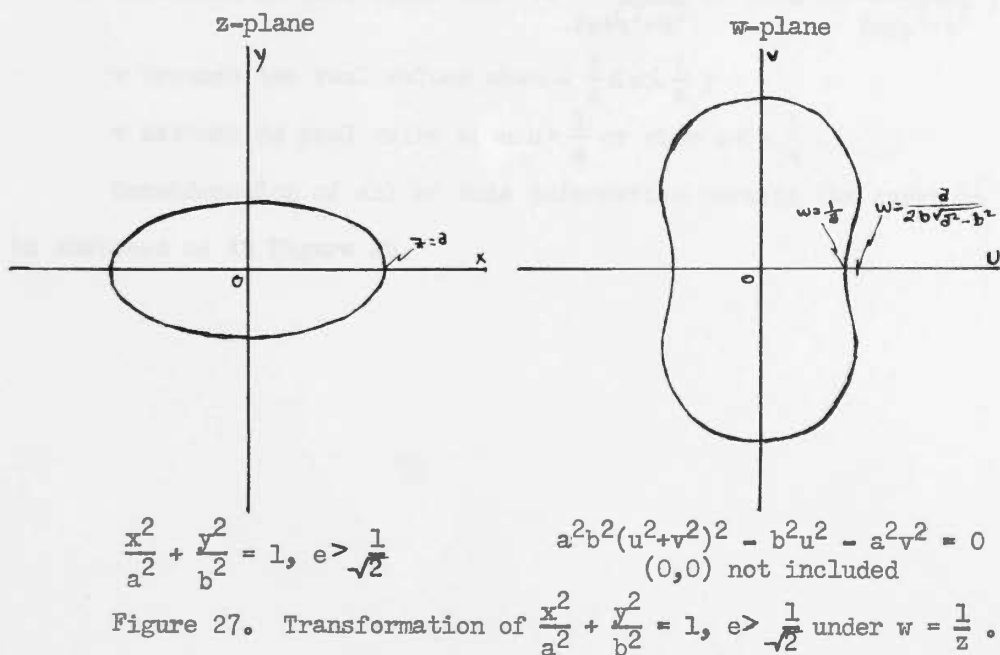
Finally, the case in which $e > \frac{1}{\sqrt{2}}$ is shown in Figure 27. There the conditions which limit the extent of the values of u on the curve are:

$$v \text{ assumes four real values when } \frac{1}{a} \leq u \leq \frac{a}{2b\sqrt{a^2-b^2}},$$

$$\text{or when } -\frac{1}{a} \geq u \geq \frac{-a}{2b\sqrt{a^2-b^2}};$$

$$v \text{ assumes two real values when } \frac{1}{a} > u > -\frac{1}{a};$$

$$v \text{ assumes no real value when } u > \frac{a}{2b\sqrt{a^2-b^2}} \text{ or when } u < \frac{-a}{2b\sqrt{a^2-b^2}}.$$



The hyperbola. Under the transformation $w = \frac{1}{z}$, the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

becomes

$$a^2b^2u^4 + 2a^2b^2u^2v^2 + a^2b^2v^4 - b^2u^2 + a^2v^2 = 0.$$

This quartic is symmetric with respect to the u -axis, the v -axis, and the origin. Its only point of intersection of the v -axis is at the origin, and the u -intercepts are at the origin and at the points $(\frac{1}{a}, 0)$ and $(-\frac{1}{a}, 0)$.

Investigation of limits of extent establishes the following set of conditions:

u assumes four real values when $\frac{-b}{2a\sqrt{a^2+b^2}} \leq v \leq \frac{b}{2a\sqrt{a^2+b^2}}$;

u assumes no real value when $v > \frac{b}{2a\sqrt{a^2+b^2}}$ or when $v < \frac{-b}{2a\sqrt{a^2+b^2}}$;

v assumes two real values when $-\frac{1}{a} \leq u \leq \frac{1}{a}$;

v assumes no real value when $u > \frac{1}{a}$ or when $u < -\frac{1}{a}$.

Consideration of all of this information permits the curve to be sketched as in Figure 28.

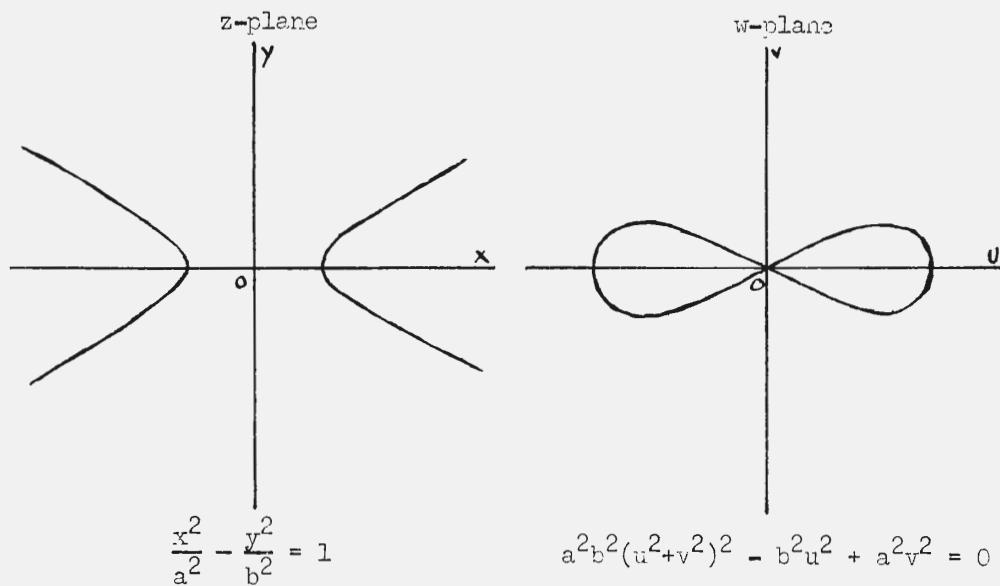


Figure 28. Transformation of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ under $w = \frac{1}{z}$.

