

## EXCELLENT RINGS WITH SINGLETON FORMAL FIBERS

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ABSTRACT. In this paper we construct a non-complete excellent local ring  $A$  such that the natural map  $\text{Spec } \widehat{A} \rightarrow \text{Spec } A$  is bijective.

### 1. INTRODUCTION

Although this paper focuses on a commutative algebra result, we shall begin by talking about algebraic geometry and its relationship with commutative algebra. An algebraic variety is a set of common zeroes of a polynomial with coefficients in some field  $k$ . Varieties are very important in algebraic geometry, and to study them we analyze the set of regular functions from the variety to  $k$ . This set turns out to be an affine ring, that is, a ring of the form  $k[x_1, \dots, x_n]/I$  where  $I$  is an ideal.

Affine rings have very nice properties, and so a question naturally arises ... which of these nice properties are the ones that make affine rings geometrically significant? Algebraists have tried to capture these nice properties by classifying nice rings through a definition. The first attempt at doing this was the definition of Noetherian. A Noetherian ring is a ring in which every ideal is finitely generated. But it turns out that this classification is much too weak and does not capture the properties that were hoped for. So a new definition was needed. In the 1950's Grothendieck presented the definition of excellent, and this is the latest attempt at answering the above question.

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A formal definition of excellent, as well as many other technical definitions, is included in section 3 of this paper, but here we shall attempt to describe the idea behind excellence.

Let  $(A, M)$  be a local ring.<sup>1</sup> By Cohen’s Structure Theorem we know the structure of  $\widehat{A}$ , the  $M$ -adic completion of  $A$ . In particular, we know that if  $\widehat{A}$  contains a field  $k$ , then  $\widehat{A}$  is of the form  $k[[x_1, \dots, x_n]]/I$ , which is quite geometric as it relates to affine rings. Furthermore, there is a nice relationship between  $A$  and  $\widehat{A}$ , namely that any prime of  $A$  can be written as  $\mathfrak{p} \cap A$ , for some prime ideal  $\mathfrak{p}$  in  $\widehat{A}$ . This property is called faithful flatness, and it is saying that every prime ideal of  $A$  can, in some sense, be described by prime ideals of the completion. So the idea behind excellence is this—since we know  $\widehat{A}$  is geometric and that the prime ideals of  $A$  can be described by the primes of  $\widehat{A}$ , we hope to carry the geometric properties of  $\widehat{A}$  down to  $A$  by controlling which primes of  $\widehat{A}$  map to which primes of  $A$ . Since the whole goal is to capture nice geometric properties of rings, to say that  $A$  is excellent is to say that it satisfies some “nice” conditions about which primes of  $\widehat{A}$  map to which primes of  $A$ , thereby transferring the geometric properties of  $\widehat{A}$  down to  $A$ .

The definition of excellent has proved to be successful in some ways. It ensures that reducedness and normality are preserved by the completion map, that is to say that if  $A$  is an excellent reduced (or normal) ring, then  $\widehat{A}$  will be reduced (or normal). This is not always the case when  $A$  is not excellent. But excellent rings do seem to fall short of their purpose; they do not appear to capture all the properties of affine rings. Matsumura has shown that if  $R$  is an affine ring, then there exist infinitely many maximal elements in the generic formal fiber of  $A$ , that is, we can find a countably infinite list of prime ideals  $\mathfrak{p}_1, \mathfrak{p}_2, \dots$  of  $\widehat{A}$  such that  $\mathfrak{p}_i \cap A = (0)$  for each  $i$ , and for any prime ideal  $\mathfrak{q}$  strictly containing any  $\mathfrak{p}_i$ ,  $\mathfrak{q} \cap A \neq (0)$  [5]. For excellent rings this is not necessarily the case. The following theorem, which was proved in [3] by Loepp, demonstrates how extremely different the situation may be for an excellent ring.

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<sup>1</sup>In this paper all rings are commutative with unity. When we write  $(R, M)$  is a quasi-local ring, we mean that  $R$  is a quasi-local ring with maximal ideal  $M$ .

