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PLANAR DOUBLE BUBBLES ON FLAT WALLS

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Abstract. In this paper, we investigate some properties of planar soap bubbles on a straight wall with a single corner. We show that when a wall has a single corner and a bubble consists of two connected regions, the perimeter minimizing bubble must be one of the types, two concentric circular arcs or a truncated standard double bubble, depending on the angle of the corner and areas of the regions.

1. Introduction

A real soap bubble tends to minimize surface area for the enclosed volume. For many years, people have been interested in determining how to enclose and separate n volumes in $\mathbf{R}^3$ with the least surface area. The sphere is well known to enclose a single volume with minimal perimeter. For $n = 2$, the minimal configuration is known only in the special case that the regions enclosed have equal volumes [H] (see Figure 1a). If the regions have different volumes, no one has yet managed to eliminate troublesome configurations such as the one pictured in Figure 1b.

We consider the simpler, and hopefully more tractable, domain of planar bubbles, where the basic problem is to enclose and separate n (not necessarily connected) areas in the plane with minimal perimeter. Of course, if we wish to enclose a single area, the classical isoperimetric inequality gives that our best bet is to use a circle. More recently, it has been shown that the shortest way to enclose and separate two areas is to use a standard double bubble, consisting of three circular arcs meeting at 120° angles at two points [F]. For $n = 3$, the solution is known [C] only with the additional (unjustified) assumption that the regions enclosed are connected. In the general case, it is conjectured that the n regions and the exterior of a perimeter minimizing bubble are each connected (see [M]), but no proof is currently known.

In this paper, we allow our bubbles to cling to walls, rigid curves that they can use for "free" perimeter. We investigate which configurations of bubbles will have the least perimeter for given areas, where the wall lengths do not contribute to perimeter, concentrating mainly on straight walls with possibly a single corner.

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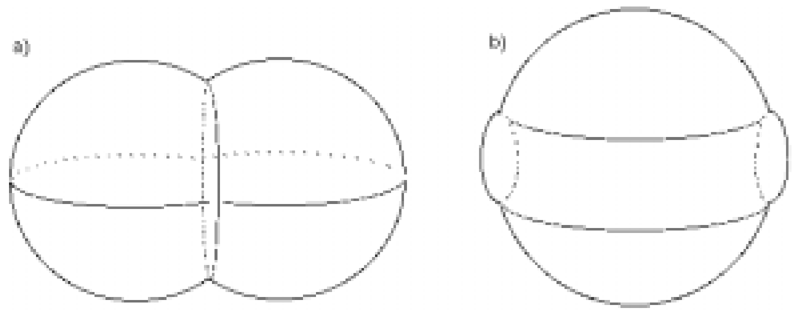

FIGURE 1. (a) The conjectured three-dimensional minimal double bubble, and (b) a possible competitor.

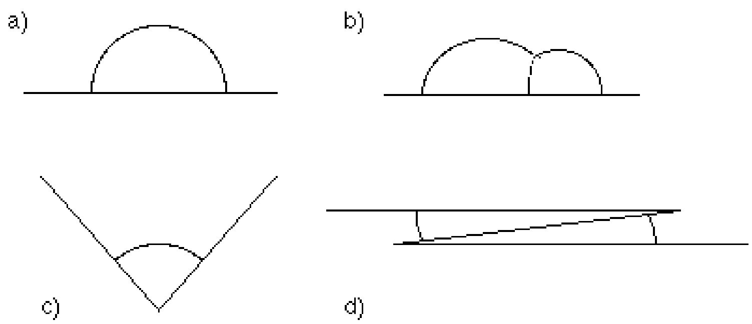

FIGURE 2. Perimeter minimizing bubbles enclosing (a) a single area and (b) two areas with a straight wall, and (c) one area with a single corner. (d) Presumably, the shortest way to enclose a single area on this wall involves disconnected regions.

If the wall is a straight line, then the perimeter minimizing bubbles for one and two areas, shown in Figure 2, are easily determined from the classical isoperimetric inequality and the standard double bubble result of [F] respectively.

The technique used for a straight wall does not easily generalize to the case where the wall has a corner. However, we can still easily determine that the minimal configuration enclosing a single region is a circular arc centered at the corner as in Figure 2c.

At this point, we need to introduce the assumption that all regions are connected. We do not know of a proof that general perimeter minimizing bubbles with walls have connected regions. In fact, for arbitrary polygonal walls with finitely many corners, perimeter minimizing bubbles can have disconnected regions. An example with a single disconnected region is illustrated in Figure 2d.

If the wall has a single corner, then we conjecture that the double bubble enclosing two given areas with minimal perimeter has two connected regions and no empty chambers. We, consequently, look for the double bubble that minimizes perimeter among all double bubbles with these additional (unjustified) properties.

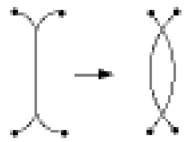

FIGURE 3. Bumping bubbles can revert to circular arcs with less perimeter when they are allowed to overlap.

When we assume connectedness, we introduce the annoying theoretical possibility that bubble edges bump up against each other or against the wall (as in [M], [C]). When edges bump against each other, they may separate regions into multiple components that are connected only by "infinitesimal" strips. This makes sense if we think of such a bubble as a limit of bubbles without bumping.

The following theorem can presumably be proved analogously to [M, 3.3].

Theorem 1.1. *Let W be the image of a piecewise linear embedding of $\mathbf{R}$ into $\mathbf{R}^2$ with finitely many points of nondifferentiability. Given $A_1, A_2, \ldots, A_n > 0$, there is a shortest graph G whose edges may overlap, but not cross, each other or the set W and whose vertices may lie in W such that $G \cup W$ has bounded faces of areas $A_1, A_2, \ldots, A_n$. (Faces are not allowed to overlap, and edges are counted with multiplicity if they overlap.) Furthermore, the edges of G consist of disjoint or coincident circular arcs or line segments*

1. *meeting in threes at angles of $\frac{2\pi}{3}$ at vertices of G not in W,*
2. *meeting in pairs at angles greater than or equal to $\frac{2\pi}{3}$ at vertices of G that lie at corners of W, so that each arc forms an angle of at least $\frac{\pi}{2}$ with W,*
3. *meeting W at right angles at degree one vertices of G,*
4. *meeting at other isolated points where the edges remain C^1.*

A single edge of a perimeter minimizing bubble with connected regions may, in theory at least, change curvature many times as it bumps against and separates from other edges. This complication makes bumping bubbles difficult to analyze.

Many arguments become easier if we allow the regions of our bubbles to overlap, as in [C]. When we allow regions to overlap, bumping edges can revert to circular arcs of less perimeter (see Figure 3). Thus, any bumping bubble has perimeter greater than or equal to the perimeter of some overlapping bubble enclosing the same areas, with a strict inequality if the bubble actually has bumping edges. Additionally, if we find that a perimeter minimizing overlapping bubble does not overlap, then it also minimizes perimeter among bumping bubbles.

To determine the perimeter minimizing bumping bubble enclosing two connected areas using a wall with a single corner, we will show that any perimeter minimizing overlapping bubble enclosing two connected areas using a wall with a single corner does not have any overlapping regions. Our main theorem will then follow from an analysis of this overlapping perimeter minimizer.

Main Theorem *Given $A_1, A_2 > 0$ and a straight wall W with a single corner of angle θ, the shortest way to enclose and separate connected regions with areas A_1 and A_2 in $\mathbf{R}^2 - W$ is either*

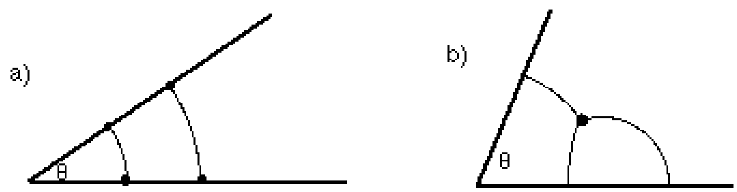

FIGURE 4. The two possibilities for a perimeter minimizing double bubble in a corner: (a) two concentric circles and (b) the truncated standard double bubble.

1. *two concentric circles inside the corner with the smaller area closer to the corner or*
2. *a "truncated standard double bubble" inside the corner, consisting of three circular arcs meeting at a single vertex at angles of $\frac{2\pi}{3}$, and meeting the wall at right angles in three distinct points (see Figure 4).*

Additionally, if

$$\theta \geq \theta_0 = \frac{A_1 \pi}{(\sqrt{A_1} + \sqrt{A_1 + A_2} - \sqrt{A_2})^2},$$

then the truncated standard double bubble has shorter perimeter. This θ_0 has a minimum of $\frac{\pi}{2}$ when $A_1 = A_2$. As $\theta \to 0$, eventually the concentric circles have shorter perimeter.

