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THE COLLATZ CONJECTURE AND INTEGERS OF THE FORM $2^k b - m$ AND $3^k b - 1$

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Abstract. One of the more well-known unsolved problems in number theory is the Collatz $(3n+1)$ Conjecture. The conjecture states that iterating the map that takes even $n \in \mathbb{N}$ to $\frac{n}{2}$ and odd n to $\frac{3n+1}{2}$ will eventually yield 1. This paper is an exploration of this conjecture on positive integers of the form $2^k b - m$ and $3^k b - 1$, and stems from the work of the first author's Senior Seminar research. We take an elementary approach to prove interesting relationships and patterns in the number of iterations, called the *total stopping time*, required for integers of the aforementioned forms to reach 1, so that our results and proofs would be accessible to an undergraduate. Our results, then, provide a degree of insight into the Collatz Conjecture.

1. Introduction

In 1931, Lothar Collatz, was the first to study the following function [7]. For any $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, let

$$C(n) = \begin{cases} 3n+1 & \text{if } n \equiv 1 \pmod 2 \\ n/2 & \text{if } n \equiv 0 \pmod 2 \end{cases} .$$

In particular, Collatz studied iterates $C^{(k)}$ of this function, for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Collatz formulated the following conjecture concerning this function, not publishing his work until 1986 [1] because he could not prove it.

Conjecture 1 (Collatz)**.** *For all* $n \in \mathbb{N}$, $\exists k \in \mathbb{N}_0$ *such that* $C^{(k)}(n) = 1$.

This tantalizing conjecture has attracted much attention due to its simplicity, yet over 80 years later, it remains open. Many mathematicians believe this conjecture to be true based on probabilistic arguments and empirical evidence. (The conjecture has been verified experimentally for all $n \leq 5 \cdot 2^{60} \approx 5.7646 \cdot 10^{18}$ [6].)

Collatz began to circulate his problem in the 1950's, introducing it to Kakutani, Ulam, and Hasse, among others. As a result, this problem came to be known by different names such as *the* $3x+1$ (or $3n+1$) *Problem*, *the Syracuse Problem*, *Kakutani's Problem*, *Hasse's Algorithm*, and *Ulam's Problem*, as it was passed around to different circles. Kakutani circulated the problem at Yale in the 1960's, stating, "For about a month everybody at Yale worked on it, with no result. A similar phenomenon happened when I mentioned it at the University of Chicago. A joke was made that this problem was part of a conspiracy to slow down mathematical research in the U.S. [2]." This caused Erdös to offer $500 for a resolution of the conjecture and remark, "Mathematics is not yet ready for such problems [3]."

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Thwaites, who formulated the conjecture independently in 1952 [8], offers a £1000 reward for a proof [9]. For a more complete introduction to the Collatz Conjecture and its extensions, see [3], and for annoted bibliographies listing other articles on this topic, see [4, 5].

Since $3n+1$ is even if n is odd, it is convenient to work with the *modified Collatz function*:

$$T(n) = \begin{cases} \frac{3n+1}{2}, & \text{if } n \equiv 1 \pmod 2 \ , \\ \frac{n}{2}, & \text{if } n \equiv 0 \pmod 2 \ . \end{cases}$$

Terras [7] defines the *total stopping time* of an integer $n \in \mathbb{N}$, denoted $\sigma_\infty(n)$, as the smallest $k \in \mathbb{N}_0$ such that $T^{(k)}(n) = 1$, or $\sigma_\infty(n) = \infty$ if no such k exists. For example, since T maps $3 \mapsto 5 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1$, we have $T^{(5)}(3) = 1$, so $\sigma_\infty(3) = 5$. Thus, the Collatz Conjecture can be restated as: for all $n \in \mathbb{N}$, $\sigma_\infty(n) < \infty$.

In this paper, we will take an elementary look at the behavior of iterates of the modified Collatz function on integers of the form $2^k b - m$ and $3^k b - 1$, so that our results and proofs would be accessible to an undergraduate. We prove new or independently discovered properties on the total stopping time of integers of the above forms. Our results establish some very curious patterns on the total stopping time for certain consecutive values of k and for certain consecutive integers n.

2. Properties of σ_∞

Given the definition of $T(n)$, we first considered total stopping times of integers of the form $2^k - 1$ and $3^k - 1$. Some examples for various k are listed in Table 1.

Table 1. Comparison of $\sigma_\infty(2^k - 1)$ and $\sigma_\infty(3^k - 1)$ and their difference.

k	$\sigma_\infty(2^k-1)$	$\sigma_\infty(3^k-1)$	$\sigma_\infty(2^k-1) - \sigma_\infty(3^k-1)$
2	5	3	2
3	11	8	3
4	12	8	4
5	67	62	5
6	68	62	6
7	31	24	7
8	32	24	8
9	41	32	9
10	42	32	10
11	100	89	11
12	101	89	12
13	102	89	13
14	103	89	14
15	85	70	15
16	86	70	16
17	144	127	17
18	145	127	18
19	116	97	19
20	117	97	20

From Table 1, it appears that $\sigma_\infty(2^k - 1) - \sigma_\infty(3^k - 1) = k$ for all $k \geq 2$ and that for even $k \geq 4$, $\sigma_\infty(3^k - 1) = \sigma_\infty(3^{k-1} - 1)$. We will prove both of these statements, first generalizing the former observation.

