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RESEARCH PAPER

FRACTIONAL GENERALIZATIONS OF FILTERING PROBLEMS AND THEIR ASSOCIATED FRACTIONAL ZAKAI EQUATIONS

Sabir Umarov [1], **Frederick Daum** [2], **Kenric Nelson** [3]

Abstract

In this paper we discuss fractional generalizations of the filtering problem. The "fractional" nature comes from time-changed state or observation processes, basic ingredients of the filtering problem. The mathematical feature of the fractional filtering problem emerges as the Riemann-Liouville or Caputo-Djrbashian fractional derivative in the associated Zakai equation. We discuss fractional generalizations of the nonlinear filtering problem whose state and observation processes are driven by time-changed Brownian motion or/and Lévy process.

MSC 2010: Primary 60H10; Secondary 35S10, 60G51, 60H05

Key Words and Phrases: time-change, stochastic differential equation, filtering problem, Zakai equation, fractional order differential equation, pseudo-differential operator, Lévy process, stable subordinator

1. Introduction

The filtering problem is formulated as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space with a state space Ω, sigma-algebra $\mathcal{F}$, and probability measure $\mathbb{P}$. Let $X_t : \Omega \to \mathbb{R}^n$ be an $\mathbb{R}^n$-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and called *a state process.* We assume that X_t is governed by the stochastic differential equation

$$dX_t = b(t, X_t)dt + g(t, X_t)dB_t, \tag{1.1}$$

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with the initial condition $X_{t=0} = X_0$, where X_0 is a random variable independent of Brownian motion B_t, the functions $b(t,x)$ and $g(t,x)$, defined for $t > 0$ and $x \in \mathbb{R}^n$, satisfy some growth and continuity conditions. The state process in the filtering problem can not be observed directly. Suppose Z_s, $s \leq t$, is $\mathbb{R}^m$-valued stochastic process called *observation process* and related to the process X_t in the noisy environment. The observation process Z_t can be expressed through a stochastic differential equation of the form

$$dZ_t = h(t, X_t)dt + dW_t, \ Z_0 = 0, \tag{1.2}$$

where $h(t,x)$, $t > 0$, $x \in \mathbb{R}^n$, is a function satisfying appropriate growth and continuity conditions, and W_t is an m-dimensional Brownian motion independent of B_t and X_0. Let $\mathcal{Z}_t$ and $\mathcal{F}_t$ be filtrations generated by the observation process Z_t, and by X_0, B_t, and W_t, respectively. We assume that X_t is $\mathcal{F}_t$-predictable stochastic process. The filtering problem is to find the best estimation of X_t at time t given Z_t, in the mean square sense. Namely, in the filtering problem one needs to find a stochastic process X_t^* such that

$$\mathbb{E}[\|X_t - X_t^*\|^2] = \inf \mathbb{E}[\|X_t - Y_t\|^2],$$

where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$ and inf is taken over all $\mathcal{F}_t$-predictable stochastic processes $Y_t \in L_2(\mathbb{P})$, given $\mathcal{Z}_t$. It follows from the abstract theory of functional analysis that X_t^* is the projection of X_t onto the space of stochastic processes $\mathcal{L}(Z_t) = \{Y \in L_2(\mathbb{P}) : Y_t \text{ is } \mathcal{F}_t - \text{predictable}\}$, given $\mathcal{Z}_t$. The process X_t^* can be written in the form $X_t^* = \mathbb{E}[X_t|\mathcal{Z}_t]$.

Filtering problems arise in many engineering models. One simple example is transmitting of modulated signals. These signals are received with an effect of noisy environment. The received signal has to be filtered in order to be realizable. Thus in this situation filtering problem is about the best estimation of the stochastic process (the modulated signal transmitted in the noisy environment) given additional information obtained via measurement of parameters of the process (of the signal).

The filtering problem is called *linear* if the functions $b(t,x)$ and $g(t,x)$ depend on x linearly. The linear filtering problem was studied by Kalman and Bucy [14] in the 1960th. They reduced the linear filtering problem to a linear SDE and a deterministic Riccati type differential equation. In the case of *non linear filtering* Kushner [16], Lipster and Shiryaev [17], and Fujisaki, Kallianpur and Kunita [6] (see also [22]) obtained a non linear infinite dimensional stochastic differential equations for the posterior conditional density of X_t, given $\mathcal{Z}_t$. However, (1) it is not easy to solve these equations, and (2) it is computationally 'expensive' due to the two stage calculation procedure (prediction and correction) in the real time. Later

Zakai [27] obtained a simpler form of the stochastic differential equation for the posterior unnormalized conditional density $\Phi(t,x) = p(t,x|\mathcal{Z}_t)$ for X_t in the following form:

$$\Phi(t,x) = \mathbb{P}(X_0 = x) + \int_0^t A^*\Phi(s,x)ds + \sum_{k=1}^{m}\int_0^t h_k(x)\Phi(s,x)dZ_s^{(k)}, \quad (1.3)$$

where the operator A^* is the dual of the infinitesimal generator A (see Section **2**) of the Markov process X_t, and $h_k(x), k = 1, ..., m$, are components of the random vector-function $h(x)$ in the observation process given by equation (1.2). Equation (1.3) can be written in the differential form as follows

$$d\Phi(t,x) = A^*\Phi(t,x)dt + \sum_{k=1}^{m} h_k(x)\Phi(t,x)dZ_t^{(k)}, \ \Phi(0,x) = \mathbb{P}(X_0 = x). \quad (1.4)$$

Equation (1.3) (or (1.4)) is a linear stochastic partial differential equation, and therefore, the methods of solution of linear equations are applicable, including some explicit forms for the solution.

Though both processes in stochastic differential equations (1.1) and (1.2) are driven by Gaussian processes (here independent Brownian motions), the solution X_t of the filtering problem may not be a Gaussian process. Daum [3, 4, 5] developed algorithms for the solution X_t of nonlinear filtering problem in the class of distributions from the exponential family. In paper [3] he reduced a solution of the Zakai equation to a solution of the Fokker-Planck equation and a deterministic matrix Riccati equation, generalizing the classical result of Kalman and Bucy. In recent works [4, 5] particle flow algorithms were suggested which provide several orders of magnitude improvement in the processing of particle filters by computing Bayes' rule as a flow of the logarithm of the conditional density from the prior to the posterior, and he derived the corresponding particle flow.

