NOTE ON GABRIEL'S HORN

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ABSTRACT. A smooth bounded solid of finite volume and infinite surface is constructed. It is a variant of the classical Gabriel's horn that is often taught in Calculus classes.

Gabriel's horn, [ABD], is a classical example from Calculus. It is a solid with finite volume and infinite surface obtained by rotating the graph of the function $f(x) = 1/x$ for $x \geq 1$ about the x-axis. As it is taught in Calculus classes, the object can be filled with paint but its surface cannot be painted.

The classical Gabriel's horn is an unbounded object. In reality, as some students point out ([L]), it cannot be filled with paint. Lynch [L] constructed a bounded variant of Gabriel's horn. He constructed such an object by rotating the graph of a certain piecewise linear function defined on the interval $[0, 1]$. The graph of the function had an infinite arclength; and thus this object could indeed be filled with paint and yet its surface could not be painted.

In this note, Lynch's example is modified to get a solid with a smooth surface. The core of the construction is the following Theorem.

Theorem 1. *There exists a bounded function f that is differentiable on a closed interval $[0, 0.5]$ and whose arclength is infinite. Moreover, f is infinitely many times differentiable on $(0, 0.5)$ and satisfies $1 \leq f(x) \leq 3$ for all $x \in [0, 0.5]$.*

Proof. Define a function

$$f(x) = \begin{cases} 2 + \dfrac{x}{\ln x} \sin\left(\dfrac{1}{x}\right), & x \in (0, 0.5], \\ 2, & x = 0. \end{cases}$$

To get a better idea about the function $f(x)$, consider the function $\sin\left(\frac{1}{x}\right)$ whose graph appears in Figure 1.

The function $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1, and has an extremely small period in the neighborhood of $x = 0$. The function f is nothing more than the function $\sin\left(\frac{1}{x}\right)$ squeezed in between the function $2 + \frac{x}{\ln x}$ and $2 - \frac{x}{\ln x}$ on the given interval $x \in [0, 0.5]$, see Figure 2.
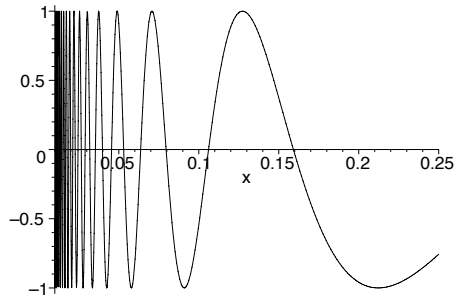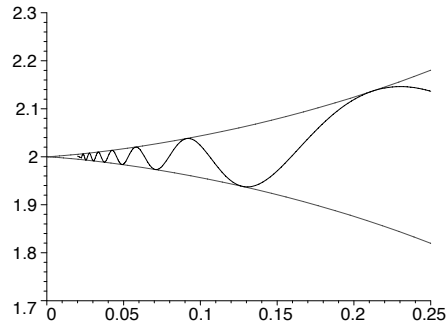Note that $1 \leq f(x) \leq 3$.

First, we will establish the differentiability of the function f. Clearly, the function is infinitely differentiable on $(0, 0.5]$. The function f is differentiable at $x = 0$ from

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FIGURE 1. Graph of $\sin\left(\frac{1}{x}\right)$.

FIGURE 2. Graph of $f(x) = 2 + \frac{x}{\ln x}\sin\left(\frac{1}{x}\right)$.

the right (and thus continuous at $x = 0$). Indeed,

$$f'_+(0) = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{\sin\left(\frac{1}{h}\right)}{\ln h} = 0,$$

where the last equality follows by the Squeezing Theorem (see [ABD]) because

$$\frac{-1}{\ln h} \leq \frac{\sin\left(\frac{1}{h}\right)}{\ln h} \leq \frac{1}{\ln h}, \quad \text{and}$$

$$\lim_{h \to 0+} \frac{\pm 1}{\ln h} = 0$$

Hence, the function f is differentiable on $[0, 0.5]$.

