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The Segregation of Real Roots of Lower Orders

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THE SEGREGATION OF REAL ROOTS

OF LOWER ORDERS

being

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by

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CONTENTS

Introduction

The purpose of this thesis is the presentation of the better algebraic methods of the segregation of real roots of equations of lower orders and a brief historical sketch of the advance made in the development of algebra. Mention is also made of a few of the men who have made notable contributions to this field of mathematics and the time, as nearly as it is possible to determine it, when new ideas were discovered.

The solution of a problem by algebraic methods has interested man through many ages. The clear skies of Egypt made possible an early development in astronomy, the annual flooding of the Nile valley, washing landmarks away, provided the necessity for surveying and both of these paved the way for the development of a form of mathematics to help those people solve the problems confronting them.

Down through the period of civilization of mankind there have developed, and been passed on, new mathematical ideas that have been helpful to man. There have also been periods of inactivity when nothing constructive was produced, but out of which the mathematical thought would awaken and go forward again, acquiring new ideas and leading in the advancement of civilization.

The Egyptians, the Greeks, the Arabs, the Romans and in modern times the Europeans have all contributed to the development of the methods for the algebraic solution of their mathematical problems.

The writer has attempted to present the different methods in a chronological order beginning with the simplest and advancing to the more difficult, therefore equations of the lowest order are considered first. In this study, 'roots of lower orders' denotes roots of equations of degree not beyond the fourth.

Chapter I

History of Roots

The study of root segregation has interested mathematicians for many centuries and they have learned much about it that is of value, not only to themselves, but to all the sciences that use mathematics as a tool. Before presenting an account of this knowledge, a brief historical sketch will be given to provide a proper understanding of its development.

The earliest known document containing a reference to roots is a papyrus, forming part of the Rhind collection in the British Museum, believed to have been written by an Egyptian priest named Ahmes. This papyrus written in hieroglyphics more than a thousand years before Christ is believed to be a copy, "with emendations",[1] of a treatise more than a thousand years older, that treats of the processes of arithmetic, geometry, and the "solution of simple numerical equations".[2] Of the many examples that Ahmes gave relative to roots the following is a good illustration:

1. Ball, W. W. R. _A Short Account of the History of Mathematics_, p. 5.
2. _Ibid._

He says, "heap, its seventh, its whole, it makes nineteen".[1] The object here is to find a number which when added to its seventh will equal nineteen. He gives 16 + 1/2 + 1/8 for his answer, which is correct. Although the papyrus does not contain theoretical results, theorems, general rules of procedure, or mathematical knowledge one would expect from the builders of pyramids, it does show "in several particulars a remarkable advanced state of mathematics at the time when Abraham visited Egypt".[2]

Nesselman[3] records three stages in the development of algebra. The first stage he calls rhetorical algebra, that is, a method of finding the roots of an equation by a reasoning process expressed in words and not using algebraic symbols. This method was used by Ahmes, the earliest Arabians, the Greek writers Iamblichus and Thymaridas, the early Italian writers, and Regiomantanus. The second period, in which the solutions are syncopated in form, begins with the later Western Arabs and Diophantus of Greece and extends to the middle of the seventeenth century with the exception of the works of Vieta and Oughtred, whose work like that of the Hindu's and the Europeans since the middle of the seventeenth century, form the third period, and are symbolic in form.

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1. Ball, W. W. R. A Short Account of the History of Mathematics, p. 5.
2. Cajori, Florian. History of Elementary Mathematics, p.19.
3. Heath, Sir Thomas L. Diophantus of Alexanderia, p. 48.

The ancient Greeks were not original in their arithmetic and algebra. They acknowledged the Egyptian priests as their teachers. It was not until the time of Nicomachus and Diophantus in the fourth century A. D. that they made substantial contributions to algebra. The nearest approach at this time to algebra is found where Thymaridus used a Greek word meaning "unknown quantity", while some of the work of Iamblichus, Theon of Smyrna and others were algebraic in meaning or principle. But in geometry they rose to heights undreamed of by their predecessors. They were acquainted with equations and could solve them geometrically. It was shown by proportion how the root of an equation of the first degree is found by the intersection of two straight lines. In the works of Heron and Archimedes are practical problems to be solved by forming linear equations, while quadratic equations are in the form of proportions. These early Greeks could "represent by geometric figures, equations of the form $a'/a'' \, x = b$, $a'/a'' \, x + b'/b'' \, y \ldots = m$. where all quantities were linear."[1] They could also solve general quadratic equations having different rational coefficients and represent their positive roots geometrically. Euclid, the great compiler of geometric knowledge, could solve linear equations and incomplete quadratics geometrically.

> "The three principal forms of equations first to be freed from geometric statement and completely solved are,

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1. Fink, Karl. A Brief History of Mathematics, p. 78.

> $x^2 = px + q, px = x^2 + q$, and $x^2 + px = q$. . . . In later times, with Heron and Diophantus, the solution of equations of the second degree was partly freed from the geometric representation, and passed into the form of an arithmetic computation proper (while desregarding the second sign in the square root)."[1]

Diophantus, who lived about the first part of the fourth century, is reputed as being the greatest algebraist of ancient Greece. He was one of the last of the Alexandrian mathematicians and had it not bëen for his work there would be no record of the Greeks making any notable accomplishment in the field of algebra. Before the discovery of the Rhind Papyrus his "Arithmetica" was the oldest known work on algebra. He worked with simple and quadratic equations, used algebraic synbols, and treated his problems analytically, being completely separated from geometry. Although he knew how to solve equations, nowhere in his "Arithmetica" does he explain the process. When his quadratic was "of the form

> $ax^2 + bx + c = 0$, he seems to have multiplied by 'a' and then 'completed the square' in much the same way as is now done."[2]

It is interesting to note that although both roots may be positive he never gives but one of them, always taking the positive value of the square. Should the root of the equation be negative or irrational, it was rejected as impossible. At that time the idea of a negative root had not been

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1. Fink, Karl. A Brief History of Mathematics, p. 81.
2. Ball, W. W. R. A Short Account of the History of Mathematics, p. 110.

conceived by the Greek, but irrational numbers were known by Pythagoras because he had discovered that the hypothenuse of a right angled isosceles triangle was incommensurable with its sides.

In the earliest period the Chinese algebra had one thing in common with the Greek. They solved their quadratic equations geometrically. The Chinese later developed a method of approximation for solving higher algebraic equations. By the seventeenth century the abacus had replaced the computing rods for business purposes, in Japan, yet it was despised by mathematicians for they were able to solve equations by use of the rods. These rods were usually of two colors, red and black. One color to designate positive and the other negative numbers.

> "This distinction between positive and negative is very old. In Chinese, cheng was the positive and fu the negative, and the same ideographs are employed in Japan today, only one of the terms having changed, sei being used for cheng. These Chinese terms are found in the Chiu-chang Suan-ahu as revised by Chang T's ng in the second century B. C., and hence are probably much more ancient even than the later date. The use of the red and black for positive and negative is found in Liu Hui's commentary on the Chiu-chang, written in 263 A. D., but there is no reason for believing that it originated with him. It is probably one of the early mathematical inheritances of the Chinese the origin of which will never be known. As applied to the solution of equations, however, we have no description of their use before the work of Ch'in Chiu-shao in 1247."

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1. Smith, D. E. and Mikami Yoshio. <u>A History of Japanese Mathematics</u>, p. 48.

Brahmagupta, a Hindu mathematician, supposed to have been born in 598 A. D., wrote a book about 660 entitled "Brahma-Sphuta-Siddhanta." Two chapters were devoted to arithmetic, algebra, and geometry. His form of writing was entirely rhetorical. In his algebra he solved quadratic equations.

The Arabian mathematician, Alkarismi, who lived in the ninth century A. D. wrote an algebra in which he solved quadratic equations, calling the unknown quantity either "the thing" or "the root"[1] (that is, of a plant). It was from him that the word "root" for the solution of an equation was obtained. In the solution of his problems he considered only the real and positive roots, but he admitted the possibility of there being two roots, which were unknown to the Greeks.

While the Hindu algebra is, in many respects, similar to that of Heron and Diophantus, it is also an improvement upon their work. Heron solved the quadratic equation $ax^2 \quad bx = c$ by a rule yielding, $x = \frac{\sqrt{ac - (b/2)^2} - b/2}{a}$ and this was followed by the Hindus until the time of Cridhara who simplified this by multiplying by "4a" not by "a" as did his predecessors and developed the rule, $x = \frac{\sqrt{4ac - b^2} - b}{2a}$ which is void of fractions under the radical. This also made possible the unifying of the three

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1. Ball, W. W. R. A Short Account of the History of Mathematics, p. 163.

cases, $ax^2 + bx = c$; $bx + c = ax^2$; $ax^2 + c = bx$, which were considered as separate forms by the Greeks. Some mathematicians believe this to have been the greatest innovation in the theory of affected quadratic equations developed by the Hindus.

