

## SOME GEOMETRY OF $\mathcal{H}(\mathbb{R}^n)$

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ABSTRACT. If  $X$  is a complete metric space, the collection of all non-empty compact subsets of  $X$  forms a complete metric space  $(\mathcal{H}(X), h)$ , where  $h$  is the Hausdorff metric. In this paper we explore some of the geometry of the space  $\mathcal{H}(\mathbb{R}^n)$ . Specifically, we concentrate on understanding lines in  $\mathcal{H}(\mathbb{R})$ . In particular, we show that for any two points  $A, B \in \mathcal{H}(\mathbb{R}^n)$ , there exist infinitely many points on the line joining  $A$  and  $B$ . We characterize some points on the lines formed using closed and bounded intervals of  $\mathbb{R}$  and show that two distinct lines in  $\mathcal{H}(\mathbb{R})$  can intersect in infinitely many points.

### 1. INTRODUCTION

Euclidean geometry has, as part of its structure, dimensionless points. With these points and the use of the Euclidean distance we can construct circles, lines, and much more. However, what happens if we broaden the points to special sets and use a different metric to measure the distance between sets? Can we still visualize lines and think of them in the same way?

In this paper we broaden the idea of dimensionless points to non-empty compact sets, and use the Hausdorff metric to measure distances between sets. The space  $(\mathbb{R}^n, d)$ , where  $d$  is the Euclidean metric, is a complete metric space. Therefore, the collection of all non-empty compact subsets of  $\mathbb{R}^n$  forms a complete metric space  $(\mathcal{H}(\mathbb{R}^n), h)$ , where  $h$  is the Hausdorff metric. A 2002 paper completely characterized circles in  $\mathcal{H}(\mathbb{R}^n)$ , as well as investigating the behavior of lines in  $\mathcal{H}(\mathbb{R}^2)$  [6]. This paper will study some geometry of the space  $\mathcal{H}(\mathbb{R}^n)$ . Specifically, we will concentrate on understanding lines in  $\mathcal{H}(\mathbb{R})$ .

### 2. $\mathcal{H}(\mathbb{R}^n)$ AND DISTANCE

This section will define and explore a metric for the space  $\mathcal{H}(\mathbb{R}^n)$ . Since  $\mathcal{H}(\mathbb{R}^n)$  is the set of all non-empty compact subsets of  $\mathbb{R}^n$ , we must first understand what the non-empty compact subsets of  $\mathbb{R}^n$  look like. Compactness in  $\mathbb{R}^n$  is described by the Heine-Borel theorem.

**Heine-Borel Theorem.** *A subset  $A$  of  $\mathbb{R}^n$  is compact if and only if  $A$  is closed and bounded.*

A proof of this theorem, along with a discussion of compactness, can be found in [3]. An example of a compact subset of the real line is the closed interval  $[0, 6]$ . However, not all non-empty compact subsets of  $\mathbb{R}$  are so nice. For example, the

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Cantor middle third set is a non-empty compact subset of  $\mathbb{R}$  [3]. Two examples of non compact subsets of  $\mathbb{R}$  are  $[0, \infty)$  and  $(2, 5)$ .

We define a metric for  $\mathcal{H}(\mathbb{R}^n)$ , called the *Hausdorff distance*.

**Definition 1.** Let  $(\mathbb{R}^n, d')$  be a complete metric space, where  $d'$  is the Euclidean metric and  $A, B \in \mathcal{H}(\mathbb{R}^n)$ .

- If  $x \in \mathbb{R}^n$ , the “distance” from the point  $x$  to the set  $B$  is

$$d(x, B) = \min_{b \in B} \{d'(x, b)\}$$

- The “distance” from the set  $A$  to the set  $B$  is

$$d(A, B) = \max_{x \in A} \{d(x, B)\}$$

- The Hausdorff distance between sets  $A$  and  $B$  is

$$h(A, B) = \max\{d(A, B), d(B, A)\}.$$

Let’s look at some examples. Let  $B = \{b_1, b_2\}$  be a two element subset of  $\mathcal{H}(\mathbb{R}^2)$ . Figure 1 shows an example where  $d(x, B) = l$ . In Figure 2, letting  $A \in \mathcal{H}(\mathbb{R}^2)$  be the grey shaded disk we have an example where  $d(A, B) = r$ .

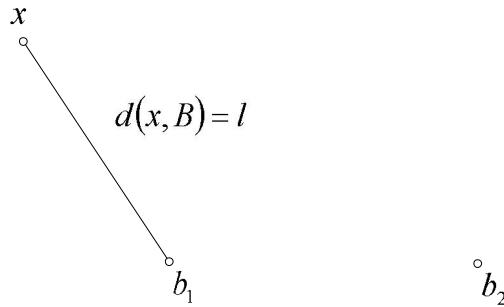


FIGURE 1.  $d(x, B) = l$ .

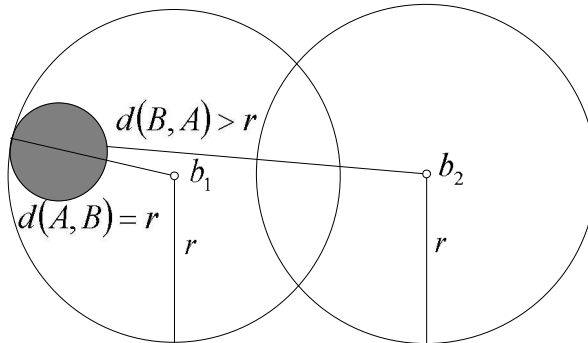


FIGURE 2.  $d(A, B) = r$  and  $d(B, A) > r$

In general, to find the distance between  $A$  and  $B$ , find the element  $x' \in A$  such that  $d(x', B) \geq d(x, B)$  for all  $x \in A$ . Then find the element  $b' \in B$  such that  $d(x', b') \leq d(x', b)$  for all  $b \in B$ . Then  $d(x', b') = d(A, B)$ . So we see that two sets  $A$  and  $B$  are within  $r$  units of each other if every point of  $A$  is within  $r$  units of some point of  $B$ . Note that the compactness of  $A$  and  $B$  guarantees the existence of the elements  $x' \in A$  and  $b' \in B$ .

It turns out that  $d(A, B)$  is not a metric for  $\mathbb{R}^n$ . In Figure 2 we have an example where  $d(A, B) = r$ , but  $d(B, A) > r$ . However, the Hausdorff distance is a metric for  $\mathcal{H}(\mathbb{R}^n)$ . Actually,  $(\mathcal{H}(\mathbb{R}^n), h)$  is a complete metric space [1]. The space  $(\mathcal{H}(\mathbb{R}^n), h)$  is the natural environment in which to study fractals.

