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Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve

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Just prior to his 19$^{\text{th}}$ birthday, the mathematical genius Carl Freidrich Gauss (1777–1855) began a "mathematical diary" in which he recorded his mathematical discoveries for nearly 20 years. Among these discoveries is the existence of a beautiful relationship between three particular numbers:

- the ratio of the circumference of a circle to its diameter, or π;
- a specific value of a certain (elliptic[1]) integral, which Gauss denoted[2] by $\varpi = 2\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$; and
- a number called "the arithmetic-geometric mean" of 1 and $\sqrt{2}$, which he denoted as $M(\sqrt{2},1)$.

Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice that historian Adrian Rice called "Gaussian Guesswork" in his *Math Horizons* article subtitled "Why 1.19814023473559220744 . . . is such a beautiful number" [Rice, November 2009].

This project is one of a set of four independent mini-projects, based on excerpts from Gauss' mathematical diary [Gauss, 2005] and related manuscripts, that looks at the power of Gaussian guesswork via the story of his discovery of this beautiful relationship. Our specific focus in this project will be on the reasons why Gauss and other mathematicians first became interested in the (elliptic) integral that defines the number ϖ in this relationship.

1 Arc Length of the Unit Circle, and the Sine Function

Before looking at the particular elliptic integral that Gauss studied, let's look at the more familiar case of the unit circle. Of course, the geometric properties of circles have been studied for millennia, and in many different cultures. The Greek mathematician Archimedes of Syracuse (287–212 BCE) is credited as the first individual to formally state and prove an 'area formula' for the circle, which he stated in his text *Measurement of a Circle* [Archimedes, 1985] as follows (as quoted in [Dijksterhuis, 1987, p. 222]):

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[1] An integral of the form $\int_0^x \frac{1}{\sqrt{1-t^n}}dt$ is called an *elliptic* integral for $n = 3$ and $n = 4$; for $n > 5$, it is called *hyperelliptic*. This terminology is historically related to the occurrence of this form in connection to the arc length of ellipses and other curves that naturally arise in astronomy and physics.

[2] The symbol 'ϖ' that Gauss used to denote this specific value is called "varpi;" it is a variant of the Greek letter π.

Proposition 1 The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other base to the circumference [of the circle].[3]

Task 1 Sketch a circle and the corresponding triangle described in Archimedes' Proposition 1, and label both using appropriate variables. Then use area formulas to verify that the relationship stated in this proposition is correct.

Letting r represent the radius of the circle and using the usual relationship between the circle's circumference and its radius, it is a straightforward task to translate Archimedes' statement into today's standard area computation formula for circles — as you just verified in Task 1! However, two things are worth noting in Archimedes' actual statement. First, it describes how we would construct a rectilinear geometrical figure (i.e., a triangle) which has the same area as the given curvilinear figure (i.e., a circle), rather than providing a computational formula of the type in common usage today. This type of construction is called a *quadrature*, or *squaring*, in keeping with the long-standing geometric tradition in which 'finding the area' of a curvilinear figure literally meant constructing a polygon, often a square or quadrilateral, with the same area as the given figure. The second noticeable thing about Archimedes' description of how to compute the area of a circle is its interesting use of the circle's arc length (or circumference). The problem of finding the arc length itself has historically been called *rectification*, from the Latin word 'rectificare' which can be translated to mean 'straighten.'

We will come back to the idea of solving quadrature problems by way of rectifications in the next section of this project. For now, let's jump forward in time from Archimedes to the actual beginnings of calculus in the late seventeenth century. At that time, mathematicians — or 'geometers' as they were then called — had just gained access to integration as a powerful tool for approaching problems in geometry and physics, and were busy exploring the interplay between these various types of problems. In keeping with the spirit (but not the notation!) of those explorations, the following task examines one specific integral from both a rectification (arc length) and a quadrature (area) perspective.

Task 2

(a) Use the equation $x^2 + y^2 = 1$ of the unit circle to set up and simplify the integral for the arc length of one fourth of this circle.

(b) Use the fact that we know the circumference of the unit circle, along with your result from part (a), to explain why $\pi = 2\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$. (You could verify this by integrating, if you are familiar with this integral — but there's no need for integration here at all!)

(c) We can re-write the integral in part (b) as $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$. Interpret this integral in terms of the area under a curve. Sketch the graph of the function that defines that curve, and shade the region associated with the area given by the integral on your sketch. Is the region in question bounded or unbounded? Do you find this surprising? Explain why or why not.

[3] Archimedes' proof of this proposition employed a strategy which can be viewed as an precursor to integration, known as the "method of exhaustion." In this method, a given curvilinear figure is 'exhausted' through a sequence of inscribed (or circumscribed) polygons, with the polygonal areas approaching the area of the given curvilinear figure. Following his proof by exhaustion of Proposition 1, Archimedes went on in this text to use circumscribed and inscribed regular polygons with 96 sides to establish the numerical estimate $3\frac{10}{71} \leq \pi \leq 3\frac{1}{7}$.

The next task continues to look at the integral from Task 2, but now in a more general way that takes us beyond seventeenth-century calculus interests.