We first enumerate the possible combinatorial types for double bubbles in a corner and use simple arguments to eliminate all but the two described in the Main Theorem. We then examine all possible regular configuarations of these two double bubbles and eliminate all overlapping cases. So we are left with two possible bubbles for the minimizer. The angle of the corner and the sizes of the areas will determine which bubble has shorter perimeter.

The obvious question one might ask at this point is, "How can one tell which configuration is shorter?" We have only shown that for certain large values of θ, the truncated standard is better, and for sufficiently small θ, the concentric circles are better. We do not know what happens in between these two points. It is not even known whether there is a unique angle at which the two configurations have equal perimeter.

Many other questions remain unanswered. Obviously, a unique "concentric circles" configuration exists for each choice of θ, A_1, and A_2. We have not shown the uniqueness of the "truncated standard double bubble" configuration. In fact, for small values of θ, this configuration does not seem to be geometrically possible. We believe that, if the configuration exists for some choice of θ, A_1, and A_2, then, in fact, a unique such configuration exists. We expect to soon have a numerical proof of this conjecture. The issue of determining when the truncated standard double bubble exists seems difficult.

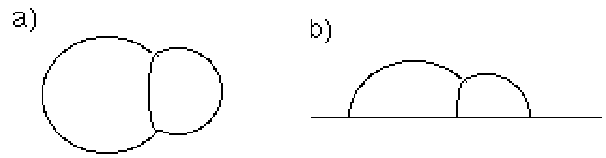

Figure 5. (a) The standard double bubble and (b) the standard split double bubble.

We also have not addressed the issue of what happens to the perimeter of the truncated standard double bubble if the areas A_1 and A_2 are exchanged. We believe that the perimeter will be less if the larger area is nestled in the corner.

If we define $P(\theta, A_1, A_2)$ to be the smallest perimeter of any double bubble enclosing areas A_1 and A_2 in a corner of angle θ, then one might ask whether P is monotonic as a function of each of its parameters. A simple argument involving pressures of regions (as defined in [C]) shows that P is monotone increasing in A_1 and A_2, but we do not know whether P is also monotonic in θ.

2. The perimeter minimizing bubble configurations with a straight wall

In this section we will determine that the perimeter minimizing bubbles enclosing one and two areas with a straight wall are respectively a semi-circle and a "split standard double bubble."

Theorem 2.1. *For any $A > 0$, a semi-circle is the unique minimal set S of smallest one dimensional measure $\mathcal{H}^1(S)$ in $\mathbf{R}^2$, up to translation, such that $S \cup \mathbf{R}$ encloses a (not necessarily connected) region of area A.*

Proof: Suppose $S \cup \mathbf{R}$ encloses a region U of area A. Without loss of generality, we may assume that all components of U lie in the upper half plane. (Any component in the lower half plane can be replaced with its reflection in the upper half plane, maybe sliding the component to the left or right before reflecting.)

Let S' be the set consisting of S and its reflection in the lower half plane. Then S' encloses a region of area $2A$. By the classical isoperimetric inequality, $\mathcal{H}^1(S') \geq 2\sqrt{2\pi A}$, with equality iff S' is a circle (plus possibly an unnecessary additional set of one-dimensional measure 0). Thus S is a semi-circle (and a one-dimensional measure 0 set which can be removed). □

The *standard double bubble* consists of three circular arcs (or two arcs and one line segment) all meeting in two points at angles of $\frac{2\pi}{3}$. The *standard split double bubble* is a standard double bubble bisected by a straight line through the common center points of the three arcs, as in Figure 5.

Theorem 2.2 (F, 2.9)**.** *For any two prescribed quantities of area A_1, $A_2 > 0$, there is a set $S \subset \mathbf{R^2}$ of least 1-dimensional Hausdorff measure $\mathcal{H}^1(S)$ such that $\mathbf{R}^2 - S$ is a disjoint union of (not necessarily connected) components R_0, R_1, and R_2, with only R_0 unbounded, and* $\text{area}(R_i) = A_i$ $(i = 1, 2)$*. S consists of a unique standard double bubble (plus possibly an additional unnecessary set of $\mathcal{H}^1$ measure 0).*

Corollary 2.3. *For any two prescribed quantities of area and a straight line, the split standard double bubble is the unique minimal perimeter-minimizing set (up to congruence) enclosing the prescribed quantities of area.*

Proof: The proof is analogous to that of Theorem 2.1. □

3. Overlapping bubbles

If φ is a piecewise smooth 1-cycle in $\mathbf{R}^2$, we define the area enclosed by φ, which we denote A(φ), by

$$A(\varphi) = \int_{\varphi} \frac{1}{2}(x\,dy - y\,dx)$$

By Green's Theorem, this definition agrees with the usual definition of area for simple closed curves, up to orientation.

A *wall* W is the image of a piecewise linear embedding $f\colon \mathbf{R} \to \mathbf{R}^2$, with finitely many points of nondifferentiability, which we call *corners*. The *inside* of a corner is defined as the sector with an angle less than π. The *outside* will be the other sector. A *wall segment* is a maximal connected subset of the wall containing no corners. We say that a map h from W to W is *nondecreasing* if the corresponding map $(f^{-1} \circ h \circ f)\colon \mathbf{R} \to \mathbf{R}$ is nondecreasing, where $f\colon \mathbf{R} \to W$ is any homeomorphism.

An *embedded graph with wall* (G, W) consists of a wall W and a finite graph G embedded in $\mathbf{R}^2$ so that G intersects W only at vertices of G. We now define an *overlapping bubble with wall* $B = (G, W, g)$ to be an embedded graph with wall (G, W) together with a piecewise C^1 map $g\colon G \cup W \to \mathbf{R}^2$ mapping W onto itself, nondecreasing. We visualize g as deforming G, preserving intersections of vertices with W, but possibly sliding these intersections around within the set W. The nondecreasing requirement means that when vertices slide around in W, their relative order "left to right" cannot change.

The *combinatorial type* of a bubble with wall (G, W, g) is the collection of all embedded graphs with wall (G', W) homotopic to (G, W) through embeddings that map W onto itself, nondecreasing. Since the type of (G, W, g) does not depend on the map g, we frequently refer to the combinatorial type of (G, W). (If one of the edges of (G, W) is mapped by g to a single point, we say that (G, W, g) is *degenerate* of type (G, W).)

Number the bounded faces of (G, W) with $1, \ldots, n$. Let φ_i be a cycle that consists of one copy of each of the curves in the boundary of the i^{th} face, such that each curve is "positively oriented" with respect to the face. Then the area of the i^{th} region enclosed by (G, W, g) is defined to be $A(g_*(\varphi_i))$. The length of (G, W, g), denoted $\ell((G, W, g))$, is defined to be the sum over all the edges γ in G (and thus not on the wall W) of the lengths of the arcs $g \circ \gamma$.

4. Existence and regularity of length minimizing overlapping bubbles with wall

In this section, we generalize the existence and regularity results proved in [C] for length minimizing overlapping bubbles without walls. We will need to use the following two lemmas from [C], presented here without proof.

Lemma 4.1. *Given an oriented line segment $\overrightarrow{PQ}$ and a real number r, the unique shortest curve α from Q to P such that $A(\overrightarrow{PQ} + \alpha) = r$ is an arc of a circle or a line segment.*

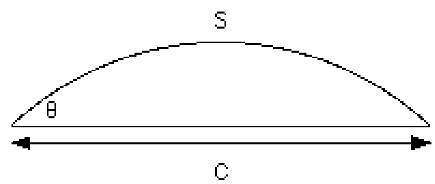

FIGURE 6. The circular arc S has curvature κ, area A, and length ℓ.

Lemma 4.2. *Suppose α is an edge of an overlapping bubble that goes from vertices P to Q. If we replace α by another curve α' such that $A(\alpha+\overrightarrow{QP}) = A(\alpha'+\overrightarrow{QP})$, then the areas of the regions enclosed by the overlapping bubble will remain unchanged.*

The following lemma allows us to bound the amount that the perimeter of a bubble must increase when we transfer areas between regions.