### 3. LINES IN $\mathcal{H}(\mathbb{R}^n)$

In this section we define lines in  $\mathcal{H}(\mathbb{R}^n)$ . Lines in  $\mathbb{R}^n$  can be defined using the triangle equality. Given two points in  $\mathbb{R}^n$  we know that we can find infinitely many points in  $\mathbb{R}^n$  that satisfy each version of the triangle equality. That is, for any  $a, b \in \mathbb{R}^n$  we can find infinitely many  $c \in \mathbb{R}^n$  satisfying  $d(a, b) = d(a, c) + d(c, b)$ ,  $d(a, c) = d(a, b) + d(b, c)$ , and  $d(c, b) = d(c, a) + d(a, b)$  (where  $d$  represents the Euclidean distance). Our goal is to extend this idea into  $\mathcal{H}(\mathbb{R}^n)$ . Instead of using the Euclidean metric, we will use the Hausdorff metric to measure distances between sets. The question that we will investigate is, given  $A, B \in \mathcal{H}(\mathbb{R}^n)$  such that  $A \neq B$ , can we find infinitely many  $C \in \mathcal{H}(\mathbb{R}^n)$  that satisfy each version of the triangle equality. That is, can we find infinitely many  $C \in \mathcal{H}(\mathbb{R}^n)$  satisfying each of the following equations:

$$\begin{aligned} h(A, B) &= h(A, C) + h(C, B), \\ h(A, C) &= h(A, B) + h(B, C), \\ h(C, B) &= h(C, A) + h(A, B). \end{aligned}$$

For  $A, B \in \mathcal{H}(\mathbb{R}^n)$  we say that  $C \in \mathcal{H}(\mathbb{R}^n)$  is on the line joining  $A$  and  $B$  if  $A$ ,  $B$ , and  $C$  satisfy one version of the triangle equality. We call the line joining two subsets of  $\mathcal{H}(\mathbb{R}^n)$  a *Hausdorff line*.

Consider the subspace of  $\mathcal{H}(\mathbb{R}^n)$  containing all the one element subsets of  $\mathbb{R}^n$ . Within this set, the Hausdorff line joining  $A = \{a\}$  and  $B = \{b\}$  will contain exactly the one-point sets corresponding to points in  $\mathbb{R}^n$  lying on the Euclidean line connecting  $a$  and  $b$ . Therefore, we see that Euclidean lines are embedded within  $\mathcal{H}(\mathbb{R}^n)$ . Theorem 1 shows that the Hausdorff line through two one point subsets of  $\mathbb{R}^n$  contain more than just the Euclidean points on the line connecting  $a$  and  $b$ .

**Theorem 1.** *The line connecting two one point subsets of  $\mathcal{H}(\mathbb{R}^n)$  contains more than just the Euclidean points that lie on the line connecting the two points.*

*Proof.* Without loss of generality let  $A = \{a\}$  and  $B = \{b\}$  where  $a, b \in \mathbb{R}^n$ , and note that  $A, B \in \mathcal{H}(\mathbb{R}^n)$ . Assume that  $h(A, B) = r$ . Then  $d(a, b) = d(b, a) = r$ . Now, construct the Euclidean ray from the point  $a$  to the point  $b$ ,

$$\vec{ab} = \{a + t(b - a) : t \geq 0\}.$$

Then, find two points on the Euclidean ray,  $c$  and  $e$  such that  $d(a, c) = 2r$  and  $d(a, e) = 3r$ , where  $c$  and  $e$  are points in  $\mathbb{R}^n$ . We let

$$C = \overline{ce} = \{tc + (1 - t)e : 0 \leq t \leq 1\} \in \mathcal{H}(\mathbb{R}^n).$$

Then,  $d(C, A) = d(e, a) = 3r$  and  $d(A, C) = d(a, c) = 2r$ . Therefore  $h(A, C) = 3r$ . Also,  $d(C, B) = d(e, b) = 2r$  and  $d(B, C) = d(b, c) = r$ . This tells us that  $h(B, C) = 2r$ . Therefore  $h(A, C) = h(A, B) + h(B, C)$ , and  $C$  is on the line joining  $A$  and  $B$ . Thus, the line connecting two one point subsets of  $\mathcal{H}(\mathbb{R}^n)$  contains more than just the Euclidean points that lie on the line connecting the two points.  $\square$

Theorem 1 shows us that  $(\mathcal{H}(\mathbb{R}^n), h)$  gives us a richer framework in which to study lines. We will illustrate Theorem 1 with an example on the real line. We let  $A = \{1\}$ ,  $B = \{3\}$ , and  $C = [5, 7]$ . See Figure 3. We see that  $h(A, C) = h(A, B) + h(B, C)$  and  $C$  is on the line connecting two one point subsets of  $\mathcal{H}(\mathbb{R})$ . Letting  $C = [5, 5 + k]$  for  $k \in \mathbb{R}$  with  $k > 0$  shows us an example where there are infinitely many  $C \in \mathcal{H}(\mathbb{R})$  on the Hausdorff line joining  $A$  and  $B$ , but not on the Euclidean line joining  $A$  and  $B$ .

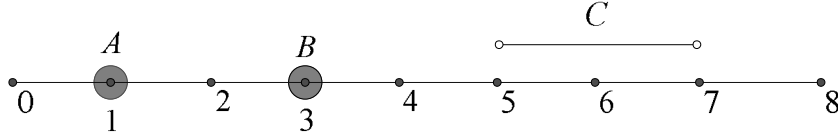


FIGURE 3. The space  $(\mathcal{H}(\mathbb{R}^n), h)$  gives us a richer framework in which to study lines.

Next we consider lines formed using closed and bounded intervals of  $\mathbb{R}$ .

**Theorem 2.** *For any two closed and bounded intervals  $A, B \in \mathcal{H}(\mathbb{R})$  we can find infinitely many  $C \in \mathcal{H}(\mathbb{R})$  that satisfy each version of the triangle equality.*

*Proof.* Let  $A, B \in \mathcal{H}(\mathbb{R})$  such that  $A$  and  $B$  are both closed and bounded intervals of  $\mathbb{R}$ . We let  $A = [a_1, a_2]$  and  $B = [b_1, b_2]$ . We will assume that  $h(A, B) = r$ . We must consider three cases:

1.  $A \subseteq B$ ,
2.  $A \cap B \neq \emptyset$ ,  $B \not\subseteq A$  and  $A \not\subseteq B$ ,
3.  $A \cap B = \emptyset$ .

For each case we must construct infinitely many  $C \in \mathcal{H}(\mathbb{R})$  that satisfy each version of the triangle equality. So there are a total of nine subcases that we must prove.

