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How to Calculate pi: Buffon's Needle (Calculus version)

Dominic Klyve

Central Washington University, Dominic.Klyve@cwu.edu

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Introduction

The challenge of estimating the value of π is one which has engaged mathematicians for thousands of years. Given this history, it's difficult to come up with a completely new method, yet this is precisely what Georges-Louis Le Clerc (1707–1788) did in 1777, in a short passage of a longer essay in which he introduced the idea of *geometric probability*. As we shall see, this is only one of two novel ideas in Le Clerc's method – the second is that his method used randomness in part of the estimation.

LeClerc is better known to history as the Comte de Buffon. A "comte" is a "count"; King Louis XVI gave LeClerc this title of nobility near the end of LeClerc's life, and it's now customary to refer to him as "Buffon". Buffon was a prolific author – he wrote an enormous 20-volume work on nature (the *Histoire Naturelle*) in which he discussed everything from the formation of the oceans to the habits of birds and foxes. At the end of one of these volumes, he included a discussion of what he called "moral arithmetic."[1] For Buffon, this was a catch-all term that encompassed the intersection of mathematics with behavior, expectation, and even morality.

Part 1: Geometric Probability

One of the many topics covered in [Buffon, 1777] is a subject now called "geometric probability", which Buffon was the first person to study. Let's start by exploring some of the basic ideas of geometric probability via examples.

*Department of Mathematics, Central Washington University, Ellensburg, WA 98926; dominic.klyve@cwu.edu.

[1]The essay was entitled *Essais d'Arithmetique Morale* (Essays on Moral Arithmetic) [Buffon, 1777]; it is a small section of this essay in which Buffon's geometric probability work appears.

Task 1 Consider a circle inscribed in a square each of whose sides is one unit in length. A dart is thrown at the square such that it will land randomly over the area of the square. What is the probability that the dart will land inside the circle?

Task 2 Consider a square inscribed in a circle of radius 1. A dart is thrown at the circle such that it will land randomly over the area of the circle. What is the probability that the dart will land inside the square?

We might ask why we should phrase the questions above (which are simply questions about the ratios of areas of two shapes) as probability. The answer, and the reason that Buffon began thinking about these problems, is similar to the answer to the question of why European mathematicians began thinking about probability in the first place: gambling. Buffon was well aware of interesting games of chance in which the probability of winning could be found only be geometric methods. Since no one had ever tried this, though, he justified the idea of using probabilistic ideas to approach geometric problems.[2]

> Analysis[3], is the only instrument that has been used up-to-now in the science of probabilities to determine and to fix the ratios of chance; Geometry appeared hardly appropriate for such a delicate matter; nevertheless if one looks at it closely, it will be easily recognized that this advantage of Analysis over Geometry is quite accidental, and that chance according to whether it is modified and conditioned is in the domain of geometry as well as in that of analysis; to be assured of this, it is enough to see that games and problems of conjecture ordinarily revolve only around the ratios of discrete quantities; the human mind, rather familiar with numbers than with measurements of size, has always preferred them; games are a proof of it because their laws are a continual arithmetic; to put therefore Geometry in possession of its rights on the science of chance is only a matter of inventing some games that revolve on size and on its ratios or to analyse the small number of those of this nature that are already found.

[2] All translations are taken, sometimes with modifications by the author in consultation of Buffon's original text, from [Hey et al., 2010].

[3] "Analysis", for an eighteenth-century mathematicians, was a term loosely corresponding to "calculus" today.

Task 3 How did Buffon justify the use of probability in geometric arguments?

Task 4 Are there any games played today (a game you have played, perhaps) in which the outcome depends on the size of areas of something, or the ratios between areas?

Buffon's first example of a geometric game was one he called the "open-tile" game.

> The "open-tile" game can serve us as example: here are its very simple terms. In a room paved with equal tiles of an unspecified shape one throws an Ecu [a French coin] in the air; one player bets that after its fall this Ecu will be located on an open-tile, that is, on [only] a single tile.

Task 5 Sketch a picture of Buffon's open-tile game.

Task 6 What information would you need to be able to calculate this probability?

We needn't be concerned with the probability of winning the open-tile game, but instead we will simply note that Buffon, in his essay, introduced a number of variations to the game, primarily by imagining tiles of different shapes. These problems have been largely forgotten over the centuries. His next example, however, became famous, and is today known as the "Buffon needle problem." It is to this which we next turn.