Lemma 4.3. *Let $B = (G, W, g)$ be an overlapping bubble with wall whose edges are all line segments or arcs of circles, and let $x \notin W$ be a point on some edge E separating regions R_i and R_j of B. We allow for the possibility that one of the regions is an exterior component. Then for any $\epsilon > 0$, there are positive constants δ and β so that, whenever $|\Delta A| \leq \delta$, the edge E may be deformed within a ball of radius ϵ about x so as to transfer an area ΔA from R_i to R_j and increase the perimeter of B by at most $\beta|\Delta A|$.*

PROOF: For any circular arc S, define C to be the distance between the endpoints, and let θ be the angle between the arc and the segment joining the endpoints (see Figure 6). Then the curvature κ of S, the area A of the region between S and the segment connecting the endpoints, and the length ℓ of S are given by the following formulas from [F]:

$$\begin{aligned} \kappa(\theta, C) &= \frac{2\sin\theta}{C}, \\ A(\theta, C) &= \frac{C^2(\theta - \sin\theta\cos\theta)}{4\sin^2\theta}, \text{ and} \\ \ell(\theta, C) &= \frac{C\theta}{\sin\theta}. \end{aligned}$$

A simple calculation shows that, if we hold C constant, and vary θ, then $\kappa = d\ell/dA$. Thus we can transfer area between two adjacent regions of a bubble by varying the curvature of a small arc about the point x, changing the perimeter of the bubble with a rate equal to the oriented curvature of the edge E. □

We can now show that a length-minimizing overlapping bubble with wall exists for any given combinatorial type. Furthermore, by variational arguments, we can determine many local properties of these length minimizing bubbles.

Proposition 4.4 (Weak Regularity)**.** *Let (G, W) be an embedded graph with wall whose bounded faces are numbered 1, 2, ... , n, and let A_1, A_2, ... , A_n be given real numbers. Then there exists an overlapping bubble with wall of type (G, W) (which may be degenerate) such that the i^{th} region has area A_i and the perimeter is minimal for the type (G, W).*

Furthermore, this minimal bubble has the following properties:

1. *All edges are line segments or arcs of circles.*
2. (a) *At any vertex not on W, the sum of the unit tangent vectors of incident edges is zero.*
   (b) *At any vertex on the straight sides of W, the sum of the unit tangent vectors of edges incident to the wall is perpendicular to the wall.*
   (c) *If a vertex lies at a corner of W, then the sum of the unit tangent vectors of the incident edges must form an angle greater than or equal to $\frac{\pi}{2}$ with each incident edge of the wall.*
3. *At any vertex not on W, the sum of the oriented curvatures of incident edges is zero.*

Proof: To show the existence of a minimizer, it suffices, by Lemmas 4.1 and 4.2, to consider only bubbles whose edges are all line segments or arcs of circles. Since the set of all such bubbles with the correct combinatorial type and enclosed areas can be parameterized by a finite number of variables, (1) follows from a standard compactness argument as in [C, 7.4]. Here, we use the fact that the wall has only a finite number of corners. Outside of some bounded set, the wall consists of just two straight rays. So any bubble far away from the corners can be slid back near the corners without changing its perimeter or enclosed areas. Thus, we see that all small perimeters are achieved in some compact collection of bubbles.

To show (2), let V be a vertex of an overlapping bubble with wall, $B = (G, W, g)$, that has minimal perimeter for its combinatorial type. Inspired by [M], we claim that the outward unit tangent vectors of all the edges of G or segments of W incident to V form a minimal network connecting the vectors' heads among all such networks of its combinatorial type that keep the set W within itself. In other words, if we deform the network without changing the shape of any nearby wall, but possibly moving the vertex V along the wall, then the total length of the network, not counting the wall, cannot decrease.

To see why this is so, suppose there were a shorter such network. Then we could deform the bubble B within a ball of some small radius r, decreasing the total perimeter by at least αr for some positive α. The areas of the incident regions change by at most πr^2. If r is small enough, we can restore the areas of these regions with a finite number of edge deformations of the type described in Lemma 4.3, increasing the total perimeter by at most βr^2 for some positive β. If r is sufficiently small, then $\alpha r - \beta r^2 > 0$, so B could not have been minimal. Thus, we must have a minimal network.

Now in the case (a), we assume that V is not on W, so we have the freedom to move each of the unit vectors $v_1, v_2, \ldots, v_k$ in our network. Consider variations moving the central vertex in the direction of some unit vector u a distance t. The total length of the network must have a local minimum when $t = 0$. *i.e.,*

$$0 = \left.\frac{dl}{dt}\right|_{t=0} = -\sum_{i=1}^{k} v_i \cdot u.$$

Since this equality holds for any unit vector u, we must have $\sum_i v_i = 0$.

The case (b) is similar to (a), except that we only have the freedom to move the vertex V in a direction u tangent to W. In this case, we can only conclude that $(\sum_i v_i) \cdot u = 0$. In other words, the sum of the unit tangent vectors at V must be normal to W.

In case (c), the vertex V is at a corner of W. So our variations can only move V a non-negative distance t in the directions u_1 and u_2 of the two incident wall segments at this corner. Thus, we find that

$$\left(\sum_i v_i\right) \cdot u_j \leq 0 \quad \text{for } j = 1, 2,$$

where the v_i's range over only bubble edges and not wall edges. This completes the proof of (2).

Part (3) can be proved exactly as in [C, 7.4]. □

Note that if a vertex lies on the wall, we have drawn no conclusion about the curvatures of the incident edges. In fact, the perimeter minimizing bubble that encloses a single area, given a straight wall, consists of a single semi-circular edge with endpoints on the wall. By varying the area of the enclosed region, the edge may achieve any desired curvature. Thus, we cannot determine the curvature of an edge incident to the wall without knowing the specific areas enclosed by the bubble.

We can now prove the existence of an overlapping double bubble with wall that minimizes perimeter for any given areas. This theorem is a stronger version of Proposition 4.4 without the restriction on combinatorial types.

Theorem 4.5 (Strong Regularity)**.** *Given a wall W and prescribed areas A_1, A_2, ... , A_n, there exists an overlapping bubble with wall, $B = (G, W, f)$, that encloses A_1, A_2, ... , A_n with minimal perimeter. Additionally, B has the following properties:*

1. *Edges of the bubble must be arcs of circles or line segments.*
2. *Vertices not on the wall must be of degree three with an angle of $\frac{2\pi}{3}$ between each incident edge.*
3. *Single edges must meet the wall at an angle of $\frac{\pi}{2}$.*
4. *The only point on W where two bubble edges can meet is on the outside of the corner, and there is no place on W where three or more bubble edges can meet. Also, the incident edges of a bubble, whose vertex is on the outside of a corner, must obey the additional two properties:*
   (a) *There must be an angle of at least $\frac{2\pi}{3}$ between the two incident bubble edges.*
   (b) *The angle between each incident bubble edge and the segment W that it is closest to must be at least $\frac{\pi}{2}$.*

Proof: By applying Proposition 4.4 to the finitely many combinatorial possibilities, we will come up with an overlapping bubble with wall, B, that will minimize perimeter. We now need to show that B obeys the above four laws. The proof of (1) comes directly from Proposition 4.4.

The proof of (2) is as follows: since this vertex does not rest on W by the proof of Theorem 2.3 in [M] we know that the angle between any two unit tangent vectors is at least $\frac{2\pi}{3}$. Therefore, there can be no more than three edges meeting at the same vertex.

The proof of (3) follows immediately from Proposition 5.4.2.b The proof of (4) is as follows: recall that from a similar argument as (2), the degree of a vertex is no more than three, where the *degree* of a vertex is the number of bubble edges, not wall segments, meeting at this vertex. It now suffices to show that, when this vertex is on W, the degree must be at most one, and at most two if the vertex is at a corner.

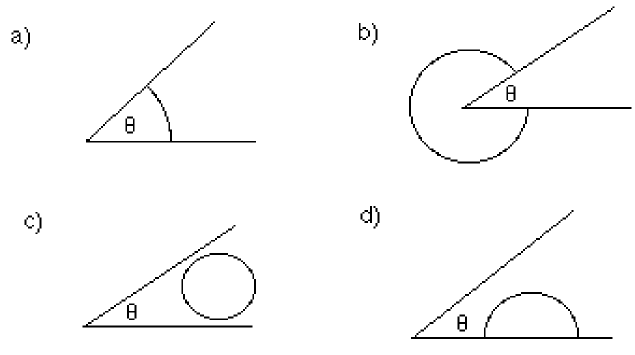

FIGURE 7. Types for enclosing one area with a wall.