When  $A \subseteq B$  we show that there exist infinitely many  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(A, B) = h(A, C) + h(C, B)$ . Since  $h(A, B) = r$  and  $A \subseteq B$ , we see that  $d(A, B) = 0$  and  $d(B, A) = r$ . Therefore  $d(b_2, a_2) = r$  or  $d(b_1, a_1) = r$ . We let  $d(b_2, a_2) = r$ , and  $d(b_1, a_1) \leq r$ . The case where  $d(b_1, a_1) = r$ , and  $d(b_2, a_2) \leq r$  is similar. We choose  $k \in \mathbb{R}$  with  $k \geq 2$ . Let  $c_1 = a_1 - \left(\frac{d(b_1, a_1)}{k}\right)$ ,  $c_2 = a_2 + \frac{r}{k}$ , and  $C = [c_1, c_2] \in \mathcal{H}(\mathbb{R})$ . See Figure 4. We see that  $d(a_1, c_1) \leq \frac{r}{k}$  and  $d(c_1, b_1) \leq \frac{(k-1)r}{k}$ . Thus

$$h(A, C) = d(C, A) = d(c_2, a_2) = \frac{r}{k} \text{ and } h(C, B) = d(B, C) = d(b_2, c_2) = \frac{(k-1)r}{k}.$$

Therefore we have that  $h(A, B) = h(A, C) + h(C, B)$ . So for any  $k \in \mathbb{R}$  with  $k \geq 2$  there exists a  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(A, B) = h(A, C) + h(C, B)$ . Thus, there exists infinitely many  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(A, B) = h(A, C) + h(C, B)$ .

The other eight subcases have a similar construction. We leave them to the reader. Therefore, for any two closed and bounded intervals  $A, B \in \mathcal{H}(\mathbb{R})$  we can find infinitely many  $C \in \mathcal{H}(\mathbb{R})$  that satisfy each version of the triangle equality.  $\square$

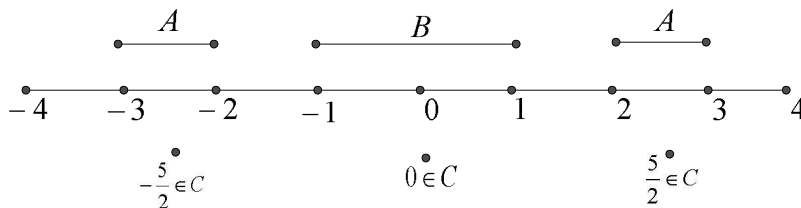
FIGURE 4. Infinitely many  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(A, B) = h(A, C) + h(C, B)$ .

We have now characterized some points on the lines joining closed and bounded intervals of  $\mathcal{H}(\mathbb{R})$ . However, we know that  $\mathcal{H}(\mathbb{R})$  contains many different kinds of sets. A closed and bounded interval is just one type of set in  $\mathcal{H}(\mathbb{R})$ . We are now ready to move on to the more general case of lines in  $\mathcal{H}(\mathbb{R})$ .

**3.1. Miscellaneous Results.** Before we study lines in general we will look at several miscellaneous results.

**Lemma 1.** *Let  $A, B \in \mathcal{H}(\mathbb{R})$  such that  $A \neq B$  and  $h(A, B) = r$ . Let  $C$  be a finite subset of  $\mathbb{R}$  such that  $d(A, C) \leq \frac{r}{2}$  and  $d(B, C) \leq \frac{r}{2}$ . If  $d(C, A) > \frac{r}{2}$  or  $d(C, B) > \frac{r}{2}$ , then by adding and subtracting a finite number of points to  $C$  we can construct a new set  $\bar{C}$  such that  $h(A, \bar{C}) \leq \frac{r}{2}$ , and  $h(B, \bar{C}) \leq \frac{r}{2}$ .*

For an illustration of sets  $A$ ,  $B$ , and  $C$  satisfying the hypothesis of Lemma 1, see Figure 5. Here we let  $A = [-3, -2] \cup [2, 3]$ ,  $B = [-1, 1]$ , and  $C = \{-\frac{5}{2}, 0, \frac{5}{2}\}$ . It can be easily verified that  $h(A, B) = r = 2$ ,  $d(A, C) = \frac{1}{2} \leq \frac{r}{2} = 1$ , and  $d(B, C) = 1 \leq \frac{r}{2}$ . We also see that  $d(C, A) = 2$  and  $d(C, B) = \frac{3}{2}$ . Throughout the proof of lemma 1 we will use this example to illustrate the construction of a new set  $\bar{C}$  such that  $h(A, \bar{C}) \leq 1$ , and  $h(B, \bar{C}) \leq 1$ .

FIGURE 5. An illustration of sets  $A$ ,  $B$ , and  $C$  which satisfy the hypothesis of Lemma 1.

*Proof.* Let  $A, B \in \mathcal{H}(\mathbb{R})$  such that  $A \neq B$ , and  $h(A, B) = r$ . Let  $C$  be a finite subset of  $\mathbb{R}$  such that  $d(A, C) \leq \frac{r}{2}$  and  $d(B, C) \leq \frac{r}{2}$ . Let  $H = \{c \in C : d(c, A) > \frac{r}{2} \text{ and } d(c, B) > \frac{r}{2}\}$  and  $C_0 = C - H$ . (In our example  $H = \emptyset$ .) If  $d(C_0, B) \leq \frac{r}{2}$  and  $d(C_0, A) \leq \frac{r}{2}$ , then let  $\bar{C} = C_0$ . If not, suppose  $d(C_0, B) > \frac{r}{2}$ . Then there exists  $c \in C_0$  such that  $d(c, B) = d(C_0, B) > \frac{r}{2}$ . Then  $d(c, b) > \frac{r}{2}$  for all  $b \in B$ . So  $B \cap [c - \frac{r}{2}, c + \frac{r}{2}] = \emptyset$ . (In our example,  $d(-\frac{5}{2}, B) > \frac{r}{2}$  and we see that  $B \cap [-\frac{7}{2}, -\frac{3}{2}] = \emptyset$ .) Since  $h(A, B) = r$ , for any  $a \in A \cap [c - \frac{r}{2}, c + \frac{r}{2}]$  there exists  $b_a \in B$  such that  $d(a, b_a) \leq r$ . Note that  $c \in C_0$ , so  $A \cap [c - \frac{r}{2}, c + \frac{r}{2}] \neq \emptyset$ .

For each  $a \in A \cap [c - \frac{r}{2}, c + \frac{r}{2}]$ , either  $b_a > c + \frac{r}{2}$  or  $b_a < c - \frac{r}{2}$ . If for some  $a \in A \cap [c - \frac{r}{2}, c + \frac{r}{2}]$ , there is a  $b_a > c + \frac{r}{2}$ , let  $b_M = \min\{B \cap [c + \frac{r}{2}, \max(B)]\}$ . Likewise, if for some  $a \in A \cap [c - \frac{r}{2}, c + \frac{r}{2}]$ , there is a  $b_a < c - \frac{r}{2}$ , let  $b_m =$

$\max\{B \cap [\min(B), c - \frac{r}{2}], c - \frac{r}{2}\}$ . (Note,  $B$  is compact so  $b_m$  and  $b_M$  will be in  $B$ .) Then, if  $b_M$  exists, let

$$D = \{a \in A \cap [c - \frac{r}{2}, c + \frac{r}{2}] : d(a, b_M) \leq \frac{r}{2}\}.$$