All the works mentioned above relate to filtering problems driven by Gaussian processes and with the solution in the exponential class. However, many processes naturally arising in the modern science (in particular, in biology, genetics, finance) and engineering do not obey Gaussian driving processes. Filtering problems with the state and/or observation processes driven by Lévy processes were discussed in recent publications [2, 19, 20].

In this paper we discuss fractional generalizations of the filtering problem to the case when the state and observation processes are driven by time-changed Brownian motion or Lévy processes. Fractional model of the filtering problem significantly extends the scope of the filtering problems both theoretically and their engineering and other applications. As a time change process T_t we consider the inverse of the Lévy stable subordinator with the stability index $\beta \in (0,1)$, or their mixtures. The associated Zakai equation then is given by the following partial stochastic differential equation driven by a semimartingale (see for details in Section **3**):

$$d\Phi(t,x) = A^*\Phi(t,x)dT_t + \sum_{k=1}^{m} h_k(x)\Phi(t,x)dZ_{T_t}^{(k)},\ \Phi(0,x) = p(x), \quad (1.5)$$

where A^* is the adjoint operator of the infinitesimal generator of the Markovian process X_t and $p(x)$ is the density function of X_0. Let $g_t(\tau)$, $\tau \geq 0$, be the density function of T_t. If one introduces a stochastic process

$$U(t,x) = \int_{R_+} \Phi(\tau,x)\mathbb{P}(T_t \in d\tau), \quad (1.6)$$

then stochastic differential equation (1.5) implies

$$D_*^{\beta}U(t,x) = A^*U(t,x)dt + \sum_{k=1}^{m} h_k(x)B_kU(t,x)dt, \quad (1.7)$$

$$U(0,x) = p(x), \quad (1.8)$$

where D_*^{β} is the fractional differentiation operator of order β in the sense of Caputo, defined as $D_*^{\beta} = J^{1-\beta}\frac{d}{dt}$, with the fractional integration operator

$$J^{\beta}f(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau)d\tau,\ t > 0,$$

and $B_k, k = 1,\ldots,m$, are linear transformations in the class of stochastic processes; see details in Section **3**. By definition, $U(t,x)$ is obtained from $\Phi(t,x)$ by conditioning on values of the time-change process T_t, and hence stochastic equation (1.7), (1.8) is a form of the stochastic differential equation (1.5) "averaged" (with the weight $g_t(\tau)$) over all the values of T_t. Equation (1.7) generalises the fractional Fokker-Planck-Kolmogorov equation. In the absence of the observation process Z_t this equation represents the fractional Fokker-Planck-Kolmogorov equation, see [7, 8]. Therefore, it is natural to call equation (1.7), (1.8) and its associated (unconditioned) stochastic differential equation (1.5) a *fractional Zakai type equations.* Obviously, the usual Zakai equation is recovered if $\beta \to 1$.

In the mathematical literature stochastic differential equations driven by a fractional Brownian motion are also called *fractional.* However, the nature of stochastic differential equations (1.5) and (1.7), that is fractional Zakai type stochastic differential equations, totally different from the nature of those fractional stochastic differential equations driven by a fractional Brownian motion.

2. Generalized filtering problems

2.1. Generalization of the filtering problem with a Lévy processes. Many processes in the modern science and engineering do not obey the Gaussian law for the state process in equation (1.1) and for the observation process in equation (3.4). Let $L_t,\ t \geq 0$, be an n-dimensional Lévy process. Lévy processes can be characterized by the Lévy-Itô decomposition theorem, which states that

$$L_t = b_0 t + \sigma B_t + \int_{|w|<1} w\tilde{N}(t, dw) + \int_{|w|\geq 1} wN(t, dw), \tag{2.1}$$

where $b_0 \in \mathbb{R}^n$, σ is an $n \times m$-matrix such that $\sigma\sigma^T = \Sigma$, B_t is an m-dimensional Brownian motion, and N_t and $\tilde{N}_t$ are a compound Poisson random measure and a compensated Poisson martingale-valued measure, respectively [25]. It is well known [25] that Lévy processes have a càdlàg modification and are semimartingales. Any Lévy process is uniquely defined by a triple (b, Σ, ν), where $b \in \mathbb{R}^n$, Σ is a nonnegative definite matrix, and a measure ν defined on $\mathbb{R}^n \setminus \{0\}$ such that $\int \min(1, |x|^2)d\nu < \infty$. Lévy processes can also be characterized by the Lévy-Khintchine formula in terms of its characteristic function $\Phi_t(\xi) = E(e^{i\xi L_t}) = e^{t\Psi(\xi)}$, with

$$\Psi(\xi) = i(b, \xi) - \frac{1}{2}(\Sigma\xi, \xi) + \int_{\mathbb{R}^n\setminus\{0\}} (e^{i(w,\xi)} - 1 - i(w, \xi)\chi_{(|w|\leq 1)}(w))\nu(dw). \tag{2.2}$$

The function Ψ is called the *Lévy symbol* of L_t. For any Lévy process, its Lévy symbol is continuous, hermitian, conditionally positive definite and $\Psi(0) = 0$. The infinitesimal generator of the Lévy process with characteristics (b, Σ, ν) is a pseudo-differential operator $A = A(\mathbf{D}_x)$ with the symbol $\Psi(\xi)$ defined in (2.2).

