3-21-2014

Existence and uniqueness of solutions for a fractional boundary value problem on a graph

John R. Graef

Lingju Kong

Min Wang
Rowan University, wangmin@rowan.edu

Dynamical Systems, Differential
Equations and Applications
AIMS Proceedings, 2015

doi:10.3934/proc.2015.0533

pp. **533–539**

EXISTENCE OF HOMOCLINIC SOLUTIONS FOR SECOND ORDER DIFFERENCE EQUATIONS WITH p-LAPLACIAN

John R. Graef

Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA

Lingju Kong

Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA

Min Wang

Equifax Inc.
Alpharetta, GA 30005, USA

Abstract. Using the variational method and critical point theory, the authors study the existence of infinitely many homoclinic solutions to the difference equation

$$-\Delta\big(a(k)\phi_p(\Delta u(k-1))\big) + b(k)\phi_p(u(k)) = \lambda f(k, u(k))), \quad k \in \mathbb{Z},$$

where $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$, $\lambda > 0$ is a parameter, $a, b : \mathbb{Z} \to (0, \infty)$, and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable. Related results in the literature are extended.

1. **Introduction.** In this paper, we are concerned with the existence of solutions of the second order difference equation with a p-Laplacian

$$\begin{cases} -\Delta\big(a(k)\phi_p(\Delta u(k-1))\big) + b(k)\phi_p(u(k)) = \lambda f(k, u(k))), & k \in \mathbb{Z}, \\ u(k) \to 0 \quad \text{as } |k| \to \infty, \end{cases} \tag{1}$$

where $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$, $\lambda > 0$ is a parameter, Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$ for $k \in \mathbb{Z}$, $a, b : \mathbb{Z} \to \mathbb{R}$ are positive real-valued functions, and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable. As in the literature, a solution of problem (1) is referred to as a *homoclinic solution* of the equation

$$-\Delta\big(a(k)\phi_p(\Delta u(k-1))\big) + b(k)\phi_p(u(k)) = \lambda f(k, u(k))), \quad k \in \mathbb{Z}.$$

In a recent paper [3], the present authors studied the existence of infinitely many solutions of problem (1) and proved the following result.

Proposition 1. ([3, Theorem 3.1]) *Let*

$$F(k,t) = \int_0^t f(k,s)ds \quad \text{for all } (k,t) \times \mathbb{Z} \times \mathbb{R}.$$

Assume that the following conditions hold:

(H1) $b(k) \ge b_0 > 0$ *for all* $k \in \mathbb{Z}$, $b(k) \to \infty$ *as* $|k| \to \infty$*;*

(H2) $\limsup_{|t|\to\infty} \frac{F(k,t)}{|t|^p} \le 0$ *uniformly for all* $k \in \mathbb{Z}$*;*

2010 *Mathematics Subject Classification.* Primary: 39A10, 37C20, 58E05.

Key words and phrases. Difference equations, homoclinic solutions, infinitely many homoclinic solutions, variational method.

(H3) $\sup_{|t|\le T}|F(\cdot,t)|\in\ell^1$ *for all* $T>0$*;*
(H4) $f(k,-t)=-f(k,t)$ *for all* $k\in\mathbb{Z}$ *and* $t\in\mathbb{R}$*;*
(H5) there exists a constant $M>0$ *such that*

$$a(k)\le Mb(k)\quad \textit{for all } k\in\mathbb{Z};$$

(H6) there exist a constant $\rho>0$ *and two positive functions* $w_1,w_2\in\ell^1$ *such that*

$$w_1(k)|t|^p\le F(k,t)\le w_2(k)|t|^p \tag{2}$$

for all $k\in\mathbb{Z}$ *and* $|t|\le\rho$*.*

Then, there exists a constant $\underline{\lambda}>0$ *such that for all* $\lambda>\underline{\lambda}$*, problem* (1) *has a sequence* $\{u_n(k)\}$ *of nontrivial solutions satisfying*

$$u_n\to 0 \textit{ in } X\quad\textit{and}\quad I_\lambda(u_n)\le 0,$$

where X *and* I_λ *are defined by* (3) *and* (6) *below, respectively.*

Our goal here is to apply the variational method and critical point theory to find new criteria for the existence of infinitely many solutions of problem (1). Our theorems extend and complement the existing results in the literature (Proposition 1, in particular). We also wish to point out that our results here do not require conditions analogous to (H5) or (H6) above. For more studies on homoclinic solutions for difference equations, we refer the reader to [1, 3, 4, 6, 7, 9, 11] and the references therein.

The following assumptions will be used in this paper.