Task 3 Define the function $a = f(t)$ by setting $f(t) = \displaystyle\int_0^t \frac{1}{\sqrt{1-x^2}}dx$.

(a) What is the domain of f? What is its range? Explain your reasoning.

(b) Use the Fundamental Theorem of Calculus to show that $f'(t) > 0$ on its domain. Use this fact to explain why f must have an inverse function f^{-1}.

(c) What is the domain of f^{-1}? What is its range?

(d) Since $a = f(t)$, we know that $f^{-1}(a) = t$, where t is the upper limit of integration in the given integral. Explain how we know that $t = \sin a$. Also explain why the domain f^{-1} from part (c) of this task is more restricted than the full domain of the sine function.

2 Arc Length of the Lemniscate

In Task 2, we began with an arc length (or rectification) problem, and then interpreted the integral that we obtained as an area (or quadrature) problem. This process might be described as 'reducing a rectification problem to a quadrature problem.' But as you likely noticed in Task 2(c), the rectification question of finding the arc length of a quarter circle is actually far easier to think about than the quadrature problem of finding the area of the unbounded region under the curve $y = \dfrac{1}{\sqrt{1-x^2}}$. In other words, it would have made more sense to have reduced the (unbounded) quadrature problem to that of the (simpler) rectification problem. This was precisely the sort of idea that led seventeenth-century mathematicians to become interested in the curve known as the *lemniscate*.

Here is an excerpt from the paper[4] [Bernoulli, September 1694] in which the mathematician Jacob Bernoulli (1655–1705) christened the lemniscate with its name, in connection with his construction of another curve called the *paracentric isochrone*.[5]

> ... because of this great desire, a curve of four dimensions is set up expressed by the equation $xx + yy = a\sqrt{xx - yy}$, and which around the axis ... [$2a$] forms a shape resembling a figure eight lying on its side ∞, a ribbon folded into a knot, a *lemniscus*; in French, *d'un nœud de ruban*.

[4] The English translation of the paper's full title is "Construction of a Curve with Equal Approach and Retreat, with the help of the rectification of a certain algebraic curve: Addenda to the June solution."

[5] The paracentric isochrone is the curve with the property that a ball rolling down it approaches or recedes from a given point with uniform velocity. Although its construction did not require the finding of an area, it was viewed as a quadrature problem because the differential equation given by the physics involved led to the evaluation of an integral (as would a quadrature problem). The paracentric isochrone problem itself was originally posed by Gottfried Leibniz in 1689 as one of a series of problems involving balls rolling along curves with which geometers of the day challenged each other, as well as the power of the new calculus techniques. In addition to the two solutions published by Jacob Bernoulli in June and September of 1694, Johann Bernoulli (1667–1748) also independently solved it in that same year by rectifying the lemniscate. This incident of independent discovery, in which Jacob beat Johann to publication by just a month, along with the surrounding debate about proper methodology in which the two brothers participated on different sides of the argument, were factors in the increasingly uncivil sibling rivalry that existed between the two brothers.

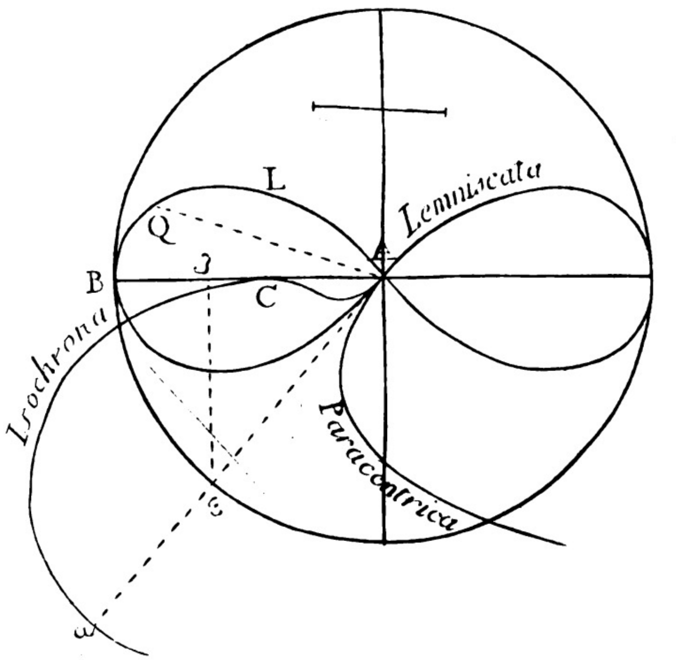

Figure 1: Diagram from [Bernoulli, December 1695] showing the lemniscate with the paracentric isochrone.

Task 4 Notice that Bernoulli, like other geometers of his time, did not use exponential notation to denote the products 'xx' or 'aa'. Rewrite the equation for the lemniscate that he gave in the preceding excerpt using x^2 and a^2 to represent these products. Then square both sides of that equation to eliminate the radical expression. How does the degree of your final equation relate to Bernoulli's description of the curve?