Part 2: Toward π: the Buffon Needle Problem

Buffon discussed several versions of his open-tile game. He calculated the probability that the thrown coin would land on an open tile, or an exactly two tiles, or on four. He even worked out what might happen if the floor was tiled not by squares, but by equilateral triangles, as well as some other shapes. Only then did he turn to the question for which he is now best known.

But if instead of throwing in the air a round piece, as an Ecu, one could throw a [. . .] needle, a stick, etc. The problem demands a little more geometry, although in general it is always possible to give its solution by space comparisons, as we will show. I suppose that in a room where the floor is simply divided by parallel joints one throws a stick in the air, and that one of the players bets that the stick will not cross any of the parallels on the floor, and that the other in contrast bets that the stick will cross some of these parallels; one asks for the chances of these two players. One can play this game on a checkerboard with a sewing needle or a headless pin.

Task 7 Sketch a picture of Buffon's needle game.

Task 8 What information would you need to be able to calculate this probability?

Buffon next carefully described the set-up of his game, assigning names and lengths to all the values in which he was interested.

To find it, I first draw between the two parallel joints on the floor, AB and CD, two other parallel lines ab and cd, at a distance from the primary ones of half the length of the stick EF, and I evidently see that as long as the middle of the stick is between these second two parallels, it never will be able to cross the primary ones in whatever position EF, ef, it may be located; and as everything that can occur above ab alike occurs below cd, it is only necessary to determine the one or the other; that is why I notice that all the positions of the stick can be represented by one quarter of the circumference of the circle of which the stick length is the diameter; letting therefore $2a$ denote the distance CA of the floor joints[4], c the quarter of the circumference of the circle of which the length of the stick is the diameter, letting $2b$ denote the length of the stick, and f the length AB of the joints, I will have $f(\overline{a-b})c$ as the expression that represents the probability of not crossing the floor-joint, or equivalently, as the expression of all cases where the middle of the stick falls below the line ab and above the line cd.

[4]Buffon is here using very poor notation. Earlier, the letter a referred to a point on the diagram; now he's using it to represent a length. This is poor mathematical practice, and should be avoided.

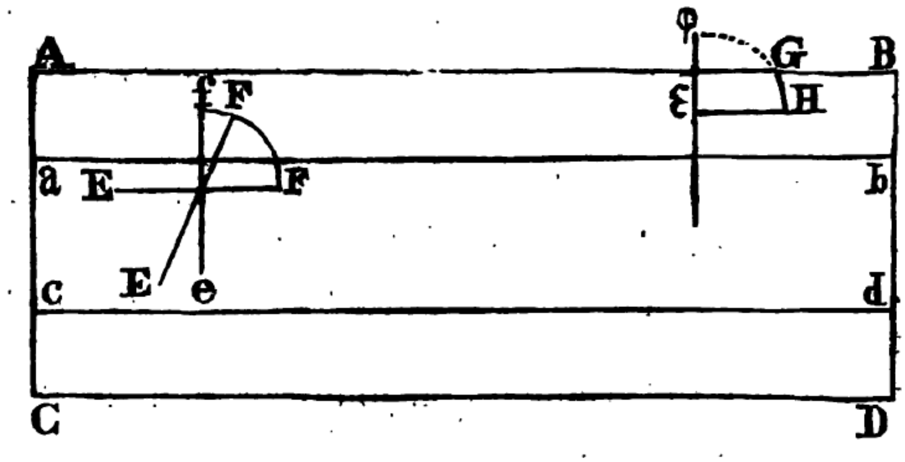

Figure 1: Buffon's sketch of the Needle Problem

Task 9 Compare the picture in Buffon's original work (Figure 1) to the description he gave of the setup for the two regions in which the needle might fall. Do they agree? If not, where do they disagree?

Task 10 Buffon described more values than he included in his picture. Redraw the picture and label all of the lengths and distances using Buffon's notation.

Task 11 Explain why "... as long as the middle of the stick is between these second two parallels, it never will be able to cross the primary ones in whatever position EF, ef, it may be located."

Task 12

a. Buffon tried to simplify his task in a few ways. First, he wrote: "and as everything that can occur above ab alike occurs below cd, it is only necessary to determine the one or the other". What does he mean by this, and why does this simplification make sense? (You may find it helpful to shade the upper half of the rectangle $abcd$.)

b. Buffon also tried to reduce the set of directions the needle might point that he would need to consider, writing "I notice that all the positions of the stick can be represented by one quarter of the circumference of the circle of which the stick length is the diameter". Explain this reasoning; use a picture if it will help.