> "The Hindus were the first to recognize the existence of absolutely negative numbers and of irrational numbers . . . . Thus Bhaskara gives x = 50 or -5 for the roots of $x^2 - 45x = 250$. 'But', says he, 'the second value is in this case not to be taken for it is inadequate; people do not approve of negative roots'. Thus negative roots were seen, but not admitted."[1]

Although Bhaskara did not accept negative roots, he did accept two positive roots. His point of view is easy to understand when one considers that his problems dealt with practical geometric form. His statement that "the square of a positive, as also a negative number, is positive; that the square root of a positive number is two fold, positive and negative. There is no square root of a negative number, for it is not a square", was far in advance at that time.

Leonardo of Pisa was one of the most traveled and learned men of his time. He studied the methods of calculation of Egypt, Syria, Greece, Sicily and India. Of these he found that of the Hindu to be unquestionably the best. In 1202 he published a mathematical book, "Liber Abaci". This book was for centuries the source of information on arith-

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1. Cajori, Florian. History of Elementary Mathematics, p.101

metic and algebra for other writers. It was written in a fluent and interesting style, containing the knowledge he had gained from the countries in which he had studied. It contained the best methods of calculation with integers and fractions used at that time. This as well as other books written by Leonardo, shows that he was a thinker, presenting his works in a new form free from conventions of the past. He proposed the universal use of the "Arabic Notation", of which the zero was the portion first adopted by the Christians. Leonardo also gave a thorough explanation of square and cube root, solved linear and quadratic equations by algebraic methods and although he realized that two values were true for the quadratic $x^2 + c = bx$, he failed to recognize negative or imaginary roots.

During the sixteenth century negative roots received considerable attention, but it seems impossible to say who first fully comprehended them. Cardan, in his treatise "Ars Magna" published in 1545, discussed negative and imaginary roots. Although Cardan mentioned negative roots and Bombelli wrote of them, they never understood their real significance and importance, speaking of them as being "false" or "fictitious". There is no doubt Cardan and Bombelli were the outstanding mathematicians of the Renaissance, yet they were no farther advanced on this phase than the Hindu, Phaskara, who as was previously stated, found negative roots but would not accept them. The expansion of the number system so as to include negative quantities was decidedly a slow, laborious process not fully conceived until well into the seventeenth century. An important contribution of Bombelli's was the recognition of the connection between the change of sign and the root of the equation.

Cardan's Ars Magna was a great advance over any algebra at that time. One of the important new items was the solution of the cubic equation, whose revelation brought forth a storm of protest from Tartaglia. Being famous as a mathematician, Tartaglia, in 1635 accepted a challenge from a certain Antonio del Fiori to a contest. According to the challenge each was to deposit a specified sum of money with a notary and the one to solve the most problems in a period of thirty days, from thirty problems proposed by his opponent, would be the winner. Fiori had learned from his instructor, Scipione Ferreo, deceased, the solution of a cubic of the type $x^3 + qx = r$. Tartaglia had perfected a solution for the general equation, $x^3 + px^2 = r$, so he prepared problems of that type. He also knew Fiori had the above mentioned solution, and guessing Fiori would construct problems accordingly he prepared a general solution for them. His guess was correct and he solved the thirty problems in less than two hours, while Fiori failed to solve a problem.

When Cardan heard of the contest he tried to get in touch with Tartaglia but every effort failed, so he hit upon the scheme of inducing the latter to visit him on the pretext that a nobleman at his home wished very much to meet such a learned man. Tartaglia succumbed to the flattery and went to visit at the house of Cardan. The nobleman failed to materialize but Cardan under the promise of strictest privacy, prevailed upon Tartaglia to reveal his method of solving the cubic, the promise being kept until the appearance of the formula in "Ars Magna."

Ferrari, the most illustrious student of Cardan, did much for mathematics even to the solving of the biquadratic equation, but since he was Cardan's pupil, Cardan appropriated his work and published it in his "Ars Magna" without giving note of its source.

The eminent French mathematician Francia Vieta is responsible for the theory of algebraic equations. He enriched algebra by innovations in notations and in method of solution. He swung away from the rule of "double fȃlse position" used by Cardan and Burgi, developing a method similar to that of ordinary root-extraction.

> "The main principle employed by him in the solution of equations is that of reduction. He solves the quadratic by making a suitable substitution which will remove the term containing 'x' to the first degree. Like Cardan, he reduces the general expression of the cubic to the form $x^3 + mx + n = 0$; then, assuming $x = \frac{(1/3a - z^2)}{z}$ and substituting, he gets $z^6 - bz^6 - 1/27a^3 = 0$. Putting $z^3 = y$, he has a quadratic. In the solution of biquadratics, Vieta still remains true to his principle of reduction. This gives him the well-known cubic resolvent. He thus adheres throughout to his favorite principle, and thereby introduces into algebra a uniformity of method which claims our lively admiration. In Vieta's algebra we discover a partial knowledge of the relation existing between the coefficients and the roots of an equa-

> tion. He shows that if the coefficient of the second term in an equation of the second degree is minus the sum of two numbers whose product is the third term, then the two numbers are roots of the equation. Vieta rejected all except positive roots; hence it was impossible for him to fully perceive the relation in question."[1]

Although Vieta rejected the negative roots of an equation, so did all other mathematicians before the Renaissance. In fact very few even understood the meaning of negative quantities. The German, Michael Stifel, wrote a treatise on numbers in 1544, where he mentions the negative quantities as being "absurd" or "fictitious below zero".

There is an indication that Fibonacci used them a very little, and while Diophantus found the product of two binominials as (a - b) (c - d) it remained for Pacioli to give the important rule "minus times minus gives plus," but used it only in obtaining the product of two binominials as did Diophantus. Pacioli's work does not show the use of purely negative quantities.

Thomas Harriot, a celebrated English mathematician born in 1560, was the first to begin separating a negative quantity from the rest of the equation and setting it in one member by itself.

Despite the assertion of Stifel that an equation could not have but one root unless they were both positive, it was stated by Alfred Girard in 1629 that the degree of an algebraic equation and the number of roots are equal. This,

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1. Cajori, Florian. A History of Mathematics, p. 138.

perhaps, was the beginning of the fundamental theorem of algebra, which states that every rational integral equation with real or complex coefficients has at least one real root. Descartes had a clearer concept of it than others of his time. In the summation of the total number of roots he distinguished between positive and negative real roots and between real and imaginary roots.

The first notable attempt to solve the theorem, of which there is a record was by d'Alembert in 1746. This proof seemed so true that it was accepted by most of the leading mathematicians and the theorem came to be known in France as d'Alembert's theorem.

It is believed that Gauss was the first to use the term, "fundamental theorem", and it is he who the world recognizes as first proving it. His first proof appeared in 1797. In the words of Gauss, this proof "had a double purpose, first, to show that all the proofs previously attempted of this most important theorem are unsatisfactory and illusory, and secondly, to give a newly constructed rigorous proof."[1] Gauss produced four proofs, the second and third being published in 1816 and the fourth in 1850.

In the early part of the seventeenth century Johann Huddle[2] developed a rule for finding equal roots.

Others who helped to enrich this phase of mathematics by their fruitful discoveries were Newton, Budan, Horner,

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1. D. E. Smith. A Source Book in Mathematics, p. 293.
2. Cajori, Florian. A History of Mathematics, p. 180.

and Sturm, so their work will be discussed in detail in the following chapters.

Chapter II

Rational Roots

Before presenting the methods of obtaining roots of equations a few of the terms used in this study will be defined and these definitions adhered to throughout the discussion.

DEFINITIONS:

If an algebraic expression containing an unknown quantity is equal, for only particular values of the unknown, to another expression differently constituted, the equality thus formed is called a conditional equation or simply an equation. An equation, then, is a statement of equality which is true only for certain values of the unknown quantity.

When the statement of equality between two expressions which become the same by the use of the permissible mathematical operations, it is called an identity and is true for all values of the unknown.

The root of an equation is any value of x that satisfies the equation.

An equation is of the nth degree in x when the highest power of x is n.

A complete equation is one containing terms involving x in all its powers from n to 0, and is incomplete when some of the terms are absent.

The term a_n, which does not contain x, is called the absolute term.

An expression involving one or several letters is called a function of these letters.

If a quantity can have different values in an expression it is called a variable. The variable to which values are assigned is called the independent variable or argument.

An absolute constant is a quantity whose value does not change.

An arbitrary constant is a quantity whose value is constant during the discussion.

NUMBER OF ROOTS OF AN EQUATION

The important theorem, every equation of the nth degree has n roots and no more, will be discussed first because after it is proved no mention will be made to the number of roots of any particular equation for the number will be determined by the degree of the equation.

If x_1 is a root of the equation $f(x) = 0$, then $f(x)$ may be divided by $x - x_1$ giving

$$f(x) = (x - x_1)\, \phi_1\, x + r$$

But since x_1 is a root of the equation, x_1 may be substituted for x and $f(x_1) = r$ is obtained.

If $r = 0$, the divisor is contained an integral number of times in the $f(x)$ and it has been shown that $x - x_1$ is a factor of $f(x)$ when x_1 is a root.

Consider the given equation,

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$$

This equation must have a root which will be denoted by x_1. When $f(x)$ is divided by $x - x_1$ the quotient will be designated by $\phi_1\,(x)$, giving the identical equation

$$f(x) = (x - x_1)\, \phi_1\, (x).$$

Again the equation $\phi_1(x) = 0$, which is of the $(n - 1)$th degree, must have a root which will be represented by x_2. Let the quotient obtained by dividing $\phi_1\,(x)$ by $x - x_2$ be $\phi_2(x)$.