Consider the case where two edges intersect the wall at the same point. We know, by an argument similar to Proposition 4.4.2.b, where we allow changes in combinatorial type, that if perimeter is minimal then the angle between the bubble edge and the wall segment that is closest to, is at least $\frac{\pi}{2}$. From a similar argument as (2), there must be an angle of at least $\frac{2\pi}{3}$ between each incident bubble edge. Now, we can conclude that the only place on W where both of these rules can be obeyed by two edges is on the outside of a corner, completing the proof of this theorem. □

5. The perimeter minimizing bubble enclosing one area within a given angle.

We wish to find the perimeter minimizing configuration enclosing one connected area A within a given angle θ where $0 < \theta < \pi$. We assume regularity as proved in Proposition 4.4. The four possibilities for bubble configurations are shown in Figure 7.

Theorem 5.1. *Given a wall with a corner of angle θ and one area A, the perimeter minimizing overlapping bubble enclosing A bounds it on the inside of the corner with a circular arc centered at the corner. For a given area A and angle θ, the perimeter is $\sqrt{2A\theta}$.*

PROOF: Straightforward calculations show that the configuration shown in Figure 7a has least perimeter given an area A and angle θ. □

In fact, similar computations show that the circular arc inside the corner has minimal perimeter even among planar bubbles with disconnected regions.

6. Combinatorial types of double bubbles

6.1. **Allowable Types.** Suppose, throughout this section, that W is a wall with a single corner. A combinatorial type represented by an embedded graph with wall (G, W) is *allowable* iff it has the following properties:

1. $G \cup W$ is connected.
2. Every edge of G is on the boundary of two distinct regions.
3. Every vertex of G not on W has degree 3.

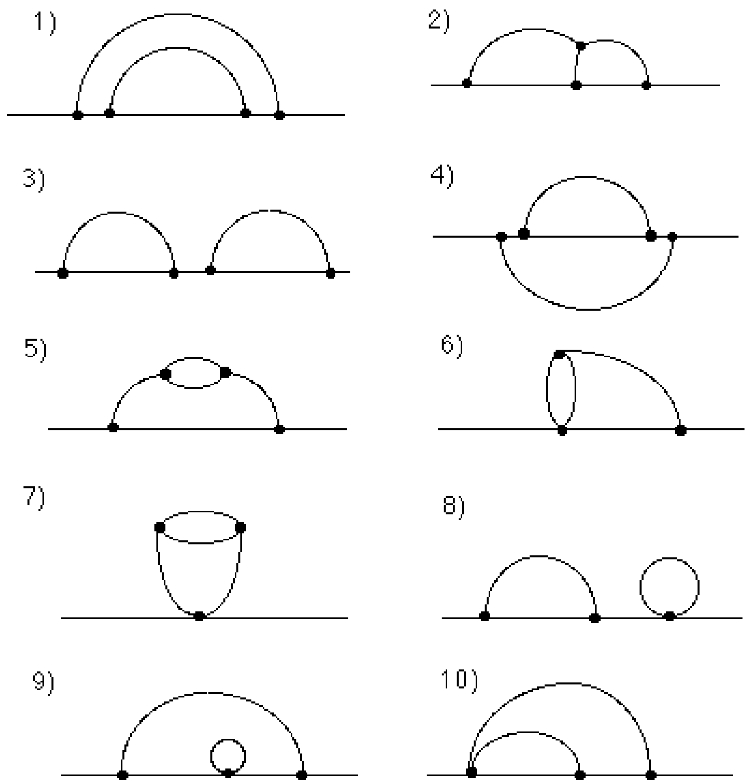

FIGURE 8. Combinatorial types.

4. Every vertex of G on W has degree 1, except for possibly a corner that can be of degree 2.

Lemma 6.1. *If an overlapping bubble with wall $B = (G, W, g)$ has minimal perimeter for its enclosed areas, then (G, W) is an allowable type.*

PROOF: If $G \cup W$ is not connected, then we can slide two components of B together until two bubble edges are tangent. If we form a new vertex at the point of tangency, we will have a new bubble B' enclosing the same areas as B with the same perimeter. But B' does not satisfy the strong regularity requirements of Theorem 4.5. Thus B was not perimeter minimizing.

If some edge of G is only on the boundary of a single region, then that edge can be deleted without changing the area of any region of B.

The requirements on the degrees of vertices follow directly from Theorem 4.5. □

6.2. **Eliminating Combinatorial Types.** Without lost of generality, we can consider combinatorial types of bubbles on a straight wall and see how they can be mapped on to a wall that forms a corner.

Proposition 6.2. *Figure 8 shows all allowable combinatorial types with two connected regions.*

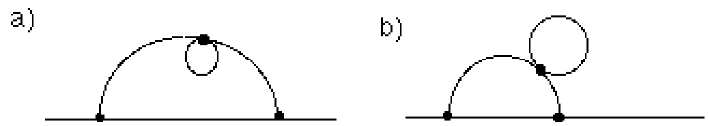

Figure 9. Cases 8 and 9.

Proof: Let R_1 and R_2 be the regions enclosing our two given areas. This argument will involve Euler's formula: $v - e + f = 1$, where v is the number of vertices in (G), e is the number of edges in (G,W), and f is the number of bounded faces in (G,W). We know that $f = 2$, so we now wish to find out what v and e are. If the regions have no edges in common, they are of type 3 or 4 in Figure 8. The two regions can share at most one edge, for if there were more than one edge that the two regions had in common, it would imply the existence of an empty chamber, which contradicts the connectness of the regions. Region R_1 cannot have four or more edges not adjacent to R_2 because the bubble would have an empty chamber. If R_1 has more than two edges that are not adjacent to R_2, the configuration must be of type 5 in Figure 8. We have an analagous argument for the number of possible edges for R_2. So now, we have at most 5 edges, and by Euler's formula, there can be no more than 4 vertices on the bubble.

Now, if one vertex were on the wall, then type 7 would be the only allowable type. If two vertices were on the wall, we would have either type 5, type 6, or have a bubble that would break regularity (two circles tangent to the wall). Three vertices on the wall result in the following types only: type 2, type 8, type 9, and type 10. Types 1, 3, 4, and bubbles that are made by sliding the vertices of type 4 around, allowing them to cross, are the only bubbles that can have four vertices on the wall. □

Proposition 6.3. *Only types 1 and 2 in Figure 8 can possibly be perimeter minimizing types for two given regions.*

Proof: In this proof, we will go through an argument for types 3 through 10 in Figure 8 to eliminate them as possible minimizers for two regions. We can eliminate types 8 and 9, because we can slide the circular area along the edge of the other area as shown in Figure 9, keeping the areas and perimeter constant, but contradicting regularity.

We can eliminate type 3, since in this case the two regions do not share any edges and the standard split double bubble of type 3 occupies less perimeter with the same areas.

For types 4, we have two possibilities: when both regions occupy a corner and when at least one of them does not. It is sufficient to consider the angle of the corner not equal to π, otherwise, reflecting one of the regions to the other side, we would get type 3.

In the first case, since edges are arcs of circles and must meet wall segments at an angle of $\pi/2$, the corner must be the center of those arcs. Results from Section 5 show that the perimeter of this bubble is bigger than the perimeter of the bubble that results when the region that occupies the outside of the corner is moved to the straight wall segment. Therefore, it is sufficient to consider only the second case.

In the second case, we can reflect the area that is on the straight wall segment to the other side of the wall, transforming to type 3 in Figure 8.

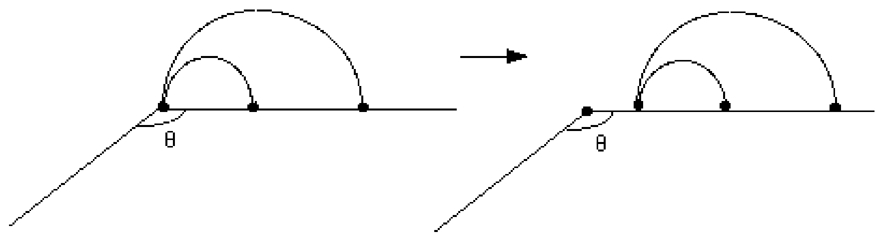

Figure 10. Sliding.

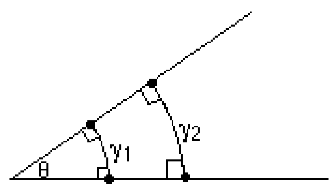

Figure 11. The perimeter minimizer of type 1.

For type 10, in order for the left vertex to be regular, it must be mapped on the outside of the corner of the wall. This configuration has all of its edges intersecting the same segment of the wall. Thus, if we slide it away from the corner, along the wall, as in Figure 10, the resulting bubble would contradict regularity, since we would have two edges meeting together on the same vertex on the wall. Similar arguments work for types 6 and 7.

For type 5, we use [C, 8.1], which states that the arcs touching each wall are arcs of the same circle. If we slide the region that does not touch either wall segment along one of the arcs, we will preserve areas and perimeter. Hence, we can slide it until it hits the wall, contradicting regularity.