CHAPTER V

OTHER TRANSFORMATIONS

The expression of a complicated transformation as the product of several successive transformations of a simpler nature often requires the use of functions other than

$$w = z^2, w = z^{\frac{1}{2}}, \text{ and } w = \frac{1}{z}.$$

Two of these other basic transformations are

$$w = e^z$$

and

$$w = \sin z.$$

Although they are usually more difficult to work with, they are still important.

The transformation $w = e^z$. If ρ and ϕ are the polar coordinates of the point w , as before, the transformation

$$w = e^z$$

can be written

$$\rho e^{i\phi} = e^x e^{iy}.$$

Then, upon the equation of the real and imaginary parts of that expression

$$\rho = e^x, \phi = y.$$

Therefore, the line $x = c$ is transformed into the circle,

$$\rho = e^c.$$

In rectangular coordinates, the equation of the circle is

$$u^2 + v^2 = e^{2c}.$$

The line $y = c$ is mapped into a ray, $\phi = c, \rho > 0$,

which, in rectangular coordinates, is the half-line

$$v = u \tan c, \quad 0 < u < \infty.$$

These curves appear in Figure 29.

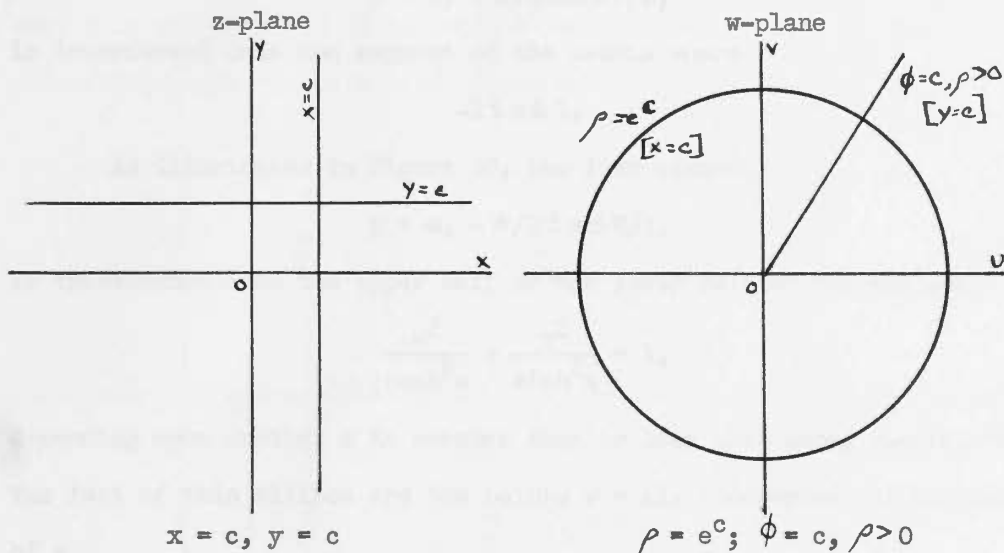


Figure 29. Transformation of $x = c$ and $y = c$ under $w = e^z$.

The transformation $w = \sin z$. It can be shown that

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

Thus the transformation

$$w = \sin z$$

can be written

$$u = \sin x \cosh y, \quad v = \cos x \sinh y.$$

Churchill¹ mentions several elementary examples of transformations by the function $w = \sin z$.

¹Op. cit., pp. 69-70.

The line $x = 0$ is transformed into the line $u = 0$ in the w -plane. The line $x = \pi/2$ is mapped into the part of the u -axis in the w -plane where $u \geq 1$. The line segment

$$y = 0, \quad -\pi/2 \leq x \leq \pi/2,$$

is transformed into the segment of the u -axis where

$$-1 \leq u \leq 1.$$

As illustrated in Figure 30, the line segment

$$y = a, \quad -\pi/2 \leq x \leq \pi/2,$$

is transformed into the upper half or the lower half of the ellipse

$$\frac{u^2}{\cosh^2 a} + \frac{v^2}{\sinh^2 a} = 1,$$

depending upon whether a is greater than or less than zero, respectively. The foci of this ellipse are the points $w = \pm 1$, independent of the value of a .