Bernoulli's reference in the preceding excerpt to 'this great desire' was part of a methodological debate in which he was involved, concerning how to best to solve integrals related to quadrature problems in general. Earlier in the same year that this paper was published, he had solved the same problem discussed in this excerpt by rectifying a different curve known as an *elastica*. Other geometers — including Bernoulli's younger brother Johann Bernoulli (1667–1748) — criticized Jacob's earlier solution of the quadrature problem, not for its use of rectification (arc length), but because the elastica itself is a transcendental equation. The lemniscate, on the other hand, is an algebraic curve since its equation involves no transcendental functions. Quoting Bernoulli once more [Bernoulli, September 1694]:

It is a better [method] to employ a construction by rectification of an algebraic curve; for curves can be more quickly and accurately rectified, using a string or small chain wrapped around them, than areas can be squared.[6]

[6] Bernoulli's full view about the best method to use was more complicated than we have presented here, as was the debate in general among geometers of the time. We have omitted these details from this project, as they would take us too far afield from the mathematics we are developing. For a thorough treatment of these issues, see [Blåsjö, 2017, 160–167].

Of course, Bernoulli did not actually use string to find the arc length of the lemniscate, but instead used the differential techniques of his time to set up its integral. Rather than follow his derivation directly, the next task will take us through that set up using polar coordinates, which were only beginning to emerge in the late seventeenth century. Recall again the equation that Bernoulli found for the lemniscate itself: $xx + yy = a\sqrt{xx - yy}$, where a is the length of the semi-axis.[7] Squaring both sides of this equation and using superscripts to represent powers, we thus have $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

In Task 6, we will use polar coordinates to set up an integral for one-fourth of its arc length. In the next task, we first derive a formula for the arc length of a curve given in polar coordinates, beginning from a formula for the arc length of a curve given in rectangular coordinates.

Task 5 Recall that for a curve in the xy-plane with arc length s, the differential ds is the hypotenuse of the right triangle with sides dx and dy. Applying the Pythagorean Theorem to this differential triangle thus gives us the formula $ds = \sqrt{(dx)^2 + (dy)^2}$. Using this expression to write an integral for the arc length s in terms of derivatives with respect to θ, we get the following:[8]

$$s = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \qquad (*)$$

Show that the formula displayed above simplifies to the following when the equation of the curve is given in polar coordinates:

$$s = \int_{a}^{b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (**)$$

Begin by differentiating the equations $\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases}$ for converting from rectangular coordinates to polar coordinates. *(Remember to use the product rule for each of these derivatives, since the radius r is often a function of θ!)* Then substitute the results into the first formula (*) for s, and algebraically simplify to obtain the polar coordinate formula (**).

Task 6 Let $a = 1$ and consider the 'unit lemniscate' of equation $(x^2 + y^2)^2 = (x^2 - y^2)$.

This task will use polar coordinates to set up an integral for one-fourth of its arc length.

(a) Begin by showing that the polar-coordinate equation for this lemniscate is $r^2 = \cos(2\theta)$.

(b) Solving the equation in part (a) explicitly for r gives us $r = \pm\sqrt{\cos(2\theta)}$. For which values of θ on the interval $[0, 2\pi]$ does this equation produce real values for the radius r? Choose several of these θ values, and use them to draw a careful sketch of the curve. Remember that each specific value of θ_0 corresponds to two signed radii, one in the positive direction and the second in the negative direction along the line $\theta = \theta_0$.

[7] The semi-axis of the lemniscate in Figure 1 is the radius of the circle in which the lemniscate is inscribed.

[8] When the curve is given by the rectangular-coordinate equation $y = f(x)$, we can also write this formula in terms of the derivative $\frac{dy}{dx}$: $s = \displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

Task 6 - continued

(c) Now complete the following steps to set up an integral in dr for the arc length of one quarter of the lemniscate.

Step 1 Use the formula (**) from Task 5 for the arc length of a curve given in polar coordinates to show that the integral $\int_0^{\pi/4} \frac{1}{r} d\theta$ gives the arc length of one quarter of the lemniscate.

Step 2 Use the polar equation of the lemniscate to explicitly write $d\theta$ in terms of r and dr.

Step 3 Now combine the results of Step 1 and Step 2, along with other appropriate substitutions, to re-write the integral from Step 1 as an integral in dr. Don't forget to change the limits of integration to radius values also!

(d) In part (c) of this task, you found an integral (in dr) for the arc length of one quarter of the lemniscate.

(i) Compare your final answer to part (c) with the integral for the arc length of one quarter of the unit circle that you found in Task 2(a). Describe at least one similarity and at least one difference.

(ii) Think about what we might do to evaluate your final integral from part (b). What difficulties do we run into?