Let  $l = \inf(D)$  and set  $c_1 = l + \frac{r}{2}$ . (In our example  $b_M = -1$ ,  $D = [-3, -2]$  and  $c_1 = -3 + 1 = -2$ .) We will now show  $l \in D$ . Since  $A \cap [c - \frac{r}{2}, c + \frac{r}{2}]$  is closed,  $l \in A \cap [c - \frac{r}{2}, c + \frac{r}{2}]$ . Construct a sequence  $\{x_n\}$  in  $D$  so that  $x_n \in (l, l + \frac{1}{n})$  for each positive integer  $n$ . Then,

$$d(l, b_M) \leq d(l, x_n) + d(x_n, b_M) \leq \frac{1}{n} + r$$

is true for each positive integer  $n$ , so  $d(l, b_M) \leq r$  and  $l \in D$ . If  $b_m$  exists, let

$$D_0 = \{a \in A \cap [c - \frac{r}{2}, c + \frac{r}{2}] : d(a, b_m) \leq \frac{r}{2}\}.$$

Let  $l_0 = \sup(D_0)$  and set  $c_2 = l_0 - \frac{r}{2}$ . (In our example  $b_m$  does not exist.) A similar argument, as used above to show  $l \in D$ , shows that  $l_0 \in D_0$ . If  $b_a > c + \frac{r}{2}$  does not exist let  $c_1 = c$  and if  $b_a < c - \frac{r}{2}$  does not exist let  $c_2 = c$ . (In our example  $c_1 = -2$  and  $c_2 = c = -\frac{5}{2}$ .) Let  $K = \{c_1, c_2\}$  and  $C_1 = (C_0 \cup K) - \{c\}$ . (In our example,  $C_1 = \{-2, 0, \frac{5}{2}\}$ .)

We will now show  $d(A, C_1) \leq \frac{r}{2}$ . Consider  $a \in A \cap [c - \frac{r}{2}, c + \frac{r}{2}]$ . Then  $a \in D$  or  $a \in D_0$ . If  $a \in D$ , then  $d(a, c_1) \leq \frac{r}{2}$  and if  $a \in D_0$ , then  $d(a, c_2) \leq \frac{r}{2}$ . Therefore,  $d(A \cap [c - \frac{r}{2}, c + \frac{r}{2}], C_1) \leq \frac{r}{2}$ , so  $d(A, C_1) \leq \frac{r}{2}$ . We now show  $d(C_1 \cap K, A) \leq \frac{r}{2}$  and  $d(C_1 \cap K, B) \leq \frac{r}{2}$ . If  $c_1 \in C_1$ , then, since  $c_1 = l + \frac{r}{2}$  and  $l \in A$ , we know that  $d(c_1, A) \leq \frac{r}{2}$ . Also, since  $l \in D$ , we have that  $d(l, b_M) < r$ . Therefore, we see that  $d(c_1, B) \leq \frac{r}{2}$ . A similar argument holds if  $c_2 \in C_1$ . Therefore, we have that  $d(C_1 \cap K, A) \leq \frac{r}{2}$  and  $d(C_1 \cap K, B) \leq \frac{r}{2}$ .

If there exists  $c \in C_1$  such that  $d(c, B) = d(C_1, B) > \frac{r}{2}$ , then  $d(c, b) > \frac{r}{2}$  for all  $b \in B$ . So  $B \cap [c - \frac{r}{2}, c + \frac{r}{2}] = \emptyset$ . Use the same construction as above to construct a new set  $C_2 = (C_1 \cup K) - \{c\}$ . (This step is necessary in our example. We have that  $B \cap [\frac{3}{2}, \frac{7}{2}] = \emptyset$ . Our new set is  $C_2 = \{-2, 0, 2\}$ .) After doing this at most a finite number of times we will have a set  $C_n$  such that  $d(A, C_n) \leq \frac{r}{2}$ ,  $d(B, C_n) \leq \frac{r}{2}$ , and  $d(C_n, B) \leq \frac{r}{2}$ . Lets consider why this is true. At each step we preserve the fact that  $d(A, C_i) \leq \frac{r}{2}$ . Therefore  $d(A, C_n) \leq \frac{r}{2}$ . Also, for  $c \in C_0$  such that  $d(c, B) \leq \frac{r}{2}$ , we have that  $c \in C_n$ . Therefore, since we assume  $d(B, C) \leq \frac{r}{2}$ , it follows that  $d(B, C_n) \leq \frac{r}{2}$ . Lastly,  $d(C_n, B) \leq \frac{r}{2}$  follows directly from the construction of the set  $C_n$ .

We have now shown that  $d(A, C_n) \leq \frac{r}{2}$ ,  $d(B, C_n) \leq \frac{r}{2}$ , and  $d(C_n, B) \leq \frac{r}{2}$ . However, it is still possible that  $d(C_n, A) > \frac{r}{2}$ . (In our example  $d(C_2, A) > \frac{r}{2}$ .) If  $d(C_n, A) > \frac{r}{2}$ , then there exists  $c \in C_n$  such that  $d(c, A) = d(C_n, A) > \frac{r}{2}$ . Then  $d(c, a) > \frac{r}{2}$  for all  $a \in A$ . So  $A \cap [c - \frac{r}{2}, c + \frac{r}{2}] = \emptyset$ . (In our example,  $A \cap [-1, 1] = \emptyset$ .) Using a construction similar to that of  $C_n$  we use a finite number of steps to construct a finite subset  $\bar{C}$  of  $\mathbb{R}$  such that  $d(A, \bar{C}) \leq \frac{r}{2}$ ,  $d(B, \bar{C}) \leq \frac{r}{2}$ ,  $d(\bar{C}, A) \leq \frac{r}{2}$ , and  $d(\bar{C}, B) \leq \frac{r}{2}$ . The fact that  $d(A, \bar{C}) \leq \frac{r}{2}$ ,  $d(B, \bar{C}) \leq \frac{r}{2}$  and  $d(\bar{C}, A) \leq \frac{r}{2}$  follow for the same reasons as in the construction of  $C_n$ . Lets consider why  $d(\bar{C}, B) \leq \frac{r}{2}$ . We know that  $d(C_n, B) \leq \frac{r}{2}$ . As we saw in the construction of  $C_1$ ,  $d(C_1 \cap K, A) \leq \frac{r}{2}$ . For the same reasons, we have that  $d(\bar{C} - C_n, B) \leq \frac{r}{2}$ . Therefore,  $d(\bar{C}, B) \leq \frac{r}{2}$ . Thus,  $h(A, \bar{C}) \leq \frac{r}{2}$ , and  $h(B, \bar{C}) \leq \frac{r}{2}$ . (In our example,

$\bar{C} = \{-2, -1, 1, 2\}$  and it is easily verifiable that  $h(A, \bar{C}) \leq 1$ , and  $h(B, \bar{C}) \leq 1$ . See Figure 6.)  $\square$

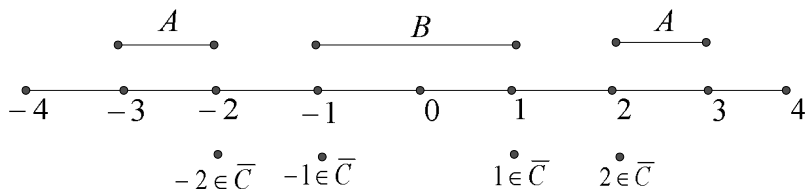


FIGURE 6. An illustration of the sets  $A$ ,  $B$ , and  $\bar{C}$ .

