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Fourier Series and Some Applications.

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FOURIER SERIES AND SOME APPLICATIONS

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TABLE OF CONTENTS

PAGE

INTRODUCTION 1

ORTHOGONAL FUNCTIONS 2

Periodicity 2

Special Integrals 3

Orthogonality of Trigonometric Functions 3

FUNDAMENTAL IDEAS CONCERNING INFINITE SERIES 4

Series With Constant Terms 4

Series With Terms Which Are Functions of x 4

Fourier Series 5

FOURIER SERIES 5

Formal Determination of Fourier Coefficients 5

Examples 7

Convergence Theorem 12

Sine Series; Cosine Series 13

PARTIAL DIFFERENTIAL EQUATIONS 15

Definition of Partial Derivatives 15

Introduction to Partial Differential Equations 17

Solutions of Certain Partial Differential Equations 18

TWO APPLICATIONS OF FOURIER SERIES 19

Problem on Distribution of Heat 19

Vibrating String Problem 24

SUMMARY 28

BIBLIOGRAPHY 30

INTRODUCTION

The object of the author in writing this thesis was to make a compilation of material necessary to work certain problems in mathematical physics with the use of Fourier series. Although whole books have been written on different aspects of this thesis, all had some intervening material which was too advanced for the average student.

Any student interested in mathematics or physics cannot fail to see the importance of Fourier series. To mention a few applications, the series is used in finding the solution to various problems in heat distribution, electrical transmission, and aircraft construction.

The information for this thesis was taken from different books containing partial differential equations, orthogonal functions, and infinite series. Also, the author's seminar notes were used to a great extent.

The writer introduces orthogonal functions and infinite series before the discussion of Fourier series. The examples help one to understand more completely the mechanics of the latter series. After a discussion of certain partial differential equations, the author proceeds to some applications of Fourier series.

ORTHOGONAL FUNCTIONS

In general, two functions u(x) and v(x) are said to be orthogonal to each other over an interval (a, b) if the integral of their product vanishes over this interval, that is

$$\int_a^b u(x) \cdot v(x)\, dx = 0.$$

This concept of orthogonality is related, not obviously, to that of perpendicularity in geometry. Since the latter relations is not needed in this paper, it will not be discussed further.

Periodicity

A function f(x) has a period p if $f(x+p) = f(x)$ for all values of x. Since $\sin(x+2\pi) = \sin x$, it follows that sin x has a period 2π. Likewise cos x has a period 2π, and cos 3x has a period $2\pi/3$. The period p may, or may not, be the smallest value of the period.

If the functions f(x) and g(x) have p for a period, it is obvious that the following have a period p:

$$f(x) + g(x),$$

$$f(x) - g(x),$$

$$f(x) \cdot g(x).$$

Special Integrals

It becomes necessary that the following special integrals be mentioned. Assume p and q to be non-negative integers.

(1) $$\int_{-\pi}^{\pi} \cos px \, dx = 0, \quad p \neq 0$$
$$= 2\pi, \quad p = 0$$

(2) $$\int_{-\pi}^{\pi} \sin px \, dx = 0,$$

(3) $$\int_{-\pi}^{\pi} \sin px \cos qx \, dx = 0,$$

(4) $$\int_{-\pi}^{\pi} \cos px \cos qx \, dx = 0, \quad p \neq q$$

(5) $$\int_{-\pi}^{\pi} \cos^2 px \, dx = \pi, \quad p \neq 0$$
$$= 2\pi, \quad p = 0$$

(6) $$\int_{-\pi}^{\pi} \sin^2 px \, dx = \pi, \quad p \neq 0$$
$$= 0, \quad p = 0$$

(7) $$\int_{-\pi}^{\pi} \sin px \sin qx \, dx = 0, \quad p \neq q.$$

These special integrals will be used from time to time throughout the paper. They will be referred to by number.

Orthogonality of Trigonometric Functions

It is obvious, from the special integrals (3), (4), and (7), that the trigonometric functions sin px, $p = 1, 2, 3, \cdots$, and cos qx, $q = 0, 1, 2, \cdots$, are orthogonal to each other over the interval $(-\pi, \pi)$.

FUNDAMENTAL IDEAS CONCERNING INFINITE SERIES

Series With Constant Terms

A sequence is a succession of terms formed according to some fixed rule or law.

A series is the indicated sum of the terms of a sequence. When the number of terms is limited, the series is said to be finite. When the number of terms is unlimited, the series is called an infinite series.

