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Spring 2016

The Cantor Set Before Cantor

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The Cantor Set before Cantor

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April 19, 2021

The Cantor set is well-known to students of topology and analysis as both a go-to source for counterexamples and an extremely interesting, yet strange, space. This set is named after mathematician Georg Cantor (1845–1918), who constructed it in his 1883 paper "Über unendliche, lineare Punktmannigfaltigkeiten, [Teil] 5" ("On infinite linear manifolds of points, Part 5"), [Cantor, 1883]. However, this set appeared as early as 1874 in a paper by the Irish mathematician Henry John Stephen Smith (1826–1883). Smith, who is best known today for introducing the Smith normal form of a matrix, was a professor at Oxford who made great contributions in matrix theory and number theory. In this project, we will explore parts of a paper he wrote entitled "On the Integration of Discontinuous Function," [Smith, 1874].

1 Background

Smith began his paper with the following words [Smith, 1874, p. 140]:

Riemann, in his *Memoir* "Ueber die Darstellbarkeit einer Function durch eine Trigonometrische Reihe,"[1] ... has given an important theorem which serves to determine whether a function $f(x)$ which is discontinuous, but not infinite, between the finite limits a and b, does or does not admit of integration[2] between those limits, the variable x, as well as the limits a and b, being supposed real. Some further discussion of this theorem would seem to be desirable ...

Task 1 Rephrase the question that Smith planned to discuss in modern notation and terminology.

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[1]The English translation of this title is "On the Representation of a Function by means of Trigonometric Series." Submitted in 1854 by Bernhard Riemann (1826–1866) to the University of Göttingen in partial fulfillment of the requirements for the extra post-doctoral qualification (or *habilitation*) that allowed one to become a lecturer in German universities, it was only published two years after his death (in 1868), thanks to the efforts of Richard Dedekind (1831–1916). Although its main focus concerned the characterization of functions that are sums of everywhere-convergent trigonometric series, Riemann's habilitation thesis is best known today as marking the first appearance of what is now called the Riemann integral. To see how (and why) the definition of integration used in the study of analysis changed relative to that given by Riemann (both before and after his work), see the primary source based-projects *The Definite Integrals of Cauchy and Riemann* (by Dave Ruch) and *Henri Lebesgue and the Development of the Integral Concept* (by Janet Heine Barnett), both available at https://digitalcommons.ursinus.edu/triumphs_analysis/.

[2]It should be noted here that when Smith discussed integration, he meant Riemann integration.

Task 2 Give an example of a function that is discontinuous but integrable.

It is well known that continuous functions are integrable and that this condition is not necessary for integrability. But how discontinuous can a function be yet still be integrable? Smith's idea was to attempt to quantify or control the discontinuities of a bounded function $f(x)$ on an interval $[a, b]$. He did this by introducing the definitions given in the following excerpts [Smith, 1874, p. 145].

A system of points is said to **fill completely** a given interval when, any segment of the interval being taken, however small, one point at least of the system lies on that segment.

A "system of points" is simply a set. We will continue to use this language throughout the project.

Today, we say that if a system of points "completely fills a given interval", then it is **dense in a given interval**. We can define this concept more generally, with respect to any topological space (not just an interval), as follows:

Let X be a topological space, $A \subseteq X$ a subspace. Then A is said to be **dense in** X if for every $x \in X$ and every neighborhood U_x of x, $U_x \cap A \neq \emptyset$.

When excerpts from Smith's paper state that "a system of points completely fills a given interval," this is the definition we want to use.

We may observe that the assertion that any given segment of an interval contains at least one point of a given system, is equivalent to the assertion that any given segment contains an infinite number ... of the points of the system.

Task 3 Generalize Smith's claim above to general topological spaces and then prove your assertion.

Smith continued:

But points may exist in an infinite number within a finite interval, without completely filling in any portion of that interval.

Task 4 Give an example of an infinite number of points within an interval that do not completely fill any portion of the interval.

Smith's next definition was the key to his construction of very discontinuous integrable functions. Today we call a system of points "in loose order" a **nowhere dense set**.

> We shall say ... that points are in **close order** on any segment when they completely fill it, and in **loose order** when they do not complectly fill it or any part of it, however small.

Task 5 Tracing back through Smith's terminology, give a self-contained definition of a nowhere dense set; that is, a set of "points in loose order".