The most crucial (and possibly the most confusing) part of Buffon's description above may be: "I will have $f(\overline{a-b})c$ as the expression that represents the probability of not crossing the floor-joint, or equivalently, as the expression of all cases where the middle of the stick falls below the line ab and above the line cd."[5] Buffon here was using a clever continuous version a "multiplicative counting principle" that is useful for finding the size of a *sample space* – a set of all the possible values and combinations something might have. For example, if you have only two shirts (red and blue) and three pairs of pants (jeans, slacks, and sweats), then the sample space of the wardrobes you could choose is the set of $2 \times 3 = 6$ combinations of shirts and pants available to you. For continuous geometric problems like the one Buffon was considering, we're not actually counting distinct objects, but measuring how much space they occupy.

Let's try to use this type of reasoning to understand Buffon's expression.

Task 13

(a) What three pieces of information would you need to completely describe the position and orientation of the stick, assuming its middle lies below the line ab and above the line cd ?

(b) Find expressions that describe all possible values of each of these three pieces of information using Buffon's notation. When you multiply them together, does your expression match the one that Buffon gave?

Buffon next turned to the question in which he was most interested: the probability that the stick would land in such a way that it would cross line AB.

> But when the middle of the stick falls outside the space $abcd$, enclosed between the second parallels, it can, depending on its position, cross or not cross the joint; so that the middle of the stick being located, for example, in ϵ, the arch ϕG represents all the positions where it will cross the joint, and the arch GH all those where it will not cross, and as it will be the same for all the points on the line $\epsilon\phi$, I denote by dx the small parts of this line, and y the arches of the circle ϕG, and I have $f \cdot (\int ydx)$ as the expression of all the cases where the stick crosses, and $f \cdot (\overline{bc - \int ydx})$ as the expression of the cases where it does not cross; I add this last expression to the one noticed above $f(\overline{a-b})c$ in order to have the entirety of the cases where the stick will not cross, and from that time I see that the chances of the first player relates to the one of the second as $ac - \int ydx : \int ydx$.

[5] Note that Buffon usually sets off an expression using both parentheses and an overline. The overline doesn't include any more information than is suggested by the parentheses.

Task 14 Go back to the picture you drew earlier, and add the labels that Buffon described here.

Buffon had some ideas that were ahead of his time, but he didn't always explain them well. Let's go through his work and see if we can understand what he did.

Task 15 Fix an angle θ between the direction of the needle and the vertical. Complete the following statement: the needle will cross line AB as long as x, the distance between ϵ and AB, satisfies $x <$ _____________?

Task 16 The same question can be asked a different way. Assume the center of the needle (ϵ) is distance x from line AB, and let θ be the angle between the direction of the needle and the vertical. Complete the following statement: the needle will cross line AB as long as $\theta <$ _____________?

Putting it all together

Buffon (and possibly you!) found that if x is the distance from the center of a needle to the parallel line, and θ is the angle between the direction of the needle and the vertical, then the needle will cross line AB as long as $x < b\cos(\theta)$ (we will use the special case that $b = 1$ in the next paragraph to simplify our calculation). Calculating the probability that this inequality will hold, though, seems tricky. If x is very small (and thus the center of the needle is close to line AB), almost any angle θ will make the needle cross the line. On the other hand, if x is large, the needle will need to be almost vertical to cross AB, and very few angles will work.

In order to "add up" all the possible states that correspond to a needle that crosses the line, Buffon integrated over the sample space for which x is bounded as in Task 16, and for which the length of the arch (GH in the diagram) was bounded by $b\cos^{-1}\frac{x}{b}$. His real interest was to find the ratio of all possible positions for the needle's center, and of all possible angles θ, that correspond to the needle's crossing line AB.

Task 17 Using your result from the previous task, set up (but don't evaluate) an integral to find the size of the sample space of crossings that work for any given θ.

Task 18 Now evaluate the integral and divide by the size of the sample space (Recall this is $\frac{\pi}{2} \cdot b \cdot a$, the length of a quarter circle multiplied by the height of the strip), to find the probability that the needle will cross the line (in terms of a and b).