An infinite series of constants would be

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

The above series is said to converge to a finite sum L if

$$\lim_{n \to \infty} S_n = L, \quad \text{where } S_n = \sum_{i=1}^{n} (a_i),$$

which is to say that the $\left| L - S_n \right|$ can be made smaller than any other preassigned small positive quantity by taking n large enough.

Series With Terms Which Are Functions of x

An infinite series whose terms are functions of x is

$$u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) + \cdots .$$

For different values of x, the above series may diverge or converge. Suppose that the series converges for $a < x < b$;

then the sum of the series is something which depends on x, say a function of x, f(x). It is said that a series of this type represents f(x) in the interval (a,b).

Fourier Series

Under certain conditions a given function f(x) may be represented in a certain interval $(-\pi, \pi)$ by a series of the form

$$(8) \qquad f(x) \sim \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots .$$

Series (8) is called a Fourier series when the coefficients are determined properly.

The symbol $\sim$ is used to mean that the series represents f(x) or corresponds to f(x). The correspondence between f(x) and its series may not always be an equality. In particular, the series is likely not to converge to f(x) at a point of discontinuity.

FOURIER SERIES

Formal Determination of Fourier Coefficients

If it is assumed that the Fourier series converges in the interval $(-\pi, \pi)$, and that the series can be integrated term by term, integration of (8) with the use of (1) and (2) gives

$$(9) \qquad \int_{-\pi}^{\pi} f(x)\, dx = a_0 \pi , \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx.$$

Each term of the series, with the exception of the constant term, reduces to zero.

To determine a_k when $k \neq 0$ let (8) be multiplied through by cos kx, and prepare to integrate in the interval $(-\pi, \pi)$. This gives

$$\int_{-\pi}^{\pi} f(x) \cos kx \, dx = \frac{a_o}{2} \int_{-\pi}^{\pi} \cos kx \, dx$$

$$+ a_1 \int_{-\pi}^{\pi} \cos x \cos kx \, dx + a_2 \int_{-\pi}^{\pi} \cos 2x \cos kx \, dx + \cdots$$

$$+ b_1 \int_{-\pi}^{\pi} \sin x \cos kx \, dx + b_2 \int_{-\pi}^{\pi} \sin 2x \cos kx \, dx + \cdots$$

for $k = 1, 2, 3, \cdots$.

Again, after integration, each integral on the right, with the exception of one, reduces to zero because of (1), (3), and (4). This one is the term containing cos kx cos kx or $\cos^2 kx$, and it is found that

$$\int_{-\pi}^{\pi} f(x) \cos kx = a_k \int_{-\pi}^{\pi} \cos^2 kx \, dx = a_k \pi .$$

Therefore

$$(10) \qquad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad k = 0, 1, 2, \cdots.$$

Similarly, if (8) is multiplied through by sin kx and integrated, the resulting expression

$$(11) \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, \quad k = 1, 2, 3, \cdots$$

can be obtained.

After the coefficients are determined, one can insert them into the series and investigate for convergence.

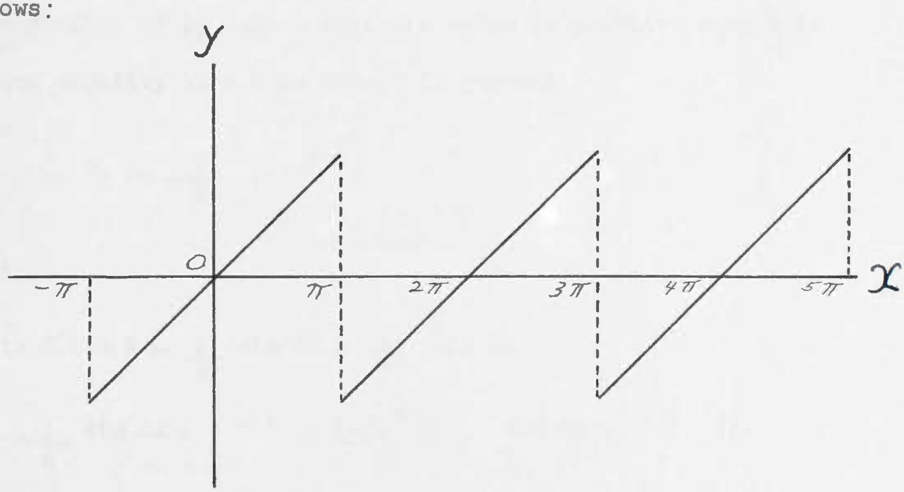
Examples

A few examples will help to get a better understanding of the series.