(A1) There exist $\alpha\ge p$, $C>D>0$, and $0<\delta<1$ such that

$$D|t|^{\alpha-1}<|f(k,t)|<C|t|^{\alpha-1}\quad\text{for } k\in\mathbb{Z} \text{ and } 0<|t|\le\delta;$$

(A2) $tf(k,t)\ge 0$ for $k\in\mathbb{Z}$ and $t\in[-\delta,\delta]$, where δ is as given in (A1).

The remainder of this paper is organized as follows. Section 2 contains some preliminary lemmas and Section 3 contains the main results and their proofs.

2. **Preliminary results.**

In this section, we will establish the variational framework for problem (1) and present some lemmas that will be used in the next section.

For each $1\le p<\infty$, let ℓ^p be the set of all functions $u:\mathbb{Z}\to\mathbb{R}$ such that

$$\|u\|_p=\left(\sum_{k\in\mathbb{Z}}|u(k)|^p\right)^{1/p}<\infty,$$

and let ℓ^∞ be the set of all functions $u:\mathbb{Z}\to\mathbb{R}$ such that

$$\|u\|_\infty=\sup_{k\in\mathbb{Z}}|u(k)|<\infty.$$

Then, we have

$$\ell^p\subseteq\ell^q\quad\text{and}\quad\|u\|_q\le\|u\|_p\quad\text{for any } 1\le p\le q<\infty.$$

In fact, for $u\in\ell^p$, by normalizing, we may assume $\|u\|_p=1$. Then, $|u(k)|\le 1$ for any $k\in\mathbb{Z}$. Hence, $|u(k)|^q\le|u(k)|^p$. This shows that $\|u\|_q\le\|u\|_p$, and so $\ell^p\subseteq\ell^q$.

The following lemma can be found in [2, pp. 3 and 429] and [4, Proposition 2].

Lemma 2.1. *For each* $1\le p<\infty$*,* $(\ell^p,\|\cdot\|_p)$ *is a reflexive and separable Banach space whose dual is* $(\ell^q,\|\cdot\|_q)$*, where* $1/p+1/q=1$*. Moreover,* $(\ell^\infty,\|\cdot\|_\infty)$ *is a Banach space, and for all* $1\le p<\infty$*, the embedding* $\ell^p\hookrightarrow\ell^\infty$ *is continuous since*

$$\|u\|_\infty\le\|u\|_p\quad\textit{for all } u\in\ell^p.$$

Let

$$X = \left\{ u : \mathbb{Z} \to \mathbb{R} \ : \ \sum_{k\in\mathbb{Z}} \big[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p\big] < \infty \right\} \tag{3}$$

and

$$\|u\| = \left(\sum_{k\in\mathbb{Z}} \big[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p\big] \right)^{1/p}. \tag{4}$$

Clearly, if (H1) holds, we have

$$\|u\|_\infty \le \|u\|_p \le b_0^{-1/p}\|u\|. \tag{5}$$

Lemma 2.2. ([3, Lemma 2.2]) *For each $1 \le p < \infty$, $(X, \|\cdot\|)$ is a reflexive and separable Banach space, and the embedding $X \hookrightarrow \ell^p$ is compact.*

For any $u \in X$ and $\lambda > 0$, let

$$\Phi(u) = \frac{1}{p}\sum_{k\in\mathbb{Z}} \big[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p\big],$$

$$\Psi(u) = \sum_{k\in\mathbb{Z}} F(k, u(k)),$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u). \tag{6}$$

Lemma 2.3. *For the functionals Φ, Ψ, and I_λ, we have the following:*

(a) Assume that (H1) holds. Then $\Phi \in C^1(X, \mathbb{R})$ with

$$\langle \Phi'(u), v\rangle = \sum_{k\in\mathbb{Z}} [a(k)\phi_p(\Delta u(k-1))\Delta v(k-1) + b(k)\phi_p(u(k))v(k)]$$

for all $u, v \in X$.

(b) Assume that (A1) holds. Then $\Psi \in C^1(\ell^p, \mathbb{R})$ with

$$\langle \Psi'(u), v\rangle = \sum_{k\in\mathbb{Z}} f(k, u(k))v(k) \quad \text{for all } u, v \in \ell^p.$$

(c) Assume that (H1) and (A1) hold. Then, for each $\lambda > 0$, every critical point $u \in X$ of I_λ is a solution of problem (1).

Remark 2.1. Part (a) of Lemma 2.3 with $a(k) \equiv 1$ on $\mathbb{Z}$ has been proved in [4, Proposition 5]; part (b) of the lemma has been shown in [4, Proposition 6] under the assumption

(C1) $\lim_{t\to 0} \frac{|f(k,t)|}{|t|^{p-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;

and part (c) of the lemma with $a(k) \equiv 1$ on $\mathbb{Z}$ has been verified in [4, Proposition 7] under (H1) and (C1). The present version of the lemma can be proved essentially by the same method used in [4]. Below, we just provide a brief sketch of the proof for part (b).