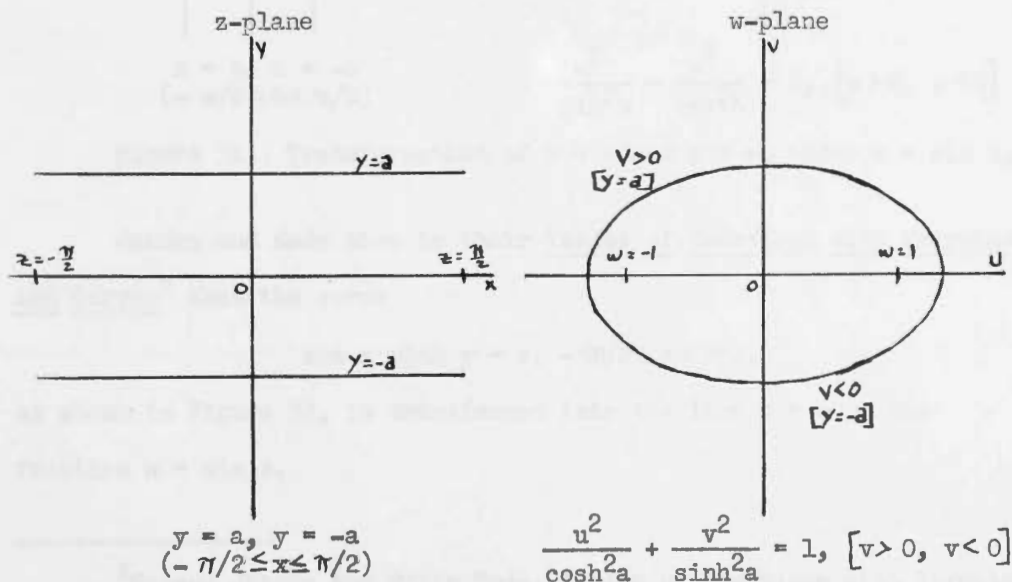


Figure 30. Transformation of $y = a$ and $y = -a$ under $w = \sin z$.

The line $x = b$, where $-\pi/2 < b < \pi/2$ is transformed into the right-hand branch of the hyperbola (Figure 31),

$$\frac{u^2}{\sin^2 b} - \frac{v^2}{\cos^2 b} = 1,$$

when b is greater than zero. When b is less than zero, the line is transformed into the left-hand branch of the same hyperbola.

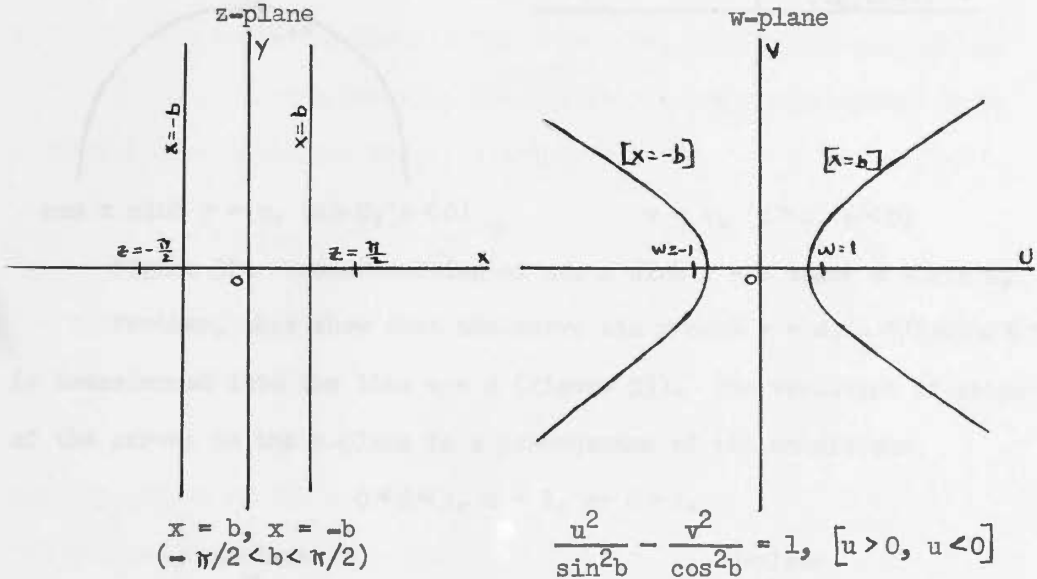


Figure 31. Transformation of $x = b$ and $x = -b$ under $w = \sin z$.

Jahnke and Emde show in their Tables of Functions with Formulae and Curves² that the curve

$$\cos x \sinh y = c, \quad -\pi/2 \leq x \leq \pi/2,$$

as shown in Figure 32, is transformed into the line $v = c$ by the function $w = \sin z$.

²Eugene Jahnke and Fritz Emde, Tables of Functions with Formulae and Curves (New York: Dover Publications, 1943), Addenda, p. 67.

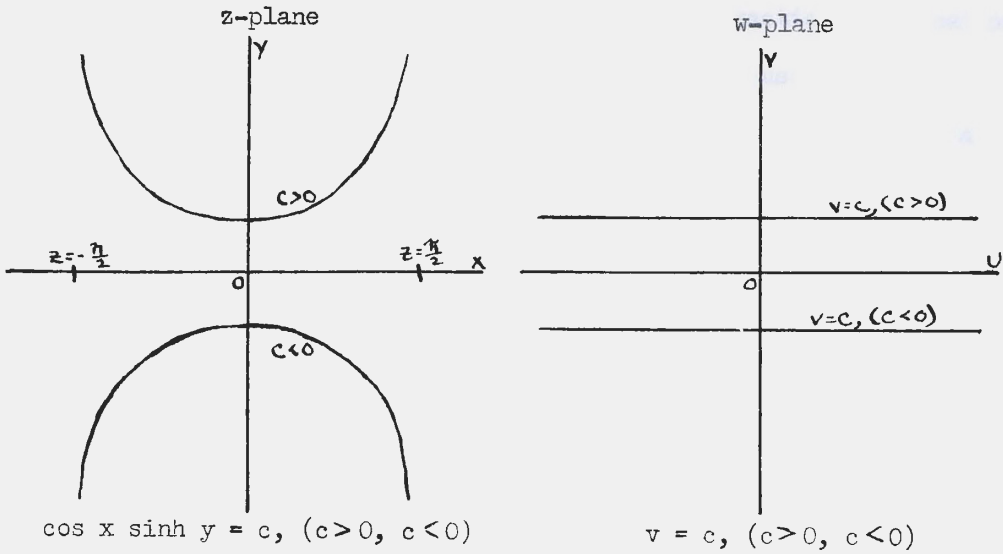


Figure 32. Transformation of $\cos x \sinh y = c$ under $w = \sin z$.

