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Sasakian metric as a Ricci soliton and related results

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Abstract: We prove the following results: (i) A Sasakian metric as a non-trivial Ricci soliton is null η-Einstein, and expanding. Such a characterization permits to identify the Sasakian metric on the Heisenberg group $\mathcal{H}^{2n+1}$ as an explicit example of (non-trivial) Ricci soliton of such type. (ii) If an η-Einstein contact metric manifold M has a vector field V leaving the structure tensor and the scalar curvature invariant, then either V is an infinitesimal automorphism, or M is D-homothetically fixed K-contact.

MSC: 53C15, 53C25, 53D10

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1 Introduction

A Ricci soliton is a natural generalization of an Einstein metric, and is defined on a Riemannian manifold (M, g) by

$$(\pounds_V g)(X, Y) + 2Ric(X, Y) + 2\lambda g(X, Y) = 0 \tag{1}$$

where $\pounds_V g$ denotes the Lie derivative of g along a vector field V, λ a constant, and arbitrary vector fields X, Y on M. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive respectively. Actually, a Ricci soliton is a generalized fixed point of Hamilton's Ricci flow [7]: $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. For details, see Chow et al. [4]. The vector field V generates the Ricci soliton viewed as a special solution of the Ricci flow. A Ricci soliton is said to be a gradient Ricci soliton, if $V = -\nabla f$ (up to a Killing vector field) for a smooth function f. Ricci solitons are also of interest to physicists who refer to them as quasi-Einstein metrics (for example, see Friedan [6]).

An odd dimensional analogue of Kaehler geometry is the Sasakian geometry. The Kaehler cone over a Sasakian Einstein manifold is a Calabi-Yau manifold which has application in physics in superstring theory based on a 10-dimensional manifold that is the product of the 4-dimensional space-time and a 6-dimensional Ricci-flat Kaehler (Calabi-Yau) manifold (see Candelas et al. [3]). Sasakian geometry has been extensively studied since its recently perceived relevance in string theory. Sasakian Einstein metrics have received a lot of attention in physics, for example, p-brane solutions in superstring theory, Maldacena conjecture (AdS/CFS duality) [9]. For details, see Boyer, Galicki and Matzeu [2].

In [12] Sharma showed that if a K-contact (in particular, Sasakian) metric is a gradient Ricci soliton, then it is Einstein. This was also shown later independently by He and Zhu [8] for the Sasakian case. Recently, Sharma and Ghosh [13] proved that a 3-dimensional Sasakian metric which is a non-trivial (i.e. non-Einstein) Ricci soliton, is homothetic to the standard Sasakian metric on nil^3. In this paper, we generalize these results and also answer the following question of H.-D. Cao (cited in [8]):"*Does there exist a shrinking Ricci soliton on a Sasakian manifold, which is not Einstein?*", by proving

Theorem 1 *If the metric of a* $(2n+1)$*-dimensional Sasakian manifold* M (η, ξ, g, φ) *is a non-trivial (non-Einstein) Ricci soliton, then (i)* M *is null* η*-Einstein (i.e.* D*-homothetically fixed and transverse Calabi-Yau), (ii) the Ricci soliton is expanding, and (iii) the generating vector field* V *leaves the structure tensor* φ *invariant, and is an infinitesimal contact* D*-homothetic transformation.*

Conversely, we consider the following question: "What can we say about an η-Einstein contact metric manifold M which admits a vector field V that leaves φ invariant?" and answer it by assuming the invariance of the scalar curvature under V, in the form of the following result.