A natural generalization of the filtering problem (1.1)-(1.2) is to replace Brownian motions B_t in (1.1) and W_t in (1.2) with Lévy processes L_t and M_t, respectively. Namely, consider a state process X_t governed by the Lévy process L_t :

$$\begin{aligned} X_t = X_0 &+ \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s \\ &+ \int_0^t \int_{|w|<1} H(X_{s-}, w)\tilde{N}(ds, dw) + \int_0^t \int_{|w|\geq 1} K(X_{s-}, w)N(ds, dw), \end{aligned} \tag{2.3}$$

where X_0 is a random variable independent of B_t and $N(t, \cdot)$; the continuous mappings $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the Lipschitz and linear growth conditions. The infinitesimal generator $\mathcal{A}$ of the process X_t is a pseudo-differential operator with the symbol

$$\Psi(x,\xi) = i(b(x),\xi) - \frac{1}{2}(\Sigma(x)\xi,\xi) + \int_{\mathbb{R}^n\setminus\{0\}} (e^{i(G(x,w),\xi)} - 1 - i(G(x,w),\xi)\chi_{(|w|<1)}(w))\nu(dw), \quad (2.4)$$

where $G(x,w) = H(x,w)$ if $|w| < 1$, and $G(x,w) = K(x,w)$ if $|w| \geq 1$ [1]. By definition, a pseudo-differential operator $\mathcal{A}$ with the symbol $\Psi(x,\xi)$ is

$$\mathcal{A}\varphi(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^n} \Psi(x,\xi)\hat{\varphi}(\xi)e^{-i(x,\xi)}d\xi, \quad (2.5)$$

where $\hat{\varphi}$ is the Fourier transform of ϕ in the domain of $\mathcal{A}$ (see details in [1, 9]). Let the observation process is given by

$$Z_t = \int_0^t \mu(s,X_{s-})ds + \int_0^t \nu(s,X_{s-})dW_s + \int_0^t\int_{|w|<1} g(X_{s-},w)\tilde{M}(ds,dw) + \int_0^t\int_{|w|\geq 1} f(X_{s-},w)M(ds,dw), \quad (2.6)$$

where Brownian motion W_t is independent of B_t in equation (2.3), the measures M_t and $\tilde{M}_t$ are a compound Poisson random measure and a compensated Poisson martingale-valued measure, and mappings $\mu(t,x)$, $\nu(t,x)$, $g(t,x,w)$, and $f(t,x,w)$ satisfy the Lipschitz and linear growth conditions. Let $\mathcal{Z}_t$ be the sigma-algebra generated by the process Z_s, $0 \leq s \leq t$. Now the generalized filtering problem is formulated as follows: *find the best estimation of X_t given $\mathcal{Z}_t$.*

Particular cases of this problem is discussed in Chapter 4 of [2] and in papers [19, 20]. Namely, consider the following two cases:

(1) the state process is driven by Lévy process and the observation process is driven by Brownian motion; and vise versa,
(2) the state process is driven by Brownian motion and the observation process is driven by Lévy process.

In the first case the filtering model is formulated as follows: The state process is given by stochastic differential equation (2.3) driven by a Lévy process, and the observation process is given by

$$Z_t = \int_0^t h(X_s)ds + W_t, \quad (2.7)$$

where X_t and Z_t are respectively an $\mathbb{R}^n$-valued and $\mathbb{R}^m$-valued stochastic processes. Then (see [19]) the unnormalized conditional distribution of

$f(X_t)$ given $\mathcal{Z}_t$, that is $\phi_t(f) = \mathbb{E}[f(X_t)|\mathcal{Z}_t]$ under some conditions, satisfies the Zakai type stochastic partial differential equation

$$\phi_t(f) = \phi_0(f) + \int_0^t \phi_s(Af)ds + \sum_{k=1}^{m} \int_0^t \phi_s(fh_k)dZ_s^{(k)}, \tag{2.8}$$

where $\mathcal{A}$ is the infinitesimal generator of the process X_t defined in (2.5), and $h_k(x)$ and $Z^{(k)}$ are components of vectors $h(x) = (h_1(x), ..., h_m(x))$ and $Z_t = (Z_t^{(1)}, ..., Z_t^{(m)})$, respectively.

In the second case of the filtering model the state process is given by

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s, \tag{2.9}$$

and the observation process is

$$Z_t = \int_0^t h(X_s)ds + W_t + \int_0^t \int_{\mathbb{R}^m} wN_\lambda(dt, dw), \tag{2.10}$$

where N_λ is an integer valued random measure with predictable compensator $\lambda(t, X_t, \omega)dtd\nu$, with a Lévy measure ν. Let $\Phi(t, x)$ be a filtering density, that is for arbitrary infinitely differentiable function f with a compact support the relation

$$\phi_t(f) = \int_{\mathbb{R}^n} f(x)\Phi(t, x)dx$$

holds. Then (see [20]) the corresponding Zakai type equation has the form

$$\Phi(t, x) = p_0(x) + \int_0^t A^*\Phi(s, x)ds + \int_0^t h(s, x)\Phi(s, x)dB_s + \int_0^t \int_{\mathbb{R}^n} (\lambda(s, x, w) - 1)\Phi(s, x)\tilde{N}(ds, dw), \tag{2.11}$$

where A^* is the dual of the infinitesimal generator A of X_t and $\tilde{N}(ds, dw) = N(ds, dw) - dsd\nu$. We note that in this case due to absence of jump components of the state process X_t its infinitesimal generator A is not a pseudo-differential operator. The operator A is a second order elliptic differential operator

$$A\varphi(x) = \frac{1}{2} \sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_k} + \sum_{k=1}^{n} b_k(x) \frac{\partial \varphi(x)}{\partial x_k}, \tag{2.12}$$

with the coefficients $a_{ik}(x)$, which are the entries of the matrix-function $a(x) = \{a_{ik}(x)\}_{i,k=1}^n$ obtained by multiplying $\sigma(x)$ by its transpose $\sigma(x)^t$.

It is not hard to verify that both cases recover the classical Zakai equation (1.3) if the jump component of the Lévy process is absent, that is if $\nu \equiv 0$.