Further, they show that the curve $\sin x \cosh y = d, -\pi/2 \leq x \leq \pi/2$ is transformed into the line $u = d$ (Figure 33). The variation of shape of the curves in the z-plane is a consequence of the conditions

$$0 < d < 1, d = 1, \text{ or } d > 1.$$

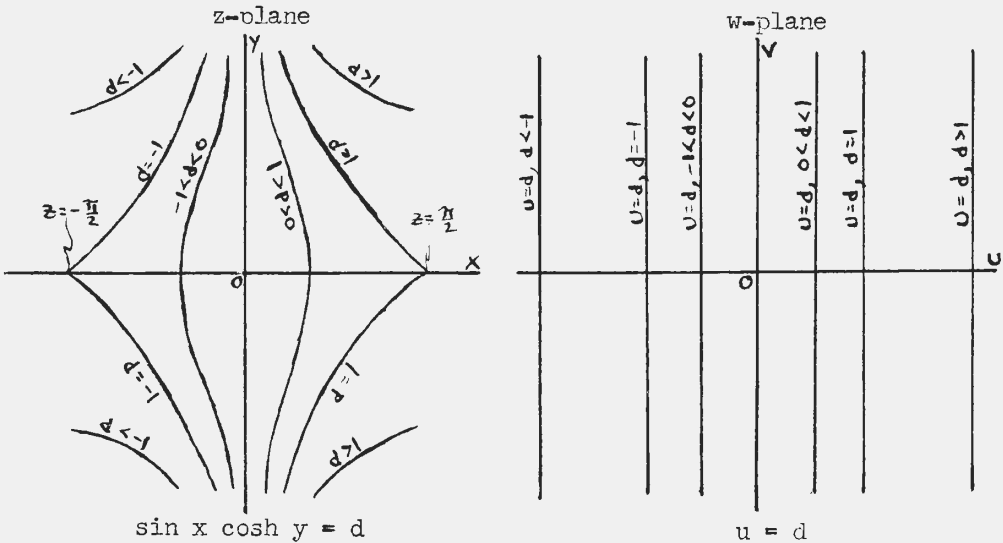


Figure 33. Transformation of $\sin x \cosh y = d$ under $w = \sin z$.

The Schwarz-Christoffel transformation.³ Practical applications of transformations in the z -plane often require the use of a generalized transformation which can be adjusted to satisfy the conditions of a certain problem. One such transformation that is used in many cases is the Schwarz-Christoffel transformation. The Schwarz-Christoffel transformation was named in honor of the two German mathematicians, H. S. Schwarz and E. B. Christoffel, who discovered it independently.

This transformation maps the entire x -axis of the z -plane into a polygon in the w -plane and is commonly written

$$w = A \int (z-x_1)^{-k_1} (z-x_2)^{-k_2} \dots (z-x_{n-1})^{-k_{n-1}} dz + B.$$

The integral sign denotes any one of the indefinite integrals of the integrand. The values

$$x_1, x_2, \dots, x_{n-1}$$

are the points on the x -axis which are transformed into the vertices of the polygon in the w -plane, and each k_j is a real constant. A and B are complex constants which depend upon the conditions imposed upon the transformation.

The linear fractional transformation. The transformation

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}; \quad \alpha\delta - \beta\gamma \neq 0,$$

where $\alpha, \beta, \gamma,$ and δ are complex constants, is called the linear fractional, or bilinear transformation. Like the transformation $w = \frac{1}{z}$,

³Churchill, op. cit., pp. 171-175.

which is actually a special case of it, the bilinear transformation always transforms circles into circles, with lines as limiting cases.

Another property of this transformation, which makes it a general type, is that it maps any three distinct points in the z -plane into any desired three points which are distinct in the w -plane.

CHAPTER VI

PRACTICAL APPLICATIONS

The theory of transformations in the z -plane has a very definite application in the fields of engineering and physics. The engineer or physicist is more often concerned with problems which involve areas or volumes than with purely mathematical considerations of points and curves. However, it is a relatively simple step from consideration of curves to areas bounded by curves. This consideration of areas leads to several applications of transformations.

Types of problems. In general, transformations in the z -plane can be helpful in solving all boundary value problems associated with the Laplace equation in two independent variables,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

or the more general Poisson equation,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = g(x,y).$$

Problems in hydrodynamics, aerodynamics, thermodynamics, and electricity and magnetism often make use of these equations. Transformations are not a method of solution of such problems, but rather a means of simplifying them. Thus, boundary value problems which involve considerations of oddly shaped areas may be transformed into similar problems involving considerations of much simpler areas.

