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# ON THE NONEXISTENCE OF SINGULAR EQUILIBRIA IN THE FOUR-VORTEX PROBLEM

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ABSTRACT. In this paper we provide a partial answer to a question recently posed by Hassan Aref et. al. in their article *Vortex Crystals*, namely whether there are certain singular equilibria of point vortices. We prove that there are no such equilibria in the four-vortex case.

## 1. Introduction

The starting point of our discussion is the set of point-vortex equations for $N$ interacting vortices $a = 1, 2, \ldots, N$ with circulations $\Gamma_a$ and (complex) positions $z_i$:

$$\overline{\frac{dz_i}{dt}} = \frac{1}{2\pi i} \sum_{j \neq i} \frac{\Gamma_j}{z_i - z_j}.$$

This system was introduced by Helmholtz [H] to model a two-dimensional slice of columnar vortex filaments, with some refinements by Lord Kelvin [T] and Kirchhoff [K]. An extensive bibliography on the subject can be found in [N]. It is worth noting that this system can be written in Hamiltonian form with Hamiltonian $H = \sum_{i<j} \Gamma_i \Gamma_j \log |z_i - z_j|$, where the symplectic pairs of variables are multiples of the real and imaginary parts of each $z_i$.

A vortex equilibrium is a configuration of vortices such that $\frac{dz_j}{dt} = 0$ for all $j$. We are concerned here with the following special type of vortex equilibrium:

**Definition 1.1.** A *singular equilibrium* is an equilibrium such that $L = \sum_{i<j} \Gamma_i \Gamma_j |z_i - z_j|^2 = 0$, $K = \sum_{i<j} \Gamma_i \Gamma_j = 0$, and $S = \sum_i \Gamma_i \neq 0$.

It is already known that there are no singular equilibria in the three-vortex problem [ANST], where it is also shown that a rigidly rotating configuration of vortices has an angular speed of $\omega = \frac{SK}{4\pi L}$. Our introduction of the term singular equilibrium refers to the indeterminancy of this expression for the angular speed.

## 2. Nonexistence of the four-vortex singular equilibria

We will prove the following theorem:

**Theorem 2.1.** *There are no four-vortex singular equilibria.*

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*Proof.* Our calculations can be greatly simplified by a few assumptions. Since setting $z_3 = 0$ and $z_4 = 1$ simply scales the relative distances between the vortices and setting $\Gamma_4 = 1$ scales the circulations, we can work under these conventions without loss of generality. From the point-vortex equations, O'Neil [O] gives the two solutions for four-vortex equilibria:

$$z_1 = \frac{2 + \Gamma_2 \pm i\sqrt{3}\Gamma_2}{2(1 + \Gamma_2 + \Gamma_3)}$$

and

$$z_2 = \frac{2 + \Gamma_1 \mp i\sqrt{3}\Gamma_1}{2(1 + \Gamma_1 + \Gamma_3)}.$$

We can use the relation $K = \sum_{i<j} \Gamma_i \Gamma_j = 0$ to eliminate $\Gamma_3$ from these equations:

$$\Gamma_3 = \frac{\Gamma_1 + \Gamma_2 + \Gamma_1 \ \Gamma_2}{1 + \Gamma_1 + \Gamma_2}.$$

Note that we cannot have $1+\Gamma_1+\Gamma_2 = 0$ since then $K$ reduces to $-(1+\Gamma_2+\Gamma_2^2)$ which cannot be zero for real vorticities.

This gives us

$$z_1 = \frac{(2 + \Gamma_2)(1 + \Gamma_1 + \Gamma_2) + i\sqrt{3}\Gamma_2(1 + \Gamma_1 + \Gamma_2)}{2(1 + \Gamma_2 + \Gamma_2^2)}$$

and

$$z_2 = \frac{(2 + \Gamma_1)(1 + \Gamma_1 + \Gamma_2) + i\sqrt{3}\Gamma_1(1 + \Gamma_1 + \Gamma_2)}{2(1 + \Gamma_1 + \Gamma_1^2)}$$

for the positions of the first two vortices in a singular equilibrium.

Now we can use these expressions for $z_1$ and $z_2$, along with our conventions $z_3 = 0$ and $z_4 = 1$ to find the squared distances $d_{ij}^2 = |z_i - z_j|^2$:

$$d_{12}^2 = \frac{(\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_2^2)(1 + \Gamma_1 + \Gamma_2)^2)}{(1 + \Gamma_1 + \Gamma_1^2)(1 + \Gamma_2 + \Gamma_2^2)}$$

$$d_{13}^2 = \frac{\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_2^2}{1 + \Gamma_2 + \Gamma_2^2}$$

$$d_{14}^2 = \frac{(1 + \Gamma_1 + \Gamma_2)^2}{1 + \Gamma_2 + \Gamma_2^2}$$

$$d_{23}^2 = \frac{\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_2^2}{1 + \Gamma_1 + \Gamma_1^2}$$

$$d_{24}^2 = \frac{(1 + \Gamma_1 + \Gamma_2)^2}{1 + \Gamma_1 + \Gamma_1^2}$$

$$d_{34}^2 = 1.$$

Now we substitute these expressions in to the original equation for $L$.