Therefore

$$\phi_1(x) = (x - x_2)\,\phi_2(x)$$

and $$f(x) = (x - x_1)(x - x_2)\,\phi_2 x$$

where $\phi_2(x)$ is of the $(n - 2)$th degree.

Continuing in this way it is proved that $f(x)$ consists of the product of n factors, each containing x to the first degree, and a numerical factor $\phi_n(x)$. The factor $\phi_n(x)$ is of degree $n - n$, because as the n factors were removed, each reduced the degree of $f(x)$ by unity, also each time x^r was divided by x the coefficient remained one, $\therefore$ $\phi_n(x) = 1$, completing the proof of the identity.

$$f(x) = (x - x_1)(x - x_2)(x - x_3) \ldots (x - x_{n-1})(x - x_n).$$

It is evident that the substitution of any one of the numbers $x_1, x_2, \ldots x_n$ for x in the right member of this equation will reduce that member to zero and will automatically reduce $f(x)$ to zero; that is the equation $f(x) = 0$ has for its roots the n values $x_1, x_2, x_3, \ldots x_{n-1}, x_n$. That the equation cannot have any other roots can be plainly seen for if any other value is substituted for x in the right member, the factors will all be different from zero; therefore the product cannot vanish.

LINEAR EQUATION

A simple equation in x, that is one in which the unknown appears only to the first power, is of the form,

$ax + b = 0,$

where a and b may have any rational positive or negative value and a shall not equal zero. Its graph is a straight line parallel to the y-axis and the line crosses the x-axis at the point where $x = -b/a$, which is the root of the equation. To prove that this is the only solution, suppose $x = c$ and $x = d$.

Substitute c in (1)

$$ac + b = 0$$

Substitute d in (1)

$$ad + b = 0$$

Substracting

$$ac - ad = 0$$

Factoring

$$a(c - d) = 0$$

By the previous statement a is not equal to 0,

$$c - d = 0$$

or $\quad c = d.$

Hence there is only one root.

QUADRATIC EQUATIONS

A quadratic equation in x is an equation which when reduced to its simplest form may be written in the form,

$$ax^2 + bx + c = 0, \qquad (1)$$

where a, b, and c may have any real rational positive or negative value, except x cannot equal 0.

When b = 0, the equation takes the form $ax^2 \quad c = 0$ and is called a pure quadratic. When c = 0, it becomes

$ax^2 + bx = 0$ and one root is zero. When a = 0 it is simply a linear equation.

To solve the quadratic transpose the c giving,

$$ax^2 + bx = -c \qquad (2)$$

Dividing by a and completing the square,

$$x^2 + bx/a + b^2/4a^2 = b^2/4a^2 - c/a \qquad (3)$$

$$= (b^2 - 4ac)/4a^2$$

Taking the root of both members,

$$x + b/2a = \pm\sqrt{b^2 - 4ac/4a^2} \qquad (4)$$

Solving,

$$x_1 = (-b + \sqrt{b^2 - 4ac})/2a \qquad (5)$$

$$x_2 = (-b - \sqrt{b^2 - 4ac})/2a \qquad (6)$$

Adding the roots

$$x_1 + x_2 = -b/a$$

Multiplying,

$$x_1 \cdot x_2 = c/a$$

Thus it is seen that the roots of an equation are a function of the coefficients. That the sum of the roots is equal to the coefficient of x^{n-1} with its sign changed divided by the coefficient of x^n.

The quantity, $b^2 - 4ac$, is known as Δ, the discriminant, of the quadratic equation. When $\Delta = 0$ the roots are real, rational, and equal. When $\Delta > 0$ the roots are real, rational or irrational and unequal. When $\Delta < 0$ the roots are conjugate imaginaries.

RELATIONSHIP OF COEFFICIENTS TO ROOTS

In the preceding paragraph it was proved that the roots of a linear or of a quadratic equation were a function of the coefficients. It shall now be established that this principle holds for equations of higher degree.

Consider the equation where

$$f(x) \equiv$$

$$a_0x_n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \ldots a_n,$$

and $a_0 \neq 0$. If in the equation ($f(x) - 0$) the coefficient a_0 of the term x^n is not unity, each term must be divided by a_0, which merely expresses the relationship in fractional form.

If the equation has the n roots $x_1, x_2, x_3, \ldots x_n$ then

$$f(x) \equiv (x - x_1)(x - x_2)(x - x_3) \ldots (x - x_n)$$

Taking n successively equal to 2, 3, and 4, the following is obtained by actual multiplication:

When n = 2

$$f(x) = (x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 x_2 = 0$$

When n = 3

$$f(x) = (x - x_1)(x - x_2)(x - x_3) =$$

$$x^3 - (x_1 + x_2 + x_3) x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3 = 0$$

When n = 4

$$f(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4) =$$

$$x^4 - (x_1 + x_2 + x_3 + x_4)x^3 + (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x^2 - (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 \\ x_2x_3x_4) + x_1x_2x_3x_4 = 0$$

Thus it is seen that

$$a_1 = -(x_1 + x_2 + x_3 \ldots x_n)$$

$$a_2 = (x_1x_2 + x_1x_3 + x_2x_3 + \ldots x_{n-1}x_n)$$

$$a_3 = -(x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + \ldots x_{n-2}x_{n-1}x_n)$$

. . . .

$$a_n = (-1)^n x_1x_2x_3x_4 \ldots x_n$$

The preceding statements may be summarized in the following manner:

In the equation $f(x) = 0$, when x is of degree n and the coefficient of x is unity, the coefficient of x^{n-1} is the negative of the sum of the roots, the coefficient of x^{n-2} is the sum of the products of the roots taken two at a time, the coefficient of x^{n-3} is the negative of the sum of the products of the roots taken three at a time, and in such manner until the last term is reached which is the product of the roots, being positive or negative, depending upon whether n is even or odd.

> "It might appear that the n distinct relations existing between the coefficients and roots of an equation of the nth degree should offer some advantage in the general solution of the equation, that one of the n roots could be obtained by the elimination of the (n-1) roots from the n equations. But this process offers no advantage, for on performing this elimination we merely reproduce the proposed equation . . . While the equations expressing the relations

> between roots and coefficients offer no advantage in the general solution of equations, they are of service in the solution of numerical equations when some special relation is known to exist among the roots. Moreover in any algebraic equation they enable us to determine the relations between the coefficients which correspond to some given relations between the roots."[1]

1. Cajori, Florian. Theory of Equations, p. 12.

THE CUBIC EQUATION

There are many different solutions of the cubic equation, but the method given is essentially the same as that given by Vieta[1] in 1591.

In the general cubic equation,

$$x^3 + bx^2 + cx + d = 0, \text{ let } x = y - b/3 \qquad (1)$$

Substituting this value for x the following reduced cubic is obtained where the unknown appears only to the first and third power.

$$y^3 - by^2 + b^2y/3 - b^3/27 + by^2 - 2b^2y/3 + b^3/9 + cy + cb/3 + d = 0,$$

or $$y^3 + (c - b^2/3)y + 2b^3/27 - cb/3 + d = 0 \qquad (2)$$

The equation is simplified by letting,

$$p = c - b^2/3$$

and $$q = 2b^3/27 - cb/3 + d,$$

which puts it in the form,

$$y^3 + py + q = 0. \qquad (3)$$

To solve this equation, let

$$y = z - p/3z, \qquad (4)$$

obtaining,

$$27z^6 + 27qz^3 - p^3 = 0$$

or $$z^6 + qz^3 - p^3/27 = 0. \qquad (5)$$

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1. Dickson, L. E. First Course in Theory of Equations, p.45.

Since this equation is in quadratic form it can be solved for z^3 by using the quadratic formula.

$$z^3 = (-q \pm \sqrt{q^2 + 4p^3/27})/2$$

$$\text{or} \qquad = -q/2 \pm \sqrt{(q/2)^2 - (p/3)^3}. \qquad (6)$$

$$\text{Then} \qquad z_1 = \sqrt[3]{-q/2 + \sqrt{(q/2)^2 + (p/3)^3}} \qquad (7)$$

$$\text{and} \qquad z_2 = \sqrt[3]{-q/2 - \sqrt{(q/2)^2 + (p/3)^3}}$$

Since there are three cube roots of any number, the cube root and the product of it by the imaginary cube roots of unity, $w = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\, i$, and $w^2 = -\frac{1}{2} - \frac{1}{2}\sqrt{3}\, i$, there will be six values of the above which are the roots of (5). z_1, $z_1 w$, $z_1 w^2$, z_2, $z_2 w$, and $z_2 w^2$. These must be paired so that the product of the two so paired is equal to $-p/3$, that is

$$z_1 z_2 = -p/3; \; z_1 w \cdot z_2 w^2 = -p/3; \; z_1 w^2 \cdot z_2 w = -p/3.$$

To each root z is paired a root equal to $-p/3z$; therefore the sum of the two is equal to y, from (4), and the three values of y will be,

$$y_1 = z_1 + z_2 \; ; \; y_2 = z_1 w + z_2 w^2 \; ; \; y^3 = z_1 w^2 + z_2 w. \qquad (8)$$

These values of the reduced cubic (3) are known as Cardan's formula. It was concerning this solution that Cardan and Tartaglia had such a controversy.