2.2. Generalization of the filtering problem with time-changed Lévy processes. We note that the driving stochastic processes in the filtering model (2.3) and (2.6) are semimartingales with independent increments. If T is a Lévy subordinator then L_T is still a Lévy process. Therefore, replacement of Lévy processes L_t and M_t in (2.3) and (2.6) with their time-changed ones L_{T_1} and M_{T_2}, where T_1 and T_2 are Lévy subordinators, does not expand the scope of the filtering models given by equations (2.3) and (2.6). If T is an inverse to a Lévy subordinator then the time-changed process L_T is no longer a Lévy process. However, it is a semimartingale [17]. Therefore, stochastic integrals driven by time-changed processes L_T, where T is the inverse to a stable Lévy subordinator, are well defined.

Consider the following model of the filtering problem driving processes of which are time-changed Lévy processes. The state process in this context is given by

$$X_t = X_0 + \int_0^t b(X_{s-})dT_s + \int_0^t \sigma(X_{s-})dB_{T_s} + \int_0^t \int_{|w|<1} H(X_{s-}, w)\tilde{N}(dT_s, dw) + \int_0^t \int_{|w|\geq 1} K(X_{s-}, w)N(dT_s, dw), \tag{2.13}$$

and the observation process is

$$Z_t = \int_0^t \mu(s, X_{s-})dE_s + \int_0^t \nu(s, X_{s-})dW_{E_s} + \int_0^t \int_{|w|<1} g(X_{s-}, w)\tilde{M}(dE_s, dw) + \int_0^t \int_{|w|\geq 1} f(X_{s-}, w)M(dE_s, dw), \tag{2.14}$$

where T_t and E_t are inverse processes to stable Lévy subordinators. Notice, that if $T_t = t$ and $E_t = t$ then the filtering model (2.13), (2.14) represents the model (2.3), (2.6) for filtering problem driven by Lévy processes. Therefore, replacement of Lévy processes L_t and M_t in the filtering model (2.3), (2.6) with L_T and M_E, respectively, where T and E are inverse Lévy subordinators does expand essentially the scope of the model (2.3), (2.6).

In Section **3** we will solve the time-changed filtering problem under certain constraints. We will need some preliminary facts on Lévy subordinators and their inverses. Let T_t be the first hitting time process for a Lévy stable subordinator D_t with stability index $\beta \in (0, 1)$. The process

T_t is also called an inverse to D_t. The relation between T_t and D_t can be expressed as $T_t = \min\{\tau : D_\tau \geq t\}$. The process $D_t,\ t \geq 0$, is a strictly increasing self-similar Lévy process with $D_0 = 0$, that is $D_{ct} = c^{\frac{1}{\beta}} D_t$ in the sense of finite dimensional distributions, and its Laplace transform is $\mathbb{E}(e^{-sD_t}) = e^{-ts^\beta}$. The density $f_{D_1}(\tau)$ of D_1 is infinitely differentiable on $(0,\infty)$, with the following asymptotics at zero and infinity [18, 26]:

$$f_{D_1}(\tau) \sim \frac{(\frac{\beta}{\tau})^{\frac{2-\beta}{2(1-\beta)}}}{\sqrt{2\pi\beta(1-\beta)}}\, e^{-(1-\beta)(\frac{\tau}{\beta})^{-\frac{\beta}{1-\beta}}},\ \tau \to 0; \tag{2.15}$$

$$f_{D_1}(\tau) \sim \frac{\beta}{\Gamma(1-\beta)\tau^{1+\beta}},\ \tau \to \infty. \tag{2.16}$$

Since D_t is strictly increasing, its inverse process T_t is continuous and non-decreasing, but not a Lévy process. Likewise for any Lévy process L_t the time-changed process L_{T_t} is also not a Lévy process (see details in [7, 8]).

Let $g_t(\tau)$ be the density function of T_t for each fixed $t > 0$. If $f_{D_1}(t)$ is the density function of D_1, then

$$g_t(\tau) = -\frac{\partial}{\partial\tau} J f_{D_1}(\frac{t}{\tau^{1/\beta}}) = -\frac{\partial}{\partial\tau}\int_0^{\frac{t}{\tau^{1/\beta}}} f_{D_1}(u)du = \frac{t}{\beta\tau^{1+\frac{1}{\beta}}} f_{D_1}(\frac{t}{\tau^{\frac{1}{\beta}}}). \tag{2.17}$$

Since $f_{D_1}(u) \in C^\infty(0,\infty)$, it follows from representation (2.17) that $g_t(\tau) = \varphi(t,\tau) \in C^\infty(\mathbb{R}^2_+)$, where $\mathbb{R}^2_+ = (0,\infty)\times(0,\infty)$. Further properties of $g_t(\tau)$ are represented in the following two lemmas.

Lemma **2.1**. *Let $g_t(\tau)$ be the function given in* (2.17). *Then:*

(*a*) $\lim_{t\to+0} g_t(\tau) = \delta_0(\tau)$ *in the sense of the topology of the space of tempered distributions $\mathcal{D}'(\mathbb{R})$;*

(*b*) $\lim_{\tau\to+0} g_t(\tau) = \frac{t^{-\beta}}{\Gamma(1-\beta)},\ t > 0;$

(*c*) $\lim_{\tau\to\infty} g_t(\tau) = 0,\ t > 0;$

(*d*) $\mathcal{L}_{t\to s}[g_t(\tau)](s) = s^{\beta-1}e^{-\tau s^\beta},\ s > 0,\ \tau \geq 0,$

where $\mathcal{L}_{t\to s}$ denotes the Laplace transform with respect to the variable t.

Lemma **2.2**. *Function $g_t(\tau)$ defined in* (2.17) *for each $t > 0$ satisfies the following equation*

$$D^\beta_{*,t} g_t(\tau) = -\frac{\partial}{\partial\tau} g_t(\tau) - \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta_0(\tau), \tag{2.18}$$

in the sense of tempered distributions.

We refer the reader to papers [7, 8] for proofs of these two lemmas.