**Lemma 2.** *Let  $A, B \in \mathcal{H}(\mathbb{R})$ . If  $d(A, B) > 0$ , then there exists  $b_0 \in \partial B$  and  $a_0 \in A$  such that  $d(A, B) = d(a_0, b_0)$ , and  $d(a_0, b_0) \leq d(a_0, b)$  for all  $b \in B$ .*

A proof of Lemma 2 can be found in [6].

Up to this point we have characterized compactness in  $\mathbb{R}^n$  with the Heine Borel Theorem. However, an alternative characterization of compactness which is based upon open covers will be useful throughout the rest of the paper. Given  $A \subset \mathbb{R}^n$ , an *open cover* of  $A$  is a collection  $O$  of open subsets of  $\mathbb{R}^n$  whose union contains  $A$ . A *subcover* derived from an open cover  $O$  is a subcollection  $O'$  of  $O$  whose union contains  $A$  [3]. The following theorem gives an alternative characterization of non-empty compact subsets of  $\mathbb{R}^n$ .

**Theorem 3.** *If  $A \neq \emptyset$  and for each open covering  $O$  of  $A$  there exists a finite open subcover  $O'$  of  $A$ , then  $A \in \mathcal{H}(\mathbb{R}^n)$ .*

A proof of Theorem 3 can be found in [3].

**3.2. General Results.** In this section we will consider more general subsets of  $\mathcal{H}(\mathbb{R})$ . We will show that for any  $A, B \in \mathcal{H}(\mathbb{R})$ , there exists a  $C \in \mathcal{H}(\mathbb{R})$ , distinct from  $A$  and  $B$ , such that  $C$  is on the line joining  $A$  and  $B$ . We then put restrictions on the sets  $A$  and  $B$  and show that there are infinitely many  $C \in \mathcal{H}(\mathbb{R})$  on the line joining  $A$  and  $B$ . We begin with the following theorem.

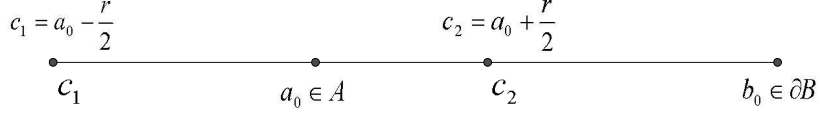
**Theorem 4.** *Let  $A, B$  be distinct elements in  $\mathcal{H}(\mathbb{R})$ . There exists  $C \in \mathcal{H}(\mathbb{R})$ , distinct from  $A$  and  $B$ , such that  $h(A, B) = h(A, C) + h(C, B)$ .*

*Proof.* Let  $A, B \in \mathcal{H}(\mathbb{R})$  such that  $A \neq B$ . We assume that  $h(A, B) = r$ . Thus  $d(A, B) = r$  or  $d(B, A) = r$ . We will let  $d(A, B) = r$  and  $d(B, A) \leq r$ . The case where  $d(B, A) = r$  and  $d(A, B) \leq r$  is similar. It is important to note that since  $d(A, B) = r$ , every point of  $A$  is less than or equal to  $r$  units away from some point of  $B$ . Likewise, since  $d(B, A) \leq r$ , every point of  $B$  is less than or equal to  $r$  units away from some point of  $A$ . Also, since  $d(A, B) = r$ , Lemma 2 tells us that there exists  $b_0 \in \partial B$  and  $a_0 \in A$  such that  $d(A, B) = d(a_0, b_0)$ , and  $d(a_0, b_0) \leq d(a_0, b)$  for all  $b \in B$ .

We will now construct  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(A, B) = h(A, C) + h(C, B)$ . We assume  $b_0 > a_0$ . The case where  $a_0 > b_0$  is similar. Let

$$c_1 = a_0 - \frac{r}{2} \quad \text{and} \quad c_2 = a_0 + \frac{r}{2}.$$

We have that  $d(a_0, c_1) = d(a_0, c_2) = d(b_0, c_2) = \frac{r}{2}$ . See Figure 7.

FIGURE 7.  $d(a_0, c_1) = d(a_0, c_2) = d(b_0, c_2) = \frac{r}{2}$ 

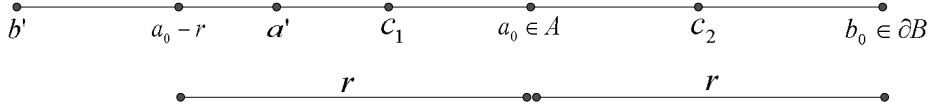
Now let

$$F = \left\{ \left( x - \frac{r}{2}, x + \frac{r}{2} \right) : x \in A \cup B \right\}.$$

We see that  $F$  is an open cover of  $A \cup B$  using intervals of length  $r$ . Since  $A, B \in \mathcal{H}(\mathbb{R})$ ,  $A \cup B \in \mathcal{H}(\mathbb{R})$ . Thus, by theorem 3 there is a finite open subcover of  $A \cup B$  in  $F$ , which we will call  $G$ .

Let  $G = \{I_1, I_2, \dots, I_k\}$  and  $C_1 = \{i_1, i_2, \dots, i_k\}$  where  $i_j$  is the midpoint of  $I_j$  for  $1 \leq j \leq k$ . Every  $a \in A$  is in some  $I_g$  and every  $b \in B$  is in some  $I_j$ . So  $d(A, C_1) \leq \frac{r}{2}$  and  $d(B, C_1) \leq \frac{r}{2}$ . However, we do not know that  $d(C_1, A) \leq \frac{r}{2}$  and  $d(C_1, B) \leq \frac{r}{2}$ . If  $d(C_1, A) > \frac{r}{2}$  or  $d(C_1, B) > \frac{r}{2}$  we apply Lemma 1 and form a new set  $\overline{C_1}$  such that  $h(A, \overline{C_1}) \leq \frac{r}{2}$  and  $h(B, \overline{C_1}) \leq \frac{r}{2}$ .