□

7. The perimeter minimizing type 1 double bubble

Now that we have eliminated all but two combinatorial types for the double bubble in a single corner, we need to examine these two types more carefully. We know that each type has a perimeter minimizer satisfying the weak regularity conditions of Proposition 4.4. If possible, we would like to determine what this perimeter minimizer looks like and whether it also satisfies the stronger regularity properties of Theorem 4.5.

For type 1, we can easily determine that the unique perimeter minimizer is the bubble pictured in Figure 11.

Theorem 7.1. *Let (G, W) be the embedded graph with wall pictured in Figure 11, where W contains a single corner of angle θ ($0 < \theta < \pi$). Let A_1, $A_2 > 0$ be given real numbers. Then the overlapping bubble with wall of type (G, W) enclosing areas A_1 and A_2 with minimal perimeter consists of two circular arcs centered at* *the corner. Additionally, these arcs lie on the inside of the corner with the larger area further from the center.*

Proof: Let γ_1 and γ_2 be the inner and outer edges respectively of (G, W), as shown in Figure 11. Notice that, if $B = (G, W, g)$ is any bubble of type (G, W), then

$$\ell(B) = \ell(g \circ \gamma_1) + \ell(g \circ \gamma_2).$$

By Theorem 5.1, the length of $g \circ \gamma_1$ cannot be less than $\sqrt{2\theta(\min\{A_1, A_2\})}$, the perimeter of the shortest way of enclosing either A_1 or A_2 against the wall W. Similarly,

$$\ell(g \circ \gamma_2) \geq \sqrt{2\theta(A_1 + A_2)}.$$

So

$$\ell(B) \geq \sqrt{2\theta}\left(\sqrt{\min\{A_1, A_2\}} + \sqrt{A_1 + A_2}\right).$$

The result follows because the bubble described in the statement of the theorem achieves this minimal perimeter. □

The technique used in the above proof is essentially that of [C, 3.3]. Also note that, for fixed A_1 and A_2, if B is the perimeter minimizing bubble of type 1, then $\ell(B) \to 0$ as $\theta \to 0$.

We now compare the perimeter minimizing bubble consisting of two concentric circles with the bubble consisting of two disjoint areas on the wall.

When the angle $\theta = \pi$, then obviously the bubble in figure b) has smaller perimeter. On the other hand, we know that perimeter in figure a) is going to 0 as $\theta \to 0$. So, there must exist an angle θ_0 when they are equal.

Note that if we reflect area A in figure *b*) over straight wall segment and slide it to the area B until it hits it in one point, the new bubble would have the same perimeter, but contradict regularity. Thus, the bubble of the type 2 in Figure 8 always has a smaller perimeter than the bubble of the type b above. Since, for any angle bigger than θ_0, the perimeter minimizing bubble of the type a has perimeter bigger than that of type b, it also has more perimeter than the perimiter minimizer of type 2.

Explicit calculations show that when $A \leq B$

$$\theta_0 = \frac{A\pi}{\left(\sqrt{A} + \sqrt{A+B} - \sqrt{B}\right)^2}.$$

If we rewrite θ_0 as

$$\theta_0 = \left(\frac{\sqrt{\pi}}{1 + \frac{\sqrt{A}}{\sqrt{A+B}+\sqrt{B}}}\right)^2,$$

we note that $\theta_0 \to \pi$ as $A \to 0$ or $B \to \infty$, and that

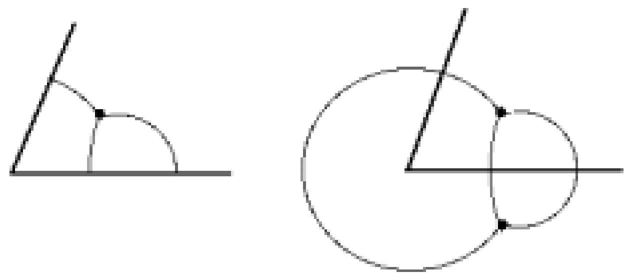

FIGURE 12. The edges of a nondegenerate perimeter minimizing bubble of type 2 can be extended (or shortened) to form a standard double bubble.

$$\theta_0 = \left(\frac{\sqrt{\pi}}{1 + \frac{1}{\sqrt{1+\frac{B}{A}} + \sqrt{\frac{B}{A}}}} \right)^2,$$

since $A \le B$, θ_0 has a minimum, equal to $\pi/2$ when $A = B$.

Proposition 7.2. *If the perimeter minimizing bubbles of types 1 and 2 in Figure 8 are equal for given areas A and B and angle α_0, then α_0 must obey the inequality $0 < \alpha_0 < \theta_0$, where*

$$\theta_0 = \frac{A\pi}{\left(\sqrt{A} + \sqrt{A+B} - \sqrt{B}\right)^2}.$$

When $A = B$, we have $0 < \alpha_0 < \pi/2$.

Corollary 7.3. *For $\theta > \theta_0$, the double bubble of type 1 is not minimizing.*

8. BUBBLES OF TYPE 2

We must analyze the geometric possibilities for a perimeter minimizing overlapping bubble of type 2. We need to examine all possible ways that a bubble of type 2 can satisfy the Weak Regularity properties of Proposition 4.4. Of these configurations, we will eliminate those that could not possibly be perimeter minimizing enclosures for two positive areas.

We begin with the following lemma, that tells us that the three edges of a nondegenerate bubble of type 2 can be extended to form three circles (or two circles and a line) that intersect in two common points. We will then just need to enumerate all possible ways that the wall can cut across this diagram to give an overlapping double bubble.

Lemma 8.1. *Suppose V is a vertex of degree three in some overlapping bubble that minimizes perimeter for its combinatorial type. Then we can extend the incident edges α_1, α_2, and α_3 to a second common point of intersection V'. Thus these three arcs are parts of three circles (or two circles and one line) with two common points of intersection.*

PROOF: This is essentially the same as the proof of [C], Lemma 8.1. □

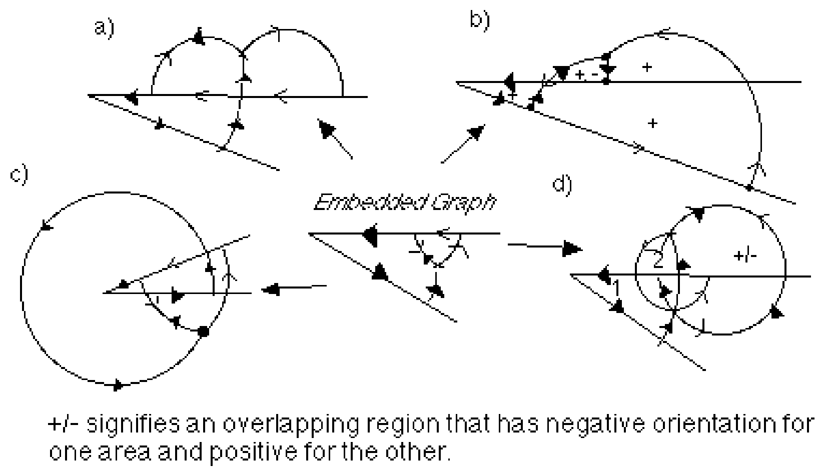

FIGURE 13. Orienting pathological cases a-d.

Corollary 8.2. *If a nondegenerate overlapping bubble B minimizes perimeter for type 3, then the edges of B can be extended or shortened to form a standard double bubble as in Figure 12.*

8.1. **Type 2 bubbles have less perimeter inside a corner than on a straight wall.**

Theorem 8.3. *Given a wall W with one corner and two areas, the best type 2 bubble enclosing these areas and minimizing perimeter is a bubble where one of the areas is bounded by a region that includes the corner.*

PROOF: Let B be a type 2 bubble on a straight part of W, where W has one corner. Slide B toward the corner until the vertex closest to the corner hits the corner, making sure that the bubble edge coming out of that vertex is on the inside of the corner, reflecting if necessary. We now have a contradiction of our regularity laws. We know that by altering the bubble edge within a circle of small radius, so that the edge comes in perpendicular to the other wall segment, perimeter will be decreased. □

Our goal now is to eliminate all overlapping type 2 bubbles and to show that if the perimeter minimizer is of type 2, then it is a non-overlapping bubble on the inside of the corner. In figures 13 and 15, we have examples of bubbles for the six methods (explained in the following subsections) used to eliminate the finite number of pathological, or overlapping, bubble configurations of type 2 (see Appendix). Each of them has been oriented according to the embedded graph of type 2 on the inside or the outside of the corner (Figures 13, 15).