3. Fractional filtering problem. Main results

First, for simplicity we consider the filtering problem with the state process given in the differential form

$$dX_t = b(X_t)dT_t + \sigma(X_t)dB_{T_t},\ X_{t=0} = X_0, \tag{3.1}$$

and driven by a time-changed Brownian motion with drift, where T_t is the inverse of the Lévy stable subordinator with the stability index $\beta \in (0, 1)$. The natural observation process associated with the state process (3.1) with invented time-change has the form

$$dV_t = h(X_t)dT_t + dW_{T_t},\ V_0 = 0. \tag{3.2}$$

In the theorem below we assume that the input data of this filtering model satisfy the following conditions:

(C1) the vector-functions $b(x)$ and $h(x)$ and $n \times m$-matrix-function $\sigma(x)$ are infinite differentiable and bounded;
(C2) the time-change process T_t and Brownian motions B_t and W_t are independent processes;
(C3) the initial random vector X_0 is independent of processes B_t, W_t, and T_t and has an infinite differentiable density function $p_0(x)$ decaying at infinity faster than any power of $|x|$.

We note that the conditions on infinite differentiability and boundedness of $b(x)$, $h(x)$, and $\sigma(x)$ in (C1), as well as of the density function $p_0(x)$ in (C3) and its decay condition at infinity can be weakened.

The filtering problem (3.1), (3.2) is closely related to the filtering problem whose state process is given by the following (non time-changed) Îto stochastic differential equation

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t,\ Y_{t=0} = X_0,\ t > 0, \tag{3.3}$$

and the corresponding observation process is given by

$$dZ_t = h(Y_t)dt + dW_t,\ Z_0 = 0. \tag{3.4}$$

Introduce the process

$$\rho(t) = \exp\{-\sum_{k=1}^{m} \int_0^t h_k(Y_s)dW_s - \frac{1}{2}\int_0^t |h(Y_s)|^2 ds\}$$

and the probability measure $d\mathbb{P}_0 = \rho(t)d\mathbb{P}$. Further, let

$$\Lambda_t = \frac{d\mathbb{P}}{d\mathbb{P}_0}\Big|_{\mathcal{Z}_t}, \tag{3.5}$$

and $\hat{E}$ be an expectation under the reference measure $\mathbb{P}_0$. Then, as is known, the optimal filtering solution of the filtering problem (3.3), (3.4) is given by the following Kallianpur-Striebel's formula [13, 19, 22])

$$\mathbb{E}[f(Y_t)|\mathcal{Z}_t] = \frac{\hat{\mathbb{E}}[f(Y_t)\Lambda_t|\mathcal{Z}_t]}{\hat{\mathbb{E}}[\Lambda_t|\mathcal{Z}_t]}.$$

It is also known (see, e.g. [19], Lemma 4.1) that under the reference measure $\mathbb{P}_0$, the process Z_t is a standard Brownian motion independent of Y_t, and Λ_t satisfies the equation

$$\Lambda_t = 1 + \sum_{k=1}^{m} \int_0^t \Lambda_s h_k(Y_s) dZ_s^k.$$

Moreover, under conditions (C1)-(C3) the unnormalized filtering measure $p_t(f) = \hat{\mathbb{E}}[f(Y_t)\Lambda_t|\mathcal{Z}_t]$ satisfies the following stochastic differential equation called the Zakai equation [22, 27]

$$p_t(f) = p_0(f) + \int_0^t p_s(Af)ds + \sum_{k=1}^{m} \int_0^t p_s(h_k f)dZ_s^k, \tag{3.6}$$

where A is a second order elliptic differential operator given by equation (2.12).

Further, introducing the filtering density $U(t,x)$ through

$$p_t(f) = \int_{\mathbb{R}^n} f(x)U(t,x)dx, \tag{3.7}$$

one can show that $U(t,x)$ solves the (adjoint) Zakai equation

$$dU(t,x) = A^*U(t,x)dt + \sum_{k=1}^{m} h_k(x)U(t,x)dZ_k(t), \tag{3.8}$$

with the initial condition $U(0,x) = p_0(x)$. Here A^* is the dual operator of A defined in (2.12).

THEOREM **3.1**. *Let T_t be a time change process and let*

$$\phi_t(f) = \hat{\mathbb{E}}[f(X_t)\Lambda_{T_t}|\mathcal{V}_t],$$

where $\mathcal{V}$ is the filtration generated by V_t. Suppose conditions (C1)-C(3) are verified. Then $\phi_t(f)$ satisfies the following Zakai type equation corresponding to filtering problem (3.1), (3.2)*:*

$$\phi_t(f) = p_0(f) + \int_0^t \phi_s(Af)dT_s + \sum_{k=1}^{m} \int_0^t \phi_s(h_k f)dZ_{T_s}^k \tag{3.9}$$

P r o o f. Let conditions (C1)-(C3) be verified. Then, in particular, the conditions for the existence of an unnormalized filtering distribution $p_t(f) = \hat{\mathbb{E}}[f(Y_t)\Lambda_t|\mathcal{Z}_t]$ which solves the Zakai equation (3.6), is also verified.

Here Y_t is a solution to stochastic differential equation (3.3). According to Theorem 3.3 in [7] the time-changed process $X_t = Y_{T_t}$ solves stochastic differential equation (3.1).

The connection $X_t = Y_{T_t}$ between the state processes X_t and Y_t implies the connection $V_t = Z_{T_t}$ between the observation processes V_t and Z_t. Indeed, letting $T_t = \tau$, or the same $D_\tau = t$, one obtains from the relation $dV_t = h(Y_{T_t})dT_t + dW_{T_t}$ and (3.4) that $Z_\tau = V_{D_\tau}$, or the same $V_t = Z_{T_t}$. It follows that the filtration $\mathcal{V}_t$ coincides with the filtration $\mathcal{Z} \circ \mathcal{T}_t \equiv \mathcal{Z}_{\mathcal{T}_t}$ generated by the time changed observation process Z_{T_t}. Hence, the unnormalized filtering distribution $\phi_t(f) = \hat{\mathbb{E}}[f(X_t)\Lambda_{T_t}|\mathcal{Z} \circ \mathcal{T}_t]$ corresponding to the filtering problem (3.1), (3.2) is the time-changed process $\phi_t(f) = p_{T_t}(f)$. Therefore, due to equation (**3.2**) the process $\phi_t(f)$ satisfies

$$\phi_t(f) = p_{T_t}(f) = p_0(f) + \int_0^{T_t} p_s(Af)ds + \sum_{k=1}^{m} \int_0^{T_t} p_s(h_k f) dZ_s^{(k)}. \quad (3.10)$$