We now let  $\overline{C_2} = \overline{C_1} \cup \{c_2\}$ . If  $c \in \overline{C_2} \cap (c_1, c_2)$  then we remove  $c$  from  $\overline{C_2}$  and form a new set  $\overline{C_3}$ . Since  $d(a_0, b_0) < d(a_0, b)$  for all  $b \in B$  we see that  $B \cap (c_1 - \frac{r}{2}, c_1 + \frac{r}{2}) = \emptyset$  and  $d(B, \overline{C_3}) \leq \frac{r}{2}$ . However, we could have that  $d(A, \overline{C_3}) > \frac{r}{2}$ . This can only happen if there exists  $a' \in [a_0 - r, a_0] \cap A$  with  $d(a', \overline{C_3}) > \frac{r}{2}$ . To fix this, suppose there exists  $a' \in [a_0 - r, a_0] \cap A$  with  $d(a', \overline{C_3}) > \frac{r}{2}$ . Note that since  $d(a_0, \overline{C_3}) = \frac{r}{2}$ , we have that  $a' < a_0$ . Since  $d(A, B) = r$ , there must exist a  $b' \in B$  such that  $b' \leq a_0 - r$  and  $d(a', b') \leq r$ . For an illustration see Figure 8.

FIGURE 8.  $a' \in [a_0 - r, a_0] \cap A$  and  $b' \in B$  such that  $b' \leq a_0 - r$  and  $d(a', b') \leq r$ .

We will now construct a set  $C$  such that  $d(A, C) \leq \frac{r}{2}$ ,  $d(B, C) \leq \frac{r}{2}$ ,  $d(C, A) \leq \frac{r}{2}$ , and  $d(C, B) \leq \frac{r}{2}$ . Let  $b_m = \max\{B \cap [\min(B), a_0 - r]\}$  and note that  $b_m \in B$ . Consider the set

$$D = \{x \in A \cap [a_0 - r, a_0] : d(x, b_m) \leq r\}.$$

Set  $a'' = \sup(D)$ ,  $c' = a'' - \frac{r}{2}$  and  $C = \overline{C_3} \cup \{c'\}$ . Note that  $C$  is a finite set of points containing  $c_2$ , so  $C \in \mathcal{H}(\mathbb{R})$ . We first show that  $d(A, C) \leq \frac{r}{2}$ . Since  $d(A, B) \leq r$ , every  $x \in A \cap [a_0 - r, a_0]$  is less than or equal to  $a''$ . Then, since  $c' = a'' - \frac{r}{2}$ , we have that  $d(x, c') \leq \frac{r}{2}$  for all  $x \in A \cap [a_0 - r, a_0]$  which implies that  $d(A, C) \leq \frac{r}{2}$ .

We now discuss why  $d(B, C) \leq \frac{r}{2}$  and  $d(C, B) \leq \frac{r}{2}$ . The fact that  $a'' \in D$  follows from an argument similar to the one used in the previous lemma. Therefore, since  $a'' \in D$ , we have that  $d(a'', b_m) \leq r$ . Since  $a'' \in D$ ,  $c' = a'' - \frac{r}{2}$  and  $d(c', b_m) \leq \frac{r}{2}$  we see that  $d(B, C) \leq \frac{r}{2}$  and  $d(C, B) \leq \frac{r}{2}$ . Lastly, we still must show that  $d(C, A) \leq \frac{r}{2}$ . Since  $c' = a'' - \frac{r}{2}$ , we have that  $d(c', A) \leq \frac{r}{2}$ . Since  $d(C - \{c'\}, A) \leq \frac{r}{2}$  and  $d(c', A) \leq \frac{r}{2}$  we have that  $d(C, A) \leq \frac{r}{2}$ .



We now will show that  $h(A, C) = \frac{r}{2}$  and  $h(C, B) = \frac{r}{2}$ . Since  $c' = a'' - \frac{r}{2}$  and  $a'' \leq a_0$ , we see that  $c' \leq c_1$ . Then, since  $\overline{C_3} \cap (c_1, c_2) = \emptyset$  and  $a'' < c_1$ , we have that  $C \cap (c_1, c_2) = \emptyset$  which implies that  $d(a_0, C) = \frac{r}{2}$ . Since  $d(A, C) \leq \frac{r}{2}$  and  $d(a_0, C) = \frac{r}{2}$ , we have that  $d(A, C) = \frac{r}{2}$ . Since there is no  $b \in B$  such that  $b \in (a_0, b_0)$  we have that  $d(C, B) = d(c_2, b_0) = \frac{r}{2}$ . Thus  $h(A, C) = \frac{r}{2}$  and  $h(C, B) = \frac{r}{2}$ . Therefore

$$h(A, C) + h(C, B) = \frac{r}{2} + \frac{r}{2} = r = h(A, B).$$

Thus we have found a  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(A, B) = h(A, C) + h(C, B)$ .  $\square$

Theorem 4 shows us that for any  $A, B \in \mathcal{H}(\mathbb{R})$  we can find a point in  $\mathcal{H}(\mathbb{R})$ , distinct from  $A$  and  $B$ , on the line joining  $A$  and  $B$ . In the following theorem we put restrictions on the sets  $A$  and  $B$  and show that there are infinitely many points in  $\mathcal{H}(\mathbb{R})$  on the line joining  $A$  and  $B$ .

**Theorem 5.** *Let  $A, B$  be distinct elements in  $\mathcal{H}(\mathbb{R})$  with  $a_1, a_2$  being the respective minimum and maximum elements of  $A$ ,  $b_1, b_2$  being the respective minimum and maximum elements of  $B$ ,  $h(A, B) = r$ ,  $d(a_1, b_1) < r$ , and  $d(a_2, b_2) < r$ . Then, there exists infinitely many  $C' \in \mathcal{H}(\mathbb{R})$  such that  $h(A, B) = h(A, C') + h(C', B)$ .*

*Proof.* We assume that  $A, B$  are distinct elements in  $\mathcal{H}(\mathbb{R})$  with  $a_1, a_2$  being the respective minimum and maximum elements of  $A$ ,  $b_1, b_2$  being the respective minimum and maximum elements of  $B$ ,  $h(A, B) = r$ ,  $d(a_1, b_1) < r$ , and  $d(a_2, b_2) < r$ . We will consider two cases:

1.  $0 < d(a_1, b_1) < r$  and  $0 < d(a_2, b_2) < r$ .
2.  $a_1 = b_1$  and  $0 < d(a_2, b_2) < r$ . (The cases where  $a_2 = b_2$  and  $0 < d(a_1, b_1) < r$ , or  $a_1 = b_1$  and  $a_2 = b_2$  are similar.)