Theorem 2 *If an* η*-Einstein contact metric manifold* M *admits a vector field* V *that leaves the structure tensor* φ *and the scalar curvature invariant,* *then either V is an infinitesimal automorphism, or M is D-homothetically fixed and K-contact.*

Remark 1 *Note that a Ricci soliton as a Sasakian metric is different from the Sasaki-Ricci soliton in the context of transverse Kaehler structure in a Sasakian manifold, for example see Futaki et al. [5]).*

Remark 2 *Boyer et al. [2] have studied η-Einstein geometry as a class of distinguished Riemannian metrics on contact metric manifolds, and proved the existence of η-Einstein metrics on many different compact manifolds. We would also like to point out that Zhang [18] showed that compact Sasakian manifolds with constant scalar curvature and satisfying certain positive curvature condition is η-Einstein.*

Remark 3 *Theorem 2 provides a generalization of the infinitesimal version of the following result of Tanno [15] "The group of all diffeomorphisms Φ which leave the structure tensor φ of a contact metric manifold M invariant, is a Lie transformation group, and coincides with the automorphism group $\mathcal{A}$ if M is Einstein." Note that the scalar curvature of an Einstein metric is constant. We also note that the set of all vector fields on a contact metric manifold M, that leave φ and scalar curvature invariant, forms a Lie subalgebra of the Lie algebra of all smooth vector fields on M.*

2 A Brief Review Of Contact Geometry

A $(2n+1)$-dimensional smooth manifold is said to be contact if it has a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. For a contact 1-form η there exists a unique vector field ξ such that $d\eta(\xi, X)= 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric g and a (1,1)-tensor field φ such that

$$d\eta(X,Y) = g(X, \varphi Y), \eta(X) = g(X, \xi), \varphi^2 = -I + \eta \otimes \xi \tag{2}$$

g is called an associated metric of η and (φ, η, ξ, g) a contact metric structure. Following [1] we recall two self-adjoint operators $h = \frac{1}{2}\pounds_\xi \varphi$ and $l = R(., \xi)\xi$. The tensors h, $h\varphi$ are trace-free and $h\varphi = -\varphi h$. We also have these formulas for a contact metric manifold.

$$\nabla_X \xi = -\varphi X - \varphi h X \tag{3}$$

$$l - \varphi l \varphi = -2(h^2 + \varphi^2) \tag{4}$$

$$\nabla_\xi h = \varphi - \varphi l - \varphi h^2 \tag{5}$$

$$Trl = Ric(\xi, \xi) = 2n - Trh^2 \tag{6}$$

where ∇, R, Ric and Q denote respectively, the Riemannian connection, curvature tensor, Ricci tensor and Ricci operator of g. For details see [1]

A vector field V on a contact metric manifold M is said to be an infinitesimal contact transformation if $\pounds_V \eta = \sigma\eta$ for some smooth function σ on M . V is said to be an infinitesimal automorphism of the contact metric structure if it leaves all the structure tensors η, ξ, g, φ invariant (see Tanno [14]).

A contact metric structure is said to be K-contact if ξ is Killing with respect to g, equivalently, $h = 0$. The contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the cone manifold $(M \times R^+, r^2 g + dr^2)$ over M, is Kaehler. Sasakian manifolds are K-contact and K-contact 3-manifolds are Sasakian. For a Sasakian manifold,

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{7}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad Q\xi = 2n\xi \tag{8}$$

For a Sasakian manifold, the restriction of φ to the contact sub-bundle D $(\eta = 0)$ is denoted by J and $(D, J, d\eta)$ defines a Kaehler metric on D, with the transverse Kaehler metric g^T related to the Sasakian metric g as $g = g^T + \eta \otimes \eta$. One finds by a direct computation that the transverse Ricci tensor Ric^T of g^T is given by

$$Ric^T(X, Y) = Ric(X, Y) + 2g(X, Y)$$

for arbitrary vector fields X, Y in D. The Ricci form ρ and transverse Ricci form ρ^T are defined by

$$\rho(X, Y) = Ric(X, \varphi Y), \quad \rho^T(X, Y) = Ric^T(X, \varphi Y)$$

for $X, Y \in D$. The basic first Chern class $2\pi c_1^B$ of D is represented by ρ^T. In case $c_1^B = 0$, the Sasakian structure is said to be null (transverse Calabi-Yau). We refer to [2] for details.

A contact metric manifold M is said to be η-Einstein in the wider sense, if the Ricci tensor can be written as

$$Ric(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y) \tag{9}$$

for some smooth functions α and β on M. It is well-known (Yano and Kon [17]) that α and β are constant if M is K-contact, and has dimension greater than 3.