Further, using the change of variable formula (see [11], Proposition 10.21)

$$\int_0^{T_t} H_s dS_s = \int_0^t H_{T_{s-}} dS_{T_s},$$

for stochastic integrals driven by a semimartingale S_t, we obtain (there are no jumps in this particular case)

$$\begin{aligned}\int_0^{T_t} p_s(Af)ds &= \int_0^t \hat{\mathbb{E}}[Af(Y_{T_s})\Lambda_{T_s}|\mathcal{Z}_{T_s}]dZ_{T_s}^{(k)} = \int_0^t \hat{\mathbb{E}}[Af(X_s)\Lambda_{T_s}|\mathcal{Z}_{T_s}]dZ_{T_s}^{(k)} \\ &= \int_0^t \phi_s(Af)dZ_{T_s}^{(k)}. \end{aligned} \quad (3.11)$$

and

$$\begin{aligned}\sum_{k=1}^{m} \int_0^{T_t} p_s(h_k f)dZ_s^{(k)} &= \sum_{k=1}^{m} \int_0^t \hat{\mathbb{E}}[h_k(Y_{T_s})f(Y_{T_s})\Lambda_{T_s}|\mathcal{Z}_{T_s}]dZ_{T_s}^{(k)} \\ &= \sum_{k=1}^{m} \int_0^t \hat{\mathbb{E}}[h_k(X_s)f(X_s)\Lambda_{T_s}|\mathcal{Z}_{T_s}]dZ_{T_s}^{(k)} \\ &= \sum_{k=1}^{m} \int_0^t \phi_s(h_k f)dZ_{T_s}^{(k)}. \end{aligned} \quad (3.12)$$

Equations (3.10), (3.11), and (3.12) imply the desired equation (3.9). □

Remark **3.1**. The process Z_{T_t} is a semimartingale, therefore SDE (3.9) is meaningful (see, e.g. [21]).

Further, we introduce a filtering density and derive the adjoint Zakai equation generalizing the equation (3.8). Let $\Phi(t,x)$ be defined as a generalized function

$$\phi_t(f) = \int_{R^n} f(x)\Phi(t,x)dx, \tag{3.13}$$

for arbitrary infinite differentiable function f with compact support. The function $\Phi(t,x)$ is called a *filtering density associated with the filtering measure* $\phi_t(f)$.

THEOREM **3.2**. *Let the conditions (C1)-(C3) be verified. Then the filtering density* $\Phi(t,x)$ *associated with the filtering measure* $\phi_t(f)$ *in equation* (3.9) *satisfies the following Zakai type equation*

$$\Phi(t,x) - \Phi(0,x) = \int_0^t A^*\Phi(s,x)dT_s + \sum_{k=1}^m \int_0^t h_k(x)\Phi(s,x)dZ^{(k)}_{T_s}. \tag{3.14}$$

The proof of this theorem immediately follows from equation (3.10) substituting $\phi_t(f)$ in (3.13).

Further, we notice that in accordance with the definitions of $p_t(f)$ and $\phi_t(f)$ the stochastic process $p_\tau(f)$ is a process obtained from $\phi_t(f)$ conditioning on the values $\tau \geq 0$ of the process T_t :

$$p_\tau(f) = \hat{\mathbb{E}}[f(Y_\tau)\Lambda_\tau|\mathcal{Z}_\tau] = \Big[\hat{\mathbb{E}}[f(Y_{T_t})\Lambda_{T_t}|\mathcal{Z}\circ\mathcal{T}_t]\Big|T_t = \tau\Big] = [\phi_t(f)|T_t = \tau].$$

Introduce the stochastic processes $\Pi_t(f)$ and $\Pi_{t,Z}(f)$ defined by

$$\Pi_t(f) = \mathcal{A}p_t(f) = \int_0^\infty g_t(\tau)p_\tau(f)d\tau, \quad \Pi_{t,Z}(f) = \mathcal{C}_t(f) = \int_0^\infty g_t(\tau)p_\tau(f)dZ_\tau, \tag{3.15}$$

where $g_t(\tau)$ is the density function of the process T_t and $p_t(f)$ is the unnormalized filtering distribution of the Zakai equation (3.6) corresponding to the filtering model (3.3)-(3.4). The theorem below shows a relationship between these processes important in applications.

THEOREM **3.3**. *Let* T_t *be the inverse to a stable Lévy subordinator* D_t *of a stability index* $\beta \in (0,1)$. *Then the following stochastic relation holds:*

$$\Pi_t(f) - p_0(f) = J_t^\beta\Big(\Pi_t(Af) + \sum_{k=1}^m \Pi_{t,Z^{(k)}}(h_k f)\Big), \tag{3.16}$$

where the processes $\Pi_t(f)$ *and* $\Pi_{t,Z}(f)$ *are defined in equation* (3.15), *and* J_t^β *is the fractional integration operator of order* β.