Proof of Lemma 2.3 (b). By (A1), we have

$$|f(k,t)| \le C|t|^{\alpha-1} \le C|t|^{p-1} \quad \text{for } k \in \mathbb{Z} \text{ and } |t| \le \delta. \tag{7}$$

Thus,

$$|F(k,t)| \le \frac{C}{p}|t|^p \quad \text{for } k \in \mathbb{Z} \text{ and } |t| \le \delta.$$

For any $u \in \ell^p$, there exists $h \in \mathbb{N}$ such that $|u(k)| \le \delta$ for all $k \in \mathbb{Z}$ with $|k| > h$. Hence,

$$\left|\sum_{k\in\mathbb{Z}} F(k, u(k))\right| \le \sum_{|k|\le h} |F(k, u(k))| + \frac{C}{p}\sum_{|k|>h} |u(k)|^p,$$

and so Ψ is well defined.

Noting the similarity between (7) and inequality (6) in [4], the remainder of the proof is almost the same as that of the proof of [4, Proposition 6]. The details are omitted. □

3. **Main results.** We first state our results in this paper. Our first theorem provides conditions for the existence of at least one solution of problem (1), and the second one is for infinitely many solutions.

Theorem 3.1. *Assume that (H1)–(H3), (A1), and (A2) hold. Then, there exists $\lambda^* > 0$ such that for any $\lambda > \lambda^*$, problem (1) has at least one nontrivial solution $\{u_\lambda(k)\}$ which is a global minimizer of the functional I_λ defined by (6).*

Theorem 3.2. *Assume that (H1)–(H4), (A1), and (A2) hold. Then, there exists $\overline{\lambda} > 0$ such that for any $\lambda > \overline{\lambda}$, problem (1) has a sequence $\{u_{\lambda,n}(k)\}$ of nontrivial solutions satisfying*

$$u_{\lambda,n} \to 0 \text{ in } X \text{ as } n \to \infty \quad \text{and} \quad I_\lambda(u_{\lambda,n}) \leq 0,$$

where X and I_λ are defined by (3) and (6), respectively.

In the remainder of this section, we prove our theorems above. Recall that a functional I defined on X is said to satisfy the Palais–Smale (PS) condition if every sequence $\{u_n\} \subset X$, such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Lemma 3.3. *Assume that (H1)–(H3) and (A1) hold. Then, for any $\lambda > 0$, I_λ is coercive and satisfies the PS condition.*

Remark 3.1. The conclusion of Lemma 3.3 with $a(t) \equiv 1$ was proved in [4, Proposition 9] under the assumptions (H1)–(H3) and (C1). Note that, in the proof, the role of (C1) is to guarantee that $\Psi \in C^1(X, \mathbb{R})$. Then, in view of Lemma 2.3 (b), the proof here is essentially the same. We omit the details.

Lemma 3.4 below can be found in [8, 10].

Lemma 3.4. *Let X be a real reflexive Banach space, and let I be a weakly lower (upper, respectively) semicontinuous functional such that*

$$\lim_{||u||\to\infty} I(u) = \infty \quad \left(\lim_{||u||\to\infty} I(u) = -\infty, \text{ respectively}\right).$$

Then, there exists $u_0 \in X$ such that

$$I(u_0) = \inf_{u\in X} I(u) \quad \left(I(u_0) = \sup_{u\in X} I(u), \text{ respectively}\right).$$

Furthermore, if $I \in C^1(X, R)$, then $I'(u_0) = 0$.

Next we prove our first main result.

Proof of Theorem 3.1. Fix $\lambda > 0$. By Lemma 3.3, the functional I_λ is coercive, that is, $\lim_{||u||\to\infty} I_\lambda(u) = \infty$. Note that I_λ is continuously differentiable and sequentially weakly lower semicontinuous and X is reflexive (by Lemma 2.2). Then, from Lemma 3.4 with $I = I_\lambda$, I_λ attains its infimum in X at some $u_\lambda \in X$ and $I'_\lambda(u_\lambda) = 0$. In view of Lemma 2.3 (c), $u_\lambda(k)$ is a solution of problem (1).