Example 1. Find the series corresponding to the function

$$f(x) = x, \quad -\pi < x < \pi,$$

and let $f(x + 2\pi) = f(x)$. The graph of this function is as follows:



Using (9) to find the coefficient a_o gives

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx,$$

$$a_o = 0$$

From (10),

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos kx \, dx = \frac{1}{\pi}\left[\frac{x}{k} \sin kx + \frac{1}{k^2} \cos kx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{k^2} \cos k\pi - \frac{1}{k^2} \cos k\pi\right],$$

$$a_k = 0.$$

From (11),

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx \, dx = \frac{1}{\pi}\left[\frac{-x}{k} \cos kx + \frac{1}{k^2} \sin kx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{k} \cos k\pi - \frac{\pi}{k} \cos k\pi\right],$$

$$b_k = -\frac{2}{k} \cos k\pi.$$

The inspection of b_k shows that its value is positive when k is odd, and negative when k is even. In general

$$b_k = \frac{2}{k} (-1)^{k-1}.$$

Hence,

$$f(x) \sim 2(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$$

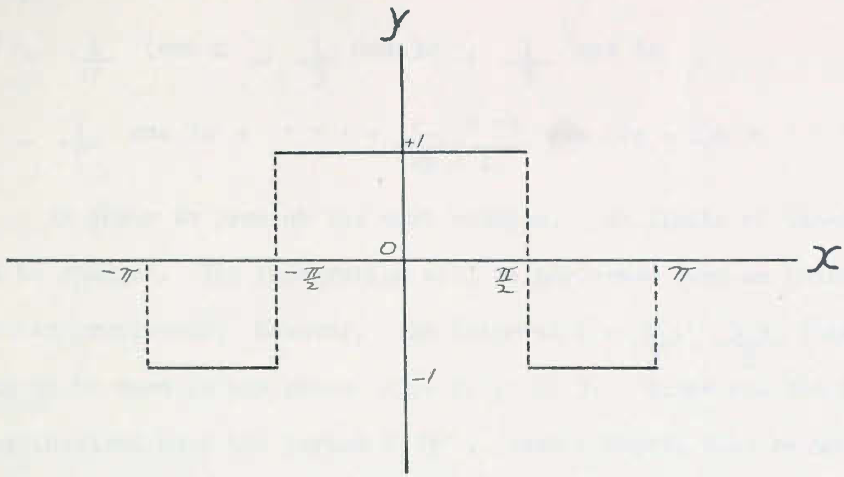
$$- \frac{1}{4} \sin 4x + \cdots + \frac{(-1)^{k-1}}{k} \sin kx + \cdots).$$

It will be shown later that the value of this series converges to the value of the function for all x except at the points of discontinuity.

Example 2. Let

$$f(x) = -1, \quad -\pi \leqslant x < -\frac{\pi}{2},$$
$$= 1, \quad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2},$$
$$= -1, \quad \frac{\pi}{2} < x \leqslant \pi.$$

The graph of this function is known as a square wave.



Formulas (9), (10), and (11) will be used in determining the coefficients of the corresponding series.

$$a_o = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} -dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx + \int_{\frac{\pi}{2}}^{\pi} - dx\right]$$

$$= \frac{1}{\pi}\left\{ - \Big[x\Big]_{-\pi}^{-\frac{\pi}{2}} + \Big[x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \Big[x\Big]_{\frac{\pi}{2}}^{\pi} \right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2} + \frac{\pi}{2} - \pi + \frac{\pi}{2}\right],$$

$$a_o = 0.$$

$$a_k = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} -\cos kx\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos kx\, dx + \int_{\frac{\pi}{2}}^{\pi} -\cos kx\, dx\right]$$

$$= \frac{1}{\pi}\left\{ - \left[\frac{1}{k}\sin kx\right]_{-\pi}^{-\frac{\pi}{2}} + \left[\frac{1}{k}\sin kx\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \left[\frac{1}{k}\sin kx\right]_{\frac{\pi}{2}}^{\pi} \right\},$$

$$a_k = \frac{4}{k\pi}\sin\frac{k\pi}{2}.$$

$$b_k = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} - \sin kx\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin kx\, dx + \int_{\frac{\pi}{2}}^{\pi} - \sin kx\, dx\right]$$

$$= \frac{1}{\pi}\left\{ \left[\frac{1}{k}\cos kx\right]_{-\pi}^{-\frac{\pi}{2}} - \left[\frac{1}{k}\cos kx\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left[\frac{1}{k}\cos kx\right]_{\frac{\pi}{2}}^{\pi} \right\},$$