The discriminant of the cubic is defined as the product of the squares of the differences of the roots where the term containing the third power of the unknown has unity for its coefficient. That is

$$\Delta = (y_1 - y_2)^2 (y_1 - y_3)^2 (y_2 - y_3)^2$$

To express this in terms of the reduced cubic the values of y_1, y_2, and y_3, from (8), will be substituted for them.

$$\begin{aligned} y_1 - y_2 &= z_1 + z_2 - z_1 w - z_2 w^2 \\ &= z_1(1 - w) - w^2 z_2 (1 - w) \\ &= (1 - w)(z_1 - w^2 z_2) \\ y_1 - y_3 &= z_1 + z_2 - z_1 w^2 - z_2 w \\ &= z_1(1 - w^2) - w z_2 (1 - w^2) \\ &= (1 - w^2)(z_1 - w z_2) \\ y_2 - y_3 &= z_1 w + z_2 w^2 - z_1 w^2 - z_2 w \\ &= z_1 (w - w^2) - z_2 (w - w^2) \\ &= (w - w^2)(z_1 - z_2) \end{aligned}$$

To obtain the product of these equivalent values of the differences of the roots it is easier to find the products of parts before obtaining the final product.

Since the cube root of unity is 1, w, and w^2,

$$(x - 1)(x - w)(x - w^2) = x^3 - 1.$$

Letting $x = z_1/z_2$

$$\begin{aligned} (z_1 - z_2)(z_1 - w z_2)(z_1 - w^2 z_2) &= z_1^3 - z_2^3 \\ &= 2\sqrt{(q/2)^2 - (p/3)^3} \end{aligned}$$

$$(1 - w)(1 - w^2) = 3\ ;\ (w - w^2) = \sqrt{3}\, i.$$

$$(y_1 - y_2)(y_1 - y_3)(y_2 - y_3) =$$

$$3 \cdot 2\sqrt{(q/2)^2 - (p/3)^3} \cdot \sqrt{3}\, i \text{ and}$$

$$(y_1 - y_2)^2 (y_1 - y_3)^2 (y_2 - y_3)^2 = -27q^2 - 4p^3$$

Since p and q are expressed in terms of the coefficients of the cubic equation one can obtain the value of Δ without solving the cubic.

When the three roots of the cubic are real, squaring the difference of any two gives a positive result, therefore Δ is positive.

Should two of the roots be conjugate imaginaries, the square of their difference is negative. If the third root is real, the square of the difference of it with each of the others gives a negative result, therefore the final product or Δ is negative.

Should two of the roots be equal and one of them imaginary, the third root would be its conjugate and they would be the roots of a real quadratic. The other equal root, helping form the third factor of the cubic, would have real coefficients, therefore the two equal roots must be real and Δ is zero.

These results lead to the very useful theorem, if Δ is positive the roots are real, if Δ is negative, one root is real and the other two conjugate imaginaries, if Δ is zero, two roots are real and equal.

To illustrate this method of solution consider the equation

$$x^3 + 4x^2 + 4x + 3 = 0$$

where $a = 1,\ b = 4,\ c = 4,\ d = 3.$

Let $x = y - b/3$. Then in the equation

$$y^3 + py + q = 0$$

$$p = c - b^2/3;$$

$$= 4 - 16/3$$

$$= -4/3.$$

$$q = d - cb/3 - 2b^3/27$$

$$= 3 - 16/3 - 128/27$$

$$= 65/27.$$

Which gives,

$$y^3 = 4y/3 - 65/27 - 0.$$

$$z_1 = \sqrt[3]{-65/54 + \sqrt{(65/54)^2 + (-4/9)^3}}$$

$$= \sqrt[3]{-65/54 + 63/54}$$

$$= -1/3$$

$$z_2 = \sqrt[3]{-65/54 - 63/54}$$

$$= -4/3$$

Then $y_1 = -1/3 - 4/3$

$$= -5/3$$

$$y_2 = (-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i)(-1/3) + (-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\,i)(-4/3)$$

$$= 5/6 + \tfrac{1}{2}\sqrt{3}\,i.$$

$$y_3 = (-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\,i)(-1/3) + (-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i)(-4/3)$$

$$= 5/6 - \tfrac{1}{2}\sqrt{3}\,i$$

Therefore,

$$x_1 = -5/3 - 4/3$$

$$= -3$$

$$x_2 = 5/6 + \tfrac{1}{2}\sqrt{3}\,i - 4/3$$

$$= -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\, i$$

$$= w.$$

$$x_3 = 5/6 - \tfrac{1}{2}\sqrt{3}\, i - 4/3$$

$$= -\tfrac{1}{2} - \sqrt{3}\, i$$

$$= w^2.$$

THE QUARTIC EQUATION

The solution of the quartic equation given here is due to Ferrari and was first published by Cardan in his "Ars Magna".

The general quartic equation is of the form,

$$x^4 + bx^3 + cx^2 + dx + e = 0. \qquad (1)$$

Adding $(mx + n)^2$ to both members of the equation gives,

$$x^4 + bx^3 + (c + m^2)\,x^2 + (d + 2mn)\,x + e + n^2 = (mn + n)^2 \qquad (2)$$

Let us assume the identity,

$$x^4 + bx^3 + (c + m^2)\,x^2 + (d + 2mn)\,x + e + n^2 = (x^2 + bx/2 + p)^2. \qquad (3)$$

It is now possible to equate the coefficients of the like powers of x,

$$c - m^2 = b^2/4 - 2p \qquad (4)$$

$$d - 2mn = bp \qquad (5)$$

$$e - n^2 = p^2 \qquad (6)$$

Eliminating m and n from (4), (5), and (6),

$$m^2 = b^2/4 + 2p - c$$

$$m = (bp - d)\,/2n$$

$$n^2 = p^2 - e.$$

$$b^2/4 + 2p - c = (b^2p^2 - 2bdp + d^2)\,/\,(4p^2 - 4e).$$

Removing fractions gives,

$$8p^3 - 4cp^2 + (2bd + 8e)p + 4ce - b^2e - d^2 = 0 \qquad (7)$$

This is a cubic in p whose solution was given in the preceding paragraph; therefore, a solution may be assumed for p. Knowing the value of p, one may readily obtain the value of m and n from (4) and (6). From (2) and (3) one may write

$$(x^2 - bx/2 - p)^2 - (mx - n)^2 \qquad (8)$$

which may be written in the following identical way,

$$x^2 + bx/2 + p - mx - n = 0, \text{ and}$$

$$x^2 + bx/2 + p + mx + n = 0.$$

The four roots obtained from these two equations are the solutions of (1).

This solution may be illustrated with the equation,

$$x^4 + x^3 - x^2 - 7x - 6 = 0, \text{ in which } b = 1;\ c = -1;$$

$$d = -7;\ e = -6. \qquad (1)$$

Adding $(mx + n)^2$ to both members of the equation,

$$x^4 + x^3 + (m^2 - 1)x^2 + (2mn - 7)x - 6 - n^2 =$$

$$(mx + n)^2. \qquad (2)$$

Assuming the identity,

$$x^4 + x^3 + (m^2 - 1)\,x^2 + (2mn - 7)\,x - 6 + n^2 \equiv$$

$$(x^2 + \tfrac{1}{2}x + p)^2. \qquad (3)$$

Squaring the right member,

$$= x^4 + x^3 + (\tfrac{1}{4} + 2p)\,x^2 + px + p^2.$$

Equating coefficients,

$$m^2 - 1 = \tfrac{1}{4} - 2p. \qquad (4)$$

$$2mn - 7 = p. \qquad (5)$$

$$m^2 - 6 = p^2. \qquad (6)$$

Eliminating m and n from these three equations,

$m^2 = 5/4 + 2p$, from (4)

$m^2 = (p + 7)^2/(2n)^2$, or $(p^2 + 14p + 49)/4n^2$, from (5).

$n^2 = p^2 + 6$, from (6)

Equating values of m^2,

$$(p^4 + 14p + 49)/(4p^2 + 24) = 5/4 + 2p$$

$$8p^3 + 4p^2 + 34p - 19 = 0. \qquad (7)$$

Solving this cubic for p, $\frac{1}{2}$ is found to be a real root.

Substituting this value for p in (4) and (6) gives,

$m^2 = 5/4 - 1$

$m = 3/2.$

$n^2 = \frac{1}{4} - 6$

$n = 5/2.$

From (2) and (3),

$$(x^2 - \tfrac{1}{2}x - \tfrac{1}{2})^2 = (3/2x - 5/2)^2.$$

Taking the square root of both members gives,

$$x^2 + \tfrac{1}{2}x + \tfrac{1}{2} + 3/2x + 5/2 = 0 \text{ and}$$

$$x^2 + \tfrac{1}{2}x + \tfrac{1}{2} + 3/2x - 5/2 = 0.$$

$x^2 + 2x + 3 = 0$ $\qquad$ $x^2 - x - 2 = 0$

$x = -1 + \sqrt{2}\, i$ $\qquad$ $x = 2$

$x = -1 - \sqrt{2}\, i$ $\qquad$ $x = -1.$

NEWTON'S METHOD FOR INTEGRAL ROOTS

Sir Issac Newton discovered a very convenient method of obtaining the integral roots of an equation when the coefficients are integers.