We now show that $u_\lambda(k) \not\equiv 0$ on $\mathbb{Z}$. Let δ be given as in (A1). Choose a function $w \in X$ satisfying $w(k) \not\equiv 0$ and $|w(k)| \leq \delta$ on $\mathbb{Z}$. Then, by Lemma 2.2 and the fact that $1 < p \leq \alpha$, we have $w \in \ell^\alpha$. From (A1) and (A2), we see that

$$F(k,t) \geq \frac{1}{\alpha} D|t|^\alpha \quad \text{for } k \in \mathbb{Z} \text{ and } |t| \leq \delta. \tag{8}$$

Then, from (4), (6), and (8), we have

$$\begin{aligned} I_\lambda(w) &= \Phi(w) - \lambda\Psi(w) \\ &< \frac{1}{p}\|w\|^p - \frac{1}{\alpha}\lambda D \sum_{k\in\mathbb{Z}} |w(k)|^\alpha \\ &= \frac{1}{p}\|w\|^p - \frac{1}{\alpha}\lambda\|w\|_\alpha^\alpha D < 0 \quad \text{if } \lambda > \lambda^*, \end{aligned} \tag{9}$$

where

$$\lambda^* = \frac{\alpha\|w\|^p}{pD\|w\|_\alpha^\alpha} > 0.$$

Then, $I_\lambda(u_\lambda) < 0$ if $\lambda > \lambda^*$. Hence, $u_\lambda(k) \not\equiv 0$ on $\mathbb{Z}$ for $\lambda > \lambda^*$, i.e, problem (1) has a nontrivial solution for $\lambda > \lambda^*$. This completes the proof of the theorem. □

To prove Theorem 3.2, we first recall the notion of genus.

Definition 3.5. Let X be a Banach space and A a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A with $0 \notin A$, the genus $\gamma(A)$ of A is defined as the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$.

We now give the following version of the symmetric mountain pass lemma, which follows from [5, Theorem 1].

Lemma 3.6. *Let X be an infinite dimensional Banach space and $I \in C^1(X, \mathbb{R})$ satisfy the following two conditions:*

(B1) $I(u)$ is even, bounded from below, $I(0) = 0$, and $I(u)$ satisfies the PS condition;
(B2) for each $n \in \mathbb{N}$, there exists an $A_n \in \Gamma_n$ such that $\sup_{u\in A_n} I(u) < 0$.

Then, $I(u)$ has a sequence of critical points u_n such that

$$I(u_n) \leq 0, \quad u_n \neq 0, \quad \text{and} \quad \lim_{n\to\infty} u_n = 0.$$

Lemma 3.7. *Assume that (H1), (A1), and (A2) hold. Then, there exists $\overline{\lambda} > 0$ such that for any $\lambda > \overline{\lambda}$ and $n \in \mathbb{N}$, there exists $A_n \in \Gamma_n$ such that $\sup_{u\in A_n} I_\lambda(u) < 0$.*

Proof. Let $\lambda > 0$ be fixed. For any $n \in \mathbb{N}$, we can choose an n-dimensional subspace $Y_n \subset X$. Let

$$S_{n-1} = \{u \in Y_n \ : \ \|u\| = 1\}.$$

Since S_{n-1} is finite dimensional and compact, we have

$$\kappa_{n-1} := \inf_{w\in S_{n-1}} \|w\|_\alpha^\alpha > 0.$$

Define

$$\kappa = \inf_{n\in\mathbb{N}} \kappa_{n-1}.$$

To show that $\kappa > 0$, assume to the contrary that $\kappa = 0$. Then, for any $l \in \mathbb{N}$, there exists $w_l \in S_{l-1}$ such that

$$\sum_{k\in\mathbb{Z}} |w_l(k)|^\alpha < \frac{1}{l}.$$

Thus,

$$\lim_{l\to\infty} w_l(k) \to 0 \quad \text{for } k \in \mathbb{Z}.$$

Then, from the facts that

$$\sum_{k\in\mathbb{Z}} b(k)|w_l(k)|^p \leq \|w_l\|^p = 1$$

and

$$\sum_{k\in\mathbb{Z}} a(k)|\Delta w_l(k-1)|^p \le \|w_l\|^p = 1,$$

we see that there exists $L \in \mathbb{N}$ such that

$$\sum_{k\in\mathbb{Z}} b(k)|w_l(k)|^p \le \frac{1}{3} \quad \text{and} \quad \sum_{k\in\mathbb{Z}} a(k)|\Delta w_l(k-1)|^p \le \frac{1}{3} \quad \text{for } l \ge L.$$

Hence,

$$\|w_l\|^p = \sum_{k\in\mathbb{Z}} \left[a(k)|\Delta w_l(k-1)|^p + b(k)|w_l(k)|^p\right] \le \frac{2}{3} \quad \text{for } l \ge L.$$

On the other hand, since $w_l \in S_{l-1}$, we have $\|w_l\| = 1$ for any $l \in \mathbb{N}$, which is a contradiction. Therefore, $\kappa > 0$.