$$b_k = 0.$$

Hence,

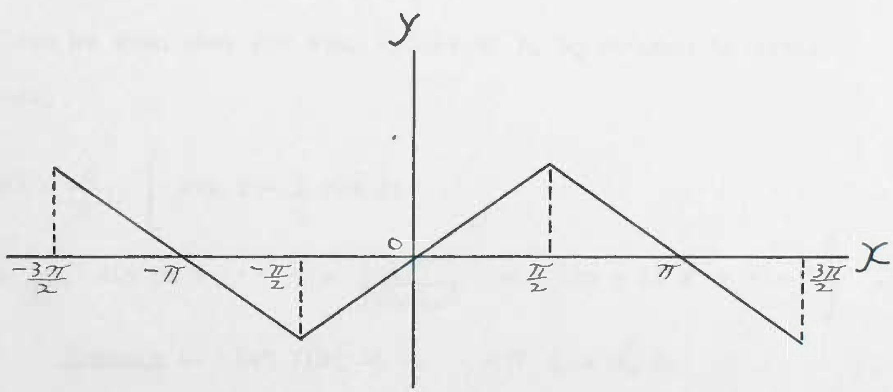
$$f(x) \sim \frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots + \frac{(-1)^{n-1}}{2n-1}\cos(2n-1)x + \cdots\right).$$

In order to present the next example, the limits of integration will be changed. The integration will be performed over an interval 2π as previously; however, the interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is going to be used in the place of $(-\pi, \pi)$. Since all the functions involved have the period 2π, each integral will be unchanged by this change in the limits of integration.

<u>Example</u> 3. Let $f(x) = \frac{2}{\pi}x, \quad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2},$

$$= -\frac{2}{\pi}x + 2, \quad \frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}.$$

The graph of this function represents an anti-symmetric saw-tooth wave.



To find the coefficients of the corresponding series of f(x), the formulas (9), (10), and (11) will be used.

From (9),

$$a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi}\, x\, dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{-2x}{\pi} + 2\right) dx,$$

$a_0 = 0.$

By using (10),

$$a_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2x}{\pi} \cos kx\, dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{-2x}{\pi} + 2\right) \cos kx\, dx,$$

$a_k = 0.$

Formula (11) gives

$$b_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi}\, x \sin kx\, dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{-2x}{\pi} + 2\right) \sin kx\, dx,$$

$$= \frac{1}{\pi k}\left(\cos \frac{3\pi k}{2} - \cos \frac{\pi k}{2}\right) + \frac{2}{\pi^2 k^2}\left(3 \sin \frac{\pi k}{2} - \sin \frac{3\pi k}{2}\right).$$

For $k = 1, 2, 3, \cdots$, the sum of the cosine terms equal zero.

Therefore,

$$b_k = \frac{2}{\pi^2 k^2}\left(3 \sin \frac{\pi k}{2} - \sin \frac{3\pi k}{2}\right).$$

It can be seen that for even values of k, b_k reduces to zero.

Hence,

$$f(x) \sim \frac{8}{\pi^2}\left[\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x + \cdots + \frac{(-1)^{n-1}}{(2n-1)^2}\sin (2n-1)x + \cdots\right].$$

<u>Example</u> 4. Let $f(x) = 0, \quad -\pi \leq x \leq 0,$

$\quad = x, \quad 0 \leq x \leq \pi,$

then using (9), (10), and (11),

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{0} 0 \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \, dx,$$

$$a_0 = \frac{\pi}{2}.$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{0} 0 \cos kx \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \cos kx \, dx,$$

$$a_k = \frac{1}{\pi k^2} (\cos k\pi - 1).$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{0} 0 \sin kx \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \sin kx \, dx,$$

$$b_k = \frac{-1}{k} \cos k\pi.$$

The inspection of a_k shows that its value is zero when k is even, and its value equals $\frac{-2}{\pi k^2}$ when k is odd. Also, for odd integers of k, b_k equals $\frac{1}{k}$; for even integers, b_k equals $\frac{-1}{k}$.
Hence, the corresponding series for f(x) is

$$f(x) \sim \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots + \frac{1}{(2n-1)^2}\cos(2n-1)x + \cdots\right]$$

$$+ \sin x - \frac{1}{2}\sin 2x$$

$$+ \frac{1}{3}\sin 3x + \cdots + \frac{(-1)^{k-1}}{k}\sin kx + \cdots.$$

Convergence Theorem

<u>Theorem</u>. Let f(x) be a function defined arbitrarily in the interval $(-\pi, \pi)$, and outside this interval defined by the equation $f(x + 2\pi) = f(x)$ so that it is periodic with period