As an example, consider the airfoil, A, in Figure 34.¹

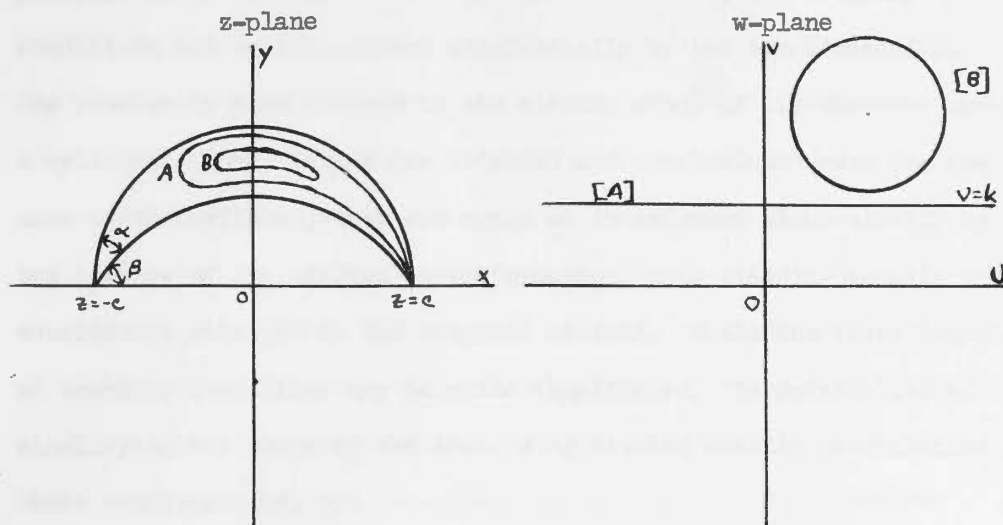


Figure 34. Transformation of an airfoil under a specialized transformation.

The shape of this airfoil is determined by the angles α and β and the value of c .

Under the transformation

$$w = c e^{i\beta\pi/\alpha} \left[\frac{c+z}{c-z} \right]^{\pi/\alpha}; \quad c > 0, \beta > 0, 0 < \alpha < \alpha + \beta < \pi,$$

the area of airfoil A, in the z -plane, is transformed into the entire area above the line $v = k$ in the w -plane, where k depends upon the values of α , β , and c . The area of a particular airfoil such as airfoil B is transformed into a circle by this same transformation.

¹Kober, op. cit., p. 50.

If the problem is to determine the behavior of a current of air passing around airfoil B under certain conditions, the boundary conditions can be transformed algebraically by the transformation. The problem is then reduced to the simpler study of air currents around a cylinder. When results are obtained and conclusions drawn for the case of the cylinder, they can again be transformed algebraically by the inverse of the original transformation, thus yielding results and conclusions relative to the original airfoil. While the transformation of boundary conditions may be quite complicated, the possibility of simplifying the shape of the area being studied usually overbalances these complications.

Applications of basic transformations. Most physical problems involve quite complicated transformations, but the simpler basic transformations find application in some special cases.

In problems of fluid mechanics, the two-dimensional steady-state type of flow is often considered; that is, the motion of the fluid is assumed to be identical in all planes parallel to the z -plane.² The velocity of the fluid is parallel to that plane and is independent of the time. Then it suffices to consider only the motion of the fluid in the z -plane.

If a problem concerns such a uniform flow to the right in the upper-half of the z -plane, its results may be transformed by the transformation $w = z^{\frac{1}{2}}$ to obtain results for a similar flow of fluid

²Churchill, op. cit., pp. 161-166.

in a quadrant of the w -plane. Reference to Figure 14, page 25, shows that the lines representing the fluid flow in the z -plane are transformed by this transformation into the branches of the equilateral hyperbolae in the first quadrant of the w -plane. These hyperbolae represent the path of a fluid flowing around a corner.

Transformations are not always used as applications to physical problems. One common use is found in the manufacture of maps for navigational purposes. If the north pole of the earth is used as a center of projection, and the surface of the sphere is projected upon a plane, a representation of the surface is obtained in which the meridians appear as rays through a single point and the parallels appear as concentric circles about the same point. If then, the inverse of the transformation $w = e^z$ is applied to these concentric circles and rays, reference to Figure 29, page 47, indicates that they will become two sets of parallel lines that are perpendicular to each other. Such a map of the earth's surface is the familiar Mercator's projection.³

Since these maps show great distortion of areas, most maps are compromises produced by a sequence of perhaps thirty transformations, each of which can be written as a transformation in the z -plane.

³E. J. Townsend, Functions of a Complex Variable (New York: Henry Holt and Company, 1915), p. 137.

CHAPTER VII

SUMMARY

In the brief discussions of the behavior of some lines and conics under various transformations, it was noted that the complexity of the algebraic operations involved increased greatly as curves of higher degree were transformed. In all cases considered, lines were transformed into conics, while conics were generally transformed into quartics. One exception to the latter case was seen when the parabola

$$y^2 = 2px$$

was transformed into a cubic by the transformation

$$w = \frac{1}{z}.$$

Other exceptions occurred when the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

were transformed under the transformation

$$w = z^2$$

and resulted in an ellipse and a hyperbola, respectively.

Several further investigations are suggested by the results of this research. First, the general forms of the conics and some representative cubics and quartics should be studied under the basic transformations. Second, a study might be made of the behavior of

lines and conics under other, more complicated transformations.

Lastly, investigations of the results obtained here should be made in order to establish practical applications of them.