3 The Leminscatic Sine Function

We are now ready to move ahead to the nineteenth-century, and Gauss' interest in the elliptic integral for the arc length of the lemniscate from Task 3. Between Bernoulli in the 1690's and Gauss' initial investigations, Leonhard Euler and other mathematicians continued to study both the lemniscate and the related elliptic integral for its arc length. From his notebooks, we know that Gauss was reading these works when he wrote his 51^{st} diary entry:

I have begun to investigate the elastic curve[9] depending on $\int (1-x^4)^{-1/2} dx$. January 8, 1796

At some later (unknown) date, Gauss crossed out the word 'elastic' in this entry and wrote in the word 'lemniscatic' in its place. In fact, a major motivation for Gauss in his study of the integral itself was the analogy that he saw between:

- the integral $\displaystyle\int_0^t \frac{dx}{\sqrt{1-x^4}}$ and its relation to the lemniscate; and
- the integral $\displaystyle\int_0^t \frac{dx}{\sqrt{1-x^2}}$ and its relation to the unit circle.

[9] This is the curve that Bernoulli called the 'elastica'.

For example, recall from Task 2 that $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$. Now take a look at the following excerpt from a note[10] written by Gauss which he entitled *Elegantiores Integralis* $\int \frac{dx}{\sqrt{1-x^4}}$ *Proprietates [Very Excellent Properties of the Integral* $\int \frac{dt}{\sqrt{1-t^4}}$*]* [Gauss, 1876a].

We always denote[11] the value of this integral $\left[\int \frac{dx}{\sqrt{1-x^4}}\right]$ from $x = 0$ to $x = 1$ by $\frac{1}{2}\varpi$. We denote the variable x [when considered] with respect to [this] integral by the symbol *sin lemn*, but [when considered] with respect to the complement of [this] integral to $\frac{1}{2}\varpi$, by *cos lemn*. Therefore, whenever

$$\text{sin lemn}\left(\int \frac{dx}{\sqrt{1-x^4}}\right) = x, \qquad \text{cos lemn}\left(\frac{1}{2}\varpi - \int \frac{dx}{\sqrt{1-x^4}}\right) = x$$

The variable x can be considered as the radius vector of the curve, but also as the integral for the arc of the curve; the true curve will be that which is called [a] lemniscate.

Task 7 Write a few sentences to describe what you think Gauss is doing in this excerpt.
How does it relate to ideas about sine/arcsine in Section 1?
How does it relate to the work you completed on the lemniscate curve in Section 2?

Task 8 Notice that one thing happening in this excerpt is the use of an integral to define a "new" function. This is how some of today's calculus textbooks define certain familiar transcendental functions. This task looks at two such functions.

(a) Define the function g by $g(t) = \int_1^t \frac{1}{x}dx$. To what familiar function is g equal?

(b) Now consider the integral from Task 3 which we used to define $f(t) = \int_0^t \frac{1}{\sqrt{1-x^2}}dx$.
To what familiar function is f equal?

The next task examines some properties of the lemniscatic sine function that Gauss defined by way of a definite integral, in exactly the same way that we defined the two familiar functions in the previous task. To see more clearly that this is what Gauss was doing, let's first modernize his notation a bit by explicitly writing the limits of integration that he only implicitly assumed, and also by clearly distinguishing the dummy variable from the actual variable in the integral that defines the lemniscatic sine. It will also be helpful in what follows to remind ourselves that we have used the *polar coordinate* equation of the

[10] All Latin translations used in the project were provided by George W. Heine III, Math and Maps (`gheine@mathnmaps.com`). The author takes full responsibility for any errors introduced while transferring those translations to this project.

[11] Recall from footnote 2 that the symbol Gauss used here is a variant of the Greek letter π which is called "varpi."

lemniscate to set up its arc length integral in terms dr, where r is a radius. Choosing our dummy variable name with this in mind, we re-state Guass's displayed equation from the previous excerpt as follows:

$$\sin \text{lemn}\left(\int_0^t \frac{dr}{\sqrt{1-r^4}}\right) = t, \qquad \cos \text{lemn}\left(\frac{1}{2}\varpi - \int_0^t \frac{dr}{\sqrt{1-r^4}}\right) = t$$

Task 9 Define the function $a = h(t)$ by setting $h(t) = \displaystyle\int_0^t \frac{1}{\sqrt{1-r^4}} dr$.

For $t \geq 0$, notice that a is the arc length of the segment of the lemniscate[12] between the two points on that curve that correspond to the radii values $r = 0$ and $r = t$ in the polar-coordinate equation of the lemniscate. For $t < 0$, this definite integral will have a negative value. *(Do you see why?)* This integral thus gives us a *signed arc length* value for $a = h(t)$.

(a) What does the first sentence of the excerpt on page 7 tell us about the value for $h(1)$? Then use that value and the ideas discussed in the introduction to this task to state a value for $h(-1)$.

(b) What is the domain of h? What is its range? Explain your reasoning.

(c) Use the Fundamental Theorem of Calculus to show that $h'(t) > 0$ on its domain. Use this fact to explain why h must have an inverse function h^{-1}.

(d) What is the domain of h^{-1}? What is its range?