Consider the equation,

$$a_0x_n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \qquad (1)$$

Transposing the last term, dividing by -1 and factoring the left member gives,

$$x(a_0x^{n-1} - a_1x^{n-2} - \ldots a_{n-1}) = a_n \qquad (2)$$

When this equation is divided by x, the x is found to be an exact divisor of a_n, because the integer in the parenthesis is the quotient. That is, an integral root is a divisor of the constant term.

Transposing the last two terms of (1) and repeating the above process gives,

$$x^2(-a_0x^{n-2} - a_1x^{n-3} - \ldots a_{n-2}) = a_{n-1}x + a_n. \qquad (3)$$

The right member must be divisible by x^2 or $a_{n-1} + a_n/x$ divisible by x. Transposing the last three terms and repeating the process gives,

$$x^3(-a_0x^{n-3} - a_1x^{n-4} - \ldots -a_{n-3}) =$$
$$a_{n-2}x^2 + a_{n-2}x + a_n \qquad (4)$$

whose sum must be divisible by x^3 or $a_{n-2} + a_{n-1}/x + a_n/x^2$ is divisible by x. By continuing the process the last sum

$a_0 + a_1/x + \ldots$ is not only divisible by x but is equal to 0, since it is the quotient of (1) by x^n. Thus a series of conditions of divisibility is produced that must be satisfied by an integral root of the equation. As an illustration take the equation,

$$x^4 + 4x^3 + 8x + 32 = 0. \qquad (1)$$

The divisor, 16, of the constant term is not a root since 8 + 32/16 = 10 is not divisible by 16. Neither is 8 a root as 8 + 32/8 = 12, which is not divisible by 8. To prove that -2 and -4 are roots and that none of the division tests fail, the work is arranged in a systematic order. Taking the sum of the coefficients, replacing the missing term with 0 gives,

$$\begin{array}{rrrrrl} 1 & + 4 & + 0 & + 8 & + 32 & \underline{(-2\quad} \\ \underline{-1} & \underline{- 2} & \underline{+ 4} & \underline{-16} & & \\ 0 & - 2 & + 4 & - 8 & & \end{array} \qquad (2)$$

First the constant term 32 is divided by -2, place the quotient under the preceding term, a + 8, and add. Dividing their sum gives 4, place it under the 0, add, dividing their sum gives a - 2, place it under the 4, add, dividing their sum gives -1, which when added to the 1 gives 0, which meets all the conditions given in the proof.

The second line of (2) is the negative of the coefficients of the quotient of (1) divided by x - 2, so the quotient is an equation of one degree lower than (1).

To show that -4 is a root, take line 2 of (2), with signs changed which is the coefficients of the depressed equations,

$$\begin{array}{r} 1 + 2 - 4 + 16 \quad \underline{(-4} \\ \underline{-1 + 2 - 4} \\ 0 + 4 - 8 \end{array}$$

Dividing 16 by -4 gives -4, add to the preceding term gives -8, dividing, gives 2, add to the preceding term, dividing their sum gives -1, add to the first term gives 0. The quotient is the polynomial $x^2 - 2x + 4$, which when set equal to zero has the imaginary roots, $1 + \sqrt{-3}$ and $1 - \sqrt{-3}$, being obtained by using the quadratic formula.

Chapter III

IRRATIONAL ROOTS

The purpose of this chapter is the explanation of the different methods used to obtain the irrational roots of equations.

Consideration will first be given to the location of roots between two stated points. Let the interval be from a to b. If a polynomial f(x) has real coefficients and if when a and b are substituted for x in the f(x) they produce opposite signs, the equation f(x) = 0 has one or more odd number of roots in the interval from a to b, when a multiple root is counted m times.

A real root of an equation f(x) = 0 is the point where the graph of the equation crosses the x-axis. Taking a and b as points on opposite sides of the x-axis, f(x) will change its sign in passing from f(a) to f(b) while f(x) varies continuously with x. As f(x) passes through all the intermediate points while changing from a to b it will pass through the zero value which causes f(x) to disappear and is a real root of the equation f(x) = 0. But f(x) need not cross the x-axis only once in going from f(a) to f(b), but may cross it any number of odd times. Also if f(x) does not change in sign when a and b are substituted for x, they represent points on the same side of the axis and the graph does not cross the axis in going from a to b, or else crosses it an even number of times. Therefore there are no roots between a and b or there is an even number of them.

DESCARTES' RULE OF SIGNS

Descartes' rule of signs cannot be used to segregate the roots of an equation but it is of value in giving some idea of the number of real roots the equation possesses.

When two or more successive signs of the coefficients of a real equation $f(x) = 0$, or any polynomial, are alike, there is said to be a continuation of sign, but if a pair of successive signs are unlike there is a variation of sign.

In the polynomial,

$$f(x) = 3x^3 + 4x^2 - 6x + 2 \qquad (1)$$

there is one continuation of signs and two variations of signs. This can be shown more clearly by writing only the signs of the coefficients. $+ + - +$.

Descartes' Rule states: The number of positive real roots of any equation $f(x) = 0$ with real coefficients does not exceed the number of its variation of sign of $f(x)$ or is less than that number by a positive even integer. A multiple root is counted as m roots.

Thus (1) has two or no positive roots. Increasing the number of positive roots of (1) by multiplying by $x - 2$ gives

$$3x^4 - 2x^3 - 14x^2 + 14x - 2$$

in which there are three variations; therefore there are three real roots or only one.

The number of negative roots of $f(x) = 0$ can be reckoned by substituting $(-x)$ for (x) in $f(x) = 0$. The negative roots being equal to the variations of sign of $f(-x)$ or less by a positive even integer.

UPPER LIMIT OF ROOTS

In the segregation of roots much time can be saved if one knows that the root lies between some definite values. To accomplish this purpose there have been developed two general theorems. The first theorem gives a better limit in some equations than the second, while in other equations the reverse is true.

Theorem 1. In the equation

$$f(x) \equiv x^n + a_1x^{n-1} + a_2x^{n-2} + \cdot \cdot \cdot + a_{n-1}x + a_n = 0$$

if the first negative term is preceded by r coefficients which are positive or zero and if the greatest negative coefficient be $-a_s$, then $\sqrt[r]{a_s}$ 1 is a superior limit of the positive roots.[1]

For example, in the equation

$$x^4 - 45x^2 + 40x + 84 = 0$$

$r = 2$ and $a_s = 45$. According to the theorem each root is less than $\sqrt{45} + 1$ and therefore less than 7.7. Although the constant term is 84, and the product of the roots, knowing that the upper limit of the roots is 7.7 will prevent one from wasting time with the larger factors of 84.

Theorem 2. If in any equation each negative coefficient be taken positively and divided by the sum of all

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1. The proof of this theorem is given by Dickson in First Course in Theory of Equations, p. 21.

the positive coefficients which precede it, the greatest quotient thus formed increased by unity is a superior limit of the positive roots.[1]

The preceding example had the upper limits 7.7 by theorem one. Applying the principle of theorem 2 the upper limit is 46, showing that theorem one gives a better upper limit for this equation. But for

$$4x^4 + 12x^3 - 17x^2 - 3x + 4 = 0$$

theorem one gives $\sqrt{17} - 1 = 5.12$ while theorem two gives $17 \div 16 + 1 = 2.06$ a much better upper limit.

1. This theorem is proven by Cajori in his Theory of Equations, p. 44.

GRAPHIC SOLUTION

The segregating of the real roots of a real equation $f(x) = 0$ by constructing its graph gives a geometric interpretation that is easy to comprehend. On an x, y axis construct the graph of the function $y = f(x)$ and measure the distance from the origin to the points of intersection of the curve with the x-axis.

To illustrate, consider the equation,

$$x^3 - 3x^2 - x - 3 = 0 \qquad (1)$$

Equate the left member to y,

$$y = x^3 - 3x^2 - x - 3,$$

and find by synthetic division the values of the function corresponding to the various assigned values of x. When the division of f(x) by the assigned positive values of x produce partial remainders that are all positive, no greater positive values of x need be considered as these values only produce increasing values for y. Similarly, when the assigned negative values of x produce partial remainders that are alternately positive and negative, no greater negative values need be used as they would produce increasing values for y.

If x =	2,	1,	0,	1,	-2,	-3,	-4
Then y =	15,	0,	-3,	0,	3,	0,	-15

For the assigned integral values of x the table gives the corresponding values for y which makes the curve shown in Fig. 1, and the roots of the equation are 1, -1, and -3. This must not be interpreted to mean that the random selection of abscisses, however numerous, will always give the true curve and all the roots of an equation.