2 Two Examples

How did Smith use these set-theoretic concepts to construct non-continuous yet integrable functions? In the quote below [Smith, 1874, p. 146–147], P is a nowhere dense set in the interval $[0, 1]$.

> Let $f(x)$ be a function, which coincides with a given continuous function $\phi(x)$ for all values of x between 0 and 1, except at the points P; and let the difference between $f(x)$ and $\phi(x)$ at these points not exceed the finite quantity σ. It may be shown that $f(x)$ is integrable between the limits 0 and 1, and that
>
> $$\int_0^1 f(x)dx = \int_0^1 \phi(x)dx.$$

Task 6 Rewrite Smith's claim above using modern notation and terminology.

We'll call Smith's claim the "Nowhere Dense Integrability Theorem," or the NDI Theorem. If you performed the above task correctly, then you might now realize that finding interesting examples of integrable but non-continuous functions boils down to finding nowhere dense sets. In other words, one way to answer the question "how discontinuous can a function be yet still be integrable?" is by constructing nowhere dense sets which are "large" in some sense. The construction of such sets will be our focus in the remainder of this project. We'll start with some basic examples that Smith discussed in his paper. In his construction of these sets, he utilized his newly defined terms.

2.1 Smith's First Example

Here is Smith's description of his first example of a nowhere dense set [Smith, 1874, p. 145].

> Let the system of points be defined by the equation $x = \frac{1}{a}, a$ being any positive integer. It will be seen that (1) these points are infinite in number; (2) that they are infinitely condensed in the vicinity of the origin; (3) that they are in loose order [nowhere dense] over the whole interval, no segment, even in the immediate vicinity of the origin, being completely filled.

Let us write Smith's system of points in set builder notation, setting $B := \{\frac{1}{a} : a \in \mathbb{Z}^+\}$. For this set to be "infinitely condensed in the vicinity of the origin," we mean that for every open set U containing 0, $U \cap B$ contains infinitely many points.

Task 7 Prove that B is nowhere dense over the whole interval.

Task 8 Find a function which is discontinuous at infinitely many points on $[0, 1]$, but which is integrable, by going thorough the following steps.

(a) Let $\phi(x) = x$ and $P = \{\frac{1}{a} \colon a = 2, 3, 4, 5, \ldots\}$ be the set of points in the above example. Define a new function $f \colon [0, 1] \to [0, 1]$ by $f(x) = \phi(x)$ if $x \notin P$ and $f(x) = x + \frac{1}{10}$ if $x \in P$. Show that f is discontinuous at infinitely many points.

(b) Use the NDI Theorem to show that f is integrable.

2.2 Smith's Second Example

Smith's next example of a nowhere dense set was similar to his first [Smith, 1874, pp. 145–146].

Let the system of points be defined by the equation $x = \frac{1}{a_1} + \frac{1}{a_2}$ where a_1 and a_2 are any positive integers. Here, it is evident that the points are infinitely condensed in the vicinity of each point of the [set B].

Recall that the set B was defined above by $B := \{\frac{1}{a} : a \in \mathbb{Z}^+\}$.

Task 9 (a) Write down "the system of points ... defined by the equation $x = \frac{1}{a_1} + \frac{1}{a_2}$ where a_1 and a_2 are any positive integers" as a set A using set builder notation. Then compute the set A' consisting of all limit points of A.

(b) Use a definition similar to the one before Task 7 to state what it means when "the points [of A to be] infinitely condensed in the vicinity of each point of the [set B]."

(c) Now prove that the points of A are infinitely condensed in the vicinity of each point of B.

Smith continued with his example as follows.[3]

But it can be shown that [the points of A] are in loose order [i.e., nowhere dense] from 0 to 1. Let $x = L_1$, $x = L_2$, $(L_1 < L_2)$ be two consecutive points of the system B; let μ be any positive quantity whatever, and consider the segment $\left(\frac{\mu L_1 + L_2}{\mu + 1}, L_2\right)$. If $x = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}$ lies on this segment, we must have $\frac{1}{\alpha_1} \leq L_1, \frac{1}{\alpha_2} \leq L_1$ because no point of the system B lies on the interval (L_1, L_2); and also $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \geq \frac{\mu L_1 + L_2}{\mu + 1}$; whence $\alpha_1 \leq \frac{\mu + 1}{L_2 - L_1}, \alpha_2 \leq \frac{\mu + 1}{L_2 - L_1}$.