(e) Since $a = h(t)$, we know that $h^{-1}(a) = t$, where t is the upper limit of integration in the given integral. Fill in the blank below with the notation that Gauss used to denote this particular inverse function; why do you think he chose this particular notation?

$$t = \underline{\hspace{3cm}}(a)$$

Task 10 Notice that the two equalities in the displayed equation above gives us two different expressions for the variable t. *(For Gauss, this was the variable 'x'.)*

Equating these two expressions and setting $a = \int_0^t \frac{dr}{\sqrt{1-r^4}}$ (as we did in Task 9) gives us the following identity for the lemniscatic sine and cosine:

$$\sin \text{lemn}\,(a) = \cos \text{lemn}\left(\frac{1}{2}\varpi - a\right)$$

Why did Gauss use the constant $\frac{\varpi}{2}$ to define the lemniscatic cosine here? Compare this to the way in which the constant $\frac{\pi}{2}$ appears in the analogous identity for the (circular) sine and cosine.

[12]This is exactly analogous to what happens with the unit circle, where the integral $f(t) = \displaystyle\int_0^t \frac{1}{\sqrt{1-x^2}} dx$ literally computes the arc length of a segment of the circle. In fact, this is why the function $f(t) = \arcsin(t)$ is called the arcsine function!

To determine the two points on the circle that bound the arc length segment associated with the arcsine function, take a look at the limits of integration of the integral $\int_0^t \frac{1}{\sqrt{1-x^2}} dx$. As you do this, remember that the arcsine integral is based on the rectangular-coordinate equation of the circle ($x^2 + y^2 = 1$). This means that the variable t does not correspond to a radius in this case. Instead, as we saw in Task 3(d), setting $a = f(t) = \int_0^t \frac{1}{\sqrt{1-x^2}} dx$ gives us $t = \sin(a)$. Sketching a diagram in which $t = \sin a$ is represented as a length, along with the associated arc length a given by $\int_0^t \frac{1}{\sqrt{1-x^2}} dx$, is an interesting exercise in the geometry of the unit circle. Check your understanding of that geometry and the arcsine integral by trying it!

Gauss himself went on to develop many quite elegant properties of the lemniscatic sine and cosine, including formulas for sums and differences that had already been discovered by Euler before him, as well as power series representations. We close this section of the project with a very short list of these taken from [Gauss, 1876a].

$\mathsf{sin\ lemn}(-a) = -\ \mathsf{sin\ lemn}(a),$	$\mathsf{cos\ lemn}(-a) = \mathsf{cos\ lemn}(a)$
$\mathsf{sin\ lemn}\ k\varpi = 0$	$\mathsf{sin\ lemn}\ (k+1/2)\varpi = \pm 1$
$\mathsf{cos\ lemn}\ k\varpi = \pm 1$	$\mathsf{cos\ lemn}\ (k+1/2)\varpi = 0$

Task 11 How do these identities for the lemniscatic sine and cosine compare[13] to identities that you know for the circular sine and cosine?

Task 12 In the excerpt from Gauss on page 7, he stated "The variable x can be considered as the radius vector of the curve ... which is called [a] lemniscate ..." This statement becomes less cryptic when we take the introductory comments in Task 9 into account. Recall in particular that the function $a = h(t)$ in that task is defined as an *arc length* integral in terms dr, where r is a *radius*:

$$\text{sin lemn}\left(\underbrace{\int_0^t \frac{dr}{\sqrt{1-r^4}}}_{a=h(t)}\right) = t$$

Since the limits of integration (0 and t) in this integral are *radius* values, this means that sin lemn(a) is actually the (signed) radius value that corresponds to a (polar coordinate) point P on the lemniscate, while a is the (signed) arc length of the lemniscate from the origin $(0, 0)$ to point P.

(a) Use this idea and the graph of the lemniscate to justify the three identities involving sin lemn in the preceding excerpt.

(b) Use the identity in Task 10 to justify the three identities involving cos lemn.

Hint: For the sin lemn identities involving ϖ, it will be helpful to remember the connection between ϖ and the arc length of the lemniscate!

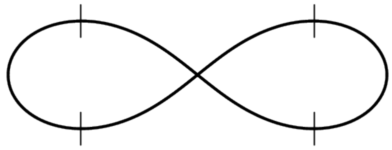

Figure 2: Sketch of the lemniscate from Gauss' notebook, from [Gauss, 1876b]

[13]To make this comparison even stronger, remember that the unit circle has the special property that the radian measure of an angle equals the *arc length* of the segment of the circle that corresponds to that angle. This follows from the formula $s = r\theta$ for a circle of radius r, where θ represents the radian measure of a central angle of a circle of radius r.

4 Why *is* 1.19814023473559220744... such a beautiful number?

The complete answer to this question requires a bit more mathematics than we will study in this short project. In this closing section, we summarize just the highlights. We begin by asking a question that most likely already occurred to you: what *is* the numerical value of ϖ? Or, in other words, what is the numerical value of $2\int_0^1 \frac{1}{\sqrt{1-t^4}}dt$?