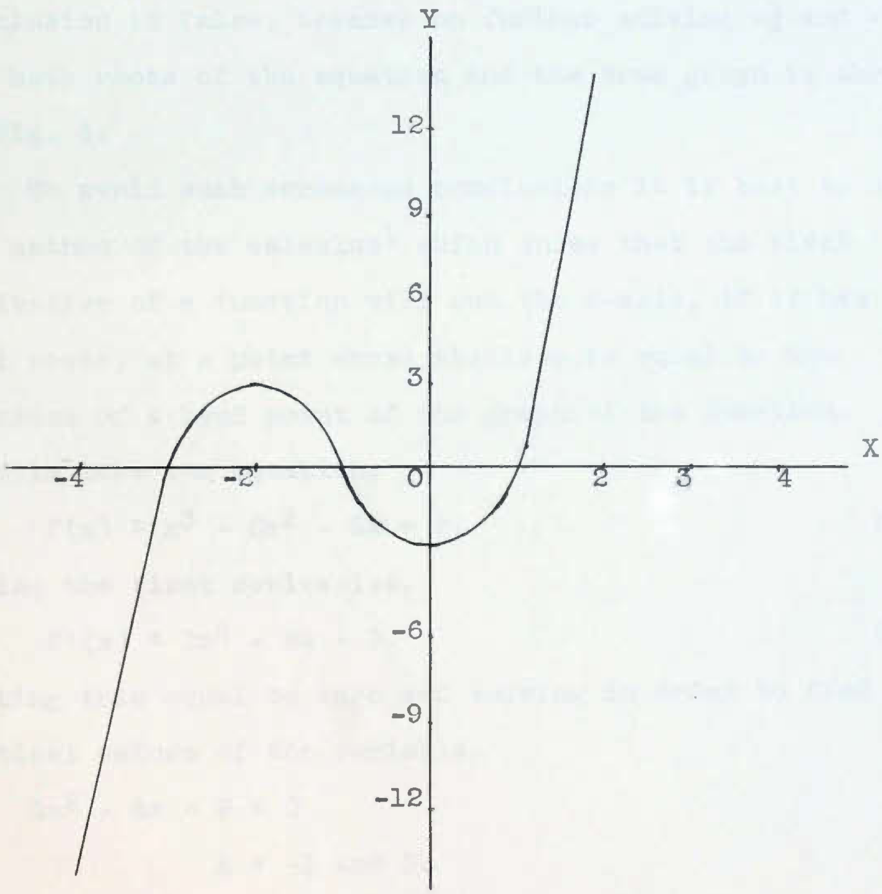

Fig. 1.

For example analyze the equation,

$y = 8x^4 + 6x^3 - 7x^2 - 6x - 1$. Assigning integral values to x produces the accompanying table.

If x =	2,	1,	0,	-1,	-2
Then y =	131,	0,	-1,	0,	63

The graph of these values is shown in Fig. 2, a U-shaped curve which indicates that the equation has but two real roots and only one bend point. However, such a conclusion is false, because on further solving $-\frac{1}{2}$ and $-\frac{1}{4}$ are both roots of the equation and the true graph is shown in Fig. 3.

To avoid such erroneous conclusions it is best to use the method of the calculus[1] which shows that the first derivative of a function will cut the x-axis, if it has real roots, at a point whose abscissa is equal to the abscissa of a bend point of the graph of the function. For example take the equation,

$$f(x) = x^3 - 3x^2 - 9x + 2. \qquad (1)$$

Taking the first derivative,

$$f'(x) = 3x^2 - 6x - 9. \qquad (2)$$

Setting this equal to zero and solving in order to find the critical values of the variable.

$$3x^2 - 6x - 9 = 0$$

$$x = -1 \text{ and } 3.$$

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1. Granville, Smith, and Longley. Elements of the Differential and Integral Calculus, p. 52.

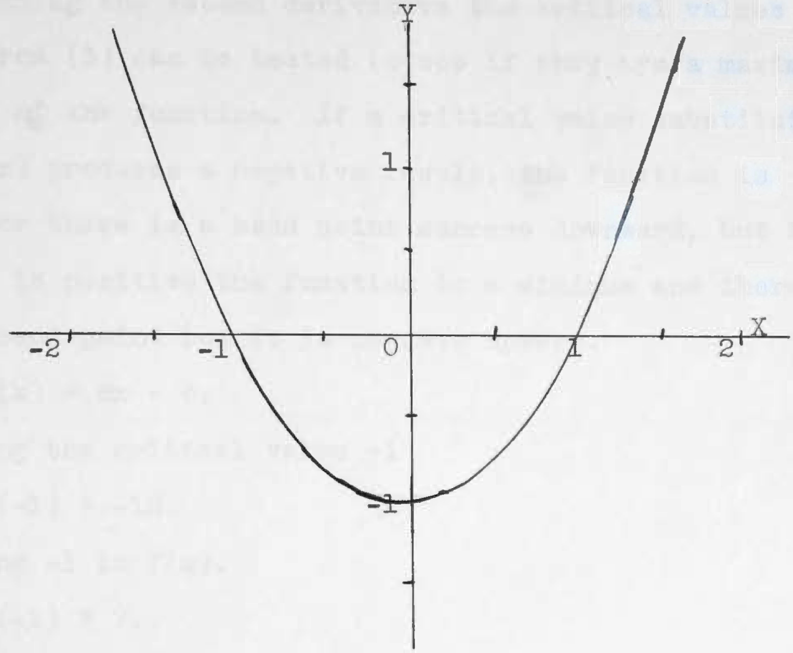

Fig. 2.

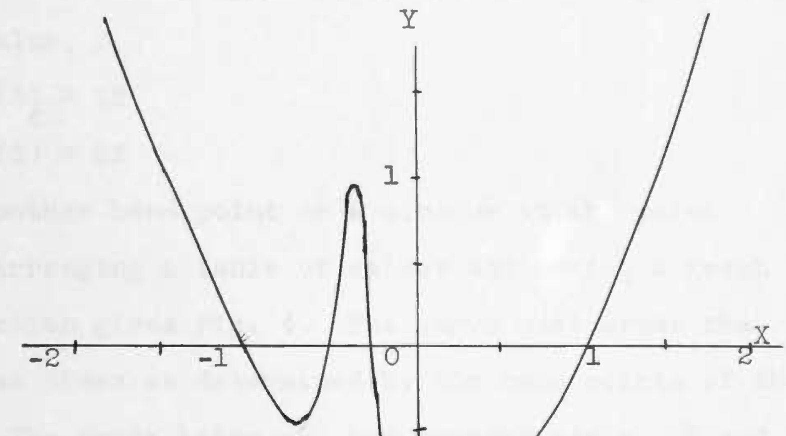

Fig. 3.

By finding the second derivative the critical values obtained from (3) can be tested to see if they are a maximum or minimum of the function. If a critical value substituted in the $f''(x)$ produces a negative result, the function is a maximum or there is a bend point concave downward, but if the result is positive the function is a minimum and there is also a bend point but it is concave upward.

$$f''(x) = 6x - 6.$$

Substituting the critical value -1

$$f''(-1) = -12.$$

Substituting -1 in $f(x)$.

$$f(-1) = 7.$$

Therefore there is a bend point or a maximum of the function concave downward at the point (-1, 7). Examining the other critical value, 3,

$$f''(3) = 12$$

$$f(3) = 25$$

There is another bend point or a minimum at the point (3, 25). Arranging a table of values and making a graph of the function gives Fig. 4. The curve must cross the x-axis three times as determined by the bend points of the function. The roots being -2, and approximately, .2 and 4.8.

If x =	5,	4,	3,	2,	1,	0,	-1,	-2,	-3
Then y =	7,	-18,	-25,	-20,	-9,	2,	7,	0,	-25

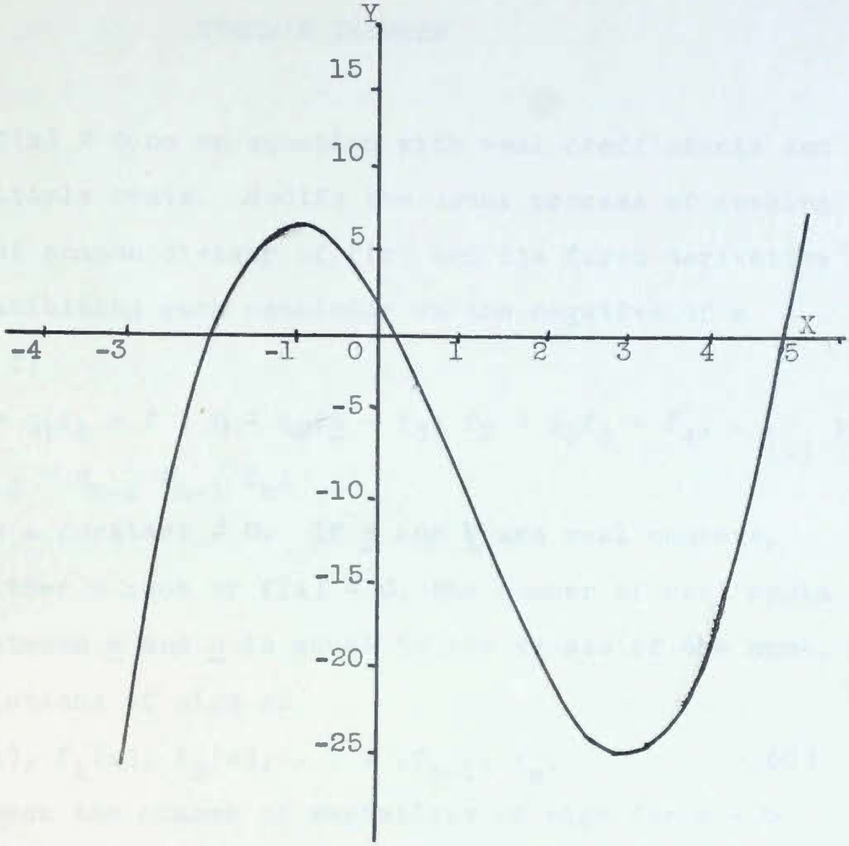

Fig. 4.