Theorem 1. *If $k, b \in \mathbb{N}$ and $2^k b - 1 > 1$, then $T^{(k)}(2^k b - 1) = 3^k b - 1$.*

Proof. Let $a \in \mathbb{N}_0$ such that $0 \leq a \leq k$. We proceed by induction. By definition,

$$T^{(0)}(2^k b - 1) = 2^k b - 1 = (3^0 \cdot 2^{k-0})b - 1 \ .$$

Thus, the base case holds. Now assume that

$$T^{(a)}(2^k b - 1) = (3^a \cdot 2^{k-a})b - 1 \ ,$$

for some $a \in \mathbb{N}_0$, $0 \leq a < k$. Since $a < k$, $(3^a \cdot 2^{k-a})b - 1$ is odd and at least 2, so

$$\begin{aligned} T^{(a+1)}(2^k b - 1) &= T\left(T^{(a)}\left(2^k b - 1\right)\right) = T((3^a \cdot 2^{k-a})b - 1) \\ &= \frac{3((3^a \cdot 2^{k-a})b - 1) + 1}{2} = \frac{(3^{a+1} \cdot 2^{k-a})b - 2}{2} \\ &= (3^{a+1} \cdot 2^{k-(a+1)})b - 1 \ . \end{aligned}$$

Thus, $T^{(a+1)}(2^k b - 1) = (3^{a+1} \cdot 2^{k-(a+1)})b - 1$, and the result follows. □

From this, we have several immediate consequences concerning integers of the form $2^k b - m$, where $k, b \in \mathbb{N}$ and $m \in \{2, 3, 4, 6\}$.

Corollary 1. *If $2^k b - 2 > 1$, then $T^{(k)}(2^k b - 2) = 3^{k-1} b - 1$.*

Proof. Since $2^k b - 2$ is even, $T^{(k)}(2^k b - 2) = T^{(k-1)}(2^{k-1} b - 1) = 3^{k-1} b - 1$, by Theorem 1. □

Corollary 2. *If $k > 2$, then $T^{(k)}(2^k b - 3) = 3^{k-2} b - 1$.*

Proof. Since $2^k b - 3$ is odd, $T^{(k)}(2^k b - 3) = T^{(k-1)}(2^{k-1} 3b - 4) = T^{(k-2)}(2^{k-2} 3b - 2) = 3^{k-3}(3b) - 1 = 3^{k-2} b - 1$, by Corollary 1. □

Corollary 3. *If $k > 1$ and $2^k b - 4 > 2$, then $T^{(k)}(2^k b - 4) = 3^{k-2} b - 1$.*

Proof. Since $2^k b - 4$ is even, $T^{(k)}(2^k b - 4) = T^{(k-1)}(2^{k-1} b - 2) = 3^{k-2} b - 1$, by Corollary 1. □

Corollary 4. *If $k > 3$, then $T^{(k)}(2^k b - 6) = 3^{k-3} b - 1$.*

Proof. Since $2^k b - 6$ is even, $T^{(k)}(2^k b - 6) = T^{(k-1)}(2^{k-1} b - 3) = 3^{k-3} b - 1$, by Corollary 2. □

We apply the definition of the total stopping time function to summarize these results.

Corollary 5. *If $k, b \in \mathbb{N}$, then:*

1. $\sigma_\infty(2^k b - 1) = \sigma_\infty(3^k b - 1) + k$ *if* $2^k b - 1 > 1$,
2. $\sigma_\infty(2^k b - 2) = \sigma_\infty(3^{k-1} b - 1) + k$ *if* $2^k b - 2 > 1$,
3. $\sigma_\infty(2^k b - 3) = \sigma_\infty(3^{k-2} b - 1) + k$ *if* $k > 2$,
4. $\sigma_\infty(2^k b - 4) = \sigma_\infty(3^{k-2} b - 1) + k$ *if* $k > 1$ *and* $2^k b - 4 > 2$, *and*
5. $\sigma_\infty(2^k b - 6) = \sigma_\infty(3^{k-3} b - 1) + k$ *if* $k > 3$.

Notice parts (3) and (4) of Corollary 5. These imply the following about total stopping times of certain consecutive integers.

Theorem 2. *If $k > 2$, $b \in \mathbb{N}$, and $n \equiv -4 \pmod{2^k b}$, then $\sigma_\infty(n) = \sigma_\infty(n+1)$.*

We turn now to our second observation from Table 1, that $\sigma_\infty(3^k - 1) = \sigma_\infty(3^{k-1} - 1)$, for certain consecutive values of k. As an aside, we notice that, as for $11 \leq k \leq 14$, we often have common values of $\sigma_\infty(3^k - 1)$ for more consecutive values of k. Table 2 shows all such runs up to $k = 102$.

TABLE 2. Values of $\sigma_\infty(3^k - 1)$ for consecutive $k \leq 102$.