2π. If f(x) has a finite number of points of ordinary discontinuity and a finite number of maxima and minima in the interval $(-\pi, \pi)$, then it can be represented by series (8), with the use of (9), (10), and (11), which converges at every point $x = x_0$ of the interval to the value

$$\frac{f(x_0 + \ ) + f(x_0 - \ )}{2}.$$

If f(x) is continuous at the point $x = x_0$, then $f(x_0 + \ ) = f(x_0 - \ ) = f(x_0)$, so that at all points of continuity the series converges to f(x). At the points of ordinary discontinuity it converges to the arithmetic mean of the values of the right-hand and left-hand limits (Sokolnikoff, 1939).

As an illustration take example 1. The function $f(x) = x$ has a sine series, and was defined to be periodic. As the number of terms is increased, the value of the series will approach the value of the function as a limit for all values of x, $-\pi < x < \pi$, but not for $x = \pm\pi$. Since the series has a period 2π, it represents a discontinuous function with discontinuities at $x = \pm (2n + 1)\pi$. At these points the series converges to zero as a consequence of the convergence theorem. At points within the period, the series converges to the value of f(x).

Sine Series; Cosine Series

From the examples given previously, one can notice that a function f(x) may have a sine series or a cosine series. In general, the series contains both sines and cosines. It is possible to determine beforehand whether the series will be a sine series or a cosine series from the idea of odd and even functions (Churchill, 1941).

An even function is defined as a function of x for which $f(-x) = f(x)$.

An odd function is defined as a function of x for which $f(-x) = -f(x)$.

If f(x) is an even function, it has the following property:

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx.$$

If f(x) is an odd function, it has the following property:

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

If f(x) is an even function, then f(x) cos kx is even. The proof of this is simple. Let

$$g(x) = f(x)\cos kx,$$

then $g(-x) = f(-x)\cos(-kx)$,

or $g(-x) = f(x)\cos kx$,

so $g(-x) = g(x)$.

Similarly, if f(x) is an even function, then f(x) sin kx is odd. As proof, let

$$g(x) = f(x)\sin kx,$$

then $g(-x) = f(-x)\sin(-kx)$,

or $g(-x) = f(x)\,(-\sin kx)$,

or $g(-x) = -f(x)\sin kx$,

so $g(-x) = -g(x)$.

It becomes obvious that if $f(x)$ is an even function in the interval $(-\pi, \pi)$ the Fourier series for $f(x)$ would contain only cosine terms, and the coefficients would be given by

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos kx \, dx,$$

$$b_k = 0.$$

Similarly, if $f(x)$ is odd, then $f(x) \sin kx$ is even and $f(x) \cos kx$ is odd. The proof for each would be similar to the previous proofs. In this case the Fourier series would contain only sine terms, and the coefficients would be given by

$$a_k = 0.$$

$$b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx \, dx.$$

PARTIAL DIFFERENTIAL EQUATIONS

Definition of Partial Derivatives

Let u be given as a function of two independent variables x and y,

$$u = f(x,y).$$

If y is given some fixed value, u will vary only when x changes. When x takes on an increment Δx, u will change by an amount Δu such that

$$\Delta u = f(x + \Delta x, y) - f(x, y).$$

and $$\frac{\Delta u}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

Now if Δx is allowed to approach zero as a limit, the quotient may approach a limit. When this limit exists, it is called the partial derivative of u with respect to x:

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

Similarly, if x is given some fixed value and y given an increment Δy, one is led, in general, to the limit

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y},$$

which is the partial derivative of u with respect to y (Miller, 1941).

If $u = f(x,y)$, its first partial derivatives with respect to x and y are functions of x and y. These functions may also be differentiated partially to obtain the four partial derivatives of second order; namely,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

When the two second-order derivatives $\partial^2 u / \partial x \partial y$ and

$\partial^2 u/\partial y \partial x$ are continuous functions of x and y, they are identical; therefore the order of differentiation is immaterial (Miller, 1941).