In order to establish that the arclength L of the graph of f is infinite we will be use the following easy observation (whose proof is straightforward and it is omitted).

Fact 2. *The function $g(x) = 1/(x\ln(x))$ is a decreasing function on $[2, \infty)$ satisfying $\lim_{x \to \infty} g(x) = 0$.*

Now, consider points

$$x_k = \frac{1}{k\pi + \pi/2}, k = 1, 2, \ldots.$$

For the arclength L_k of the graph between points x_k and x_{k+1} we have the following estimates

$$
\begin{aligned}
L_k &\geq |f(x_k) - f(x_{k+1}|\\
&= \left| \frac{-\sin(k\pi + \pi/2)}{(k\pi + \pi/2)\ln(k\pi + \pi/2)} - \frac{-\sin((k+1)\pi + \pi/2)}{((k+1)\pi + \pi/2)\ln((k+1)\pi + \pi/2)} \right|\\
&= \left| (-1)^k g((k\pi + \pi/2) - (-1)^{k+1} g((k+1)\pi + \pi/2) \right|\\
&\geq 2g((k+1)\pi + \pi/2)\\
&\geq 2g((k+2)\pi),
\end{aligned}
$$

where we used (twice) the monotonicity of the function g. Hence, the arclength L of the graph of f is at least

$$
L \geq \sum_{k=1}^{\infty} L_k \geq \sum_{k=1}^{\infty} 2g((k+2)\pi) = \sum_{k=1}^{\infty} \frac{2}{(k+2)\pi \ln((k+2)\pi)}.
$$

The series diverges by the limit comparison test (see [ABD]) with the series

$$
\sum_{k=3}^{\infty} \frac{1}{k \ln k}
$$

where the later series diverges by the integral criteria because g is a strictly decreasing function and

$$
\int_2^{\infty} \frac{1}{x \ln x} \, \mathrm{d}x \overset{\substack{\text{substitution} \\ u = \ln x \\ \mathrm{d}u = \frac{1}{x}\mathrm{d}x}}{=} \int_{\ln 2}^{\infty} \frac{1}{u} \, \mathrm{d}u = \infty.
$$

Thus, the arclength L of the graph of f is infinite.

$\square$

Corollary 3. *There exists a bounded smooth solid with finite volume and infinite surface.*

Proof. Consider the solid given by rotating the graph of f (given in the above Theorem) about the x-axis. The volume of the solid is finite. Indeed,

$$
f(x) \leq 3
$$

and thus the solid is contained in a cylinder of radius 3 and height 0.5. It remains to show that the surface S of the solid is infinite. This is done by the following estimate, where we used the inequality $f \geq 1$.

$$
S = \int_0^{0.5} 2\pi f(x)\sqrt{1 + f'(x)^2}\mathrm{d}x \geq 2\pi \int_0^{0.5} \sqrt{1 + f'(x)^2}\mathrm{d}x = 2\pi L = \infty.
$$

$\square$

Remark. Note, as already observed in Lynch's paper [L], that the solid could not be constructed by using a function $h : [a,b] \mapsto \mathbb{R}$ with a continuous derivative. Indeed, we would have

$$
S = \int_a^b 2\pi h(x)\sqrt{1 + h'(x)^2}\mathrm{d}x \leq (b-a)2\pi M \sqrt{1 + D^2} < \infty,
$$

where

$$M = \max\{h(x), x \in [a, b]\},$$
$$D = \max\{|h'(x)|, x \in [a, b]\},$$

with both M and D being finite by the continuity of h and h'.

References

[ABD] H. Anton, I. Bivens and S. Davis, Calculus Eight edition, John Wiley & Sons, Inc., 2005.
[L]   M. Lynch: *A Paradoxical Paint Pail*, The College Mathematics Journal, No. 5, 2005, 402–403.

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