Given a contact metric structure (η, ξ, g, φ), let $\bar{\eta} = a\eta, \bar{\xi} = \frac{1}{a}\xi, \bar{\varphi} = \varphi, \bar{g} = ag + a(a-1)\eta \otimes \eta$ for a positive constant a. Then $(\bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ is again a contact metric structure. Such a change of structure is called a D-homothetic deformation, and preserves many basic properties like being K-contact (in particular, Sasakian). It is straightforward to verify that, under a D-homothetic deformation, a K-contact η-Einstein manifold transforms to a K-contact η-Einstein manifold such that $\bar{\alpha} = \frac{\alpha+2-2a}{a}$ and $\bar{\beta} = 2n - \bar{\alpha}$. We remark here that the particular value: $\alpha = -2$ remains fixed under a D-homothetic deformation, and as $\alpha + \beta = 2n$, β also remains fixed. Thus, we state the following definition.

Definition 1 *A K-contact η-Einstein manifold with $\alpha = -2$ is said to be D-homothetically fixed.*

3 Proofs Of The Results

Proof Of Theorem 1: Using the Ricci soliton equation (1) in the commutation formula (Yano [16], p.23)

$$\begin{aligned} &(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = \\ - \;& g((\pounds_V \nabla)(X,Y), Z) - g((\pounds_V \nabla)(X,Z), Y) \end{aligned} \tag{10}$$

we derive

$$\begin{aligned} g((\pounds_V \nabla)(X,Y), Z) &= (\nabla_Z Ric)(X,Y) \\ &- (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) \end{aligned} \tag{11}$$

As ξ is Killing, we have $\pounds_\xi Ric = 0$ which, in view of (3), the last equation of (8) and $h = 0$, is equivalent to $\nabla_\xi Q = Q\varphi - \varphi Q$. But for a Sasakian manifold, Q commutes with φ, and hence Ric is parallel along ξ. Moreover, differentiating the last equation of (8), we have $(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X$. Substituting ξ for Y in (11) and using these consequences we obtain

$$(\pounds_V \nabla)(X, \xi) = -2Q\varphi X + 4n\varphi X \tag{12}$$

Differentiating this along an arbitrary vector field Y, using (7) and the last equation of (8), we find

$$(\nabla_Y \pounds_V \nabla)(X, \xi) - (\pounds_V \nabla)(X, \varphi Y) = -2(\nabla_Y Q)\varphi X + 2\eta(X)QY - 4n\eta(X)Y$$

The use of the foregoing equation in the commutation formula [16]:

$$(\pounds_V R)(X, Y)Z = (\nabla_X \pounds_V \nabla)(Y, Z) - (\nabla_Y \pounds_V \nabla)(X, Z) \tag{13}$$

for a Riemannian manifold, shows that

$$\begin{aligned}(\pounds_V R)(X, Y)\xi - (\pounds_V \nabla)(Y, \varphi X) + (\pounds_V \nabla)(X, \varphi Y) = -2(\nabla_X Q)\varphi Y \\ +2(\nabla_Y Q)\varphi X + 2\eta(Y)QX - 2\eta(X)QY + 4n\eta(X)Y - 4n\eta(Y)X\end{aligned}$$

Substituting ξ for Y in the foregoing equation, using (12) and the formula $\nabla_\xi Q = 0$ noted earlier, we find that

$$(\pounds_V R)(X, \xi)\xi = 4(QX - 2nX) \tag{14}$$

Equation (1) gives $(\pounds_V g)(X, \xi) + 2(2n + \lambda)\eta(X) = 0$, which in turn, gives

$$(\pounds_V \eta)(X) - g(\pounds_V \xi, X) + 2(\lambda + 2n)\eta(X) = 0 \tag{15}$$

$$\eta(\pounds_V \xi) = (2n + \lambda) \tag{16}$$

where we used the Lie-derivative of $g(\xi, \xi) = 1$ along V. Next, Lie-differentiating the formula $R(X, \xi)\xi = X - \eta(X)\xi$ [a consequence of the first formula in (8)] along V, and using equations (14) and (16) provides

$$4(QX - 2nX) - g(\pounds_V \xi, X)\xi + 2(2n + \lambda)X = -((\pounds_V \eta)(X))\xi$$