Case(1): We assume that  $0 < d(a_1, b_1) < r$  and  $0 < d(a_2, b_2) < r$ . Also, we will assume  $b_1 < a_1$  and  $b_2 < a_2$ . The other possibilities are similar. Since  $h(A, B) = r$ , we know that  $d(A, B) = r$  or  $d(B, A) = r$ . We assume that  $d(A, B) = r$  and the case where  $d(B, A) = r$  is similar. Recall that Lemma 2 states that for  $A, B \in \mathcal{H}(\mathbb{R})$ , if  $d(A, B) > 0$ , then there exists  $b_0 \in \partial B$  and  $a_0 \in A$  such that  $d(A, B) = d(a_0, b_0)$ , and  $d(a_0, b_0) \leq d(a_0, b)$  for all  $b \in B$ . Since  $d(A, B) = r > 0$  we can apply Lemma 2. The  $a_0$  used cannot be  $a_1$  or  $a_2$  since  $d(a_1, b_1) < r$  and  $d(a_2, b_2) < r$ . Therefore, there exists  $a_0 \in A \cap (a_1, a_2)$  and  $b_0 \in \partial B$  satisfying Lemma 2. Now let  $C$  be constructed as it is in Theorem 4. Note that in the construction of  $C$ ,  $c_1 = a_0 - \frac{r}{2}$  and  $c_2 = a_0 + \frac{r}{2}$ . We now introduce points  $c_3$  and  $c_4$  to help construct sets  $C'$  so that  $h(A, B) = h(A, C') + h(C', B)$ . If  $a_1 < c_1$ , then let  $c_3 = \max\{b_1, a_1 - \frac{r}{2}\}$  and  $c_4 = \min\{a_1, b_1 + \frac{r}{2}\}$ . See Figure 9.

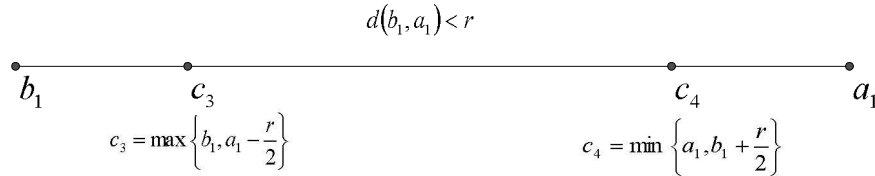


FIGURE 9. An illustration of the first part of Case 1.

Suppose  $c_1 < a_1$ . Since  $a_1 < a_0$ ,  $d(a_1, a_0) < \frac{r}{2}$ . Let  $c_3 = a_1 - \frac{r}{2}$  and  $c_4 = \min\{c_1, b_1 + \frac{r}{2}\}$ . It is important to note  $c_3 < c_4 \leq c_1$ . The fact that  $c_3 < c_4$  is fairly obvious. We will show that  $c_4 \leq c_1$ . Note,  $b_1 \leq a_0 - r$ , so  $b_1 + \frac{r}{2} \leq a_0 - \frac{r}{2}$ . Thus  $c_4 \leq c_1$ . See Figure 10

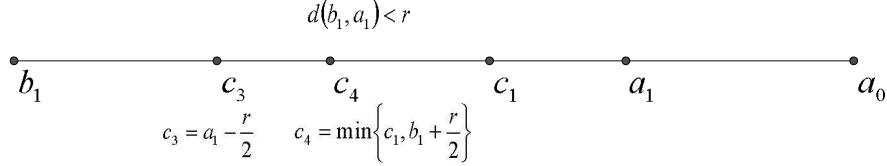


FIGURE 10. An illustration of the second part of Case 1.

Case(2): We assume that  $a_1 = b_1$  and  $0 < d(a_2, b_2) < r$ . Since  $h(A, B) = r$ , we know that  $d(A, B) = r$  or  $d(B, A) = r$ . We assume that  $d(A, B) = r$ . The case where  $d(B, A) = r$  is similar. Since  $d(A, B) = r > 0$  we can apply Lemma 2. The  $a_0$  used cannot be  $a_1$  since  $d(a_1, b_1) = 0$ . Since  $d(a_0, B) = r$ , we see that  $a_0 \geq a_1 + r$ . Let  $C$  be constructed as in Theorem 4. In the construction of  $C$ ,  $c_1 = a_0 - \frac{r}{2}$  and  $c_2 = a_0 + \frac{r}{2}$ . We now introduce points  $c_3$  and  $c_4$  to help construct sets  $C'$  so that  $h(A, B) = h(A, C') + h(C', B)$ . Let  $c_3 = a_1 - \frac{r}{2}$  and  $c_4 = a_1 - \frac{r}{4}$ . Since  $a_0 \geq a_1 + r$  and  $c_1 = a_0 - \frac{r}{2}$  it is important to note  $c_3 < c_4 < c_1$ .

We now will use the points  $c_3$  and  $c_4$  which were introduced in cases 1 and 2 to construct infinitely many sets  $C'$ . Let  $D = [c_3, c_4]$ . We see that for each  $k \in D$ ,  $d(k, A) \leq \frac{r}{2}$  and  $d(k, B) \leq \frac{r}{2}$ . For each  $k \in D$ , let  $C' = C \cup \{k\}$ . Since  $D \cap (c_1, c_2) = \emptyset$  and  $h(A, C) = h(C, B) = \frac{r}{2}$  we see that  $h(A, C') = h(C', B) = \frac{r}{2}$ . Thus  $h(A, B) = h(A, C') + h(C', B)$  for each  $k \in D$ . Therefore there exists infinitely many  $C' \in \mathcal{H}(\mathbb{R})$  such that  $h(A, B) = h(A, C') + h(C', B)$ .  $\square$

Here is another special case.

**Theorem 6.** *Let  $A, B \in \mathcal{H}(\mathbb{R})$  such that  $A \neq B$ ,  $h(A, B) = r$ , with  $a_1, a_2$  being the respective minimum and maximum elements of  $A$ ,  $b_1, b_2$  being the respective minimum and maximum elements of  $B$ , and  $d(a_1, b_1) = r$  or  $d(a_2, b_2) = r$ . Then, there exists infinitely many  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(B, C) = h(A, B) + h(A, C)$ .*

*Proof.* Let  $A, B \in \mathcal{H}(\mathbb{R})$  such that  $A \neq B$ ,  $h(A, B) = r$ , with  $a_1, a_2$  being the respective minimum and maximum elements of  $A$ ,  $b_1, b_2$  being the respective minimum and maximum elements of  $B$ , and  $d(a_1, b_1) = r$  or  $d(a_2, b_2) = r$ . We will assume that  $d(a_1, b_1) = r$ . The case where  $d(a_2, b_2) = r$  is similar. We will also assume that  $a_1 \leq b_1$ . The case where  $b_1 \leq a_1$  is similar.

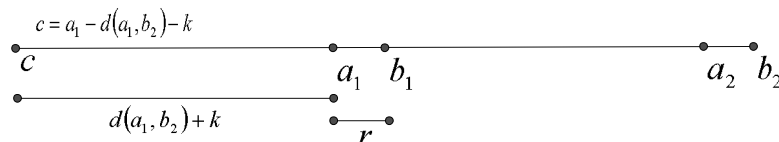
We will now construct sets  $C$  so that  $h(B, C) = h(A, B) + h(A, C)$ . Let  $k \in \mathbb{R}$  such that  $k \geq 0$  and set

$$c = a_1 - d(a_1, \max\{a_2, b_2\}) - k.$$

Note that

$$d(c, a_1) = d(a_1, \max\{a_2, b_2\}) + k \text{ and } d(c, b_1) = d(a_1, \max\{a_2, b_2\}) + k + r.$$

Figure 11 gives a general illustration of the point  $c$  when  $a_2 < b_2$ . Let  $C = A \cup B \cup \{c\} \in \mathcal{H}(\mathbb{R})$ .