STURM'S THEOREM

"Let $f(x) = 0$ be an equation with real coefficients and without multiple roots. Modify the usual process of seeking the greatest common divisor of $f(x)$ and its first derivative $f_1(x)$ by exhibiting each remainder as the negative of a polynomial f;

$$f = q_1f_1 - f \; ; f_1 = q_2f_2 - f_3; \; f_2 = q_3f_3 - f_4, \; . \; . \; . \; ; \qquad (1)$$
$$f_{n-2} = q_{n-1} \; f_{n-1} \; f_n;$$

Where f_n is a constant $\neq 0$. If a and b are real numbers, $a < b$, neither a root of $f(x) = 0$, the number of real roots $f(x) = 0$ between a and b is equal to the excess of the number of variations of sign of

$$f(x), \; f_1(x), \; f_2(x), \; . \; . \; . \; , f_{n-1}, \; f_n, \qquad (2)$$

for x - a over the number of variations of sign for x - b. Terms which vanish are to be dropped out before counting the variations of sign."[1]

The purpose of this theorem is the isolation of the roots between consecutive integers, or narrower limits should there be more than one root between two consecutive integers. Then the root can be found to as many decimal places as are required by the use of Horner's or Newton's method.

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1. Dickson, L. C. First Course in the Theory of Equations, p. 76.

To illustrate Sturm's theorem take the equation

$$f(x) = x^3 - 3x^2 - 4n + 13 \qquad (3)$$

$$f_1(x) = 3x^2 - 6x - 4$$

Dividing $f(x)$ by $f_1(x)$ there is a remainder of

$-14x/3 + 35/3$ which is designated by $-f_2$

Dividing $f_1(x)$ by f_2 there is a constant remainder of $-\frac{1}{4}$, designated by $-f_3$, therefore;

$$f(x) = x^3 - 3x^2 - 4x + 13 = (x/3 - 1/3)\, f_1 - f_2$$

$$f_1(x) = 3x^2 - 6x - 4 = (9x/14)\, f_2 - f_3$$

$$f_2 = 14x/3 - 35/3$$

$$f_3 = 1/4$$

From Descartes' rule (3) has two or no real positive roots and it has 3 or one real negative root.

For $x = -\infty$, the signs of f, f_1, f_2, f_3, are $- + - +$, showing 3 variations of sign. For $x = 0$, the signs are $+ - - +$, showing two variations. From the theorem there must be 3-2 = 1 real root between $-\infty$ and 0. Therefore according to Descartes' rule if there are any more real roots they must be two positive ones. For when $x = +\infty$ the signs are $+ + + +$ showing no variation. For $x = 0$ to $x = +\infty$ there is a difference of two in variation of signs; therefore there are two positive roots between those values.

Arranging a tabular form for the

x	signs	variations	
-3	- + - +	3	
-2	- + - +	3	1 root
-1	+ + - +	2	
0	+ - - +	2	
1	+ - - +	2	
2	+ - - +	2	
2.2	+ - - +	2	1 root
2.4	- - - +	1	
2.6	- + + +	1	1 root
2.8	+ + + +	0	
3	+ + + +	0	

Values of x, the signs of f, f_1, f_2, f_3, and the variations of sign, there is seen to be a change in the number of variations for the values of x = -2 and x = -1 for x = 2.2 and x = 2.4, and for x = 2.6 and x = 2.8; therefore there are three real roots between those values, and they are isolated between integers according to the statement of the theorem.

BUDAN'S THEOREM

> "Let a and b be real numbers, a $<$ b, neither a root of f(x) = 0, an equation of degree n with real coefficients. Let V_a denote the number of variations of sign of
> $$f(x),\ f'(x),\ f''(x),\ \ldots,\ f^{n}(x) \qquad (1)$$
> for x = a, after vanishing terms have been depleted. Then $V_a - V_b$ is either the number of real roots of f(x) = 0 between a and b or exceeds the number of those roots by a positive even integer. A root of multiplicity m is here counted as m roots."[1]

The purpose of this Theorem is the same as that of Sturm's, the isolation of real roots between integers. The fundamental difference lies in the process of obtaining functions to obtain variations of signs. With Sturm's method one would find the first derivative, there by the process of division obtain the remainders until the last one was a constant, while with this method of Budan's one merely takes the successive derivatives until the last one is a constant. While this method of obtaining the functions is the less laborious, it is not as specific in determining the number of roots. The roots may be equal to the variation of sign or less by an even integer.

For example,

$$f(x) = x^3 - 3x - 1$$

$$f'(x) = 3x^2 - 3$$

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1. Dickson, L. C. First Course in the Theory of Equations, p. 83. The proof of this theorem is given by Dickson.

$f''(x) = 6x$

$f'''(x) = 6$

Using Descartes' rule there is one positive and two or no negative roots.

Arranging the values of x in a tabulated form with the signs of f, f', f'', f''', and their variations.

x	f	f'	f''	f'''	variations	
3	+	+	+	+	0	
2	+	+	+	+	0	1 root (between 2 and 1)
1	-	-	+	+	1	
0	-	-	+	+	1	1 root (between 0 and -1)
-1	+	+	-	+	2	1 root (between -1 and -2)
-2	-	+	-	+	3	
-3	-	+	-	+	3	

There is seen to be a change in the number of variations between the values where x = 2 and x = 1, where x = 0 and x = -1, and where x = -1 and x = -2. Therefore since there was a difference of one in the variation of sign, there is one positive root and two negative roots between the above values of x.

HORNER'S METHOD

The approximation of irrational roots may be determined to as many decimal places as required, after the root has been isolated by one of the preceding methods, by a process perfected by Horner and named for him.

A geometric interpretation will be given to help clarify the algebraic explanation of Horner's Method.

It is assumed that the equation $f(x) = 0$ has been graphed and found to cross the x-axis in the unit interval between and from a and b, as shown in Fig. 1, and passes through the points P and P_1 whose coordinates are (a, h) and (b, k) respectively.

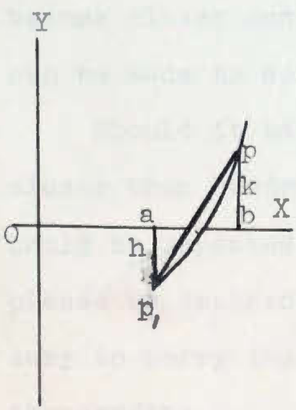

Fig. 1.

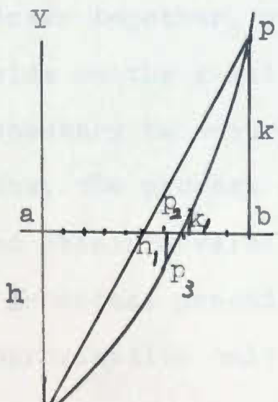

Fig. 2.

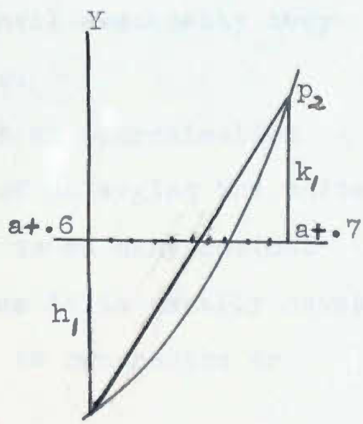

Fig. 3.

The secant is drawn from P to P_1 and the unit from a to b is enlarged and separated into tenths as shown in Fig. 2. The point a is used as the origin or it may be thought of as moving the y-axis a units to the right. This is accomplished algebraically by dividing $f(x) = 0$ by $x - a$, by synthetic division, giving the first transformed equation, $f(x_1) = 0$.

The graph is now seen to cross the x-axis between .6 and .7. The secant is drawn from P_2 to P_3 and by computing the $f(a + .6)$ and the $f(a + .7)$ the ordinates of P_3 and P_2 are found to be h_1 and k_1 respectively.

The unit from .6 to .7 (Fig. 3) is enlarged and divided into ten segments which correspond to hundredths in Fig. 1. $\underline{a} + .6$ is used as the origin, or the y-axis is moved again which means, algebraically, that the $f(x_1)$ is divided by $x - (a + .6)$ giving the second transformed equation $f(x_2)$.

The root of the equation is now seen to be $\underline{a} + .67$ and as the approximation becomes closer the secant keeps drawing closer to the root of the equation, the points P and P_1 become closer and closer together, until eventually they can be made to coincide on the x-axis.

Should it be necessary to secure an approximation closer than hundredths, the process of enlarging the units could be repeated and obtain a value to as many decimal places as desired. In actual practice it is usually necessary to carry the approximation only to hundredths or thousandths.

For the algebraic solution consider the equation mentioned in a previous paragraph,

$$f(x) = x^3 - 3x^2 - 4x + 13 \qquad (1)$$

where a real root was located between 2.2 and 2.4 by Sturm's method.