Stuck on how to evaluate this numerically? If so, then you're in good company! Evaluating integrals of the form $\int_0^x \frac{1}{\sqrt{1-t^n}}dt$ for $n > 2$ is notoriously difficult. Bernoulli, Euler, and others all used power series techniques to estimate such values, which was in turn quite a challenge due to the slow rate of convergence of the series in question. Using a technique for speeding up the convergence rate developed by James Stirling (1692–1770), along with his astounding computational ability, Gauss succeeded in calculating ϖ to twenty decimal places, finding that

$$\varpi = 2.62205755429211981046\ldots.$$

Gauss also used his computational prowess to similar advantage in his work on a type of average called "the arithmetic-geometric mean of two numbers," which he denoted by $M(a,b)$. His computed value of $M(\sqrt{2},1) = 1.19814023473559220744$ joins the value of $\pi = 3.14159265358979323846\ldots$ and that of ϖ (given above) to give the trio of constants mentioned in the introduction of this project. Take a look at the numerical values of these three constants and see if you can find a relationship between them. *(Pull out your calculator and pause in your reading for a minute to try this!)*

Here is what Gauss had to say when he discovered the rather unexpected relationship between these three constants:

We have established that the arithmetic-geometric mean between 1 and $\sqrt{2}$ is π/ϖ to 11 places; the proof of this fact will certainly open up a new field of analysis.

May 30, 1799

It took Gauss another year to fully prove that his guesswork about the numerical relationship $M(\sqrt{2},1) = \frac{\pi}{\varpi}$ was correct. The 'new field of analysis' that opened up in connection with this proof led him well beyond the study of elliptic functions of a single real-valued variable, and into the realm of functions of several complex-valued variables. Today, a special class of such functions known as the 'theta functions' provides a powerful tool that is used in a wide range of applications throughout mathematics — providing yet one more piece of evidence of Gauss' extraordinary ability as a mathematician and a guesswork genius!

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Notes to Instructors

This mini-Primary Source Project (mini-PSP) is one of a set of four mini-PSPs designed to consolidate student proficiency of the following traditional topics from a first-year Calculus course:[14]

- Gaussian Guesswork: Arc Length and the Numerical Approximation of Integrals
- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution
- Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve
- Gaussian Guesswork: Sequences and the Arithmetic-Geometric Mean

Each of the four mini-PSPs can be used either alone or in conjunction with any of the other three. All four are based on excerpts from Gauss' mathematical diary [Gauss, 2005] and related primary texts that will introduce students to the power of numerical experimentation via the story of his discovery of a relationship between three particular numbers: the ratio of the circumference of a circle to its diameter (π), a specific value (ϖ) of the elliptic integral $u = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^4}}$; and the Arithmetic-Geometric Mean of 1 and $\sqrt{2}$. Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice referred to as "Gaussian Guesswork" by historian Adrian Rice in his *Math Horizons* article subtitled "Why 1.19814023473559220744... is such a beautiful number" [Rice, November 2009].

The primary content goals of this particular mini-PSP are to consolidate students' understanding of integration by substitutions, and to extend their understanding of arc length to the polar coordinate case.

Student Prerequisites

Given the project's first content goal, familiarity with integration by substitution is thus naturally required (but used only in Task 5), and familiarity with the arcsine integral $\int_0^t \frac{1}{\sqrt{1-x^2}}dx$ is also needed for the tasks in Section 1 which lay the groundwork for the latter part of the project. In light of the project's second goal, it is assumed that students have had some basic introduction to polar coordinates, and that they are familiar with the integral for arc length for a function given in rectangular coordinates. There are also two tasks (Task 3 and Task 9) that assume familiarity with the Fundamental Theorem of Calculus and basic facts about invertible functions (e.g., If f is strictly increasing on I, then f is 1-to-1 and, therefore, invertible on I.). Instructors who choose not to complete Section 3 could (and likely should) omit both these two tasks.

PSP Design, and Task Commentary

Following a brief introduction to set the historical stage, Section 1 sets the mathematical stage by examining the integral for arc length of a circle (Task 2), and how it can be used to define the arcsine function by setting $f(t) = \displaystyle\int_0^t \frac{1}{\sqrt{1-x^2}}dx$ (Task 3). Using the history of the lemniscate curve as a backdrop, Section 2 then turns to the problem of setting up the arc length integral for a lemniscate — a problem that is best handled via polar coordinates. Students first transform the general arc length integral formula from rectangular to polar coordinates in Task 5. Task 6 then leads them through the steps of showing that

[14]As of June 2019, the first of these four mini-PSPs is not yet completed.

the arc length of one-quarter of the lemniscate defined by $r^2 = \cos(2\theta)$ is given by $\int_0^1 \frac{1}{\sqrt{1-r^4}}dr$. This task concludes by having students compare this result with the integral they obtained for the arc length of one quarter of the unit circle, and to think about what the difficulties involved in evaluating the lemniscate integral.

Section 3 turns to an examination of Gauss' introduction of a non-elementary function, the lemniscatic sine, which he defined as the inverse of the non-elementary function that is itself defined as the arc length integral $f(t) = \displaystyle\int_0^t \frac{1}{\sqrt{1-x^4}}dx$. The lemniscatic cosine function is then defined by an analogous relationship with the circular trigonometric functions. Although these particular non-elementary functions themselves are not part of the standard Calculus II curriculum, they provide an excellent opportunity for Calculus II students to apply and consolidate core concepts and techniques, and to witness their interplay within the context of some amazingly beautiful, and important, mathematics!