8.2. **Pathological case a.** For case a) we use an orientation argument to note that the edges cannot be consistently oriented so as to give both areas positive. Since we are not concerned with negative areas, we can eliminate these bubble configurations.

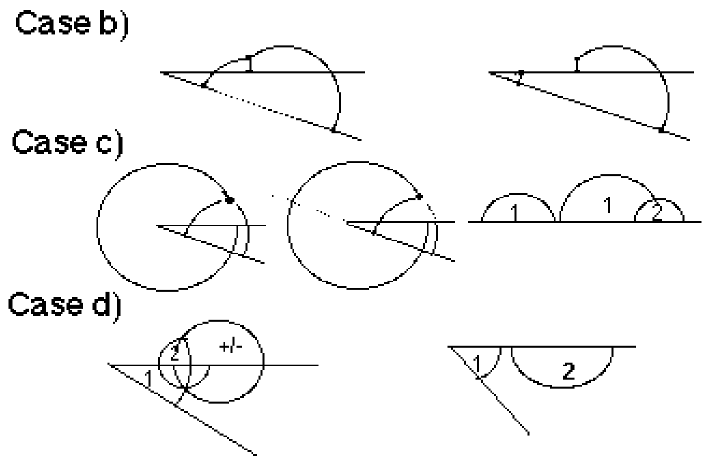

FIGURE 14. Eliminating pathological cases b-d.

8.3. **Pathological case b.** For the configuration in Figure 13b, we notice that one of the areas consist of two components one with positive area and the other containing negative area. The positive area has to be bigger than the negative one in order for the total region to be positive. Thus, if we subtract the negative area from the positive, we can fit alll the area inside the corner (see Figure 14). Therefore, the area of the regions is preserved, but we have less perimeter. It follows that the case b bubble is not minimizing.

8.4. **Pathological case c.** To eliminate the configuration in Figure 13c, we note that by extending the wall segment that two bubble edges form vertices with, you form a split standard double bubble on that wall together with a sector of angle $\pi + \theta$. If we erase the other wall segment and form a semi-circle on the new wall containing the same area as the sector, we decrease perimeter. By Corollary 2.3, we can form a better bubble on a straight wall by a split standard double bubble containing the original two areas, necessarily decreasing perimeter, proving that the case c bubble is not perimeter minimizing.

8.5. **Pathological case d.** This case has region 1 divided into a positive area at the corner and a negative area partially overlapping the other region, region 2, which is completely positive. Note that the area at the corner is a sector of a circle centered at the corner. Consolidate region 1 by decreasing the radius of the sector, moving the arc closer to the center and decreasing perimeter. Clearly region 2 is divided up into a split standard double bubble on the outside of the corner and two semi-circles on the inside. Put all of region 2 inside one semi-circle on the inside of the corner, necessarily decreasing perimeter and slide it until it hits region 1, breaking regularity.

8.6. **Pathological case e.** For the configuration in figure 16i), note that the orientation gives one positive component and one negative component for region 2, where the positive component is on segment 1. Subtract the negative area from the positive area and create a split standard double bubble from the components of the area on the outside of segment 1, decreasing perimeter. The component of region

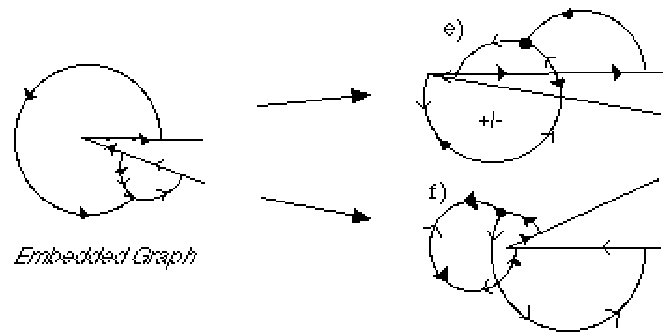

FIGURE 15. Orienting pathological cases e, f.

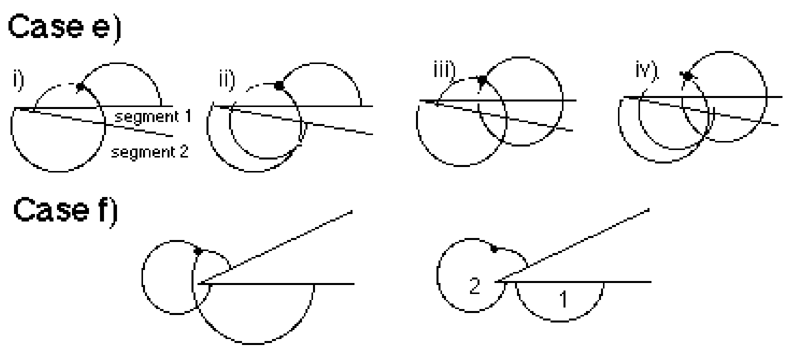

FIGURE 16. Eliminating pathological cases e, f.

1 on the outside of segment 2 can be made into a semi-circle, decreasing perimeter again. Now, pull the component of region 1 that is inside the corner so that it is bounded by the corner and a circular arc, decreasing perimeter. Reflecting the semicircle about segment 2 and sliding it to the circular arc, we break regularity, and a better bubble is formed by a larger circular arc bounded by the corner. Reflect the split bubble about segment 1 so that the two components of the region meet again and perimeter is decreased by connecting the regions, therefore, figure 16i) is not perimeter minimizing.

For the configuration in figure 16 ii), we simply have another component of region 1 bounded by a semi-circle on the outside of segment 2. The proof follows similarly as above.

Figure 16 iii), has a positive component of region 2 bounded by a semi-circle on the inside of the corner. Simply reflect this component about segment 1 and slide it to the other component of region 2. Connecting the regions we decrease perimeter and we note that the figure 16iii) bubble is not perimeter minimizing. The figure 16iv) can be eliminated in the same manner.

8.7. **Pathological case f.** Note that one area has a positive component and a negative component (see Figure 15). If we extend the wall segment that two bubble edges are incident upon, a semi-circle of positive area is formed on the new wall. Absorb the negative component into the positive one and decrease the radius of

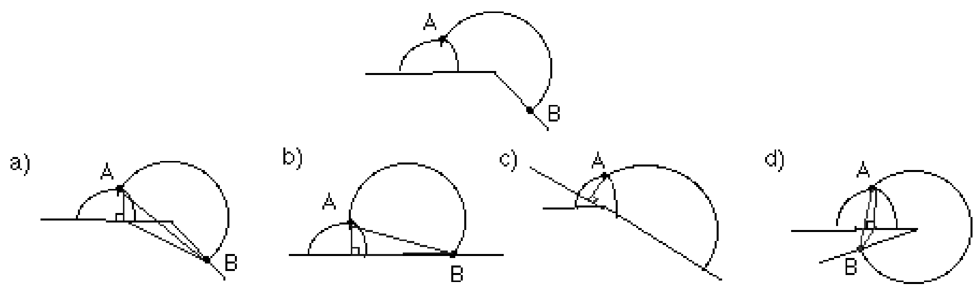

FIGURE 17. Fixing bubbles on the outside.

the semicircle. Since we have two separate enclosed areas with less perimeter, case f could not be the minimizer.

Eliminating all overlapping bubbles, we are left with two non- overlapping type 2 bubble cases; on the inside or the outside of the corner. The following theorem can now be proven for the two remaining cases.

Theorem 8.4. *The Split Standard Double Bubble has less perimeter than the truncated double bubble on the outside of the corner.*

PROOF: The strategy in the method that follows is to bound, with a split standard double bubble of perimeter less than or equal to the original bubble, two areas, one that has the same area as the original one and the other with area greater than the original. Then, since we know that decresing either area enclosed by a standard double bubble decreases perimeter [F], we can decrease the area in the split standard double bubble until the original areas are bounded with necessarily less perimeter, proving that the bubble on the outside of the corner is not perimeter minimizing.

Extend the wall segment touching two bubble edges, segment 1, and make it the new wall, drawing a line orthogonal from the new wall to point A, the vertex off the wall. Point A must lie on the same side of the new wall as the two bubble edges that are incident to segment 1, for, if not, the new wall will not intersect those two bubble edges perpendicularly, but this is impossible because that line contains the centers of all three arcs (see Figure 18. Now, draw line segments from the ends of that line segment to the end of the arc not on the new wall, creating a triangle (see Figure 17a,c,d).