Task 19 In the case that $a = b$, what is the probability that the stick will cross a line?

Part 3: Calculating π

Ideally, your work above has shown that the probability the needle will cross the line (assuming the length of the needle, $2b$, satisfies $b \leq a$) is $\frac{2b}{\pi a}$.

In 1901, Italian mathematician Mario Lazzarini reported that he had performed an experiment using Buffon's game, and that the results had allowed him to approximate π to 6 decimal places [Lazzarini, 1901]. Let's look at what he did.

First, Lazzarini chose needles with a length that was 5/6 of the distance between parallel lines on the floor.

Task 20 What proportion of Lazzarini's throws should he have expected to land crossing a line?

Lazzarini knew that the value of π is very well approximated by the fraction 355/113, and he hoped to use this fact in his work.

Task 21 Check how closely 355/113 approximates π.

Task 22 If Lazzarini dropped n needles and C crossed a line, what would he calculate as his value of π?

Task 23 If Lazzarini were to have a chance to be successful at achieving the desired fraction 355/113, what is the smallest number of throws he could make?

It seems likely that Lazzarini tried to use the number of throws you calculated above, but the value C (the number of needles that crossed) was not what he hoped, so he kept going. Eventually, however, after 3408 throws, he stopped the experiment, having observed 1808 times when a needle crossed a line.

Task 24 What value of π did Lazzarini deduce from his experiment?

Task 25 Why do you think Lazzarini stopped at 3408 throws? Why was it necessary that this be a multiple of 213?

Task 26 Based on your answers to the previous tasks in this section, what problems do you see with Lazzarini's experiment?

Even if his methods were a bit questionable,[6] Lazzarini's work used Buffon's eighteenth-century game to do something that it seems had not been done before – estimate an important mathematical constant using a randomized experiment. You may want to try this yourself – grab some toothpicks, draw parallel lines on a piece of paper, and start throwing and counting!

References

Lee Badger. Lazzarini's lucky approximation of π. *Mathematics Magazine*, 67(2):83–91, 1994.

George LeClerc Buffon. *Essai d'Arithmétique morale, Supplément à l'Histoire Naturelle*, volume 4. 1777.

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Mario Lazzarini. Un'applicazione del calcolo della probabilita alla ricerca esperimentale di un valore approssimato di π. *Periodico di Matematica*, 4:140–143, 1901.

[6] For an interesting discussion of Lazzarini's experiment, and an analysis of whether he may have fudged his numbers, see [Badger, 1994].

Notes to Instructors

Goals

This is one of a series of Primary Source Projects (PSPs) that explore ways that mathematicians have used material now in the undergraduate curriculum to estimate π. In this particular (mini-)PSP, students explore the "geometric probability" of Georges LeClerc, Compte de Buffon, who wrote the first true work in this field. The project has been prepared in two versions – this version assumes that students have knowledge of integral calculus. When working with non-calculus-ready students, instructors are encouraged to use the project entitled "How to Calculate π: Buffon's Needle (Non-calculus version)", in which students solve Buffon's needle problem without calculus.

As it uses excerpts from the first work in the field, the project is an ideal introduction to geometric probability in any course that treats it. More generally, it can be used in an Integral Calculus course as a non-standard example of the scope of calculus, and an unusual way to practice reasoning with integrals. It is also designed to be sufficiently self-contained as to be usable in a capstone or history of mathematics class.

Background

In this particular (mini-)PSP, students explore the "geometric probability" of Georges LeClerc, Compte de Buffon, whose 18th-century work *Essais d'Arithmetique Morale* was the first true work in this field. Buffon's goal wasn't to calculate π; rather, he was interested in a large number of questions about expectation – both mathematical (the probability of an event) and personal (what a reasonable human might expect). The *Essais* may be most famous for Buffon's question: "if a needle of length $2b$ is thrown on a floor, marked with parallel lines of distance $2a$ apart, what is the probability that the needle crosses one of the lines"?

Buffon actually began with a similar question, in which a coin (not a needle) is thrown on the floor, and later went on to corresponding problems in which the square grid is replaced by other grids (in the form of triangles, lozenges and hexagons) ([Holgate, 1981] notes that not all of Buffon's answers are correct.) In most case, Buffon calculated the probability of falling on at least one, and then two or three tiles. He investigated more problems, including using a square (rather than a circular) coin, and finally by replacing the coin with a needle.