Now, choose $\mu > 0$ small enough so that $\mu b_0^{-1/p} \le \delta$, where δ is given in (A1). Then, for any $w \in S_{n-1}$, in view of (5), we see that

$$\|\mu w\|_\infty \le \mu b_0^{-1/p} \|w\| \le \delta.$$

Then, as in (9), from (4), (6), and (8), we have that, for any $w \in S_{n-1}$,

$$\begin{aligned}
I_\lambda(\mu w) &= \Phi(\mu w) - \lambda \Psi(\mu w) \\
&< \frac{1}{p}\|w\|^p \mu^p - \frac{1}{\alpha}\lambda \mu^\alpha D \sum_{k\in\mathbb{Z}} |w(k)|^\alpha \\
&\le \frac{1}{p}\mu^p - \frac{1}{\alpha}\lambda \kappa_{n-1}\mu^\alpha D \\
&\le \frac{1}{p}\mu^p - \frac{1}{\alpha}\lambda \kappa \mu^\alpha D < 0 \quad \text{if } \lambda > \overline{\lambda},
\end{aligned}$$

where

$$\overline{\lambda} = \frac{\alpha \mu^{p-\alpha}}{p\kappa D}.$$

Let $A_n = \mu S_{n-1}$. Then, $\gamma(A_n) = n$ and $\sup_{u\in A_n} I_\lambda(u) < 0$ for $\lambda > \overline{\lambda}$. This completes the proof of the lemma. $\square$

Proof of Theorem 3.2. Let $\lambda > \overline{\lambda}$ be fixed. Then, by (H4) and Lemmas 3.3 and 3.7, conditions (B1) and (B2) of Lemma 3.6 with $I = I_\lambda$ are satisfied. Hence, Lemma 3.6 and Lemma 2.3 (c) imply that, for every $\lambda > 0$, problem (1) has a sequence $\{u_n(k)\}$ of nontrivial solutions satisfying the required properties. This completes the proof of the theorem. $\square$

We conclude this paper with the following remark.

Remark 3.2. It is not hard to see that if $\alpha = p$, then the results here are independent from those of Iannizzotto and Tersian [4]. That is, our condition (A1) does not imply their condition (F_1) and vice-versa. If $\alpha > p$, then our condition (A1) does imply their (F_1). On the other hand, our Theorem 3.2 guarantees the existence of an infinite number of solutions, whereas in [4] the authors obtain the existence of only two solutions.

Acknowledgments. The authors would like to thank the referee who pointed out some errors in the original version of our paper and made several suggestion that lead to an improvement of our results.

REFERENCES

[1] P. Chen, X. Tang, and R. P. Agarwal, Existence of homoclinic solutions for $p(n)$-Laplacian Hamiltonian systems on Orlicz sequence spaces, *Math. Comput. Modelling*, **55** (2012), 989–1002.
[2] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory*, Springer, New York, 2011.
[3] J. R. Graef, L. Kong, and M. Wang, Infinitely many homoclinic solutions for second order difference equations with p-Laplacian, *Commun. Appl. Anal.*, to appear.
[4] A. Iannizzotto and S. A. Tersian, Multiple homoclinic solutions for the discrete p-Laplacian via critical point theory, *J. Math. Anal. Appl.*, **403** (2013), 173–182.
[5] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, *J. Funct. Anal.*, **225** (2005), 352–370.
[6] M. Ma and Z. Guo, Homoclinic orbits for second order self-adjoint difference equations, *J. Math. Anal. Appl.*, **323** (2006), 513–521.
[7] M. Mihăilescu, V. Rădulescu, and S. Tersian, Homoclinic solutions of difference equations with variable exponents, *Topol. Methods Nonlinear Anal.*, **38** (2011), 277–289.
[8] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, *Appl. Math. Sci.*, Springer, New York, **74** 1989.
[9] X. Tang and X. Lin, Existence of infinitely many homoclinic orbits in discrete Hamiltonian systems, *J. Math. Anal. Appl.* **373** (2011), 59–72.
[10] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. III*, Springer, New York, 1985.
[11] Z. Zhang and R. Yuan, Homoclinic solutions for a class of non-autonomous subquadratic second order Hamiltonian systems, *Nonlinear Anal.*, **71** (2009), 4125–4130.

Received August 2014; revised December 2014.

E-mail address: John-Graef@utc.edu
E-mail address: Lingju-Kong@utc.edu
E-mail address: mwang808@gmail.com