**Theorem 1.1.** *Let  $(T, \mathfrak{m})$  be a complete regular local ring of dimension at least two containing the rationals, and let  $|T/\mathfrak{m}| \geq |\mathbb{R}|$ . Let  $\mathfrak{p} \neq \mathfrak{m}$  be a prime ideal of  $T$ . Then there exists an excellent regular local ring  $A$  with the following properties:*

- $\widehat{A} = T$ .
- *The generic formal fiber of  $A$  is local with  $\mathfrak{p}$  its maximal ideal.*
- *For all  $\mathfrak{q} \in \text{Spec } T$ , if  $\mathfrak{q} \not\subseteq \mathfrak{p}$ , then  $(\mathfrak{q} \cap A)T = \mathfrak{q}$ .*

The above  $A$  is very strange for two reasons. First, the generic formal fiber of  $A$  has exactly one maximal element, whereas there were infinitely many in the affine case. Second, by the third property of the theorem, there is a one-to-one correspondence between the primes of  $A$  and the primes of  $T$  not contained in the generic formal fiber of  $A$ . Although this property is nice in some sense, it is far from the behavior exhibited by affine rings. Such an odd ring hardly deserves to be called “excellent,” and yet  $A$  is an excellent ring.

In this paper, we use the above theorem to help us construct an excellent regular local ring  $B$  such that there is a one-to-one correspondence between the primes of  $B$  and the primes of  $\widehat{B}$  while  $B \neq \widehat{B}$ . Since  $B$  is excellent, there are strong conditions on the formal fibers of  $B$ , and yet *all* of the formal fibers turn out to be singleton sets.

## 2. MAIN THEOREM

**Theorem 2.1.** *Let  $T$  be a complete regular local ring with maximal ideal  $\mathfrak{m}$  such that  $\dim T \geq 2$ ,  $\mathbb{Q} \subseteq T$ , and  $|T/\mathfrak{m}| \geq |\mathbb{R}|$ . Suppose  $x \in T$  is a prime element. Then there exists an excellent regular local ring  $B$  such that  $\widehat{B} = T$ , the natural map  $\text{Spec } T \rightarrow \text{Spec } B$  is bijective, and  $x \notin B$ .*

*Proof.* Since  $T$  is a regular local ring,  $T$  is a UFD by (20.3) of [4]. Since  $\dim T \geq 2$ ,  $\mathfrak{m}$  is a prime ideal generated by at least two elements, so by factoring the generators into prime elements, we see that there are at least two prime elements in  $T$ . Therefore let  $y \neq x$  be a prime element. Using Theorem 1.1, we construct  $A$  so that its generic formal fiber is  $\{(0), xT\}$ .

We now define  $B = A[xy]_{\mathfrak{m} \cap A[xy]}$ . We will prove that this  $B$  satisfies the conclusion of the theorem. Clearly  $B$  is quasi-local. Moreover, by the Hilbert Basis Theorem,  $A[xy]$  is Noetherian, and since the localization of a Noetherian ring is Noetherian, it follows that  $B$  is Noetherian. Thus  $B$  is local. Obviously,  $A \subseteq B$ , and since elements in  $A[xy] - (\mathfrak{m} \cap A[xy]) \subseteq T - \mathfrak{m}$  are all units in  $T$ , it follows that  $B \subseteq T$ . By completeness of  $T$ , every Cauchy sequence in  $T$  converges in  $T$ , so in particular, every Cauchy sequence in  $B$  converges in  $T$ . Moreover,  $\widehat{A} = T$  implies that every element of  $T$  can be written as the limit of a Cauchy sequence in  $A$ , and therefore also as the limit of a Cauchy sequence in  $B$ . Hence  $\widehat{B} = T$ . Also note that  $B$  is a regular local ring since  $T$  is a regular local ring (see (10.15)(iii) and (11.12)(ii) of [1]). Note that this implies that  $B$  is a UFD.

**Claim 2.1.**  $x \notin B$

*Proof.* Assume  $x \in B$ . Then  $x = \frac{a_0 + a_1 xy + \cdots + a_m (xy)^m}{a'_0 + a'_1 xy + \cdots + a'_n (xy)^n}$ , where the denominator is not in  $\mathfrak{m}$ . So  $x(a'_0 + a'_1 xy + \cdots + a'_n (xy)^n) = a_0 + a_1 xy + \cdots + a_m (xy)^m$ . So  $a_0 \in xT \cap A = (0)$ . Then since  $T$  is a domain, we can cancel  $x$  on both sides to conclude that  $a'_0 + a'_1 xy + \cdots + a'_n (xy)^n \in yT \subseteq \mathfrak{m}$ , which is a contradiction.  $\square$

Since  $T$  is the completion of  $B$ , we know that the map  $\text{Spec } T \rightarrow \text{Spec } B$  is surjective by faithful flatness. We also want to show that it is injective.