Prerequisite knowledge

Basic notions of probability are introduced in this project; students are not expected to have anything more than the mathematical maturity to understand probabilistic thinking. Students are assumed to have a working knowledge of basic trigonometry, and of integral calculus (in particular, they need a knowledge of integration by parts, to allow them to integrate an arccosine function.

Suggestions for PSP Implementation

One possible implementation, based on 50-minute classes, follows. Using this will take only 1.5 days of class. If time allows, the homework can be eliminated and replaced with in-class time, in which case it may take 2-2.5 days to complete the PSP.

- **Day 0.** Introduce the project, and the big question asked within – if you throw a needle into the air, what is the probability that it will cross a line on the floor? If you are fortunate enough to teach in a room that actually has parallel lines on the floor, you could consider trying a few actual experiments.

  Day 0 Homework: Assign Part 1 as homework.

- **Day 1.** Students meet in groups to discuss their answers to Task 7, and work together to complete Part 2 through Task 13. Assign the remaining tasks can be completed as homework.

  Day 1 Homework: Assign students to look at, and take notes on, Tasks 14–19. These are fairly difficult, and students may not be able to answer them alone, but it's reasonable for students to write up and turn in their preliminary work on these problems. In a more advanced class (say, an upper-level course in probability theory), these could be assigned as homework.

- **Day 2.** Allow students to work in groups on Tasks 14–19, then spend some class time going over these. Debrief the project as a class.

- **Day 2 Homework:** Assign Part 3. After the previous Tasks have been completed, these are likely to be rather straightforward. If you like, you may want to discuss Lazzarini's experiment in class on Day 3.

- **Day 3 (Optional)**: Discuss Part 3 (on Lazzarini's experiment) in class. In some classes, you may want to simulate the experiment with toothpicks.

LaTeX code of the entire PSP is available from the author by request to facilitate preparation of reading guides or other assignments related to the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Commentary on Selected Student Tasks

- Task 4 doesn't have any obvious answer, and I don't mind if the students can't think of any such games. Pondering the question for a little while may still improve their understanding of what Buffon was trying to do.

- Task 13 asks for three pieces of information to completely describe the position and orientation of the stick. This could be a good opportunity to discuss the value of different coordinate systems. The stick could be described, for example, by giving the x and y coordinates for its two endpoints (four values), but if we specify the center of the stick and its orientation (angle from vertical, say), then we need only three. (A student could argue that the left-right value is irrelevant in this problem. While true, Buffon considered it is his calculations, and we follow him here.)

- Task 15. I'm here hoping for $x < b\cos\theta$.

- Task 16. The student should find $\theta < \cos^{-1}(x/b)$.

- Task 18: Buffon's idea here, improper as it looks to the modern eye, is to integrate over all possible values of x, and to divide by the size of his sample space (which is itself a product of the height of the strip, a, and the circumference of the quarter circle, $b\pi/2$. He thus used (in modern notation):

$$\frac{\int_0^b b\arccos\left(\frac{x}{b}\right)}{ba\frac{\pi}{2}} = \frac{1}{a}\int_0^b \frac{2}{\pi}\arccos\left(\frac{x}{b}\right) = \frac{2b}{\pi a}.$$

Recommendations for Further Reading

Much has been written about the needle problem, but almost everything I found modernized the solution quite a bit. A rare exception is [Holgate, 1981], who carefully studied Buffon's solution and whose paper provides a nice (if short) history of interpretation of Buffon's text. In fact, rather than evaluate the integral as we do in this project, Buffon wrote:

> If one wants therefore that the game is fair, one will have $ac = 2\sin ydx$ or $a = \frac{\sin ydx}{(1/2)c}$, that is, equal to the area of the part of the cycloid whose generating circle has as diameter the stick-length $2b$; now, one knows that this cycloid-area is equal to the square of the radius, therefore $a = \frac{b^2}{(1/2)c}$, that is, the length of the stick must be almost three quarters of the distance of the floor joints.

Holgate performs a clever exegesis in which he explains what the cycloid is doing in Buffon's paper, and what he was likely to have known about the area under the cycloid. After much consideration, and at the risk of anachronism, I cut this section from the PSP. It's historically interesting, but does little to help students understand either geometric probability or the investigation of the value of π.

Acknowledgments

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References

Lee Badger. Lazzarini's lucky approximation of π. *Mathematics Magazine*, 67(2):83–91, 1994.

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