P r o o f. To prove the theorem we need an additional property of $g_t(\tau)$, $\tau \in (0,\infty)$, the density function of the time-change process T_t for each $t \in (0,\infty)$, defined in equation (2.17). Applying the fractional integration operator J^β to equation (2.18), we have

$$g_t(\tau) - \lim_{t\to 0+} g_t(\tau) = -\frac{\partial}{\partial\tau} J_t^\beta g_t(\tau) - \frac{\delta_0(\tau)}{\Gamma(1-\beta)} J_t^\beta t^{-\beta},$$

in the sense of distributions. Due to part (a) of Lemma **2.1** we have $\lim_{t\to 0+} g_t(\tau) = \delta_0(\tau)$. This fact together with the equation $J^\beta t^{-\beta} = \Gamma(1-\beta)$ implies

$$g_t(\tau) = -\frac{\partial}{\partial\tau} J_t^\beta g_t(\tau). \tag{3.17}$$

Now conditioning equation (3.10) on the process $T_t = \tau$ and integrating over the interval $(0,\infty)$ with respect to the measure $\mathbb{P}(T_t \in d\tau) = g_t(\tau)d\tau$, one obtains

$$\begin{aligned}\Pi_t(f) - p_0(f) &= \int_0^\infty g_t(\tau)\Big[\int_0^\tau p_s(Af)ds\Big]d\tau \\ &+ \sum_{k=1}^m \int_0^\infty g_t(\tau)\Big[\int_0^\tau p_s(h_k f)dZ_s^{(k)}\Big]d\tau. \end{aligned}\tag{3.18}$$

Due to relation (3.17) the first term on the right side of (3.18) can be written in the form

$$\begin{aligned}\int_0^\infty g_t(\tau)\Big[\int_0^\tau p_s(Af)ds\Big]d\tau &= -\int_0^\infty \frac{\partial}{\partial\tau} J_t^\beta g_t(\tau)\Big[\int_0^\tau p_s(Af)ds\Big]d\tau \\ &= -\int_0^\infty \Big[\int_0^\tau p_s(Af)ds\Big]d\Big[J_t^\beta g_t(\tau)\Big]. \end{aligned}\tag{3.19}$$

The integration by parts in (3.19) implies

$$\int_0^\infty g_t(\tau)\Big[\int_0^\tau p_s(Af)ds\Big]d\tau = J_t^\beta \int_0^\infty g_t(\tau)p_\tau(Af)d\tau = J_t^\beta \Pi_t(Af), \tag{3.20}$$

since $J_t^\beta g_t(\tau) \to 0$ as $\tau \to \infty$, and $\int_0^\infty p_s(Af)ds$ is bounded due to conditions (C1)-(C3). Similarly, for each $k = 1,\dots,m$, one has

$$\begin{aligned}\int_0^\infty g_t(\tau)\Big[\int_0^\tau p_s(h_k f)dZ_s^{(k)}\Big]d\tau &= J_t^\beta \int_0^\infty g_t(\tau)p_\tau(h_k f)dZ_\tau^{(k)} \\ &= J_t^\beta \Pi_{t,Z^{(k)}}(h_k f). \end{aligned}\tag{3.21}$$

Now equation (3.16) follows from equalities (3.18), (3.20), and (3.21). □

Let B maps the class of stochastic processes $\Pi_t(f)$ to the class of processes $\Pi_{t,Z}(f)$, that is $\Pi_{t,Z}(f) = B\Pi_t(f)$. One can verify easily that the operator B can be expressed with the help of operators $\mathcal{A}$ and $\mathcal{C}$ in equation (3.15). Namely,

$$B = \mathcal{C}\mathcal{A}^{-1}. \tag{3.22}$$

Using $L^2(\mathbb{P})$-norm and calculus of stochastic processes one can show that $\mathcal{A}$ is a one-to-one bounded linear operator and $\mathcal{C}$ is a bounded linear operator. Therefore, it follows that operator B is well defined bounded linear operator. We note that equation (3.16) can be written in the form

$$\Pi_t(f) - p_0(f) = J_t^{\beta}\Big(\Pi_t(Af) + \sum_{k=1}^{m} B_k \Pi_t(h_k f)\Big), \tag{3.23}$$

where

$$B_k \Pi_t(f) = \Pi_{t,Z^{(k)}}(f). \tag{3.24}$$

The differential form of (3.23) involves a fractional derivative in the Riemann-Liouville sense

$$d\Pi_t(f) = \mathcal{D}_t^{1-\beta}\Pi_t(Af)dt + \sum_{k=1}^{m} \mathcal{D}_t^{1-\beta} B_k \Pi_t(h_k f)dt, \ \Pi_{t=0}(f) = p_0(f). \tag{3.25}$$

Here the fractional derivative of order β, $0 < \beta < 1$ in the Riemann-Liouville sense is $\mathcal{D}_t^{\beta} = \frac{d}{dt}J^{1-\beta}$. It follows from equation (3.25) that the unnormalized density $U(t,x)$ associated with the process $\Pi_t(f)$ satisfies the equation

$$D_*^{\beta}U(t,x) = A^*U(t,x) + \sum_{k=1}^{m} h_k(x)B_k U(t,x), \ U(0,x) = f(x), \tag{3.26}$$

where D_*^{β} is the fractional derivative in the sense of Caputo, which by definition is $D_*^{\beta} = J^{1-\beta}d/dt$.

Remark **3.2**. The time-changed processes B_T and W_T are not Markovian and has no independent increments. Therefore, the model (3.1), (3.2) can be applied to a class of correlated filtering processes. We note also that the classical Zakai equation is recovered when $\beta \to 1$. The stochastic equation (3.26) generalizes the fractional order Fokker-Planck-Kolmogorov type equations [7, 8] to the case when the associate stochastic differential equation is connected with an observation process Z_t. In the absense of the observation process Z_t equation (3.26) coincides with *the fractional Fokker-Planck-Kolmogorov equation*, see [7, 8]. Therefore, the associated stochastic partial differential equations (3.9) or (3.14) can be called *fractional Zakai type equations.*

Now we generalize the results of Theorems **3.1**, **3.2**, and **3.3** to two different cases of filtering problems whose either state or observation processes are driven by a time-changed Lévy process. Without time-changed driving processes these two cases were discussed in Section **2**. Let L_t be the Lévy process given by equation (2.1). The first case is the fractional filtering problem the state process of which is given by stochastic differential equation (2.13) and the observation process of which is given by equation (3.2). Suppose that the input data of this filtering model satisfy the conditions:

$(C1)'$ the vector-functions $b(x)$, $h(x)$, $H(x,w)$, $K(x,w)$ and the matrix-function $\sigma(x)$ are infinite differentiable and bounded;

$(C2)'$ the time-change process T_t, Brownian motions B_t and W_t, and Poisson random measures $\tilde{N}(t,\cdot)$ and $N(t,\cdot)$ are independent processes;

$(C3)'$ the initial random vector X_0 is independent of processes B_t, W_t, $\tilde{N}(t,\cdot)$, $N(t,\cdot)$, T_t and has an infinite differentiable density function $p_0(x)$ decaying at infinity faster that any power of $|x|$.