**Claim 2.2.** For all  $\mathfrak{p} \in \text{Spec } T$  such that  $\mathfrak{p} \not\subseteq xT$ , we have  $(\mathfrak{p} \cap B)T = \mathfrak{p}$ .

*Proof.* By our construction of  $A$  and by Theorem 1.1, we know that  $(\mathfrak{p} \cap A)T = \mathfrak{p}$ . So  $\mathfrak{p} = (\mathfrak{p} \cap A)T \subseteq (\mathfrak{p} \cap B)T \subseteq \mathfrak{p}$ .  $\square$

The only primes contained in  $xT$  are  $(0)$  and  $xT$ , so in order to get injectivity we need to check the following four cases: for all distinct  $\mathfrak{p}, \mathfrak{q} \in \text{Spec } T$  not contained in  $xT$ :

- (i)  $\mathfrak{p} \cap B \neq (0) \cap B$
- (ii)  $\mathfrak{p} \cap B \neq xT \cap B$
- (iii)  $xT \cap B \neq (0) \cap B$
- (iv)  $\mathfrak{p} \cap B \neq \mathfrak{q} \cap B$

We know  $\mathfrak{p} \cap A \neq (0)$  by the construction of  $A$ , so (i) follows. For (ii), suppose  $\mathfrak{p} \cap B = xT \cap B$ . Then by Claim 2.2,  $\mathfrak{p} = (\mathfrak{p} \cap B)T = (xT \cap B)T \subseteq xT$ . This is a contradiction, so we have (ii). Clearly  $xy$  is nonzero and  $xy \in xT \cap B$ , thus we get (iii). Case (iv) follows directly from Claim 2.2. Therefore  $\text{Spec } T \longrightarrow \text{Spec } B$  is a bijection.

We also want  $B$  to be excellent. Because  $T$  is an integral domain and contains  $\mathbb{Q}$ , it suffices to show that all of its formal fiber rings are regular rings (see “excellent” in section 3). In fact, we will show that in this case each formal fiber ring is a field. This will complete the proof of the theorem.

Fix a prime ideal  $P$  of  $B$ . Let  $k(P) = B_P/PB_P$  be the residue field of  $B_P$ . We need to show that the formal fiber ring at  $P$ ,  $T \otimes_B k(P)$ , is a field.

As noted on page 56 of [2],  $T \otimes_B k(P) \cong S^{-1}T/((PT)(S^{-1}T))$ , where  $S = \overline{B - P}$  and the bar denotes saturation (see section 3 for the definition of saturation). Moreover, the prime ideals of this formal fiber ring correspond to the prime ideals of  $T$  which contract down to  $P$ , so by the (1-1) correspondence proved above, this ring has a unique prime ideal. Thus it suffices to show that for each  $P \in \text{Spec } B$ ,  $(PT)(S^{-1}T)$  is a prime ideal of  $S^{-1}T$ . This would show that  $S^{-1}T/((PT)(S^{-1}T))$  is an integral domain, and therefore the unique prime ideal must be the zero ideal, proving that the formal fiber ring at  $P$  is a field. Now we divide into cases depending on  $\mathfrak{p}$ , where  $\mathfrak{p}$  is the unique prime ideal of  $T$  such that  $\mathfrak{p} \cap B = P$ .

(a) Suppose  $\mathfrak{p} = xT$ .

$P = xT \cap B$  is a nonzero prime ideal as it contains  $xy$ . Since the completion map is flat, the Going Down Theorem holds (see exercise 5.11 in [1]). This implies that  $xT \cap B$  has height at most 1, since  $xT$  has height 1. Because  $xT \cap B$  is nonzero, it follows that  $xT \cap B$  is a height 1 prime, and since  $B$  is a UFD, it follows that  $xT \cap B = bB$  for some prime element  $b \in B$ .