[3] This excerpt was modified slightly in order to continue to use the notation introduced above for the sets A and B.

We will verify Smith's final two inequalities in the following task.

Task 10 Take L_1, L_2 and μ as described by Smith in the preceding excerpt. Further suppose (with Smith) that $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}$ lies on the segment $\left(\frac{\mu L_1+L_2}{\mu+1}, L_2\right)$; that is, suppose $\frac{1}{\alpha_1}+\frac{1}{\alpha_2} \in \left(\frac{\mu L_1+L_2}{\mu+1}, L_2\right)$.

(a) Prove that $\frac{\mu L_1+L_2}{\mu+1} = L_2 - \frac{\mu}{\mu+1}(L_2 - L_1)$.
Explain how this shows that $\left(\frac{\mu L_1+L_2}{\mu+1}, L_2\right) \subseteq (L_1, L_2)$, so that $L_1 < \frac{\mu L_1+L_2}{\mu+1}$.

(b) Smith asserted that $\frac{1}{\alpha_1} \leq L_1, \frac{1}{\alpha_2} \leq L_1$. What reason did he give to support this claim? Explain why that reasoning is valid or provide any additional detail needed to fully justify these inequalities.

(c) By assumption, we know that $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} > \frac{\mu L_1+L_2}{\mu+1}$. Use this inequality together with those in part (b) to prove that $\frac{\mu L_1+L_2}{\mu+1} \leq L_1 + \frac{1}{\alpha_1}$ and $\frac{\mu L_1+L_2}{\mu+1} \leq L_1 + \frac{1}{\alpha_2}$.

(d) Subtract L_1 from both sides of each inequality in part (c) and simplify the left-hand side of each inequality.

(e) Use the results of part (d) to conclude Smith's claim that $\alpha_1 \leq \frac{\mu+1}{L_2-L_1}$ and $\alpha_2 \leq \frac{\mu+1}{L_2-L_1}$.

Having verified Smith's algebra, we will now see how Smith showed that these inequalities imply that our set is nowhere dense.

These inequalities show that, if, from the beginning of any free segment in the [set B], we cut off as small a part as we please (which we may do by taking μ great enough), the remaining portion of that segment will contain only a finite number of points belonging to [set A]. And this suffices to prove that the points of the system are in loose order; for if d be any segment, however small, situated anywhere in the interval from 0 to 1, we can certainly find on this segment a part free from points of [set B], and, by what has just been proved, parts of that part will be free from points of the [set A].

Task 11 Re-write Smith's above argument using modern notation and terminology. It may be helpful to first carefully define what Smith meant by "free segment" and to "be free from."

Task 12 Use the same method as in Task 8 to construct a function which is integrable on $[0, 1]$ but discontinuous on A.

3 A Very Discontinuous Integrable Function

The final example that we will consider is Smith's construction of a set that allows us to define a function that is integrable, but nevertheless, extremely non-continuous [Smith, 1874, p. 147].

Let m be any given integral number greater than 2. Divide the interval from 0 to 1 into m equal parts; and exempt the last segment from any subsequent division [by removing it]. Divide each of the remaining $m-1$ segments into m equal parts; and exempt the last segment of each from any subsequent division [by removing it]. If this operation be continued *ad infinitum*, we shall obtain an infinite number of points of division P upon the line from 0 to 1. These points are in loose order [nowhere dense].

First we should try to establish more modern and rigorous notation.

Task 13 Let $m = 3$ and denote by $P_0 = [0, 1]$, the closed unit interval. Let P_1 denote the construction after the first division, P_2 after the second, etc. Write down a formula for P_i and finally for P.

Task 14 Prove that "these points are in loose order;" that is, prove that the set P is nowhere dense in $[0, 1]$.

The set that Smith defined in Task 13 is similar to what today is known as a **generalized Cantor set**. While Smith removed the "last segment," the generalized Cantor set of today removes the "middle" segment, where the middle removal for cases in which the segment is broken up into an even number of pieces is handled by taking half from each of the two middle segments. What most people today call **the Cantor set** corresponds to the special case $m = 3$.

Task 15 Recall that a point-set X is called **perfect** if the set X' consisting of all limit points of X satisfies $X = X'$. Prove that the Cantor set ($m = 3$) is perfect.