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BIBLIOGRAPHY

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APPENDIX A

ROTATIONAL PROPERTIES OF THE TRANSFORMATION $w = z^2$

Suppose that: The points z_1 and z_2 are in the z -plane; z_2 is obtained by rotating z_1 through an angle Ψ about $z = 0$;
 amp $z_1 = \theta$; amp $z_2 = \omega$; w_1 is the image of z_1 and w_2 is the image of z_2 under the transformation $w = z^2$;
 amp $w_1 = \phi$; and amp $w_2 = \lambda$.

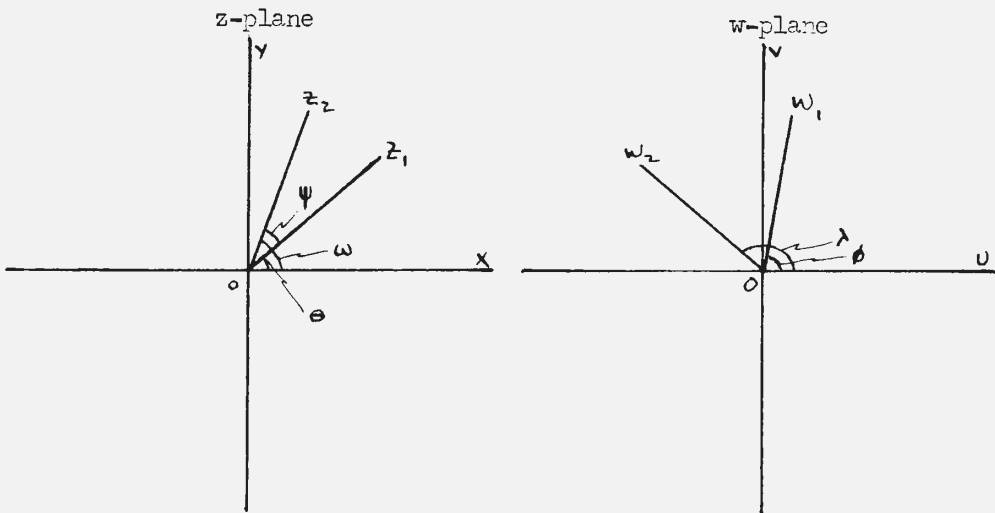


Figure 35. Rotational properties of the transformation $w = z^2$.

The transformation $w = z^2$ may be written

$$\rho = r^2, \quad \phi = 2\theta.$$

Then

$$w_1 = \rho_1 e^{i\phi} = z_1^2 = (r_1 e^{i\theta})^2 = r_1^2 e^{2i\theta},$$

or

$$\rho_1 = r_1^2, \quad \phi = 2\theta;$$

and

$$w_2 = \rho_2 e^{i\lambda} = z_2^2 = (r_2 e^{i\omega})^2 = r_2^2 e^{2i\omega},$$

or

$$\rho_2 = r_2^2, \quad \lambda = 2\omega.$$

Since z_2 is obtained by rotating z_1 through an angle Ψ about $z = 0$,

$$|z_1| = |z_2|, \text{ or } r_1 = r_2,$$

and

$$\omega - \theta = \Psi.$$

Whence,

$$\rho_1 = \rho_2 = r_1^2,$$

or

$$|\bar{w}_1| = |\bar{w}_2|;$$

and

$$\lambda - \phi = 2\omega - 2\theta = 2(\omega - \theta),$$

or

$$\lambda - \phi = 2\Psi.$$

The conditions

$$|\bar{w}_1| = |\bar{w}_2|, \lambda - \phi = 2\Psi$$

show that w_2 is obtained by rotating w_1 through an angle 2Ψ about $w = 0$.

Now, consider two curves in the z -plane, C_1 and C_2 . C_2 is obtained by rotating C_1 through an angle Ψ about $z = 0$. Then, if z_{1i} are points on C_1 , and z_{2i} are points on C_2 , when $i = 1, 2, \dots$, then by the rotation,

$$|z_{1i}| = |z_{2i}| \text{ and } \text{amp } z_{2i} = \text{amp } z_{1i} + \Psi.$$

These points are transformed into the points w_{1i} and w_{2i} ,

where $i = 1, 2, \dots$. Then from the above proof,

$$|w_{1i}| = |w_{2i}| \text{ and } \text{amp } w_{2i} = \text{amp } w_{1i} + 2\Psi.$$

But these points, w_{1i} and w_{2i} , form the curves, K_1 and K_2 , that are the images of the curves C_1 and C_2 . Since the amplitudes of corresponding points on the curves K_1 and K_2 differ by 2Ψ , the result can be stated:

If C_2 is a curve in the z -plane which is obtained by rotating another curve, C_1 , through an angle Ψ about $z = 0$, then K_2 , the image of C_2 under the transformation $w = z^2$, can be obtained by rotating K_1 , the image of C_1 under the same transformation, through an angle 2Ψ about $w = 0$.

ROTATIONAL PROPERTIES OF THE TRANSFORMATION $w = z^{\frac{1}{2}}$

Suppose that: The points z_1 and z_2 are in the z -plane; z_2 is obtained by rotating z_1 through an angle Ψ about $z = 0$;
 amp $z_1 = \theta$; amp $z_2 = \omega$; w_1 is the image of z_1 and w_2 is the image of z_2 under the transformation $w = z^{\frac{1}{2}}$;
 amp $w_1 = \phi$; and, amp $w_2 = \lambda$.