By the direct application of (15) to the the above equation we find

$$Ric(X, Y) = (n - \frac{\lambda}{2})g(X, Y) + (n + \frac{\lambda}{2})\eta(X)\eta(Y) \tag{17}$$

which shows that M is η-Einstein with scalar curvature

$$r = 2n(n+1) - n\lambda \tag{18}$$

At this point, we recall the following integrability formula [12]:

$$\pounds_V r = -\Delta r + 2\lambda r + 2\left|Q\right|^2 \tag{19}$$

for a Ricci soliton, where $\Delta r = -divDr$. A straightforward computation using (17) gives the squared norm of the Ricci operator as $\left|Q\right|^2 = 2n(n^2 - n\lambda + \frac{\lambda^2}{4} + 4n^2)$. Using this and (18) in (19), we obtain the quadratic equation $(2n+\lambda)(2n+4-\lambda) = 0$. As $\lambda = -2n$ corresponds to g becoming Einstein, we must have $\lambda = 2n+4$ and hence the soliton is expanding, which proves part (ii). Moreover, equation (18) reduces to $r = -2n$. Thus equation (17) assumes the form

$$Ric(Y,Z) = -2g(Y,Z) + 2(n+1)\eta(Y)\eta(Z) \tag{20}$$

Hence, as defined in Section 2, M is a D-homothetically fixed null η-Einstein manifold, proving part (i). Using (20) in (11) provides

$$(\pounds_V \nabla)(Y,Z) = 4(n+1)\{\eta(Y)\varphi Z + \eta(Z)\varphi Y\} \tag{21}$$

Differentiating this along X, using equations (3) and (7), incorporating the resulting equation in (13), and finally contracting at X we get

$$(\pounds_V Ric)(Y,Z) = 8(n+1)\{g(Y,Z) - (2n+1)\eta(Y)\eta(Z)\} \tag{22}$$

Equation (20) reduces the soliton equation (1) to the form

$$(\pounds_V g)(Y,Z) = -4(n+1)\{g(Y,Z) + \eta(Y)\eta(Z)\} \tag{23}$$

Next, Lie-differentiating (20) along V, and using (23) shows

$$\begin{aligned}(\pounds_V Ric)(Y,Z) &= 8(n+1)\{g(Y,Z) + \eta(Y)\eta(Z)\}\\ &+ 2(n+1)\{\eta(Z)(\pounds_V \eta)(Y) + \eta(Y)(\pounds_V \eta)Z\}\end{aligned} \tag{24}$$

Comparing equations (22) with (24) and substituting ξ for Z leads to

$$\pounds_V \eta = -4(n+1)\eta \tag{25}$$

Therefore, substituting ξ for Z in (23) and using (25) we immediately get $\pounds_V \xi = 4(n+1)\xi$. Operating (25) by d, noting d commutes with $\pounds_V$ and using the first equation of (2) we find

$$(\pounds_V d\eta)(X,Y) = -4(n+1)g(X,\varphi Y)$$

Its comparison with the Lie-derivative of the first equation of (2) and the use of (23) yields $\pounds_V \varphi = 0$, completing the proof.

Before proving Theorem 2, we state and prove the following lemma.

Lemma 1 *If a vector field V leaves the structure tensor φ of the contact metric manifold M invariant, then there exists a constant c such that (i)$\pounds_V \eta = c\eta$, (ii)$\pounds_V \xi = -c\xi$, (iii) $\pounds_V g = c(g + \eta \otimes \eta)$.*

Though this lemma was proved by Mizusawa in [10], to make the paper self-contained, we provide a slightly different proof as follows.