Example. Find the second partial derivatives of $u = f(x,y) = 2x^2y - 4xy^2$,

$$u = 2x^2y - 4xy^2,$$

$$\frac{\partial u}{\partial x} = 4xy - 4y^2, \quad \frac{\partial u}{\partial y} = 2x^2 - 8xy,$$

$$\frac{\partial^2 u}{\partial x^2} = 4y, \quad \frac{\partial^2 u}{\partial y^2} = -8x,$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = 4x - 8y.$$

Introduction to Partial Differential Equations

A partial differential equation is a relation between any number of independent variables x_1, x_2, x_3, $\cdots x_n$, a dependent variable u depending upon them, and the partial derivatives of u with respect to the independent variables. The order of the equation is that of the derivative of highest order contained in it. Thus two partial differential equations of the first order are

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0,$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}.$$

A partial differential equation of the second order would be

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Solutions of Certain Partial Differential Equations

Taking the partial differential equation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

from the last paragraph, one can see that a solution would be $u = x + y$. For

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 1,$$

satisfies the equation. Other solutions are $u = \sin(x + y)$, $u = e^{x+y}$, $u = (x + y)^n$, and, in general, $u = f(x + y)$. The equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

is satisfied by the particular solution $u = \tan^{-1} y/x$. As proof,

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2 + y^2}, \qquad \frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2},$$

therefore

$$\frac{-xy}{x^2 + y^2} + \frac{xy}{x^2 + y^2} = 0.$$

TWO APPLICATIONS OF FOURIER SERIES

Problem on Distribution of Heat

Now it is possible to work a problem with the application of Fourier series. As an illustration take a thin rectangular plate of infinite length and width equal to π. Let the long edges of the plate be kept at constant temperature zero, and one of the short edges be the base with temperature equal to some function f(x). The temperature decreases with increasing distances from the base to the temperature zero at an infinite distance from the base. Assume that the temperature has reached a steady state. The problem is to find the temperature at any point on the plate (Byerly, 1893).

It is convenient to use the rectangular coordinate system for this problem. Let the base be on the x-axis with one end at the origin, and let the long edges of the plate extend in the direction of the y-axis.

To solve a problem of two-dimensional steady-state heat flow, one has to deal with Laplace's equation (Miller, 1941). The equation has the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{12}$$

Also the following boundary conditions will be needed to get the temperature u(x,y):

(1) $u(0,y) = 0$, when $y > 0$,

(2) $u(\pi,y) = 0$, when $y > 0$,

(3) $\lim_{y \to +\infty} u(x,y) = 0, \quad 0 \leqslant x \leqslant \pi,$

(4) $u(x,0) = f(x),$

where f(x) is a given function assigned in advance.

Using the method of separation, one can assume that a particular solution of (12) is some function u(x,y) of the form X(x) · Y(y), where X(x) is a function of x alone, and Y(y) is a function of y alone. If the assumption is correct, it will lead to a solution; otherwise it is not justifiable. Beginning with u(x,y) = X · Y and taking partial derivatives with respect to x and y, the following

$$\frac{\partial u}{\partial x} = X' \cdot Y, \qquad \frac{\partial u}{\partial y} = X \cdot Y',$$

$$\frac{\partial^2 u}{\partial x^2} = X'' \cdot Y, \qquad \frac{\partial^2 u}{\partial y^2} = X \cdot Y''$$

is obtained. Substitution in (12) gives

$$X'' \cdot Y + X \cdot Y'' = 0$$

or

$$-\frac{X''}{X} = \frac{Y''}{Y} .$$

Since the left member is independent of y and the right member is independent of x, each member must be equal to the same constant. As far as equation (12) is concerned the constant could be positive or negative. However, to satisfy the boundary conditions of this problem it is necessary to have the constant positive, in which case it might be represented by $+\lambda^2$, λ being assumed to be real (Jackson, 1941). This can be seen by setting

$$\frac{-X''}{X} = \frac{Y''}{Y} = -\lambda^2$$

and solving for Y. Then

$$Y = C_1 \sin \lambda y + C_2 \cos \lambda y;$$

but this cannot satisfy boundary condition (3), $\lim_{y \to \infty} u(x,y) = 0$. The constants C_1 and C_2 would have to be zero. This cannot happen for a proper solution to the problem at hand. Therefore setting both members of

$$\frac{-X''}{X} = \frac{Y''}{Y}$$

equal to $+\lambda^2$, and not to $-\lambda^2$, is the best policy.