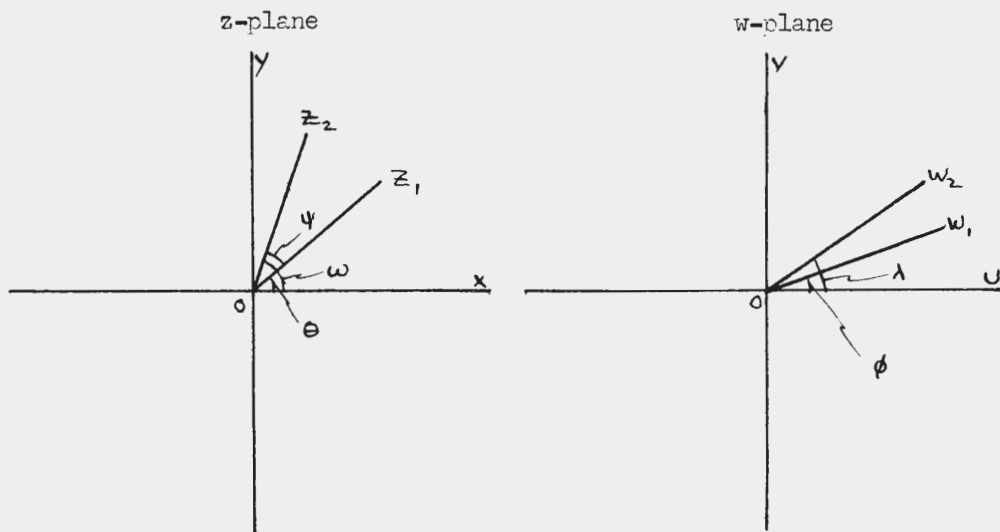


Figure 36. Rotational properties of the transformation $w = z^{\frac{1}{2}}$.

The transformation $w = z^{\frac{1}{2}}$ may be written

$$\rho = \sqrt{r}, \quad \phi = \frac{\theta}{2} + \pi k; \quad k = 0, 1.$$

Then $w_1 = \rho_1 e^{i\phi} = z_1^{\frac{1}{2}} = (r_1 e^{i\theta})^{\frac{1}{2}} = \sqrt{r_1} \exp \left[i \left(\frac{\theta}{2} + \pi k \right) \right],$

or $\rho_1 = \sqrt{r_1}, \quad \phi = \frac{\theta}{2} + \pi k;$

and $w_2 = \rho_2 e^{i\lambda} = z_2^{\frac{1}{2}} = (r_2 e^{i\omega})^{\frac{1}{2}} = \sqrt{r_2} \exp \left[i \left(\frac{\omega}{2} + \pi k \right) \right],$

or $\rho_2 = \sqrt{r_2}, \quad \lambda = \frac{\omega}{2} + \pi k.$

Since z_2 is obtained by rotating z_1 through an angle Ψ about $z = 0$,

$$|z_1| = |z_2|, \text{ or } r_1 = r_2,$$

and

$$\omega - \theta = \Psi.$$

Whence,

$$\rho_1 = \rho_2 = \sqrt{r_1},$$

or

$$|w_1| = |w_2|;$$

and

$$\lambda - \phi = \frac{\omega}{2} + \pi k - \frac{\theta}{2} - \pi k = \frac{\omega - \theta}{2},$$

or

$$\lambda - \phi = \Psi/2.$$

The conditions

$$|w_1| = |w_2|, \lambda - \phi = \Psi/2$$

show that w_2 is obtained by rotating w_1 through an angle $\Psi/2$ about $w = 0$.

Now, consider two curves in the z -plane, C_1 and C_2 . C_2 is obtained by rotating C_1 through an angle Ψ about $z = 0$. Then, if z_{1_i} are points on C_1 and z_{2_i} are points on C_2 , when $i = 1, 2, \dots$, then by the rotation,

$$|z_{1_i}| = |z_{2_i}| \text{ and } \text{amp } z_{2_i} = \text{amp } z_{1_i} + \Psi.$$

These points are transformed into the points w_{1_i} and w_{2_i} ,

where $i = 1, 2, \dots$. Then from the proof above,

$$|w_{1_i}| = |w_{2_i}| \text{ and } \text{amp } w_{2_i} = \text{amp } w_{1_i} + \Psi/2.$$

But these points, w_{1_i} and w_{2_i} , form the curves, K_1 and K_2 , that are the images of the curves C_1 and C_2 . Since the amplitudes of corresponding points on the curves K_1 and K_2 differ by $\Psi/2$, the result can be stated:

If C_2 is a curve in the z -plane which is obtained by rotating another curve, C_1 , through an angle Ψ about $z = 0$, then K_2 , the image of C_2 under the transformation $w = z^2$, can be obtained by rotating K_1 , the image of C_1 under the same transformation, through an angle $\Psi/2$ about $w = 0$.

ROTATIONAL PROPERTIES OF THE TRANSFORMATION $w = \frac{1}{z}$

Suppose that: The points z_1 and z_2 are in the z -plane; z_2 is obtained by rotating z_1 through an angle Ψ about $z = 0$;
 amp $z_1 = \Theta$; amp $z_2 = \omega$; w_1 is the image of z_1 and w_2 is the image of z_2 under the transformation $w = \frac{1}{z}$;
 amp $w_1 = \phi$; and, amp $w_2 = \lambda$.