$$L = 3(\Gamma_1^2 + \Gamma_1^3 + \Gamma_1^4 + \Gamma_1\Gamma_2 + \Gamma_1^2\Gamma_2 + \Gamma_1^3\Gamma_2 + \Gamma_1^4\Gamma_2 + \Gamma_2^2 + \Gamma_1\Gamma_2^2 + \Gamma_1^3\Gamma_2^2 + \Gamma_1^4\Gamma_2^2 + \Gamma_2^3 + \Gamma_1\Gamma_2^3 + \Gamma_1^2\Gamma_2^3 + \Gamma_1^3\Gamma_2^3 + \Gamma_2^4 + \Gamma_1\Gamma_2^4 + \Gamma_1^2\Gamma_2^4)/((1 + \Gamma_1 + \Gamma_1^2)(1 + \Gamma_2 + \Gamma_2^2)).$$

The expression in the denominator is always positive. Now all that remains is to determine the sign of the numerator in parentheses, $N(\Gamma_1, \Gamma_2)$. If it is always positive on $\mathbb{R}^2 - (0,0)$ we will have proven our claim, namely that there are no four-vortex stationary equilibria with $L = 0$. We start with a lemma:

**Lemma 2.2.** *$\frac{\partial^2 N}{\partial \Gamma_1^2}$ and $\frac{\partial^2 N}{\partial \Gamma_2^2}$ are non-negative.*

*Proof.* Since $N$ is symmetric in $\Gamma_1$ and $\Gamma_2$ it suffices the prove the lemma for $\frac{\partial^2 N}{\partial \Gamma_1^2}$. This is a quadratic function of $\Gamma_1$, whose minimum (for a fixed $\Gamma_2$) is

$$\frac{(1+\Gamma_2)^2(5+2\Gamma_2^2+5\Gamma_2^4)}{8(1+\Gamma_2+\Gamma_2^2)} \geq 0.$$

□

This lemma implies that $\frac{\partial N}{\partial \Gamma_1}$ and $\frac{\partial N}{\partial \Gamma_2}$ are monotone functions of $\Gamma_1$ and $\Gamma_2$ respectively. Thus they have at most one zero for each fixed $\Gamma_2$ (for $\frac{\partial N}{\partial \Gamma_1}$) and $\Gamma_1$ (for $\frac{\partial N}{\partial \Gamma_2}$).

We need a further lemma to reach our goal:

**Lemma 2.3.** *For each fixed $\Gamma_2$, $\frac{\partial N}{\partial \Gamma_1}$ has its unique zero between $\Gamma_1 = \Gamma_2$ and $\Gamma_1 = -\Gamma_2$.*

*Proof.* We simply compute that

$$\frac{\partial N}{\partial \Gamma_1}(\Gamma_2, \Gamma_2) = \Gamma_2(3 + 6\Gamma_2 + 8\Gamma_2^2 + 10\Gamma_2^3 + 9\Gamma_2^9).$$

Using Sturm's theorem it is not hard to show that the above polynomial is always positive for $\Gamma_2 > 0$ and always negative for $\Gamma_2 < 0$. Likewise, from the calculation

$$\frac{\partial N}{\partial \Gamma_1}(-\Gamma_2, \Gamma_2) = -\Gamma_2(1 + 2\Gamma_2 + 2\Gamma_2^3 + 3\Gamma_2^9)$$

we can find that $\frac{\partial N}{\partial \Gamma_1}(-\Gamma_2, \Gamma_2)$ is always negative for $\Gamma_2 > 0$ and positive for $\Gamma_2 < 0$. Combined with the monotonicity of $\frac{\partial N}{\partial \Gamma_1}$ as a function of $\Gamma_1$ this completes the lemma. □

Since $N$ is symmetric, $\frac{\partial N}{\partial \Gamma_1}(\Gamma_1, \Gamma_2) = \frac{\partial N}{\partial \Gamma_2}(\Gamma_2, \Gamma_1)$. Lemma 2.3 then implies that the gradient of $N$ can only be zero at the origin, since otherwise the two partials can only vanish in the disjoint open cones bounded by the lines $\Gamma_1 = \Gamma_2$ and $\Gamma_1 = -\Gamma_2$. Thus the origin is the only critical point of $N$. It is elementary to compute that the origin is a minimum of $N$, and thus the unique global minimum.

□

## 3. Conclusion

The nonexistence of singlar equilibria in the three- and four-vortex problems naturally prompts the question of whether such an equilibrium can exist for a larger number of vortices. It seems quite possible that there is a more general argument which would show the nonexistence of singular equilibria for any number of vortices, but we are unaware of a strategy for conducting such a proof.

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