Then X and Y separately satisfy the differential equations

$$X''(x) = -\lambda^2 X(x), \quad Y''(y) = \lambda^2 Y(y).$$

The first has the solutions $\cos \lambda x$ and $\sin \lambda x$; whereas the second has solutions $e^{\lambda y}$ and $e^{-\lambda y}$. Hence, four possibilities for $u(x,y) = X(x) \cdot Y(y)$ are:

(a) $e^{\lambda y} \sin \lambda x$,

(b) $e^{\lambda y} \cos \lambda x$,

(c) $e^{-\lambda y} \sin \lambda x$,

(d) $e^{-\lambda y} \cos \lambda x$.

Of the four possibilities, functions (a) and (c) satisfy boundary condition (1); however functions (b) and (d) do not. Function (a) does not satisfy boundary condition (3), but function (c) does. Now, that leaves only $e^{-\lambda y} \sin \lambda x$ to work with. It can be seen that this function will satisfy boundary condition (2) if λ is any integer. Therefore $e^{-y}\sin x$, $e^{-2y}\sin 2x$, $e^{-3y}\sin 3x$, $\cdots$, $e^{-ny} \sin nx$, $(n = 1, 2, 3, \cdots)$, are solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and they satisfy boundary conditions (1), (2), and (3).

It follows that if k_n is any constant for n an integer

$$k_n e^{-ny} \sin nx$$

is a solution. This follows from the fact that a solution of a homogeneous differential equation multiplied by a constant is also a solution (Byerly, 1893).

Assuming convergence, the following is true:

$$u(x,y) = k_1 e^{-y}\sin x + k_2 e^{-2y}\sin 2x + \cdots + k_n e^{-ny}\sin nx + \cdots.$$

This follows from the fact that if one has several solutions of a homogeneous differential equation the sum of such solutions is also a solution (Byerly, 1893).

By using the boundary condition $u(x,0) = f(x)$, one can determine the coefficients or k's.

$$f(x) = u(x,0) = k_1 \sin x + k_2 \sin 2x + \cdots + k_n \sin nx + \cdots.$$

The above series will represent f(x) if the coefficients are given by

$$k_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad (n = 1, 2, 3, \cdots).$$

Finding this expression for the coefficients is accomplished by taking

$$\int_0^{\pi} f(x) \sin nx \, dx = k_1 \int_0^{\pi} \sin x \sin nx \, dx + k_2 \int_0^{\pi} \sin 2x \sin nx \, dx$$

$$+ \cdots + k_n \int_0^{\pi} \sin nx \sin nx \, dx,$$

or $$\int_0^{\pi} f(x) \sin nx \, dx = k_n \int_0^{\pi} \sin^2 nx \, dx = \frac{k_n \pi}{2}.$$

Therefore

$$k_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

This system consisting of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the four boundary conditions then has the solution

(13) $$u(x,y) = k_1 e^{-y} \sin x + k_2 e^{-2y} \sin 2x + k_3 e^{-3y} \sin 3x$$

$$+ \cdots + k_n e^{-ny} \sin nx + \cdots$$

where

$$k_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad (n = 1, 2, 3, \cdots).$$

Vibrating String Problem

The vibrations of a stretched uniform elastic string fastened at both ends are described by a second-order partial differential equation.

Let the string have a length of π units for simplicity in working the problem. Assume that one end of the string is at the origin, and that the string is along the x-axis. Assume further that the motion of the plucked string is in the (x,y)-plane. The problem is to get a function y(x,t) which will give the displacement y from the equilibrium position for any x and t. Again Fourier series can be applied.

The function y(x,t) must satisfy the partial differential equation of the vibrating string (Churchill, 1941),

$$(14) \qquad \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2},$$

where a is a positive constant depending on the units of measurement and the physical properties of the string.

If the string is displaced initially into the shape $y = f(x)$ at time $t = 0$, the function y(x,t) must also satisfy the boundary conditions

$$(1) \qquad y(0,t) = 0,$$

$$(2) \qquad y(\pi,t) = 0,$$

$$(3) \qquad \frac{\partial y}{\partial t}(x,0) = 0, \quad 0 < x < \pi,$$

$$(4) \qquad y(x,0) = f(x), \quad 0 < x < \pi,$$

where f(x) is assigned in advance. Boundary condition (3) states that the velocity is zero at the initial displacement.