Note that line segment AB in the figures are chords of circles. Redraw the chords and their arcs so that they touch the new wall as shown in 17b). Recall that keeping two sides of a triangle constant, the largest area is obtained when there is an angle of $\frac{\pi}{2}$ between them and that the area decreases as that angle is increased or decreased. So, the area enclosed has increased but the perimeter has not; then form the split standard double bubble for the new areas and decrease the area until the original given areas are enclosed. Note that perimeter has not increased. □

From the above, we can conclude the following theorem:

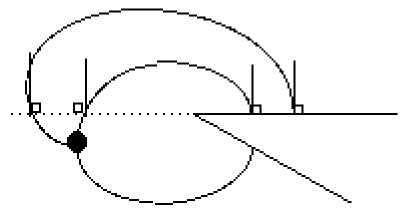

FIGURE 18. The vertex cannot be below the new wall.

Theorem 8.5. *If the perimeter minimizing overlappping bubble with areas A_1 and A_2 on a wall with one corner of angle θ is of type 2, then the bubble is on the inside of the corner with one area bounded by the corner.*

We have now established our main theorem.

Appendices

Appendix A. Calculations for the Truncated Standard Double Bubble

In this section we find the explicit formula for a standard double bubble on a corner with angle θ given two areas, A_1 and A_2.

Following Figure 19, number the curves, beginning with the one farthest from the corner, and continue counter-clockwise. Label the radius of the i^{th} curve and angle subtended by r_i and θ_i. Note that all the radii and angles can be found if one radius and one angle is known. Since we have two equations for areas and all variables can be expressed in terms of two, it appears simple to solve for both variables. As can be seen, however, only an approximate numerical answer can be given to solve for the angle θ_1 explicitly. Once the angle is found, it is only a trivial matter to find the perimeter of the bubble.

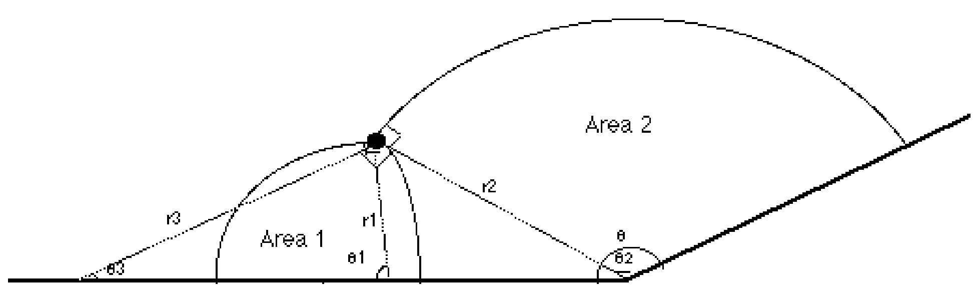

Figure 19. A truncated standard double bubble.

A.1. **Relations.**

$$\theta_3 = \frac{2\pi}{3} - \theta_1 \qquad \theta_2 = \theta + \frac{\pi}{3} - \theta_1$$

$$\frac{r_1}{\sin\left[\theta_1 + \frac{\pi}{3}\right]} = \frac{r_3}{\sin[\theta_1]} \qquad \frac{r_1}{\sin\left[\theta_1 - \frac{\pi}{3}\right]} = \frac{r_2}{\sin[\theta_1]}$$

$$T_{ri} = \frac{1}{2}{r_i}^2 \sin[\theta_i]\cos[\theta_i]$$

$$T_{ri} + T_{rj} = \frac{1}{2} r_i r_j \sin\left[\frac{\pi}{3}\right] = \frac{\sqrt{3}}{4} r_i r_j$$

A.2. **Area 1.**

$$A_1 = \frac{1}{2}{r_3}^2\theta_3 - T_{r3} + \frac{1}{2}{r_1}^2\theta_1 - T_{r1}$$

$$= \frac{1}{2}\left(\frac{r_1 \sin[\theta_1]}{\sin\left[\frac{\pi}{3} + \theta_1\right]}\right)^2 \left(\frac{2\pi}{3} - \theta_1\right) + \frac{1}{2}{r_1}^2\theta_1 - \quad \frac{\sqrt{3}}{4}\frac{r_1 \sin[\theta_1]}{\sin\left[\theta_1 + \frac{\pi}{3}\right]}$$

A.3. **Area 2.**

$$A_2 = \frac{1}{2}{r_2}^2\theta_2 + (T_{r3} + T_{r2}) - \frac{1}{2}{r_3}^2\theta_3$$

$$= \frac{1}{2}\left(\frac{r_1 \sin[\theta_1]}{\sin\left[\theta_1 - \frac{\pi}{3}\right]}\right)^2 \left(\theta + \frac{\pi}{3} - \theta_1\right) + \quad \frac{\sqrt{3}}{4}\frac{r^2 \sin[\theta_1]}{\sin\left[\theta_1 - \frac{\pi}{3}\right]\sin\left[\theta_1 + \frac{\pi}{3}\right]} -$$

$$\frac{1}{2}\left(\frac{r_1 \sin[\theta_1]}{\sin\left[\frac{\pi}{3} + \theta_1\right]}\right)^2 \left(\frac{2\pi}{3} - \theta_1\right)$$

Appendix B. Regular configurations of type 2 overlapping bubbles

By Corollary 8.2, we can enumerate all regular configurations of type 2 bubbles by drawing a standard double bubble and then considering all possible ways that a wall can be drawn across the picture. We then extend the edges from one of the vertices until they intersect the wall.

In the diagrams below, dashed edges represent optional edges. A regular bubble exists both with and without each dashed edge.

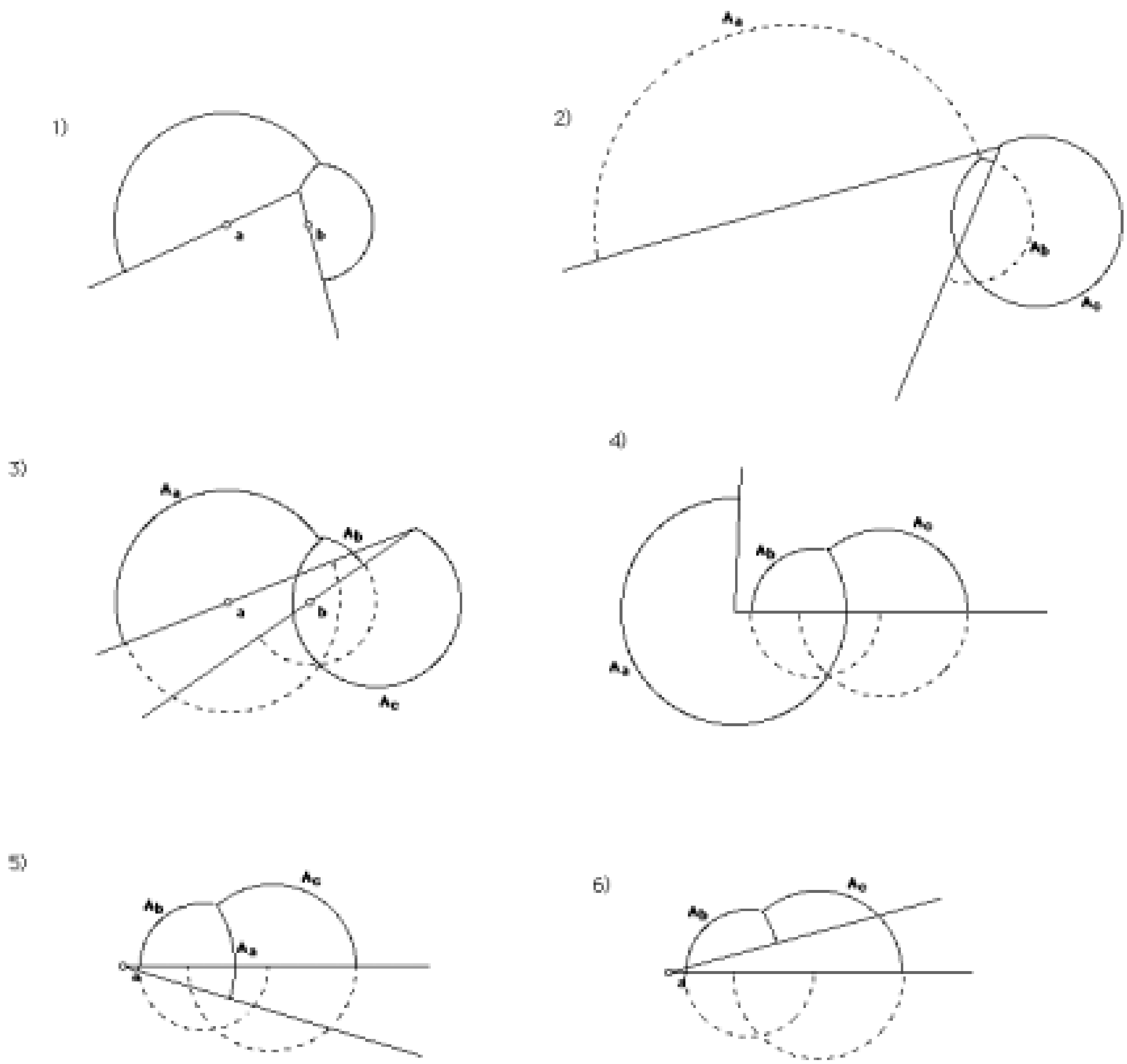

FIGURE 20. Cases 1 to 6.