The short conclusion in Section 4 that ties the mathematical details of this particular project to the larger story of Gauss' amazing discovery does so in a way that is independent of the material in Section 3. Instructors with more limited class time could thus choose to omit Section 3 in its entirety.

Suggestions for Classroom Implementation

Classroom implementation of this and other mini-PSPs in the collection may be accomplished through individually assigned work, small-group work and/or whole-class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project.

To reap the full mathematical benefits offered by the PSP approach, students should be required to read assigned sections in advance of in-class work, and to work through primary source excerpts together in small groups in class. The author's method of ensuring that advance reading takes place is to require student completion of "Reading Guides" (or "Entrance Tickets"). These Reading Guides typically include "Classroom Preparation" exercises (drawn from the PSP Tasks) for students to complete prior to arriving in class; they may also include "Discussion Questions" that ask students only to read a given task and jot down some notes in preparation for class work. On occasion, tasks are also assigned as follow-up to a prior class discussion. In addition to supporting students' advance preparation efforts, these guides provide helpful feedback to the instructor about individual and whole-class understanding of the material. The author's students receive credit for completion of each Reading Guide (with no penalty for errors in solutions). Sample guides (based on the Advanced Preparation Work suggested in the Sample Implementation Schedule given below) are appended to the end of these Notes.

Sample Implementation Schedule (based on a 75-minute class period)

- **Advance Preparation Work for Day 1** (to be completed before class): Read pages 1–5, completing Tasks 1, 2 and 4 for class discussion along the way, per the sample Reading Guide on pages 15–17.
- **Class Work for Day 1**
  - Brief whole group or small-group comparison of answers to Tasks 1, 2 and 4.
  - Small-group work on Tasks 3, 5 and 6 (supplemented by whole-class discussion as deemed appropriate by the instructor). Note that Tasks 5 & 6 form the core of the material on polar coordinate arc length in this mini-PSP.

- Time permitting, individual or small-group reading of pages 6–7, perhaps beginning Task 7–8.

- **Day 1 Homework (optional)**: A complete formal write-up of student work on Tasks 3, 5 and/or 6 could also be assigned, to be due at a later date (e.g., one week after completion of the in-class work). As noted earlier, instructors with more limited class time could choose to omit Section 3 in its entirety; in this case, students should also be assigned to read Section 4 (pages 9–10) as follow-up to their Day 1 work.

- **Advance Preparation Work for Day 2** (to be completed before class, assuming Section 3 will also be implemented in class): Read pages 6–7, including the introduction to Task 9, completing Tasks 7 and 8 for class discussion along the way, per the sample Reading Guide on pages 16–17 below. As desired, the concluding section (pages 9–10) could also be assigned for reading, since students do not need to complete all tasks in the penultimate section in order to understand the concluding section.

- **Class Work for Day 2**
  - Brief whole group or small-group comparison of answers to Task 7; answers to Task 8 could also be discussed, or simply reviewed by the instructor following class.
  - Small-group work on Tasks 9–12 (supplemented by whole-class discussion as deemed appropriate by the instructor). Tasks 11 and 12 could also be done as individual homework outside of class, if the instructor prefers not to spend an entire second class period on the project.

- **Follow-up Assignment** (to be completed prior to the next class period): As needed, complete work on Tasks 9–12; read Section 4 (pages 9–10). This assignment could also be made part of a *Reading Guide*, and scored for completeness only.

- **Day 2 Homework (optional)**: A complete formal write-up of student work on Tasks 9 and 12 could also be assigned, to be due at a later date (e.g., one week after completion of the in-class work). Footnote 11 (page 8) also contains an optional task related to the arcsine integral that could be assigned as homework.

LaTeX code of the entire PSP is available from the author by request to facilitate preparation of reading guides or 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Connections to other Primary Source Projects

Links to all available "Gaussian Guesswork" mini-PSPs (described earlier in these Notes) are as follows:

- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution
`https://digitalcommons.ursinus.edu/triumphs_calculus/8/`
- Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve
`https://digitalcommons.ursinus.edu/triumphs_calculus/3/`
- Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean
`https://digitalcommons.ursinus.edu/triumphs_calculus/2/`

Additional mini-PSPs intended for use in a Calculus II course include the following; the PSP author name (and general content focus) for each is given.

- How to Calculate π: Buffon's Needle - Calculus version, Dominic Klyve (integration by parts)
  `https://digitalcommons.ursinus.edu/triumphs_calculus/7/`
- How to Calculate π: Machin's Inverse Tangents, Dominic Klyve (infinite series)
  `https://digitalcommons.ursinus.edu/triumphs_calculus/6/`
- Euler's Calculation of the Sum of the Reciprocals of Squares, Kenneth M. Monks (infinite series)
  `https://digitalcommons.ursinus.edu/triumphs_calculus/9/`

The following projects based on primary sources are also available for use in other courses in the standard calculus sequence. The content focus of each is indicated in the PSP title. The first three projects listed are mini-PSPs that can be completed in 1–2 class days; the fourth is a full-length PSP that requires approximately 2 full weeks for implementation.