Proof: Lie-differentiating the formulas $\varphi\xi = 0$ and $\eta(\varphi X) = 0$ and using $\pounds_V \varphi = 0$, we find $\pounds_V \xi = -c\xi$, and $\pounds_V \eta = c\eta$ for a smooth function c on M. Next, Lie-derivative of the formula $\eta(X) = g(X,\xi)$ along V gives

$$(\pounds_V g)(X,\xi) = 2c\eta(X) \tag{26}$$

The Lie-derivative of the first equation of (2) along V provides

$$(\pounds_V g)(X,\varphi Y) = ((dc) \wedge \eta)(X,Y) + cg(X,\varphi Y) \tag{27}$$

Substituting ξ for Y in the above equation we get $dc = (\xi c)\eta$. Taking its exterior derivative, and then exterior product with η shows $(\xi c)(d\eta) \wedge \eta = 0$. By definition of the contact structure, $(d\eta) \wedge \eta$ is nowhere zero on M, and so $\xi c = 0$. Hence $dc = 0$, i.e. c is constant. Using this consequence, and equations (26) and (27) we obtain (iii), completing the proof.

Proof Of Theorem 2 : By virtue of Lemma 1, we have

$$(\pounds_V g)(Y,Z) = c\{g(Y,Z) + \eta(Y)\eta(Z)\} \tag{28}$$

Differentiating this and using (3) we get

$$(\nabla_X \pounds_V g)(Y,Z) = -c\{\eta(Z)g(Y,\varphi X + \varphi hX) + \eta(Y)g(Z,\varphi X + \varphi hX)\} \tag{29}$$

Equation (10) can be written

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y) \tag{30}$$

A straightforward computation using (29) and (30) shows

$$(\pounds_V \nabla)(Y,Z) = -c\{\eta(Z)\varphi Y + \eta(Y)\varphi Z + g(Y,\varphi h Z)\xi\}$$

Its covariant differentiation and use of (2) provides

$$\begin{aligned}(\nabla_X \pounds_V \nabla)(Y,Z) &= -c\{\eta(Z)(\nabla_X\varphi)Y + \eta(Y)(\nabla_X\varphi)Z \\ &- g(Z,\varphi X + \varphi h X)\varphi Y - g(Y,\varphi X + \varphi h X)\varphi Z \\ &- g(\varphi h Y, Z)(\varphi X + \varphi h X) + g((\nabla_X \varphi h)Y,Z)\xi\}\end{aligned}$$

Using this in the commutation formula (13) for a Riemannian manifold, contracting at X, and using equations (2), (3) and also the well known formula: $(div\varphi)X = -2n\eta(X)$ for a contact metric (see [1]), we find

$$\begin{aligned}(\pounds_V Ric)(Y,Z) &= c\{-2g(Y,Z) + 2g(hY,Z) \\ &+ 2(2n+1)\eta(Y)\eta(Z)\} - cg((\nabla_\xi \varphi h)Y,Z)\end{aligned} \tag{31}$$

Also, Lie-differentiating (9) along V and using Lemma 1 we have

$$(\pounds_V Ric)(Y,Z) = (V\alpha + c\alpha)g(Y,Z) + (V\beta + c(\alpha+2\beta))\eta(Y)\eta(Z) \tag{32}$$

Comparing the previous two equations shows that

$$\begin{aligned}&[V\alpha + c(\alpha+2)]g(Y,Z) + [V\beta + c(\{\alpha + 2\beta - 2(2n+1)\}]\eta(Y)\eta(Z) \\ &\qquad -c[2g(hY,Z) - g((\nabla_\xi\varphi h)Y,Z)] = 0\end{aligned}$$

On one hand, we substitute $Y = Z = \xi$ in the above equation getting one equation, and on the other hand, we contract the above equation (noting that both h and φh are trace-free) getting another equation. Solving the two equations we obtain

$$V\alpha + c(\alpha+2) = 0, \quad V\beta + c(\alpha + 2\beta - 4n - 2) = 0 \tag{33}$$