Task 16 Use the same method as in Task 8 to construct a function which is integrable on $[0, 1]$ but discontinuous on any generalized Cantor set.

4 Conclusion

Smith set out to construct integrable functions which were highly discontinuous in the sense of being discontinuous at many points. As we saw above, he showed that finding such functions reduces to finding a subset of $[0, 1]$ that is nowhere dense. Hence, a highly discontinuous function will be found by constructing "big" nowhere dense sets. The most interesting example Smith gave, and one that has continued to impress itself upon mathematicians to this day, was the Cantor set. Thus, this problem in analysis can be viewed as a problem in topology.

References

G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten, [Teil] 5 (On infinite linear manifolds of points, Part 5). *Mathematische Annalen*, 21:545–586, 1883.

H. J. S. Smith. On the Integration of Discontinuous Functions. *Proceedings of the London Mathematical Society*, 6(1):140–153, 1874.

Notes to Instructors

PSP Content: Topics and Goals

The purpose of this Primary Source Project (PSP) is not only to introduce students to the Cantor set, but also to expose the student to how topological ideas naturally flow from analysis or calculus questions. It is based on a paper by John Henry Smith [Smith, 1874] in which Smith was investigating discontinuous yet integrable functions. Roughly speaking, the question is "how discontinuous can a function be yet still be integrable?" One possible way to interpret this question is to construct functions which are integrable but discontinuous on as many points as possible. The method that Smith uses to construct such functions has nice connections to topology. The connection is most clearly seen in what we are calling the NDI Theorem (stated directly above Task 6). This theorem of Smith states that if a function on $[0, 1]$ agrees with an integrable function everywhere other than a nowhere dense set, then the function is integrable. It is thus seen that finding discontinuous yet integrable functions boils down to constructing big nowhere dense sets. Once this is realized, the rest of the PSP is devoted to constructing such sets.

Student Prerequisites

The PSP is intended for a topology or analysis course, so prerequisites for these courses should be met by the student. In particular, the student should be familiar with basic proof techniques. Furthermore, the project assumes some familiarity with basic concepts in topology: student should have worked with topological spaces, open sets, limit points, and neighborhoods. For a course in analysis that does not develop these concepts in their full generality, the instructor can make the appropriate substitutions, e.g. "open set in $\mathbb{R}$" for "open set."

PSP Design, and Task Commentary

The PSP consists of three main sections, plus a short introduction and conclusion. Section 1 sets the terms and notation for the problem at hand. Tasks 1 and 5 ask the students to struggle through Smith's words by rewriting some of his ideas in modern notation and terminology. This is essential for the rest of the project, especially Task 5 where the student defines a nowhere dense set. The instructor should be aware that if students do not have the correct definition here, there could be major difficulties and frustrations with the rest of the project. Thus, care should be taken to ensure that students have this definition correct before moving on.

In the next section, Section 2, the main theorem that Smith used to construct integrable but discontinuous functions is given, along with two examples. Students should work through Smith's first example to ensure that they are able to make the connection between nowhere dense sets and Smith's theorem. Task 7 has the student prove that the set $\{\frac{1}{a} : a \in \mathbb{Z}^+\}$ is nowhere dense, while Task 8 takes the student step-by-step through the construction of a function based on the nowhere dense set from Task 7. Smith's second example (Section 2.2) is more substantive than the first, in that he used some fairly delicate inequality estimates in order to show that a certain set is nowhere dense. The student is walked through this argument in Tasks 10–11, the former consisting of several parts. Although this is a substantive and illustrative example, it may be skipped without sacrificing the main goals of the project.

Finally, Section 3 examines a version of the generalized Cantor set. It may be a good idea to explicitly point out, as the note after Task 14 discusses, that this set isn't quite the same as what we call the Cantor set today. While Smith removed the "final" third, today's definition of the Cantor set removes the middle third. Several tasks are given for the student, and the instructor should feel free to add additional tasks on the Cantor set.

Suggestions for Classroom Implementation and Sample Schedule (based on 75-minute class periods)

This PSP can be completed over two class periods lasting about 75 minutes, as outlined below.