$\sigma_\infty(3^k - 1)$	Range of k	length
89	$11 \leq k \leq 14$	4
257	$29 \leq k \leq 32$	4
303	$33 \leq k \leq 42$	8
333	$43 \leq k \leq 52$	10
490	$57 \leq k \leq 64$	8
528	$69 \leq k \leq 74$	6
528	$77 \leq k \leq 80$	4
837	$81 \leq k \leq 86$	6
837	$89 \leq k \leq 102$	14

Theorem 3. *If $k \geq 4$ is even, then*

- $\sigma_\infty(2^k - 1) = \sigma_\infty(2^{k-1} - 1) + 1$ *and*
- $\sigma_\infty(3^k - 1) = \sigma_\infty(3^{k-1} - 1)$.

Proof. We will show that $T^{(k+1)}(2^{k-1} - 1) = T^{(k+2)}(2^k - 1)$, from which the first result follows. The second result follows from the first via Theorem 1. By Theorem 1, $T^{(k+1)}(2^{k-1} - 1) = T^{(2)}(3^{k-1} - 1)$. Since k is even, $3^{k-1} - 1 \equiv 2 \pmod 4$, so

$$T^{(2)}(3^{k-1} - 1) = \frac{3\left(\frac{3^{k-1}-1}{2}\right) + 1}{2} = \frac{3^k - 1}{4} \quad .$$

Now by Theorem 1, $T^{(k+2)}(2^k - 1) = T^{(2)}(3^k - 1)$. Since k is even, $3^k - 1 \equiv 0 \pmod 4$, so $T^{(2)}(3^k - 1) = \frac{3^k-1}{4}$. Thus, $T^{(k+1)}(2^{k-1} - 1) = T^{(k+2)}(2^k - 1)$. □

Furthermore, Theorem 3 can be generalized.

Theorem 4. *If $b \geq 3$ is odd, then $\sigma_\infty(2^k b - 1) = \sigma_\infty(2^{k-1} b - 1) + 1$ and $\sigma_\infty(3^k b - 1) = \sigma_\infty(3^{k-1} b - 1)$ if*

- $b \equiv 1 \pmod 4$ *and $k \geq 2$ is even and*
- $b \equiv 3 \pmod 4$ *and $k \geq 3$ is odd.*

Proof. As in Theorem 3, $\sigma_\infty(2^k b - 1) = \sigma_\infty(2^{k-1} b - 1) + 1$ implies $\sigma_\infty(3^k b - 1) = \sigma_\infty(3^{k-1} b - 1)$ by Theorem 1. We will show that $T^{(k+1)}(2^{k-1} b - 1) = T^{(k+2)}(2^k b - 1)$, from which it follows that $\sigma_\infty(2^k b - 1) = \sigma_\infty(2^{k-1} b - 1) + 1$.

By Theorem 1, $T^{(k+1)}(2^{k-1} b - 1) = T^{(2)}(3^{k-1} b - 1)$. If either $b \equiv 1 \pmod 4$ and k is even or $b \equiv 3 \pmod 4$ and k is odd, then $3^{k-1} b - 1 \equiv 2 \pmod 4$, so

$$T^{(2)}(3^{k-1} b - 1) = \frac{3\left(\frac{3^{k-1}b-1}{2}\right) + 1}{2} = \frac{3^k b - 1}{4} \quad .$$

Now by Theorem 1, $T^{(k+2)}(2^k b - 1) = T^{(2)}(3^k b - 1)$. If either $b \equiv 1 \pmod 4$ and k is even or $b \equiv 3 \pmod 4$ and k is odd, then $3^k b - 1 \equiv 0 \pmod 4$, so $T^{(2)}(3^k b - 1) = \frac{3^k b - 1}{4}$. Thus, $T^{(k+1)}(2^{k-1} b - 1) = T^{(k+2)}(2^k b - 1)$. □

3. Conclusions

At first glance, the total stopping time function appears to have random and unpredictable behavior. However, our results show that this function is not random and that there are several relationships between its values for integers of the form $2^k b - m$ and $3^k b - 1$. One observation that has been made on the total stopping time function is that it takes on very few values on a restricted domain [3]. Our results help to explain that phenomenon. For example, $134 = 3^3 \cdot 5 - 1$ and $404 = 3^4 \cdot 5 - 1 = 2^3 \cdot 51 - 4$, so by Theorems 3 and 4, it must be the case that $\sigma_\infty(134) = \sigma_\infty(404) = \sigma_\infty(405)$, and in fact they are all 20.

4. Open Problems and Future Work

Obviously, the Collatz Conjecture is the most significant open problem at hand. One unresolved question that has resulted from our work is if one could find an explicit formula for $\sigma_\infty(2^k b - m)$ or $\sigma_\infty(3^k b - 1)$, where $k, b \in \mathbb{N}$ and $m \in \{1, 2, 3, 4, 6\}$. Such a result would serve as a partial proof of the Collatz Conjecture. It also remains open if one could find and prove a pattern for $\sigma_\infty(3^k b - 1)$ or explain the common values of $\sigma_\infty(3^k b - 1)$ for longer ranges of values of k. We would also like to find larger families of integers with common total stopping times.

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