The g-trace of equation (9) gives the scalar curvature

$$r = (2n+1)\alpha + \beta \tag{34}$$

The divergence of (9) along with the contracted second Bianchi identity yields $dr = 2d\alpha + 2(\xi\beta)\eta$. Taking its exterior derivative, and then exterior product with η we have $(\xi\beta)\eta \wedge d\eta = 0$. As $\eta \wedge d\eta$ vanishes nowhere on M, we find $\xi\beta = 0$ whence $dr = 2d\alpha$. Hence $V\alpha = Vr = 0$, by hypothesis. Thus, it follows from (34) that $V\beta = 0$. Consequently, equations (33) reduce to: $c(\alpha + 2) = 0$ and $c(\alpha + 2\beta - 4n - 2) = 0$, and hence imply that, either $c = 0$ in which case V is an infinitesimal automorphism, or $\alpha = -2$ and $\alpha + 2\beta = 4n + 2$. In the second case, adding the two equations gives $\alpha + \beta = 2n$. But, from equation (9) we have $\alpha + \beta = Tr.l$. Therefore, $Tr.l = 2n$, and applying equation (6) we obtain $h = 0$, i.e. M is K-contact. As $\alpha = -2$, the η-Einstein structure is D-homothetically fixed, completing the proof.

4 An Explicit Example

An explicit example of non-trivial Ricci soliton as a Sasakian metric is the (2n+1)-dimensional Heisenberg group $\mathcal{H}^{2n+1}$ (which arose from quantum mechanics) of matrices of type $\begin{bmatrix} 1 & Y & z \\ O^t & I_n & X^t \\ 0 & O & 1 \end{bmatrix}$, where $X = (x_1, ..., x_n), Y = (y_1, ..., y_n), O = (0, ..., 0) \in R^n, z \in R$. As a manifold, this is just R^{2n+1} with coordinates (x^i, y^i, z) where $i = 1, ..., n$, and has the left-invariant Sasakian structure (η, ξ, φ, g) defined by $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i)$, $\xi = 2\frac{\partial}{\partial z}$, $\varphi(\frac{\partial}{\partial x^i}) = -\frac{\partial}{\partial y^i}$, $\varphi(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}$, $\varphi(\frac{\partial}{\partial z}) = 0$, and the Riemannian metric $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^n((dx^i)^2 + (dy^i)^2)$. Its φ-sectional curvature (i.e. the sectional curvature of plane sections orthogonal to ξ) is equal to -3, so its Ricci tensor satisfies equation (20), as shown by Okumura [11], and hence $\mathcal{H}^{2n+1}$ is a D-homothetically fixed null η-Einstein manifold. Setting $V = \sum_{i=1}^n(V^i\frac{\partial}{\partial x^i} + \bar{V}^i\frac{\partial}{\partial y^i}) + V^z\frac{\partial}{\partial z}$, using equations: $\pounds_V\xi = 4(n+1)\xi$, $\pounds_V\varphi = 0$ obtained in the proof of Theorem 1, and the aforementioned actions of φ on the coordinate basis vectors, shows that V^i and $\bar{V}^i$ do not depend on z and yields the PDEs:

$$\begin{aligned} \frac{\partial V^i}{\partial x^j} &= \frac{\partial \bar{V}^i}{\partial y^j}, \quad \frac{\partial V^i}{\partial y^j} = -\frac{\partial \bar{V}^i}{\partial x^j}, \quad y^i\frac{\partial V^i}{\partial y^j} = \frac{\partial V^z}{\partial y^j} \\ \bar{V}^j &= y^j\frac{\partial V^z}{\partial z} - y^i\frac{\partial \bar{V}^i}{\partial y^j}, \quad \frac{\partial V^z}{\partial z} = -4(n+1) \end{aligned}$$

The last equation readily integrates as $V^z = -4(n+1)z + F(x^i, y^i)$. For a special solution, assuming $F = 0$, $V^i = cx^i$, $\bar{V}^i = cy^i$ and substituting in the above PDEs, we get $c = -2(n+1)$, and hence the Ricci soliton vector field $V = -2(n+1)(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} + 2z\frac{\partial}{\partial z})$. For dimension 3, this reduces to $V = -4(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z})$ which occurs on p. 37 of [4] without the factor 4, but gets adjusted with our $\lambda = 6$ which is 4 times their $\lambda = 3/2$.

Remark 4 *Another conclusion that we draw for Theorem 1 is the following: The value $-2n$ for the scalar curvature r obtained during the proof, and the equation (17) show that the generalized Tanaka-Webster scalar curvature [1] $W = r - Ric(\xi,\xi) + 4n$ vanishes.*

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