FIGURE 21. Cases 7 to 8.

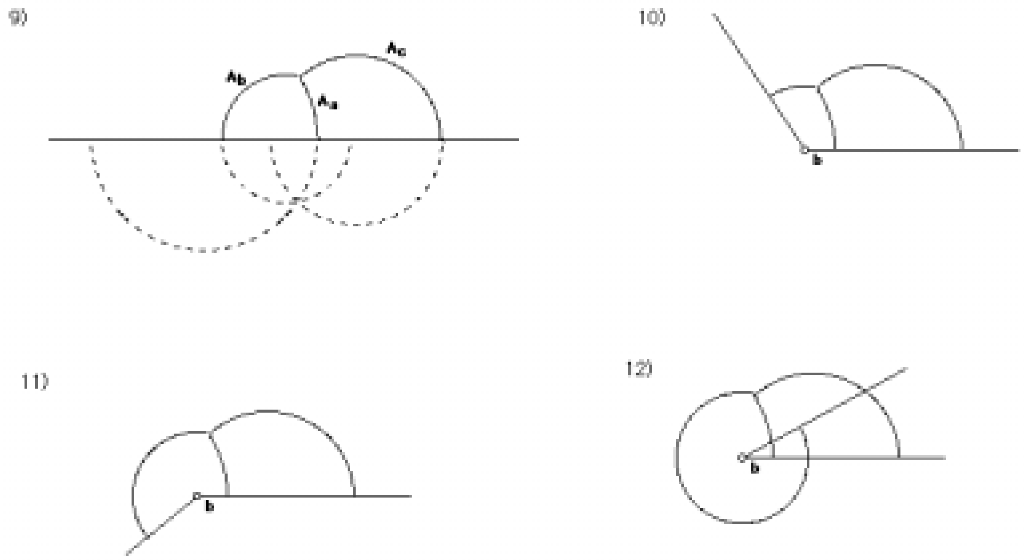

FIGURE 22. Cases 9 to 12.

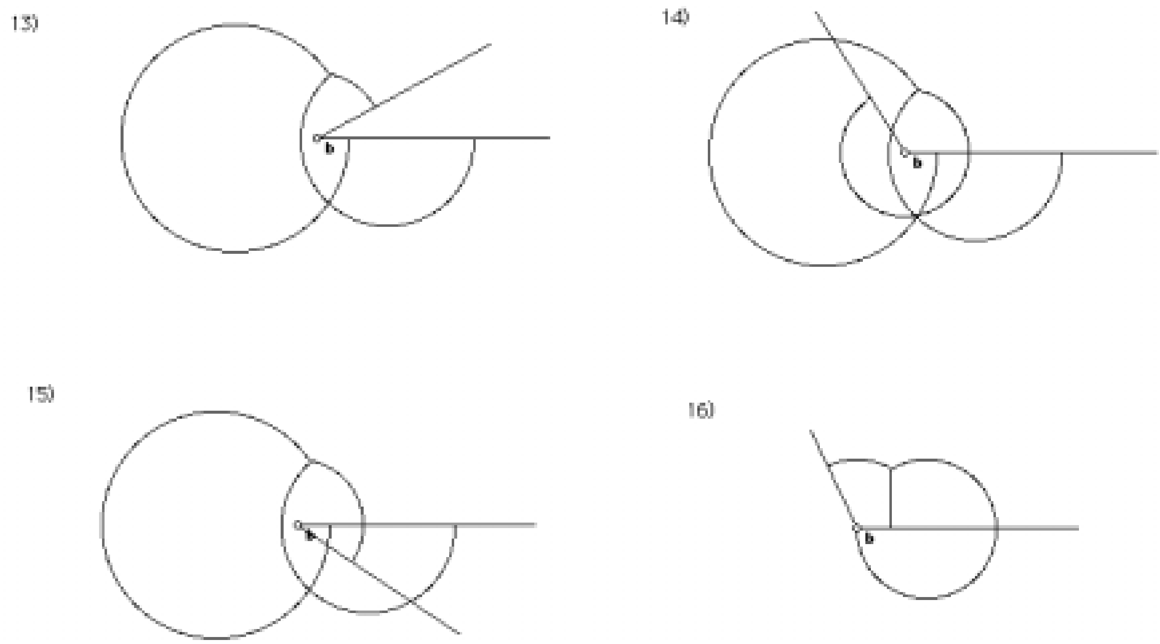

FIGURE 23. Cases 13 to 16.

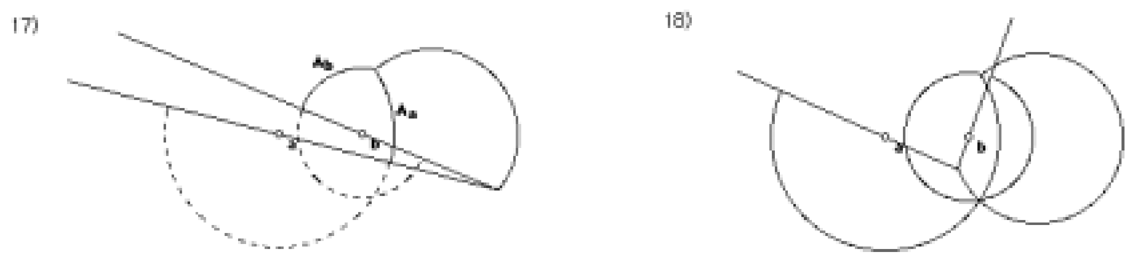

FIGURE 24. Cases 17 to 18.

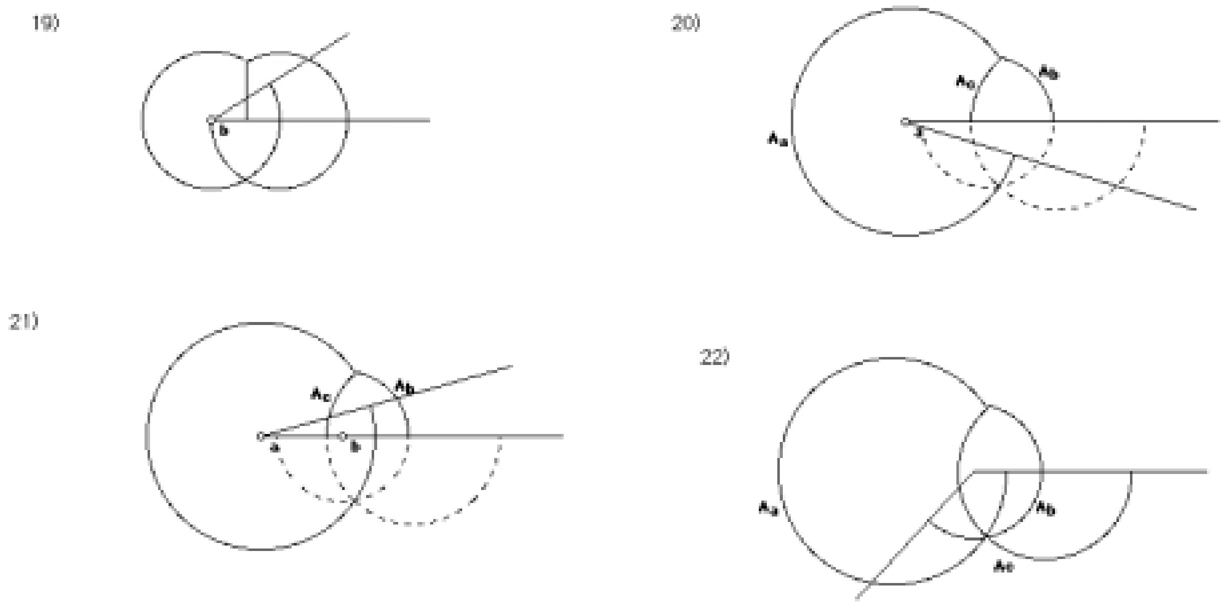

Figure 25. Cases 19 through 22.

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