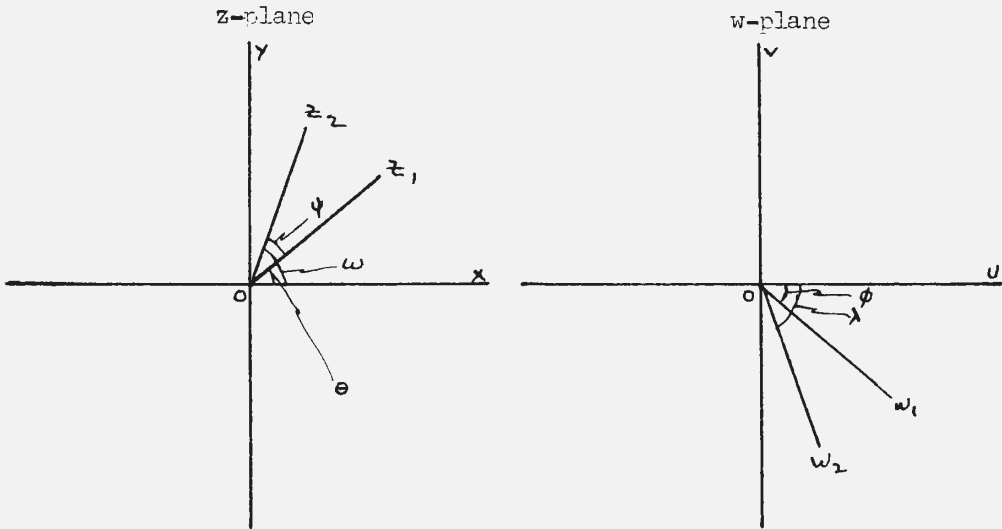


Figure 37. Rotational properties of the transformation $w = \frac{1}{z}$.

The transformation $w = \frac{1}{z}$ may be written

$$\rho = \frac{1}{r}, \quad \phi = -\theta.$$

Then

$$w_1 = \rho_1 e^{i\phi} = \frac{1}{z_1} = \frac{1}{r_1} e^{-i\theta},$$

or

$$\rho_1 = \frac{1}{r_1}, \quad \phi = -\theta;$$

and

$$w_2 = \rho_2 e^{i\lambda} = \frac{1}{z_2} = \frac{1}{r_2} e^{-i\omega},$$

or

$$\rho_2 = \frac{1}{r_2}, \quad \lambda = -\omega.$$

Since z_2 is obtained by rotating z_1 through an angle Ψ about $z = 0$,

$$|z_1| = |z_2|, \text{ or } r_1 = r_2,$$

and

$$\omega - \theta = \Psi.$$

Whence,

$$\rho_1 = \rho_2 = \frac{1}{r_1},$$

or

$$|w_1| = |w_2|;$$

and

$$\lambda - \phi = -\omega - (-\theta) = -(\omega - \theta),$$

or

$$\lambda - \phi = -\Psi.$$

The conditions

$$|w_1| = |w_2|, \quad \lambda - \phi = -\Psi$$

show that w_2 is obtained by rotating w_1 through an angle $-\Psi$ about $w = 0$.

Now, consider two curves in the z -plane, C_1 and C_2 . C_2 is obtained by rotating C_1 through an angle Ψ about $z = 0$. Then, if z_{1i} are points on C_1 , and z_{2i} are points on C_2 , when $i = 1, 2, \dots$, then by the rotation,

$$|z_{1i}| = |z_{2i}| \text{ and } \text{amp } z_{2i} = \text{amp } z_{1i} + \Psi.$$

These points are transformed into the points w_{1i} and w_{2i} , where $i = 1, 2, \dots$. Then from the above proof,

$$|w_{1i}| = |w_{2i}| \text{ and } \text{amp } w_{2i} = \text{amp } w_{1i} - \Psi.$$

But these points, w_{1i} and w_{2i} , form the curves, K_1 and K_2 , that are the images of the curves C_1 and C_2 . Since the amplitudes of corresponding points on the curves K_1 and K_2 differ by $-\Psi$, the result can be stated:

If C_2 is a curve in the z -plane which is obtained by rotating another curve, C_1 , through an angle Ψ about $z = 0$, then K_2 , the image of C_2 under the transformation $w = 1/z$, can be obtained by rotating K_1 , the image of C_1 under the same transformation, through an angle $-\Psi$ about $w = 0$.

APPENDIX B

A METHOD OF ANALYZING AN EQUATION

Step One. Axes of symmetry are determined by the following characteristics:

1. If the equation obtained by replacing v by $-v$ is identical with the original equation, the curve is symmetrical with respect to the u -axis.
2. If the equation obtained by replacing u by $-u$ is identical with the original equation, the curve is symmetrical with respect to the v -axis.
3. If the equation obtained by replacing both v by $-v$ and u by $-u$ is identical with the original equation, the curve is symmetrical with respect to the origin.
4. If the equation obtained by replacing u by v and v by u is identical with the original equation, the curve is symmetrical with respect to the line $u = v$.

Step Two. The intercepts can be found by setting $v = 0$ and solving for u , and setting $u = 0$ and solving for v .

Step Three. The equations of any vertical asymptotes can be found by equating to zero the real linear factors of the coefficient of the highest power of v . The equations of any horizontal asymptotes can be found by equating to zero the real linear factors of the coefficient of the highest power of u .

Step Four. If the equation of the curve can easily be solved for one variable in terms of the other, it may be possible to establish limits of extent. With this determination of limits of extent it is also possible to ascertain the number of real values of one variable that correspond to certain values of the other.

Step Five. Special points on the curve are located and the nature of the curve is investigated at those points; that is, points of inflection, cusps, etc., are found.