The root of the equation is first decreased by 2.2 which means moving the graph of the equation so that it crosses the x-axis between .2 and .4. This is accomplished by synthetic division.

$$\begin{array}{llll} 1 & -3 & -4 & +13 \quad \underline{2.2} \\ & 2.2 & -1.76 & -12.672 \\ \hline 1 & -.8 & -5.76 & +.328 \\ & 2.2 & +3.08 & \\ \hline 1 & +1.4 & -2.68 & \\ & +2.2 & & \\ \hline 1 & +3.6 & & \end{array}$$

The transformed equation is

$$x^3 + 3.6x^2 - 2.68x + .328 \qquad (2)$$

which has a root between .2 and .4. It is important that the sign of the known term in each transformed equation be the same as that of the original equation.

To obtain an approximation to the root of (2) ignore the terms x^3, and $3.6x^2$. Then if $-2.68x_1 - .328 = 0$, $x_1 = .1+$, but before accepting this it must be verified. When it is tested the result is just positive; therefore it is acceptable.

If the result was negative it would mean that the point was on the left side of the origin; therefore that value would be too large. The coefficients of the transformed equations appear on the first lines of the following scheme which shows the procedure for obtaining the approximate root to six decimal places. Since the root is taken to only six decimal places, the fractions are rounded off at the sixth place.

```
1 + 3.6 - 2.68 + .328        .1
 + .1 + .37 - .231
1 + 3.7 - 2.31 + .097
 + .1 + .38
1 + 3.8 - 1.93
 + .1
1 + 3.9                              .097
                                     ----
                                     1.93

1 + 3.9 - 1.93 + .097              = .05
 + .05 + .1975 - .086625
1 + 3.95 - 1.7325 + .010375
 + .05 + .20
1 + 4.00 - 1.5325
 + .05
1 + 4.05                                  .010375
                                          -------
                                          1.5325

1 + 4.05 - 1.5325 + .010375              = .006
 + .006 + .024336 - .009049
1 + 4.056 - 1.508164 + .001326
 + .006 + .024372
1 + 4.062 - 1.483792
 + .006
1 + 4.068
```

$$\frac{.001326}{.483792} = .0008$$

1 + 4.068 - 1.483792 + .001326

+ .0008 + .003255 - .001184

1 + 4.0688 - 1.480537 + .000142

+ .0008 + .003256

1 + 4.0696 - 1.477281

+ .0008

1 + 4.0704

$$\frac{.000142}{1.477281} = .00009$$

1 + 4.0704 - 1.477281 + .000142

+ .00009 + .000366 - .000133

1 + 4.07049 - 1.476915 + .000009

+ .00009 + .000366

1 + 4.07058 - 1.476549

+ .00009

1 + 4.07049

Taking the sum of the roots of the transformed equations, the approximate value of the root of (1) is

$x = 2.215689+$

The other two roots of (1) could be approximated in like manner. If the root is negative a -x can be substituted for x in f(x) and the procedure for positive roots followed.

NEWTON'S METHOD

There is a great deal of similarity between the methods of Newton and Horner. In both a root is isolated before their respective methods are used. Also there is quite a difference in part of the plan of procedure. Newton's method is more applicable than Horner's in that it can be used to solve other types of equations, while Horner's is only for algebraic equations.

It is taken for granted that a real root of the equation $f(x) = 0$ has been isolated between a and b on the x-axis where $0 \leqq a < b$. These values of a and b must be taken so close together that the $f'(x) = 0$ does not have a root between a and b, because if it did there would be a bend point in the $f(x) = y$. Also the $f''(x) = 0$ must not have a root between the limits a and b, for if it did there would be an inflexion point in the graph of $f(x) = y$.

Since neither $f'(x)$ nor $f''(x)$ have a root between the limits a and b, $f''(x)$ will have the same sign throughout the interval, while $f(x)$ changes sign; therefore they will both have the same sign at one end of the interval and that value should be taken to work from in approximating the root of $f(x) = 0$. If this value is a, then a better approximation will be $a + h$. To find h Newton[1] used Taylor's Theorem,

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1. Burnside and Panton. Theory of Equations, p. 226.

substituting a for x so

$$f(a+h) = f(a) + f'(a)h + f''(a)h^2/2! + f'''(a)h^3/3! ..$$

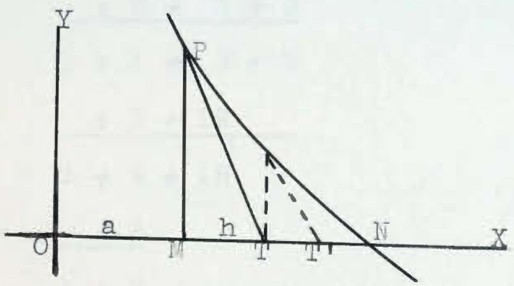
Ignoring the powers of h^2, h^3, . . . and taking

$$f(a) \quad f'(a)h = 0, \; h = -f(a) / f'(a).$$

To obtain the next approximation he let $a_1 = a + h$ in the first transformed equation and found h_1.

$$h_1 = -f(a_1) / f'(a_1).$$

Perhaps a graphic presentation will help give a better understanding of the discussion. Consider the graph of y = f(x) as shown in Fig. 1 where the point P has the abscissa OM = a and the tangent at P and the graph cut the x-axis at T and N respectively.



Let the subtangent MT = h and MP = f(a). Since the first derivative of the function equals the tangent

$$f'(a) = \tan XTP$$
$$= -MP/h$$
$$= -f(a)/h$$
$$h = -f(a)/f'(a)$$

A better approximation to the root ON is OT = a + h. A still closer approximation would be OT', which is bringing the tangent nearer to N as the approximation comes nearer to the root.

By way of illustration take the equation

$$f(x) = x^3 - 2x^2 - 2$$

where a root has been isolated between 2 and 3. Which of the values to use for a will first be determined.

$$f'(x) = 3x^2 - 4x$$

$$f''(x) = 6x - 4$$

f (2) = a - value. f (3) = a + value.

f"(2) = a + value. f"(3) = a + value.

Since the f(3) and the f"(3) have the same sign, 3 will be used for a. To find the first approximation transform f(x) by dividing it by 3 by synthetic division

```
1 - 2 + 0 - 2  |_3_
  + 3 + 3 + 9
---------------
1 + 1 + 3 + 7
  + 3 + 12
---------------
1 + 4 + 15
  + 3
---------
1 + 7
```

giving

$$f(x_1) = x^3 + 7x^2 + 15x + 7$$

$$f'(x_1) = 3x^2 + 14x + 15$$

$$\therefore \quad h = -f(3)/ f''(3) = -7/15 = -.4$$

The process of performing the substitutions to obtain the transformed equations is done by synthetic division just as in Horner's method, except some of the values of h, h_1, h_2, . . . will be negative instead of always positive.

1 + 7 + 15 + 7   -.4

\- .4 - 2.64 - 4.944

1 + 6.6 + 13.36 + 2.056

\- .4 - 2.48

1 + 6.2 + 9.88

\- .4

1 + 5.8

-2.056

9.88

1 + 5.8 + 9.88 + 2.056   = -.2

\- .2 - 1.12 - 1.752

1 + 5.6 + 8.76 + .304

\- .2 - 1.08

1 + 5.4 + 7.68

\- .2

1 + 5.2

-.304

7.68

1 + 5.2 + 7.68 + .304   = -.04

\- .04 - .2064 - .298944

1 + 5.16 + 7.4736 + .005056

\- .04 - .2048

1 + 5.12 + 7.2688

\- .04

1 + 5.08

-.005056

7.2688

$$1 + 5.08 \quad + 7.2688 \quad + .005056 \qquad = -.0007$$

$$- .0007 - .00355551 - .005085671143$$

$$1 + 5.0792 + 7.26524449 - .000029671143$$

$$- .0007 - .00355502$$

$$1 + 5.0786 + 7.26168947$$

$$- .0007$$

$$1 + 5.0779$$

$$g = \frac{.000029671143}{7.26168947} = .0000041$$

> "It can be proved that if the final fraction g, when expressed as a decimal, has k zeros between the decimal point and the first significant figure, the division may be safely carried to 2 k decimal places."[1]

In this problem k is 5; therefore there is no need of further division and the root of the equation to seven decimal places is

$$x = 3 - .4 - .2 - .04 - .0007 + .0000041 = 3.3393041$$

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1. Dickson, L. C. Elementary Course in Equations, p. 95.

Conclusion

The different methods of segregation of roots have their advantages as well as their disadvantages. There is not much difficulty encountered in equations of the second degree, for the quadratic formula will obtain the solution regardless of whether the root is rational or irrational, real or imaginary.

In the solution of cubic and quartic equations Newton's Method for integral roots is convenient when the roots are integers. When solving a cubic with Tartaglia's Method some equations are solved very easily but when the discriminant is positive it requires the solution of the cube root of a complex number, which in most cases must be done trigonometrically. Ferrari's Method of solving the quartic equation is a laborious task.

When isolating the roots, Budan's Method is the most convenient, but Sturm's Method often gives the best result. The approximation of the root may be accomplished very accurately with Horner's or Newton's Method but they are long and laborious, and when the approximation is not required beyond one or two decimal places it can be accomplished more quickly and easily by the graphic method.

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