Theorem **3.4**. *Let T_t be the inverse to a stable Lévy subordinator D_t and $\phi_t(f) = \hat{\mathbb{E}}[f(X_t)\Lambda_{T_t}|\mathcal{Z}\circ\mathcal{T}_t]$, where $\mathcal{T}$ is the filtration generated by T_t and Λ_t is defined by (3.5). Let $\mathcal{P}$ be a pseudo-differential operator with the symbol $\Psi(x,\xi)$ given by (2.4). Suppose the conditions $(C1)' - (C3)'$ are verified. Then:*

(1) *$\phi_t(f)$ satisfies the following Zakai type equation corresponding to filtering problem (2.13), (3.2):*

$$\phi_t(f) = p_0(f) + \int_0^t \phi_{s-}(\mathcal{P}f)dT_s + \sum_{k=1}^m \int_0^t \phi_{s-}(h_k f)dZ^k_{T_s}; \qquad (3.27)$$

(2) *the filtering density $\Phi(t,x)$ associated with the filtering measure $\phi_t(f)$ in equation (3.27) satisfies the following Zakai type equation*

$$\Phi(t,x) - \Phi(0,x) = \int_0^t \mathcal{P}^*\Phi(s-,x)dT_s + \sum_{k=1}^m \int_0^t h_k(x)\Phi(s-,x)dZ^{(k)}_{T_s}; \quad (3.28)$$

(3) *the unnormalized filtering density $U(t,x)$ associated with the process $\Pi_t(f)$ defined as in equation (3.15) with $p(t,x)$ satisfying the Zakai equation (2.8), solves the following Cauchy problem for fractional order stochastic equation*

$$D^\beta_* U(t,x) = \mathcal{P}^*U(t,x) + \sum_{k=1}^m h_k(x)B_kU(t,x), \ U(0,x) = f(x), \qquad (3.29)$$

where D^β_* is the fractional derivative of order β in the sense of Caputo and the operator B_k is defined in (3.24).

The second case is the filtering problem whose state process is driven by a time-changed Brownian motion and the observation process is driven by a time-changed Lévy process. Namely, let the state process be given by equation (3.1) and the observation process be given by the following stochastic differential equation

$$dH_t = h(X_t)dT_t + dW_{T_t} + \int_{\mathbb{R}^m} wN_\lambda(dT_t, dw),\ V_0 = 0, \tag{3.30}$$

where the random measure N_λ has the predictable compensator $\lambda(t, X_t, w)dt\nu(dw)$ with a Lévy measure ν. Introduce the process

$$\begin{aligned}\mathcal{L}_t = \exp\{&-\sum_{k=1}^m \int_0^t h_k(Y_{s-})dW_s \\ &- \frac{1}{2}\int_0^t |h(Y_{s-})|^2 ds + \int_0^t \int_{\mathbb{R}^m\setminus\{0\}} \ln \lambda(s, X_{s-}, w)N_\lambda(ds, dw) \\ &+ \int_0^t \int_{\mathbb{R}^m\setminus\{0\}} (1 - \lambda(s, X_{s-}, w))ds\nu(dw)\}.\end{aligned} \tag{3.31}$$

THEOREM **3.5**. *Let T_t be the inverse to a stable Lévy subordinator D_t and $\phi_t(f) = \hat{\mathbb{E}}[f(X_t)\mathcal{L}_{T_t}|\mathcal{H}_t]$, where $\mathcal{H}$ is the filtration generated by the process H_t in equation (3.30) and $\mathcal{L}_t$ is defined by (3.31). Suppose conditions $(C1), (C2)^{'}$ and $(C3)^{'}$ are verified. Then:*

(1) $\phi_t(f)$ satisfies the following Zakai type equation corresponding to filtering problem (3.1), (3.30):

$$\begin{aligned}\phi_t(f) = p_0(f) + \int_0^t \phi_{s-}(Af)dT_s + \sum_{k=1}^m \int_0^t \phi_{s-}(h_k f)dZ^k_{T_s} \\ + \int_0^t \int_{\mathbb{R}^m\setminus\{0\}} \phi_{s-}\Big((\lambda(s, \cdot, w) - 1)f\Big)\hat{N}_T(ds, dw),\end{aligned} \tag{3.32}$$

where $\hat{N}_T(ds, dw) = N_T(ds, dw) - dT_s\nu(dw)$ and the operator A is defined in equation (2.12);

(2) the filtering density $\Phi(t, x)$ associated with the filtering measure $\phi_t(f)$ in equation (3.32) satisfies the following Zakai type equation

$$\begin{aligned}\Phi(t, x) - \Phi(0, x) = \int_0^t A^*\Phi(s-, x)dT_s + \sum_{k=1}^m \int_0^t h_k(x)\Phi(s-, x)dZ^{(k)}_{T_s} \\ + \int_0^t \int_{\mathbb{R}^m\setminus\{0\}} \Big(\lambda(s, x, w) - 1\Big)\Phi(s-, x)\hat{N}_T(ds, dw);\end{aligned} \tag{3.33}$$

(3) *the unnormalized filtering density* $U(t,x)$ *associated with the process* $\Pi_t(f)$ *defined as in equation* (3.15) *with* $p(t,x)$ *satisfying the Zakai equation* (2.11)*, solves the following Cauchy problem for fractional order stochastic equation*

$$D_*^{\beta} U(t,x) = A^* U(t,x) + \sum_{k=1}^{m} h_k(x) B_k U_{Z^{(k)}}(t,x), \ U(0,x) = f(x).$$

The proofs of Theorems **3.4** and **3.5** are similar to the proof of Theorem **3.1**.

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