Particular solutions of (14) may be found by the method assumed for the heat problem. Therefore

$$y(x,t) = X(x) \cdot T(t),$$

$$\frac{\partial y}{\partial x} = X' \cdot T, \quad \frac{\partial y}{\partial t} = X \cdot T'$$

$$\frac{\partial^2 y}{\partial x^2} = X'' \cdot T, \quad \frac{\partial^2 y}{\partial t^2} = X \cdot T''.$$

Substitution in (14) gives

$$X \cdot T'' = a^2 X'' \cdot T$$

or

$$\frac{T''}{a^2 T} = \frac{X''}{X} \cdot$$

Since the first member is independent of x and the second member is independent of t, the quantity must be equal to a constant, say $-\lambda^2$, λ being assumed to be real. A positive constant would not be useful for the present problem (Jackson, 1941). This can be seen by setting

$$\frac{T''}{a^2 T} = \frac{X''}{X} = \lambda^2.$$

The solution

$$X = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

will not satisfy the boundary conditions (1) and (2). There are no values of C_1 and C_2 for which $X(0) = 0$ and $X(\pi) = 0$. Therefore, one must use the negative constant $-\lambda^2$ (Churchill, 1941).

Now X and T separately satisfy the differential equations

$$T''(t) = -\lambda^2 a^2 T(t), \qquad X''(x) = -\lambda^2 X(x).$$

The first has the solutions $\sin \lambda at$ and $\cos \lambda at$; whereas the second has solutions $\sin \lambda x$ and $\cos \lambda x$. Hence, the four possible solutions for $y(x,t) = X(x) \cdot T(t)$ are:

(a) $\sin \lambda x \sin \lambda at,$

(b) $\sin \lambda x \cos \lambda at,$

(c) $\cos \lambda x \sin \lambda at,$

(d) $\cos \lambda x \cos \lambda at.$

As in the last problem, it may be supposed that λ is a positive number.

Of the four possible solutions, (a) and (b) satisfy boundary condition (1); whereas functions (c) and (d) do not. Function (a) does not satisfy boundary condition (3), but function (b) does. Now function (b) will satisfy (2) if λ is an integer. Then

$\sin x \cos at, \ \sin 2x \cos 2at, \ \cdot \cdot \cdot, \ \sin nx \cos nat,$

$(n = 1, 2, 3, \cdot \cdot \cdot),$

are solutions to (14), and they also satisfy the boundary conditions

(1), (2), and (3). If k_n is any constant for n an integer

$$k_n \sin nx \cos nat$$

is a solution (Byerly, 1893). Furthermore, assuming convergence, the following is true (Jackson, 1941).

$$y(x,t) = k_1 \sin x \cos at + k_2 \sin 2x \cos 2at + \cdots$$
$$+ k_n \sin nx \cos nat + \cdots.$$

Using the boundary condition (4), one can determine the coefficients or k's.

$$f(x) = y(x,0) = k_1 \sin x + k_2 \sin 2x + k_3 \sin 3x + \cdots$$
$$+ k_n \sin nx + \cdots.$$

The above series will represent f(x) if the coefficients are given by

$$k_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad (n = 1, 2, 3, \cdots).$$

Finding this expression for the coefficients is done by taking

$$\int_0^{\pi} f(x) \sin nx \, dx = k_1 \int_0^{\pi} \sin x \sin nx + k_2 \int_0^{\pi} \sin 2x \sin nx$$
$$+ \cdots + k_n \int_0^{\pi} \sin nx \sin nx \, dx,$$

or

$$\int_0^{\pi} f(x) \sin nx \, dx = k_n \int_0^{\pi} \sin^2 nx \, dx = \frac{k_n \pi}{2}$$

Therefore $k_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx \, dx.$

This system consisting of the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

and the four boundary conditions then has the solution

$$(15) \qquad y(x,t) = k_1 \sin x \cos at + k_2 \sin 2x \cos 2at + \cdots + k_n \sin nx \cos nat + \cdots$$

where $k_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad (n = 1, 2, 3, \cdots).$

SUMMARY

The representation of a function $f(x)$ by a Fourier series within a certain interval is not uncommon. Although the writer has taken the interval $(-\pi, \pi)$ as his limits in integration, it is not necessary to do so. The limits depend on the period of the function at hand. The period 2π may be replaced by one of arbitrary length; however the formulas will not be as simple.

One should notice that the solution to the heat problem contained exponential functions of y and sine functions of x. Since the plate was assumed to have reached a steady state, the time element did not enter into the solution. Other problems of this type could be worked where the time element is a factor. In fact one could have a different surface with different boundary conditions.

The vibrating string problem presents a very interesting case for the individual interested in vibratory motion. Using a special function f(x), one could find the amplitude and period for a fixed x. For other functions, a person might be interested in the nodes and overtones. When the string is struck instead of plucked, the boundary conditions would be different than those given.

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