We know that  $b = xt$  for some  $t \in T$  and we also know that  $xy = bb'$  for some  $b' \in B$ , since  $xy \in xT \cap B = bB$ . Then we have  $xy = bb' = xtb'$ , so  $y = tb'$  since  $T$  is an integral domain. Further, since  $y$  is a prime element in  $T$ , it follows that either  $t$  or  $b'$  must be a unit in  $T$ . We claim that  $t$  can

never be a unit. Assume on the contrary that  $t$  is a unit. Then  $xt = b \in B = A[xy]_{\mathfrak{m} \cap A[xy]}$ , so we can write

$$xt = \frac{a_0 + a_1xy + \cdots + a_m(xy)^m}{a'_0 + a'_1xy + \cdots + a'_n(xy)^n},$$

where the denominator is not in  $\mathfrak{m}$ . Clearing the denominator, we get

$$xt(a'_0 + a'_1xy + \cdots + a'_n(xy)^n) = a_0 + a_1xy + \cdots + a_m(xy)^m.$$

Thus,  $a_0 \in xT \cap A = (0)$ , so  $a_0 = 0$ . Now, observe that the RHS is divisible by  $y$ ,  $y$  is a prime element in  $T$ , and  $y \nmid xt$ . So  $a'_0 + a'_1xy + \cdots + a'_n(xy)^n \in yT \subseteq \mathfrak{m}$ . But this is a contradiction since  $a'_0 + a'_1xy + \cdots + a'_n(xy)^n$  was assumed not to be in  $\mathfrak{m}$ . So we have proved that  $t$  cannot be a unit. Thus  $b'$  must be a unit. Now,  $PT = (xT \cap B)T = bT = xy(b')^{-1}T = xyT$ . Note that  $yT \not\subseteq xT$ . So by the construction of  $A$ , it follows that  $yT \cap A \neq (0)$ . So let  $0 \neq yt' \in A$ . If  $yt'$  were in  $P$ , then  $yt' \in PT \cap A \subseteq xT \cap A = (0)$ , yielding a contradiction. So  $yt' \in B - P$ , and thus by definition of saturation,  $y \in S$ . Thus, in  $S^{-1}T$ ,  $y$  is a unit, so  $(PT)(S^{-1}T) = (xyT)(S^{-1}T) = (xT)(S^{-1}T)$ . Since  $xT$  is a prime ideal of  $T$ , this is a prime ideal of  $S^{-1}T$  if we can show that  $xT \cap S = \emptyset$ . But this is clear, since if  $xu \in S$ , then  $xut'' \in B - P$  for some  $t'' \in T$ , so  $xut'' \in xT \cap B = P$ , giving a contradiction. Thus, we have shown that  $(PT)(S^{-1}T)$  is a prime ideal, so we are done with this case.

(b) Suppose  $\mathfrak{p} = (0)$ .

Since  $\mathfrak{p} = (0)$ ,  $P = (0)$ , and so  $PT = (0)$ . So  $(PT)(S^{-1}T) = (0)$  and is therefore prime, since  $S^{-1}T$  is an integral domain.

(c) Suppose  $\mathfrak{p} \not\subseteq xT$ .

Recall that  $P = \mathfrak{p} \cap B$ . Then  $PT = (\mathfrak{p} \cap B)T = \mathfrak{p}$  by Claim 2.2. Therefore  $PT$  is prime in  $T$ . Thus, to show that  $(PT)(S^{-1}T)$  is prime, we only need to show that  $PT \cap S = \emptyset$ . Assume  $t \in PT \cap S$ . Then since  $t \in S$ ,  $tt' \in B - P$  for some  $t' \in T$ , which implies  $tt' \in PT \cap B = P$ . This is a contradiction. Therefore  $(PT)(S^{-1}T)$  is prime.

This concludes the proof of Theorem 2.1.  $\square$

## 3. DEFINITIONS

In this section we define some necessary terms. See [7], [1], or [4] for more explanation.