Calculus I

- The Derivatives of the Sine and Cosine Functions, Dominic Klyve
  `https://digitalcommons.ursinus.edu/triumphs_calculus/1/`
- Fermat's Method for Finding Maxima and Minima, Kenneth M. Monks
  `https://digitalcommons.ursinus.edu/triumphs_calculus/11/`

Multivariable or Vector Calculus

- Braess' Paradox in City Planning: An Application of Multivariable Optimization, Kenneth M. Monks
  `https://digitalcommons.ursinus.edu/triumphs_calculus/10/`
- The Radius of Curvature According to Christiaan Huygens, Jerry Lodder
  `https://digitalcommons.ursinus.edu/triumphs_calculus/4/`

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For more information about the NSF-funded project TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), visit:

`https://blogs.ursinus.edu/triumphs/`

APPENDIX

This appendix provides two 'Sample Reading Guides' that illustrates th author's method for assigning advance preparation work in connection with classroom implementation of primary source projects. More detail concerning these guides is included in the subsection "Suggestions for Classroom Implementation" of the Notes to Instructors for this project.

SAMPLE READING GUIDE: Advance Preparation Work for Day 1

Background Information: The goal of the advance reading and tasks assigned in this 3-page reading guide is to prepare students for in-class small-group work on Tasks 5–6, which address arc length of the lemniscate in polar coordinates, by first laying the groundwork by having them consider arc length of the unit circle in rectangular coordinates (Tasks 2–3).

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Reading Assignment: *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve*, pp. 1–5

1. Read the introduction on page 1.
   Any questions or comments?

2. In Section 1, read pages 1–2, stopping at Task 1.
   Any questions or comments?

3. **Class Prep** Complete **Task 1** from page 2 here:

   Task 1 Sketch a circle and the corresponding triangle described in Archimedes' Proposition 1 (top of page 2), and label both using appropriate variables. Then use area formulas to verify that the relationship stated in this proposition is correct.

4. Continue reading the rest of page 2.
   Any questions or comments?

5. **Class Prep** Complete **Task 2** from page 2 here:

Task 2

(a) Use the equation $x^2 + y^2 = 1$ of the unit circle to set up and simplify the integral for the arc length of one fourth of this circle.

(b) Use the fact that we know the circumference of the unit circle, along with your result from part (a), to explain why $\pi = 2\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$. (You could verify this by integrating, if you are familiar with this integral — but there's no need for integration here at all!)

(c) We can re-write the integral in part (b) as $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$. Interpret this integral in terms of the area under a curve. Sketch the graph of the function that defines that curve, and shade the region associated with the area given by the integral on your sketch. Is the region in question bounded or unbounded? Do you find this surprising? Explain why or why not.

6. SKIP Task 3 for now, and begin reading Section 2, pages 3–4, stopping at Task 4.
   Any questions or comments?

7. **Class Prep** Complete **Task 4** from page 4 here:

   > **Task 4** Notice that Bernoulli, like other geometers of his time, did not use exponential notation to denote the products 'xx' or 'aa'. Rewrite the equation for the lemniscate that he gave (in the excerpt at the bottom of page 3) using x^2 and a^2 to represent these products. Then square both sides of that equation to eliminate the radical expression. How does the degree of your final equation relate to Bernoulli's description of the curve?

8. Continue your reading of Section 2, pages 4–5, stopping at Task 5
   Any questions or comments?

SAMPLE READING GUIDE: Advance Preparation Work for Day 2

Background Information: The goal of the advance reading and tasks assigned in this 2-page reading guide are to prepare students for in-class small-group work on Tasks 9–12.

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Reading Assignment: *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve*, pp. 6–7

1. In Section 3, read pages 6–7, stopping at Task 7.
   Any questions or comments?

2. **Class Prep** Complete **Task 7** from page 7 here:

   Task 7
   Write a few sentences to describe what you think Gauss is doing in the excerpt at the top of p. 7.
   How does it relate to ideas about sine/arcsine in Section 1?
   How does it relate to the work you completed on the lemniscate curve in Section 2?

3. **Class Prep** Complete **Task 8** from page 7 here:

   Task 8
   Notice that one thing happening in this excerpt is the use of an integral to define a "new" function. This is how some of today's calculus textbooks today define certain familiar transcendental functions. This task looks at two such functions.

   (a) Define the function g by $g(t) = \int_1^t \frac{1}{x} dx$. To what familiar function is g equal?

   (b) Now consider the integral from Task 3 which we used to define $f(t) = \int_1^t \frac{1}{\sqrt{1-x^2}} dx$.
   To what familiar function is f equal?

4. Continue your reading of Section 3 by reading the rest of page 7.

   Write at least one question or comment about this part of the reading.

5. **Class Prep** Complete the following in Task 9 at the top of page 8.

   - Read the introduction to this task (above part (a)).
     Write at least one question or comment about this part of the reading.

   - Complete part (a) of Task 9.
     Begin by writing the two questions that are asked in Task 9(a) here; then give your answers to both questions.