Day 1

- **Whole-class discussion (15 minutes):** One way that I have found success in introducing this project is to begin the class by asking for examples of integrable functions. Usually, polynomials and trig functions are offered by students. If these are suggested, I remind the students that these are all differentiable, and I ask if they can find an example of an integrable but non-differentiable function. After the absolute value of x is suggested, I ask if they can create a function which is not differentiable at 2 points. 3 points? 4 points? n points? They are usually able to figure out a kind of "saw tooth" function. I then ask if they can break continuity while maintaining integrability. Some students may suggest something like $\frac{1}{x}$, which provides a good opportunity to remind the students that in order for a function to be discontinuous at the point a in its domain, the function needs to be defined at a. Usually students will arrive at some form of the step function. At this point, they may foresee my next question and point out that a step function can be constructed with any finite number of discontinuities. But the next question is the hard one. Can you find a function with infinitely many discontinuities on $[0, 1]$ that is still integrable? This may also be a good time to talk about how mathematicians work. When they discover a helpful concept, they like to push it to its breaking point: how bad can a function be and still be integrable? This is, then, the main question of the PSP, a question that is answered generally with topology and in particular, most satisfyingly with the Cantor set.

- **Working in groups (20 minutes):** Students next work in small groups on Section 2. This section develops the background and sets the stage for the work in the next section. The goal is to get to the concept of a nowhere dense set, or as Smith calls it, a set in "loose order." This section is written with the student who has the minimal prerequisites in mind; if students have seen nowhere dense sets already, this section may either be skipped or assigned as reading to be done as homework before the class begins.

- **Debrief (10 minutes):** After students work in groups on Section 2, the next 10 minutes can be spent as a class regrouping and making sure everyone is on the same page. Do students adequately understand the concepts of dense and nowhere dense sets? If students believe they have adequately addressed Tasks 3 and 5, they can be invited to share their answers on the board, and a class discussion may ensue.

- **Working in groups (20 minutes):** Students can spend the next 15 minutes working on Section 3. The goal here is for students to finish Subsection 2.1.

- **Debrief (10 minutes):** The main purpose of this debrief is to ensure that students see how constructing a large nowhere dense set yields an integrable function which is discontinuous on that nowhere dense set. It is recommended that a student group shares its solution to Task 8 during this debrief. The class as a whole should discuss it and ultimately, it should be clear what a full solution to this task looks like.

- **Homework:** Write up solutions to all tasks completed in-class and begin working on Section 2.2.

Day 2

- **Whole-class discussion (25 minutes):** The first 20 minutes of this class period can be spent as a class reviewing what was covered the previous day as well as the student work on Section 2.2.

- **Working in groups (25 minutes):** This time should be spent working in groups on Section 4 on the Cantor set.

- **Debrief (25 minutes):** As before, the class as a whole can work through the Cantor set example. All tasks can be completed and written up for homework.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Connections to other Primary Source Projects

The following additional primary source-based projects by the author are also freely available for use in teaching courses in point-set topology. The first three projects listed are full-length PSPs that require 10, 5 and 3 class periods respectively to complete. All others are designed for completion in 2 class periods.

- *Nearness without Distance*
- *Connectedness: Its Evolution and Applications*
- *From Sets to Metric Spaces to Topological Spaces*
- *Topology from Analysis* (Also suitable for use in Introductory Analysis courses.)
- *Connecting Connectedness*
- *The Closure Operation as the Foundation of Topology*
- *A Compact Introduction to a Generalized Extreme Value Theorem*

Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_topology`. They can also be obtained (along with their LaTeX code) from the author.

For a PSP that discusses both the Riemann and Lebesgue integrals, see *Henri Lebesgue and the Development of the Integral Concept* by Janet Heine Barnett (available at `https://digitalcommons.ursinus.edu/triumphs_analysis/2/`). This two-day project contrasts the Riemann integral with the Lebesgue integral, as these are described by Lebesgue himself in a relatively non-technical paper published in 1927. The project's primary goal is to consolidate students' understanding of the

Riemann integral and its relative strengths and weaknesses, while providing students with an introduction to the Lebesgue integral, an important concept that is rarely discussed in an undergraduate analysis course. By offering an overview of the evolution of the integral concept, this project also exposes students to the ways in which mathematicians hone various tools of their trade (e.g., definitions, theorems).

Acknowledgments

The development of this student project has been partially supported by the TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Program with funding from the National Science Foundation's Improving Undergraduate STEM Education Program under Grant Nos. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily represent the views of the National Science Foundation.

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