FIGURE 11.  $A \cup B \cup \{c\}$  is a point on the line joining  $A$  and  $B$ .

We will now consider the Hausdorff distances between the sets  $A$ ,  $B$ , and  $C$ . Since  $A$  and  $B$  are both subsets of  $C$  we see that  $d(A, C) = d(B, C) = 0$ . We also have

$$d(C, A) = d(c, a_1) = d(a_1, \max\{a_2, b_2\}) + k$$

and

$$d(C, B) = d(c, b_1) = d(a_1, \max\{a_2, b_2\}) + k + r.$$

Therefore we see that

$$h(B, C) = d(a_1, \max\{a_2, b_2\}) + k + r,$$

and

$$h(A, C) = d(a_1, \max\{a_2, b_2\}) + k.$$

Therefore, there exists infinitely many  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(B, C) = h(A, B) + h(A, C)$ .  $\square$

Given any two  $A, B \in \mathcal{H}(\mathbb{R})$  with  $h(A, B) = r$ , the distance between the respective minimum and maximum elements of  $A$  and  $B$  will be  $r$  or less than  $r$ . Thus, combining Theorems 5 and 6 gives the following corollary.

**Corollary 1.** *Let  $A$  and  $B$  be distinct elements in  $\mathcal{H}(\mathbb{R})$ . There exist infinitely many  $C \in \mathcal{H}(\mathbb{R})$  on the line joining  $A$  and  $B$ .*

The proof of the following theorem is very similar to Theorems 4 and 5. We state the theorem without proof.

**Theorem 7.** *Let  $A, B$  be distinct elements in  $\mathcal{H}(\mathbb{R})$ . If  $h(A, B) = r$  and there exists an  $a_0 \in \partial A$  and  $b_0 \in \partial B$  such that  $d(a_0, b_0) = r$  and  $d(b_0, A) = r$  then there exists infinitely many  $C \in \mathcal{H}(\mathbb{R})$  such that  $h(B, A) + h(A, C) = h(B, C)$ .*

#### 4. INTERSECTION PROPERTIES OF LINES

Euclidean Lines in  $\mathbb{R}^n$  only intersect in one, zero, or infinitely many points. We think of two lines as being parallel if they do not intersect at any point. Also, in Euclidean geometry, if two lines intersect in infinitely many points, then they are the same line. In the geometry of  $\mathcal{H}(\mathbb{R})$  do distinct lines only intersect in zero or one points? The following theorem answers this question as no.

**Theorem 8.** *Two distinct lines in  $\mathcal{H}(\mathbb{R})$  can intersect in infinitely many points.*

*Proof.* We will construct such an example. Let  $k \in \mathbb{R}$ , such that  $k > 2$ ,  $A = [0, 2]$ ,  $B = [8, 10]$ ,  $C = [-1, 1] \cup [9, 11]$ , and  $D = [4, 5 - \frac{1}{k}] \cup [5 + \frac{1}{k}, 6]$ . See Figure 12. We see that  $A, B, C$ , and  $D$  are all closed and bounded sets of  $\mathbb{R}$ , and therefore are in  $\mathcal{H}(\mathbb{R})$ . We will first consider the line joining  $A$  and  $B$ . We see that  $d(A, B) = d(B, A) = 8$ . Therefore  $h(A, B) = 8$ . Consider the point  $C \in \mathcal{H}(\mathbb{R})$ . We see that  $d(A, C) = 1$  and  $d(C, A) = 9$ . Therefore  $h(A, C) = 9$ . Similarly, we see that  $d(B, C) = 1$  and

$d(C, B) = 9$ . Therefore  $h(B, C) = 9$ . Therefore  $C$  is not on the line joining  $A$  and  $B$ , and  $B$  is not on the line joining  $A$  and  $C$ .

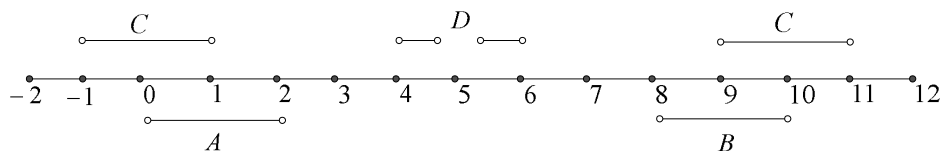


FIGURE 12. Distinct lines intersecting in infinitely many points.

Now consider the point  $D$ . We see that  $d(A, D) = d(D, A) = d(B, D) = d(D, B) = 4$ . Therefore  $h(A, D) = h(B, D) = 4$ . Thus  $h(A, B) = h(A, D) + h(D, B)$  and  $D$  is on the line joining  $A$  and  $B$  for any value of  $k$ . Now consider the point  $D$  and the line joining  $C$  and  $A$ . We have already shown that  $h(C, A) = 9$ , and  $h(A, D) = 4$ . We see that  $d(C, D) = 5$ , and  $d(D, C) = 3 + \frac{k-1}{k}$ . Therefore  $h(C, D) = 5$ . Thus,  $h(A, C) = h(A, D) + h(D, C)$  and  $D$  is on the line joining  $A$  and  $C$  for any value of  $k$ . Therefore we see that the line joining  $A$  and  $B$  intersects the line joining  $A$  and  $D$  at infinitely many points in  $\mathcal{H}(\mathbb{R})$ .  $\square$

## 5. CONCLUSION

This paper has looked at some special cases when dealing with lines in  $\mathcal{H}(\mathbb{R})$ , as well as looking at  $\mathcal{H}(\mathbb{R})$  in general. We have shown that given any two points  $A$  and  $B$  in  $\mathcal{H}(\mathbb{R})$  there exists infinitely many point in  $\mathcal{H}(\mathbb{R})$  on the line joining  $A$  and  $B$ . We also have looked at several special cases. For example, given any two closed and bounded intervals of  $\mathbb{R}$ ,  $A$  and  $B$ , there exists infinitely many  $C \in \mathcal{H}(\mathbb{R})$  satisfying each version of the triangle equality.

There is still much work to be done on this topic. Further work includes finishing to classify all lines in  $\mathcal{H}(\mathbb{R})$ . Once this is done, a characterization of lines needs to be generalized to  $\mathcal{H}(\mathbb{R}^n)$ . We know that distinct lines can intersect in infinitely many points. Given  $n \in \mathbb{N}$ , can we find distinct lines that intersect in exactly  $n$  points? (This question is partially answered in [6]. It is shown that in  $\mathcal{H}(\mathbb{R}^n)$  we can find distinct lines that intersect in 0, 1, or infinitely many points. Can we find distinct lines that intersect in 7 points?) What type of geometry does this form? We have seen that Euclidean lines are embedded within  $\mathcal{H}(\mathbb{R}^n)$ . Do special subsets of  $\mathcal{H}(\mathbb{R}^n)$  form even different types of geometries, or maybe other familiar types of geometries?

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