- **Spec R:** In a ring  $R$ ,  $\text{Spec } R$  denotes the set of all prime ideals in  $R$ .
  - **Quasi-local:** A ring  $R$  is called *quasi-local* when it has exactly one maximal ideal.
  - **Local:** A ring  $R$  is called *local* when it is quasi-local and also Noetherian.
  - **Localization:** Let  $S$  be a multiplicatively closed subset of a commutative ring  $R$ . Then the ring of fractions  $S^{-1}R$  is the ring we obtain by inverting all elements of  $S$ . When  $S = R - P$  for some  $P \in \text{Spec } R$ , we write  $R_P$  for  $S^{-1}R$ . In this case,  $R_P$  is a quasi-local ring, called the *localization* of  $R$  at  $P$ , with maximal ideal  $PR_P$ .
  - **Saturation:** Let  $S$  be a multiplicatively closed subset of  $R$ . Define the *saturation* of  $S$  to be  $\overline{S} = \{r \in R \mid rr' \in S \text{ for some } r' \in R\}$ . Note that  $S^{-1}R = (\overline{S})^{-1}R$ .
  - **Regular Local Ring:** Let  $(R, M)$  be a local ring. Then  $(R, M)$  is called a *regular local ring* if  $M$  can be generated by  $\dim R$  elements.
  - **Regular Ring:** A ring  $R$  is called a *regular ring* when the localization at every prime ideal is a regular local ring.
  - **Complete:** We can define a metric on a local ring  $(R, M)$  as follows. For all  $x, y \in R$ , let  $n$  be the largest integer such that  $x - y \in M^n$ , where  $M^0 = R$ . Then we define the metric  $d(x, y) = \frac{1}{2^n}$  when  $n$  exists and 0 otherwise.
- A local ring  $R$  is called *complete* when it is complete with respect to the above metric, that is, when all its Cauchy sequences converge.
- **Completion:** The completion of a local ring  $A$  is the ring of all Cauchy sequences modded out by the following equivalence relation: two Cauchy sequences are equivalent if their difference converges to zero. The completion of  $A$  is denoted  $\widehat{A}$ ; this is always complete.
  - **Formal Fiber:** Let  $(A, \mathfrak{m})$  be a local ring,  $\widehat{A}$  its completion, and  $P$  a prime ideal in  $A$ . We define the *formal fiber* of  $A$  at  $P$  to be the inverse image of  $P$  under the map  $\text{Spec } \widehat{A} \longrightarrow \text{Spec } A$ . That is, prime ideals in the formal fiber at a prime ideal  $P$  are the prime ideals of  $\widehat{A}$  that lie over  $P$ .

• **Generic Formal Fiber:** The *generic formal fiber* of a local integral domain is the formal fiber of  $(0)$ .

• **Formal Fiber Ring:** We call the ring  $\widehat{A} \otimes_A (A_P/PA_P)$  the *formal fiber ring* of  $A$  at  $P$ . The prime spectrum of the formal fiber ring of  $A$  at  $P$  naturally corresponds to the formal fiber of  $A$  at  $P$ .

• **Flat:** Let  $A$  be a ring and let  $M, N$ , and  $N'$  be  $A$ -modules. We say that  $M$  is *flat* over  $A$  if for all injections  $f : N' \rightarrow N$ ,  $f \otimes 1 : N' \otimes_A M \rightarrow N \otimes_A M$  is also injective.

• **Faithfully Flat:** A ring  $B$  is said to be *faithfully flat* over  $A$  if  $B$  is flat over  $A$  and  $\text{Spec } B \rightarrow \text{Spec } A$  is surjective. For a local ring  $A$ ,  $\widehat{A}$  is faithfully flat over  $A$ .

• **Excellent:** Let  $(A, \mathfrak{m})$  be a local ring. For each  $P \in \text{Spec } A$ , let  $k(P)$  denote the field  $A_P/PA_P$ . Then  $A$  is said to be *excellent* if it satisfies the following two conditions: (a) For each  $P \in \text{Spec } A$ , the formal fiber ring  $\widehat{A} \otimes_A k(P)$  is geometrically regular over  $k(P)$ , that is,

$$\left( \widehat{A} \otimes_A k(P) \right) \otimes_{k(P)} L \cong \widehat{A} \otimes_A L$$

is regular for every finite field extension  $L$  over  $k(P)$ , and (b)  $A$  is universally catenary.

In this paper, it is easier to check excellence because of additional conditions. Specifically, assume that  $(T, \mathfrak{m})$  is a complete local domain containing  $\mathbb{Q}$  and let  $A$  be a local ring such that  $T = \widehat{A}$ . Then by Theorem 31.6 in [4], condition (b) is automatically satisfied for  $A$ . Moreover, since every integer in  $T$  is invertible, no integer belongs to  $\mathfrak{m}$ . For  $P \in \text{Spec } A$ ,  $P \subseteq \mathfrak{m} \cap A \subseteq \mathfrak{m}$ , so it follows that  $k(P) = A_P/PA_P$  must be a field of characteristic zero. So the only purely inseparable extension of this field is itself. Thus, by Remark 1.3 in [6], for (a), we only need to check that each formal fiber ring  $\widehat{A} \otimes_A (A_P/PA_P)$  is a regular ring.



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